NETWORKS WITH VERY SMALL AND VERY LARGE PARASITICS:
NATURAL FREQUENCIES AND STABILITY

by

C. A. Desoer and M. J. Shensa

Memorandum No. ERL-M276

13 April 1970

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

April 13, 1970
Revised July 6, 1970

NETWORKS WITH VERY SMALL AND VERY LARGE PARASITICS: NATURAL FREQUENCIES AND STABILITY

C. A. DESOER and M. J. SHENSA

Department of Electrical Engineering and Computer Sciences
Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

This paper considers a nonlinear time-invariant network $\mathcal{N}$ (of order $n + h + \ell$) which contains, in addition to the usual elements, h stray elements (stray capacitances and lead inductances) and ℓ sluggish elements (chokes and coupling capacitors). We prove that the asymptotic stability of any equilibrium point of $\mathcal{N}$ is guaranteed once the simplified (i.e. with stray and sluggish elements neglected) linearized network, and two other linear networks S_H and S_L are asymptotically stable. The networks S_H and S_L are obtained by both a physically intuitive argument and by a rigorous one. We also prove that if any one of the three linear networks is exponentially unstable, then the equilibrium point of $\mathcal{N}$ is unstable. Thus our theory explains the commonly occurring fact that $\mathcal{N}$ is unstable even though the simplified linearized network is asymptotically stable. An example illustrates the several possibilities. In the proof we obtain the asymptotic behavior of the natural frequencies of $\mathcal{N}$ (valid in a neighborhood of the equilibrium point). In Appendix II, we show how the natural modes of the three simple networks are related to those of the given network $\mathcal{N}$.

Received ____________________________

Research sponsored by the National Science Foundation, Grant GK-10656X and a National Science Foundation Fellowship.

NETWORKS WITH VERY SMALL AND VERY LARGE PARASITICS: NATURAL FREQUENCIES AND STABILITY

I. INTRODUCTION

This paper deals, in a circuit theoretic context, with a basic problem of System Theory. It is usual that in a study of a given physical system one considers several models depending on the problem considered. For example, in amplifier design one uses low-frequency, mid-frequency, and high-frequency models in the design for sinusoidal-steady state specifications. For this problem, the relations between the models is pretty straightforward. When other problems are considered (say, transient behavior, stability, etc.), it is known that sometimes the simplified model gives completely erroneous answers. In this paper we focus our attention on the stability of equilibrium points and we obtain conditions under which the simplified model will give correct (or totally erroneous) predictions. This paper extends by algebraic methods the results of [1].

To be specific we consider a lumped network so that we benefit from the established conceptual framework and terminology of circuit theory [2]. Let this network consist of nonlinear time-invariant elements and of no independent sources; therefore, it is usually described by an autonomous system of differential equations. In addition to the usual elements, we introduce in the analysis small elements such as stray capacitances and lead inductances: we refer to these elements as <u>stray elements</u> and assume that their values are proportional to a <u>small</u> positive number ε. This number ε indicates the degree of smallness of the stray elements. To neglect the stray elements in the analysis amounts to setting $\varepsilon = 0$. We know intuitively that the stray elements affect mostly the "high frequency" behavior of the network. At the other end of the scale, there are elements like chokes and coupling capacitors that have very large values and affect mostly the "low frequency" behavior of the network. We call them the <u>sluggish elements</u> and assume that their values are proportional to a <u>large</u> positive number μ. This number μ determines the degree of bigness of the sluggish elements. To neglect these elements amounts to setting $\mu = \infty$. Given any pair of values for ε and μ, we denote the corresponding network by $\mathcal{N}_{\varepsilon\mu}$. We call $\mathcal{N}_{0\infty}$ the <u>simplified</u> network, i.e. the network obtained by neglecting the stray and sluggish elements.

In this paper we investigate the relation between the behavior of $\mathcal{N}_{\varepsilon\mu}$ and $\mathcal{N}_{0\infty}$ about equilibrium points when $\varepsilon \ll 1$ and $\mu \gg 1$. We give a complete description of the behavior of the natural frequencies of the <u>small-signal equivalent circuit</u> $S_{\varepsilon\mu}$ in terms of those of $S_{0\infty}$ and of two auxiliary linear networks S_H and S_L. In particular, for small ε and large μ, we partition the set of natural frequencies of $S_{\varepsilon\mu}$ into three sets related to the sets of natural frequencies of three simpler networks ($S_{0\infty}$, S_H, S_L). We obtain also the asymptotic behavior of the natural frequencies of $S_{\varepsilon\mu}$ in terms of those of these three networks. The first result states that for small ε and for large μ, $S_{\varepsilon\mu}$ is asymptotically stable if <u>all</u> three simpler networks are asymptotically stable. The second result says that if <u>any</u> of the three simpler networks is unstable so is the network $S_{\varepsilon\mu}$ for small ε and large μ. The results are illustrated by a simple example (see sec. IV) which illustrates the several possibilities. The reader might find it helpful to use this example to illustrate step by step the general discussion which follows.

II. ANALYSIS

We assume that the equations of $\mathcal{N}_{\varepsilon\mu}$ can be written in the form:

$$\begin{aligned} \dot{x} &= f_1(x, y, z) \\ \varepsilon\dot{y} &= f_2(x, y, z) \\ \mu\dot{z} &= f_3(x, y, z) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^h$, $z(t) \in \mathbb{R}^\ell$; ε and μ are positive numbers, typically $\varepsilon \ll 1$ and $\mu \gg 1$. General conditions under which such equations can be written are given in the literature, see for example the review paper [3]. We assume that f_1, f_2, f_3 are defined and twice continuously differentiable on an appropriate open set of $\mathbb{R}^n \times \mathbb{R}^h \times \mathbb{R}^\ell$. The components of $x(t)$ are the state variables associated with the simplified network $\mathcal{N}_{0\infty}$, those of $y(t)$ ($z(t)$, resp.) are associated with the stray (sluggish, resp.) elements.

$\mathcal{N}_{\varepsilon\mu}$ may have one or more equilibrium points, let $P = (0,0,0)^T$ be one of them. Then except for critical cases the stability of $\mathcal{N}_{\varepsilon\mu}$ about this equilibrium point P can be completely decided on the basis of the linearized equations [4]

$$\begin{bmatrix} \dot{x} \\ \varepsilon\dot{y} \\ \mu\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{1H} & A_{1L} \\ A_{H1} & A_{HH} & A_{HL} \\ A_{L1} & A_{LH} & A_{LL} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{matrix} (2.1) \\ (2.2) \\ (2.3) \end{matrix}$$

where, in particular, A_{11}, A_{HH} and A_{LL} are constant square matrices of dimension n, h and ℓ, respectively. These (linearized) equations represent the network equations of the small-signal equivalent circuit (about equilibrium point P) which we denote by $S_{\varepsilon\mu}$. It is a well known fact that if all natural frequencies of S are in the open left half-plane then the equilibrium point P of $\mathcal{N}_{\varepsilon\mu}$ is asymptotically stable. Furthermore if any one or more of the natural frequencies of $S_{\varepsilon\mu}$ is in the open right half-plane, then the equilibrium point $\underline{P}$ of $_{\varepsilon\mu}$ is unstable [4]. The set of natural frequencies of $S_{\varepsilon\mu}$ is precisely the set of zeros of the polynomial $\Delta(\lambda,\varepsilon,\mu^{-1})$ defined by

$$\Delta(\lambda,\varepsilon,1/\mu) = \det \begin{bmatrix} A_{11} - \lambda I & A_{1H} & A_{1L} \\ A_{H1} & A_{HH} - \varepsilon\lambda I & A_{HL} \\ \frac{1}{\mu}A_{L1} & \frac{1}{\mu}A_{LH} & \frac{1}{\mu}A_{LL} - \lambda I \end{bmatrix} \qquad (3)$$

Δ is a polynomial in λ of degree $n+h+\ell$ and whose coefficients are polynomials in ε and $1/\mu$. In particular, the leading term is $(-1)^{n+h+\ell}\,\varepsilon^h\,\lambda^{n+h+\ell}$. It is well known that as long as the polynomial Δ has $n+h+\ell$ distinct λ-roots, then each root is locally a holomorphic function of ε and of $1/\mu$ [5,6]. In the general case, when for a fixed ε and $1/\mu$, say, ε_0 and $1/\mu_0$, Δ has a k-multiple λ-root, say λ_0 then, given any $r > 0$, there exists an $\varepsilon_r > 0$ and a $\mu_r < \infty$ such that $|\varepsilon - \varepsilon_0| < \varepsilon_r$ and $|\frac{1}{\mu} - \frac{1}{\mu_0}| < \frac{1}{\mu_r}$ imply that $\Delta(\lambda,\varepsilon,1/\mu)$ has k zeros counting multiplicities in the disc $|\lambda - \lambda_0| < r$. [9, p. 13-14; also 6, Theor. 9.17.4 and Probl. 4 p. 245].

Consider now the simplified network $S_{0\infty}$ in which both the sluggish and the stray elements are neglected. First, let $\mu \to \infty$ in Eq. (3). The last row of the determinant becomes $[0 \vdots 0 \vdots - \lambda I]$. Second, as $\varepsilon \to 0$, h zeros of Δ go to infinity, since the degree of the polynomial Δ drops from $n+h+\ell$ to $n+\ell$ [7]. Thus, the set of natural frequencies of $S_{0\infty}$ is the set of zeros of the polynomial $\Delta_M(\lambda)$:

$$\Delta_M(\lambda) = \lim_{\varepsilon \to o} \lim_{\mu \to \infty} (-\lambda)^{-\ell} \Delta(\lambda,\varepsilon,1/\mu) = \det \begin{bmatrix} A_{11}-\lambda I & A_{1H} \\ A_{H1} & A_{HH} \end{bmatrix} \tag{4}$$

This shows in particular that $S_{0\infty}$ is a network of order n.

The physical interpretation of this reduction is the following: as $\mu \to \infty$, the magnitude of the natural frequencies associated with the sluggish elements get smaller and smaller; in the limit, the corresponding modes become constant; hence they do not affect the natural frequencies of $S_{\varepsilon\infty}$, and, a fortiori, those of $S_{0\infty}$. As $\varepsilon \to 0$, the natural frequencies associated with the strays get larger and larger; if <u>all</u> eigenvalues of A_{HH} are in the <u>open left</u> half-plane, then, as $\varepsilon \to 0$, the solution of

$$\varepsilon \dot{y} = A_{H1} x + A_{HH} y,$$

expressed in terms of $x(\cdot)$, namely,

$$y(t) = \left[\exp A_{HH}\, t\varepsilon^{-1}\right] y(0) + \int_0^t \left[\exp A_{HH}\tau\varepsilon^{-1}\right] A_{H1}\, x(t-\tau)\varepsilon^{-1} d\tau$$

can easily be shown to tend to $-A_{HH}^{-1}\ A_{H1}\ x(t)$ on (o,∞) [8]. Thus provided that A_{HH} has all its eigenvalues in the open left half-plane, the relation between y(t) and x(t) is obtained by setting $\varepsilon\dot{y}$ to zero in (2.2). Physically, this means setting to zero all the currents through the stray capacitors and all the voltages across the stray inductors. Thus we are again led to the conclusion that λ is a natural frequency of $S_{0\infty}$ if and only if $\Delta_M(\lambda) = 0$.

The nonlinear time-invariant network $\mathcal{N}_{\varepsilon\mu}$ (described by (1)) is approximately represented, about its equilibrium point P, by a <u>linear</u> network $S_{\varepsilon\mu}$ (described by (2)). Call S_H the <u>high-frequency</u> approximation to $S_{\varepsilon\mu}$: physically it is obvious that it is obtained from $S_{\varepsilon\mu}$ by open-circuiting all inductors except the stray inductors and by short-circuiting all capacitors except the stray capacitors. S_H is a network of order h. Since, with respect to the short time-constant modes of S_H, the other state variables (x and z) are essentially constant, the natural frequencies of S_H are completely determined by A_{HH}. This can also be seen mathematically by the following change of variables: $t = \varepsilon\tau$. The τ scale is a stretched-out time-scale which emphasizes the high frequencies. After the change of variables we have

$$\begin{cases} \dfrac{dx}{d\tau} = \varepsilon A_{11}x + \varepsilon A_{1H}y + \varepsilon A_{1L}z \\[2ex] \dfrac{dy}{d\tau} = A_{H1}x + A_{HH}y + A_{HL}z \\[2ex] \dfrac{dz}{d\tau} = \dfrac{\varepsilon}{\mu} A_{L1}x + \dfrac{\varepsilon}{\mu} A_{LH}y + \dfrac{\varepsilon}{\mu} A_{LL}z \end{cases}$$

Now as $\varepsilon \to 0$, the first and last equation show that, in the limit, x and z are constants. Therefore, the second equation shows that, in the τ-scale, the natural frequencies of S_H are the eigenvalues of A_{HH}.

Call S_L the low-frequency approximation to $S_{\varepsilon\mu}$: physically it is obvious that it will be obtained from $S_{\varepsilon\mu}$ by short-circuiting all inductors except the sluggish inductors and open-circuiting all capacitors except the sluggish capacitors. Thus S_L is a network of order ℓ whose equations are obtained by deleting $\dot{x}$ and $\dot{y}$ from the equations (2). Indeed $\dot{x}$ and $\dot{y}$ represent the currents (the voltages, resp.) through the non sluggish capacitors (across the non sluggish inductors, resp.). Mathematically this can be checked by changing the time scale according to $t = \mu t'$: since μ is large, this will emphasize the large time-constants i.e. the low frequencies. After the change of variables we have

$$\begin{cases} \dfrac{1}{\mu}\dfrac{dx}{dt'} = A_{11}x + A_{1H}y + A_{1L}z \\ \dfrac{\varepsilon}{\mu}\dfrac{dy}{dt'} = A_{H1}x + A_{HH}y + A_{HL}z \\ \dfrac{dz}{dt'} = A_{L1}x + A_{LH}y + A_{LL}z \end{cases}$$

As $\mu \to \infty$, we see that the derivatives of x and y drop out and, therefore, the natural frequencies of S_L (in the new time scale) are the zeros of

$$\Delta_L(\lambda) = \det \begin{bmatrix} A_{11} & A_{1H} & A_{1L} \\ A_{H1} & A_{HH} & A_{HL} \\ A_{L1} & A_{LH} & A_{LL}-\lambda I \end{bmatrix} \tag{5}$$

In summary, we state the following:

$S_{0\infty}$, which can be thought of as the mid frequency approximation to $S_{\varepsilon\mu}$ and which is obtained by neglecting all stray and sluggish elements, is characterized by

$$S_{0\infty} \begin{cases} \dot{x} = A_{11}x + A_{1H}y \\ 0 = A_{H1}x + A_{HH}y \end{cases} \tag{6}$$

S_H is the high-frequency approximation to $S_{\varepsilon\mu}$, it is obtained by neglecting all normal and sluggish energy-storing elements, the equation of S_H is

$$S_H: \quad \varepsilon\dot{y} = A_{HH}y \tag{7}$$

S_L is the low frequency approximation to $S_{\varepsilon\mu}$; it is obtained by neglecting all normal and sluggish energy-storing elements; the equations of S_L are

$$S_L \begin{cases} 0 = A_{11}x + A_{1H}y + A_{1L}z \\ 0 = A_{H1}x + A_{HH}y + A_{HL}z \\ \mu\dot{z} = A_{L1}x + A_{LH}y + A_{LL}z \end{cases} \tag{8}$$

III. MAIN RESULTS

It turns out that the asymptotic stability of the equilibrium point P of $\mathcal{N}_{\varepsilon\mu}$ can be ascertained by checking the asymptotic stability of $S_{0\infty}$, S_H and S_L. Also P may be unstable even though $S_{0\infty}$ is asymptotically stable!

This is made precise in the following two theorems.

Theorem I.

Suppose that the equations of $\mathcal{N}_{\varepsilon\mu}$ can be put in normal form valid for a neighborhood of the equilibrium point P, as in (1). If the three linear networks $S_{0\infty}$, S_H and S_L (of respective order n, h and ℓ) are asymptotically stable, then there is an $\varepsilon_0 > 0$ and a $\mu_0 > 0$ such that the equilibrium point P of the autonomous nonlinear network $\mathcal{N}_{\varepsilon\mu}$ is asymptotically stable for any $\varepsilon \in [0,\varepsilon_0]$ and any $\mu \in [\mu_0,\infty]$.

Theorem II.

Suppose that the equations of $\mathcal{N}_{\varepsilon\mu}$ can be put in normal form valid for a neighborhood of the equilibrium point P, as in (1). If one or more of the three linear networks $S_{0\infty}$, S_H and S_L has one or more natural frequencies in the open right half-plane then there is an $\varepsilon_0 > 0$ and a $\mu_0 > 0$ such that the equilibrium point P of $\mathcal{N}_{\varepsilon\mu}$ is unstable for any $\varepsilon \in [0,\varepsilon_0]$ and any $\mu \in [\mu_0,\infty]$.

Theorem I says that for small ε and large μ, the asymptotic stability of $S_{\varepsilon\mu}$ (which is of order $n+h+\ell$) is guaranteed once it is shown that $S_{0\infty}$, S_H and S_L (which are of order n, h, ℓ, respectively) are asymptotically stable. Theorem II says that if any one of the three is exponentially unstable so is $S_{\varepsilon\mu}$ for small ε and large μ. The theorems are proved in Appendix I. The asymptotic behavior of the natural frequencies of $S_{\varepsilon\mu}$ are given in the proof. In Appendix II, we exhibit the relation between the natural modes of $S_{0\infty}$, S_L, S_H and the corresponding ones of $S_{\varepsilon\mu}$.

IV. EXAMPLE

Instabilities due to stray elements (high-frequency singing) and due to sluggish elements (mororboating) are a frequent experience to circuit designers. Examples of this sort are usually complicated and to make them intelligible require lots of detailed explanations. So, to avoid burdening the reader with a complicated example we present a very simple example which illustrates all the features we wish to exhibit. Let the small-signal equivalent circuit about the equilibrium point P be the third order circuit $S_{\varepsilon\mu}$ shown in Fig. 1(a). The simplified circuit $S_{o\infty}$, the high-frequency approximate circuit S_H and the low-frequency approximate circuit S_L are shown in Fig. 1(b), (c) and (d). A little calculation shows that the equations of $S_{\varepsilon\mu}$ are

$$\begin{cases} \dot{x} = -\frac{R}{R+1}x - y + \frac{1}{R+1}z \\ \varepsilon\dot{y} = x - \frac{1}{r}y \\ \mu\dot{z} = -\frac{1}{R+1}x - \frac{1}{R+1}z \end{cases} \tag{9}$$

The equations for $S_{o\infty}$ are: (see Fig. 1b)

$$\begin{cases} \dot{x} = -\frac{R}{R+1}x - y \\ 0 = x - \frac{1}{r}y \end{cases} \tag{10}$$

The equation for S_H is (see Fig. 1c)

$$\varepsilon\dot{y} = -\frac{1}{r}y \tag{11}$$

The equations for S_L are: (see Fig. 1d)

$$\begin{cases} 0 = -\frac{R}{R+1}x - y + \frac{1}{R+1}z \\ 0 = x - \frac{1}{r}y \\ \mu\dot{z} = -\frac{1}{R+1}x - \frac{1}{R+1}z. \end{cases} \qquad (12)$$

Observe that (10), (11), (12) can be obtained directly from (9) by respectively, (a) setting $z = 0$ and $\varepsilon = 0$, (b) setting $x = 0$ and $z = 0$, (c) setting $\dot{x} = 0$ and $\dot{z} = 0$. This corresponds precisely to the physical approximations done in Sec. II above.

The table below exhibits three sets of values of r and R for which $S_{\varepsilon\mu}$ is unstable even though the simplified network $S_{o\infty}$ is stable in each case. The instability is due to high frequency instability or to low frequency instability or to both.

r	R	$S_{\varepsilon\mu}$	$S_{o\infty}$	S_H	S_L
-.5	2	unstable	stable	unstable	stable
-.5	-2	unstable	stable	unstable	unstable
.5	-2	unstable	stable	stable	unstable

CONCLUSION

From a practical point of view, Theorem II is probably more useful than Theorem I: because once it is determined that either S_H or S_L is exponentially unstable, then no matter how small the parasitics can be made, $\mathcal{N}_{\varepsilon\mu}$ will be unstable; therefore a design change is required. In some cases S_H or S_L have purely imaginary natural frequencies, it is possible that suitable refinements of our theory may lead to some useful conclusions for the designer. Extensions to the time-varying case are possible using Desoer's results [10].

ACKNOWLEDGEMENT

The authors benefitted greatly from the comments of Professor Leon Chua on a preliminary version of the paper. A reviewer contributed comments which streamlined the proofs.

APPENDIX I

Proof of Theorem I.

It is well known that if all the natural frequencies of $S_{\varepsilon\mu}$ are in the open left half-plane, then the equilibrium point P of $\mathcal{N}_{\varepsilon\mu}$ is asymptotically stable [4]. To prove that all natural frequencies of $S_{\varepsilon\mu}$ are in the open left half-plane we establish three assertions.

Assertion I: for ε small and μ large, n natural frequencies of $S_{\varepsilon\mu}$ are close to those of $S_{0\infty}$. Let λ_M be any natural frequency of $S_{0\infty}$, so $\mathrm{Re}\lambda_M < 0$ since $S_{0\infty}$ is asymptotically stable. Also λ_M is a zero of the polynomial $\Delta_M(\lambda)$ and of the polynomial $\Delta(\lambda,0,0)$. By the continuity result referred to above, if k is the multiplicity of λ_M as a zero of $\Delta(\lambda,0,0)$, then for any $r > 0$ with $r < |\mathrm{Re}\lambda_M|$, there exist an $\varepsilon_M > 0$ and a $\mu_M > 0$ such that $\varepsilon \in [0,\varepsilon_M]$ and $\mu \in [\mu_M,\infty]$ imply that $\Delta(\lambda,\varepsilon,1/\mu)$ has k zeros (counting multiplicities) in the disc centered at λ_M and of radius r. Note also that for the allowed values of ε and μ, all these k natural frequencies of $S_{\varepsilon\mu}$ are in the open left half plane. By applying successively this reasoning to all natural frequencies of $S_{0\infty}$ we prove that Assertion I is true.

Assertion II: for ε small and μ large, h natural frequencies of $S_{\varepsilon\mu}$ are close to λ_H/ε, where λ_H denotes any natural frequency of S_H with ε set equal to one. More precisely, if λ_H is any eigenvalue of A_{HH} which is of multiplicity k as a zero of $\det[\lambda I - A_{HH}]$, then, for ε small and μ large $S_{\varepsilon\mu}$ has k natural frequencies (counting multiplicities) close to λ_H/ε.

Note that $\mathrm{Re}\lambda_H < 0$ by asymptotic stability.

Let $\lambda = \xi/\varepsilon$ in the right hand side of (3) and multiply the first and last block of rows by ε, then we conclude from (3) that λ is a natural frequency of $S_{\varepsilon\mu}$ if and only if

$$p(\xi,\varepsilon,\tfrac{\varepsilon}{\mu}) = \det \begin{bmatrix} \varepsilon A_{11}-\xi I & \varepsilon A_{1H} & \varepsilon A_{1L} \\ A_{H1} & A_{HH}-\xi I & A_{HL} \\ \frac{\varepsilon}{\mu} A_{L1} & \frac{\varepsilon}{\mu} A_{LH} & \frac{\varepsilon}{\mu} A_{LL}-\xi I \end{bmatrix} = 0 \tag{A1}$$

For $\varepsilon = 0$, we have

$$p(\xi,0,0) = (-\xi)^{n+\ell} \det (A_{HH}-\xi I) \tag{A1a}$$

Any λ_H is a zero of $p(\xi,0,0)$; suppose that the λ_H we consider is of multiplicity k. Since $p(\xi,\varepsilon,\varepsilon/\mu)$ is a holomorphic function of ξ, ε and ε/μ, we apply again the quoted result to assert that for any $r > 0$ with $r < |\mathrm{Re}\lambda_H|$, there exist an $\varepsilon_H > 0$ and $\mu_H > 0$ such that $\varepsilon \in [0,\varepsilon_H]$ and $\mu \in [\mu_H,\infty]$ imply that $p(\xi,\varepsilon,\varepsilon/\mu)$ has k zeros (counting multiplicities) in a disc of radius r centered on λ_H. Equivalently, $\Delta(\lambda,\varepsilon,1/\mu)$ has k zeros satisfying $|\lambda-(\lambda_H/\varepsilon)| < r/\varepsilon$. Considering successively all eigenvalues of A_{HH}, Assertion II is proved. Note that all these h natural frequencies are in the open left half-plane, provided ε and $\frac{1}{\mu}$ are small, indeed the asymptotic stability of S_H implies that all the λ_H are in the open left half-plane.

Assertion III: for ε small and μ large, ℓ natural frequencies of $S_{\varepsilon\mu}$ are close to λ_L/μ, where λ_L is any natural frequency of S_L with μ set equal to one. More precisely, if λ_L is any natural frequency of S_L, which is a zero of mulitplicity k of $\Delta_L(\lambda)$, then $S_{\varepsilon\mu}$ has k natural frequencies which are close to λ_L/μ. Note that $\text{Re}\lambda_L < 0$ by asymptotic stability.

Let $\eta = \mu\lambda$ in the right hand side of (3) and multiply the last row of blocks by μ, then from (3) we conclude that λ is a natural frequency of $S_{\varepsilon\mu}$ if and only if

$$q(\eta,\frac{1}{\mu},\frac{\varepsilon}{\mu}) = \det \begin{bmatrix} A_{11}-\frac{1}{\mu}\eta I & A_{1H} & A_{1L} \\ A_{H1} & A_{HH}-\frac{\varepsilon}{\mu}\eta I & A_{HL} \\ A_{L1} & A_{LH} & A_{LL}-\eta I \end{bmatrix} = 0 \qquad \text{(A2)}$$

From (5) and (A2) we note that the set of λ_L's is equal to the set of zeros of $q(\eta,0,0)$. Consider any specific λ_L and let k be its multiplicity. Again q is a holomorphic function of η, $\frac{1}{\mu}$, $\frac{\varepsilon}{\mu}$. Hence we can assert that for any given $r > 0$ with $r < |\text{Re}\lambda_L|$, there exist $\varepsilon_L > 0$ and $\mu_L > 0$ such that $\varepsilon \in [0,\varepsilon_L]$ and $\mu \in [\mu_L,\infty]$ imply that k zeros of $q(\eta, \frac{1}{\mu}, \frac{\varepsilon}{\mu})$ are in a disc centered on λ_L and of radius r. Equivalently, $\Delta(\lambda,\varepsilon,1/\mu)$ has k zeros satisfying $|\lambda-(\lambda_L/\mu)| < r/\mu$. Considering successively all zeros of $\Delta_L(\lambda)$, Assertion III follows. Note that all these ℓ natural frequencies are in the open left half-plane provided ε is small and μ is large, indeed the assumed asymptotic stability of S_L implies that all the λ_L's are in the open left half-plane.

Now observe that the three assertions describe a partition of the natural frequencies of $S_{\varepsilon\mu}$ in three sets of respective size n, h and ℓ. For ε small and μ large, these three sets are disjoint since the first one is bounded and bounded away from zero, the second one recedes to infinity and the third one tends to zero. Therefore all $n+h+\ell$ natural frequencies of $S_{\varepsilon\mu}$ are accounted for and the assumptions on $S_{0\infty}$, S_H and S_H imply the asymptotic stability of the equilibrium point P of $\mathcal{N}_{\varepsilon\mu}$.

Q.E.D.

Proof of Theorem II.

It is well known that if one or more natural frequency of $S_{\varepsilon\mu}$ is in the <u>open right</u> half-plane, then the equilibrium point P of the nonlinear network $\mathcal{N}_{\varepsilon\mu}$ is unstable [4]. Theorem II follows from this fact and the three assertions above.

APPENDIX II: NATURAL MODES

Intuitively one would expect that the natural modes of $S_{0\infty}$, S_H and S_L are closely related to those of $S_{\varepsilon\mu}$ for ε small and μ large. We propose to exhibit this relationship for the case where $S_{0\infty}$, S_H and S_L have, respectively, n, h and ℓ distinct natural frequencies. In fact we shall calculate the leading term of the asymptotic expansion of the modes of $S_{\varepsilon\mu}$.

We say that the (constant) vector $(\bar{x},\bar{y},\bar{z}) \in \mathbb{C}^{n+h+\ell}$ represents a natural mode of $S_{\varepsilon\mu}$ with natural frequency λ iff the function $t \mapsto (\bar{x},\bar{y},\bar{z})^T \exp(\lambda t)$ is a solution of eq. (2) (superscript T denotes "transpose", and $\mathbb{C}$ denotes the complex plane). With suitable modifications, this concept applies to $S_{0\infty}$, S_H, S_L.

I. Mid-frequency modes. Let λ_M be any natural frequency of $S_{0\infty}$ and let $(x_M,y_M)^T$ be the corresponding natural mode; thus

$$\begin{bmatrix} A_{11}-\lambda_M I & A_{1H} \\ A_{H1} & A_{HH} \end{bmatrix} \begin{bmatrix} x_M \\ y_M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A3}$$

Let λ'_M be the zero of $\Delta(\lambda,\varepsilon,1/\mu)$ close to λ_M. By the implicit function theorem [9, p. 14; 6, Theor. 10.2.4] we have

$$\lambda'_M = \lambda_M + v(\varepsilon,1/\mu) \tag{A4}$$

where v is holomorphic in a neighborhood of (0,0), $v(0,0) = 0$, and $v(\varepsilon,1/\mu) = O(\varepsilon,1/\mu)$. The leading term of the expansion of the natural mode of

$S_{\varepsilon\mu}$ corresponding to λ_M' is given by $(x_M, y_M, z_M)^T$ exp $(\lambda_M' t)$ where

$$z_M = \frac{1}{\mu\lambda_M'} (A_{L1} x_M + A_{LH} y_M) \tag{A5}$$

Note that $\|z_M\| = O(1/\mu)$. To check this natural mode substitute it into (2) and obtain

$$\begin{bmatrix} A_{11}-\lambda_M' I & A_{1H} & A_{1L} \\ A_{H1} & A_{HH}-\varepsilon\lambda_M' I & A_{HL} \\ A_{L1} & A_{LH} & A_{LL}-\mu\lambda_M' I \end{bmatrix} \begin{bmatrix} x_M \\ y_M \\ z_M \end{bmatrix} = \begin{bmatrix} O(\varepsilon, 1/\mu) + O(1/\mu) \\ \varepsilon O(\varepsilon, 1/\mu) + O(1/\mu) \\ O(1/\mu) \end{bmatrix}$$

The right hand side consists of terms of first and higher order in ε and $1/\mu$. Interpretation: when $S_{0\infty}$ is changed to $S_{\varepsilon\mu}$, λ_M changes to λ_M' according to (A4), and the motion starts affecting the sluggish elements' in the order $O(1/\mu)$ as specified by (A5).

II. High frequency modes. Let λ_H be any natural frequency of S_H with ε set equal to one; λ_H is an eigen value of A_{HH} (see (7)); call y_M the corresponding (normalized) eigenvector of A_{HH}. λ_H is also a simple zero of $p(\xi,0,0)$, and the zero of $p(\xi,\varepsilon,1/\mu)$ close to λ_H is of the form (see (A1))

$$\lambda_M + \phi(\varepsilon, \varepsilon/\mu)$$

where ϕ is holomorphic, $\phi(0,0) = 0$ and $\phi(\varepsilon,\varepsilon/\mu) = O(\varepsilon,\varepsilon/\mu)$. Denormalizing the time (and frequency) scale we have

$$\lambda'_H = \frac{\lambda_H}{\varepsilon} + r(\varepsilon,1/\mu)$$

where $r(\varepsilon,1/\mu)$ is holomorphic but $r(0,0) = O(1)$, uniformly for $\mu \geq 1$.

We claim that the leading term of the expansion of the mode of $S_{\varepsilon\mu}$ corresponding to λ'_H is given by $(x_H,y_H,z_H)^T \exp(\lambda'_H t)$ where

$$x_H \triangleq \frac{1}{\lambda'_H} A_{1H}\, y_H \quad (\text{hence, } \|x_H\| = O(\varepsilon))$$

$$z_H \triangleq \frac{1}{\mu\lambda'_H} A_{LH}\, y_H \quad (\text{hence, } \|y_H\| = O(\varepsilon/\mu))$$

By substituting into (2) we observe that

$$\begin{bmatrix} A_{11}-\lambda'_H I & A_{1H} & A_{1L} \\ A_{H1} & A_{HH}-\varepsilon\lambda'_H I & A_{HL} \\ A_{L1} & A_{LH} & A_{LL}-\mu\lambda'_H I \end{bmatrix} \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix} = \begin{bmatrix} O(\varepsilon) + O(\varepsilon/\mu) \\ O(\varepsilon) + O(\varepsilon/\mu) \\ O(\varepsilon) + O(\varepsilon/\mu) \end{bmatrix}$$

Thus the right hand side $\to 0$ as $\varepsilon \to 0$, uniformly in μ for $\mu \geq 1$. In conclusion, when the normal elements and the sluggish elements are brought into the analysis, the natural frequency shifts from λ_H/ε to $(\lambda_H/\varepsilon) + r(\varepsilon,1/\mu)$, and the motion affects the normal elements in the order $O(\varepsilon)$ and the sluggish ones in the order $O(\varepsilon/\mu)$.

III. <u>Low frequency modes</u>. Let λ_L be any natural frequency of S_L with μ set equal to one; λ_L is a zero of $\Delta_L(\lambda)$ and call $(x_L,y_L,z_L)^T$ the (normalized) solution of the homogeneous equations whose matrix is given by (5) with $\lambda = \lambda_L$. λ_L is also a zero of $q(\eta,0,0)$ (see (A2)); so the zero of $q(\eta,1/\mu,\varepsilon/\mu)$ close to λ_L is of the form

$$\lambda_L + \psi(1/\mu,\varepsilon/\mu)$$

where ψ is holomorphic, $\psi(0,0) = 0$ and $\psi(1/\mu,\varepsilon/\mu) = 0(1/\mu,\varepsilon/\mu)$. Denormalizing the time (and frequency) scale we have

$$\lambda_L' = \frac{\lambda_L}{\mu} + s(1/\mu,\varepsilon/\mu)$$

where s is holomorphic but $s(1/\mu,\varepsilon/\mu) = 0(1/\mu^2)$, uniformly for $\varepsilon \leq 1$. We claim that the leading term of the expansion of the mode of $S_{\varepsilon\mu}$ corresponding to λ_L' is given by $(x_L,y_L,z_L)^T \exp(\lambda_L' t)$. Indeed, direct substitution into (2) gives

$$\begin{bmatrix} A_{11}-\lambda_L'I & A_{1H} & A_{1L} \\ A_{H1} & A_{HH}-\varepsilon\lambda_L'I & A_{HL} \\ A_{L1} & A_{LH} & A_{LL}-\mu\lambda_L'I \end{bmatrix} \begin{bmatrix} x_L \\ y_L \\ z_L \end{bmatrix} = \begin{bmatrix} 0(1/\mu) \\ 0(\varepsilon/\mu) \\ 0(1/\mu) \end{bmatrix}$$

In this case the leading terms of the components of the mode are not affected.

As a final comment, it should be stressed that the only advantage that accrued from the assumption of having distinct natural frequencies is that the functions v, ϕ, ψ were holomorphic and that typically they were $0(\varepsilon) + 0(1/\mu)$, i.e. their leading terms are linear in ε and $1/\mu$. In case of multiple natural frequencies these functions have usually algebraic singularities and tend to zero as $\varepsilon \to 0$ and $1/\mu \to 0$. However the rate at which they do so may be much slower, e.g. they may behave as $\sqrt{\varepsilon}$.

REFERENCES

[1] M. J. Shensa, "Parasitics and the Stability of Equilibrium Points of Nonlinear Networks," Paper submitted to the IEEE Trans. on Circuit Theory.

[2] C. A. Desoer and E. S. Kuh, Basic Circuit Theory, McGraw Hill Book Co., New York, 1970.

[3] E. S. Kuh and R. A. Rohrer, "The State Variable Approach to Network Analysis," Proc. IEEE, 53, 7, p. 672-686, July 1965.

[4] W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath Mathematical Monographs, D. C. Heath and Co., Boston, 1965. [p. 68-69, p. 75]

[5] E. Hille, Analytic Function Theory, Volume I and II, Ginn and Co., New York, 1959. (Chapt. 9, in particular)

[6] J. Dieudonné, Foundation of Modern Analysis, Academic Press, 1960. (Theor. 10.2.4)

[7] E. Hille, Ibid. (Sec. 12.2)

[8] C. A. Desoer, "Singular Perturbation and Bounded-Input Bounded-State Stability," Electronics Letters (in press).

[9] R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall, Englewood Cliffs, N. J., 1965.

[10] C. A. Desoer, "Slowly-varying System $\dot{x} = A(t)x$," Trans. IEEE, AC-14, 6, p. 780-781, December 1969.

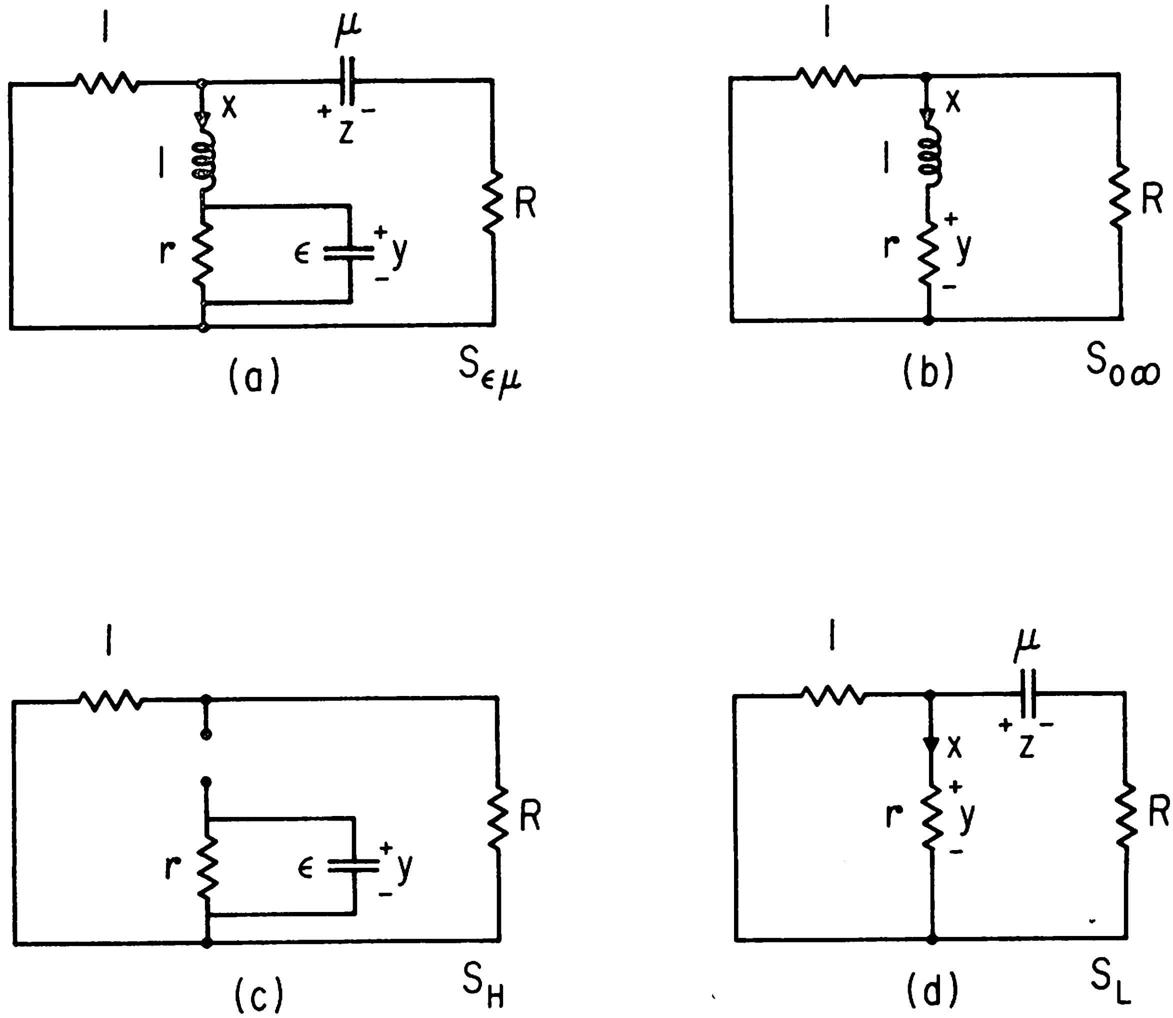

FIGURE CAPTIONS

All elements are in ohms, henrys and farads. Fig. (1a) shows the linear network $S_{\varepsilon\mu}$: the capacitor of ε F is the stray capacitor and the capacitor μ F is the sluggish one. Fig. (1b) shows $S_{0\infty}$ where both the stray and sluggish elements have been neglected. Fig. (1c) shows S_H where all but the stray elements have been neglected. Fig. (1d) shows S_L where all but the sluggish elements have been neglected.