REPRESENTATION OF MARTINGALES, QUADRATIC VARIATION AND APPLICATIONS

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1. Introduction

Let $\{X_t, t \geq 0\}$ be a sample-continuous second-order martingale. Then $\{X_t^2, t \geq 0\}$ is a sample-continuous first-order submartingale and the conditions for Meyers' decomposition[1] are always satisfied so that we can write

$$X_t^2 = M_t + A_t, \quad t \geq 0 \tag{1}$$

where M is a martingale, A is an increasing process, and both are sample-continuous. The decomposition is unique if we set $M_0 = X_0^2$. Following Kunita and Watanabe[2], we shall adopt the suggestive notation $\langle X \rangle_t$ for A_t.

In this paper we present two related results. First, we shall obtain a sufficient condition under which a second-order sample-continuous martingale can be represented as a stochastic integral in terms of a Brownian motion. Secondly, we shall show that if X and Y are sample-continuous second-order martingales (not necessarily with respect to the same family

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of σ-algebras) and if either $X + Y$ or $X - Y$ is almost surely of bounded variation then $\langle X \rangle_t = \langle Y \rangle_t$. This rather simple result has some surprising consequences.

2. Martingales and Stochastic Integrals

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, and let $\{\mathcal{A}_t, t \geq 0\}$ be an increasing family of sub-σ-algebras. A process $\{X_t, t \geq 0\}$ is said to be adapted to $\{\mathcal{A}_t\}$ if for each t X_t is $\mathcal{A}_t$-measurable. We say that $\{X_t, \mathcal{A}_t\}$ is a martingale if X is adapted to $\{\mathcal{A}_t\}$ and for every $t > s$

$$E^{\mathcal{A}_s} X_t = X_s \tag{2}$$

almost surely. If $\{X_t, \mathcal{A}_t\}$ is a sample-continuous second-order martingale then the increasing process $\langle X \rangle_t$ introduced earlier is well-defined and $E \langle X \rangle_t < \infty$.

If $\{W_t, \mathcal{A}_t\}$ is a sample-continuous second-order martingale such that for $t > s$

$$E^{\mathcal{A}_s}(W_t - W_s)^2 = t - s \tag{3}$$

then W is necessarily a Brownian motion and for each s $\{W_t - W_s, t \geq s\}$ is independent of $\mathcal{A}_s$ [3, p. 384]. We describe this situation by saying that $\{W_t, \mathcal{A}_t\}$ is a Brownian motion. Let $\{W_t, \mathcal{A}_t\}$ be a Brownian motion and let $\{\phi_t, t \geq 0\}$ be a measurable process adapted to $\{\mathcal{A}_t\}$ such that

$$\int_0^t E\, \phi_s^2\, ds < \infty \tag{4}$$

for each t. The stochastic integral $\int_0^t \phi_s \, dW_s$ is well-defined as the quadratic limit of a sequence of sums $\sum_\nu \phi_{t_\nu^{(n)}} \left[W_{t_{\nu+1}^{(n)}} - W_{t_\nu^{(n)}} \right]$, where $\{t_\nu^{(n)}\}$ is a sequence of partitions of [0, t] such that

$$\max_\nu \left(t_{\nu+1}^{(n)} - t_\nu^{(n)} \right) \xrightarrow[n \to \infty]{} 0$$

If we define

(5) $$X_t = \int_0^t \phi_s \, dW_s$$

and choose a separable version for X then $\{X_t, \mathcal{A}_t\}$ is a second-order sample-continuous martingale, with

(6) $$E(X_t - X_s)^2 = \int_s^t E^{\mathcal{A}_s} \phi_\tau^2 \, d\tau \quad , \; 0 \leq s \leq t$$

If, instead of (4), ϕ merely satisfies

(7) $$\int_0^t \phi_s^2 \, ds < \infty \quad , \text{ a.s.}$$

then $\int_0^t \phi_s \, dW_s$ can be defined as follows: Let $\tau_n(\omega)$ be defined by

(8) $$\tau_n(\omega) = \begin{cases} \inf \left(t : \int_0^t \phi_s^2(\omega) \, ds \geq n \right) \\ \infty \text{ if } \int_0^t \phi_s^2(\omega) \, ds < n \text{ for all } t \end{cases}$$

and set

$$\text{(9)} \qquad \phi_{ns}(\omega) = \phi_s(\omega) \qquad s \leq \tau_n(\omega)$$
$$= 0 \qquad s > \tau_n(\omega)$$

For each n $\int_0^t \phi_{ns}\, dW_s$ is well-defined. It can be shown that $\int_0^t \phi_{ns}\, dW_s$ converges in probability as $n \to \infty$, and we define $\int_0^t \phi_s\, dW_s$ as this limit. Now, the process $X_t = \int_0^t \phi_s\, dW_s$ need no longer be second-order or a martingale, but it is still sample-continuous if a separable version is chosen. Moreover, if we denote min(t,s) by $t \wedge s$ then for each n $\{X_{t\wedge\tau_n}, \mathcal{A}_t\}$ is a second-order martingale. By definition X is a local martingale.[2]

If X is a stochastic integral of the form

$$\text{(10)} \qquad X_t = X_0 + \int_0^t \phi_s\, dW_s$$

and f is a twice continuously differentiable function of a real variable then Ito's differentiation formula[4] yields

$$\text{(11)} \qquad f(X_t) = f(X_0) + \int_0^t f'(X_s)\, \phi_s\, dW_s + \frac{1}{2}\int_0^t f''(X_s)\, \phi_s^2\, ds$$

In particular

$$\text{(12)} \qquad X_t^2 - X_0^2 = 2\int_0^t X_s\, \phi_s\, dW_s + \int_0^t \phi_s^2\, ds$$

If ϕ satisfies (4) then X is a second-order martingale. Furthermore, the term $2\int_0^t \phi_s X_s \, dW_s$ in (12) is also a martingale, though not necessarily second-order. It follows that (12) is in the form of the Meyer decomposition of X_t^2 with $X_0^2 + 2\int_0^t \phi_s X_s \, dW_s$ being the martingale term and with the increasing process given by

(13) $$\langle X \rangle_t = \int_0^t \phi_s^2 \, ds$$

If X is a martingale of the form (10) then we can define stochastic integral $\int_0^t \psi_s \, dX_s$ by

$$\int_0^t \psi_s \, dX_s = \int_0^t \psi_s \, \phi_s \, dW_s$$

provided that $\int_0^t \psi_s^2 \, \phi_s^2 \, ds < \infty$ almost surely. More generally, if $Z = X + Y$, where X is of the form (10) and $\{Y_t, \, t \geq 0\}$ is a process with sample functions almost surely of bounded variation, then we can define

$$\int_0^t \psi_s \, dZ_s = \int_0^t \psi_s \, dX_s + \int_0^t \psi_s \, dY_s$$

provided that the first integral exists as a stochastic integral and the second as a Stieltjes integral. If Y is also sample continuous and F is any twice continuously differentiable function, then Ito's differentiation rule is extended to read

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_s) \, dZ_s + \frac{1}{2} \int_0^t F''(Z_s) \phi_s^2 \, ds$$

We note that in particular

$$Z_t^2 - Z_0^2 - \int_0^t 2Z_s \, dZ_s = \int_0^t \phi_s^2 \, ds = \langle X \rangle_t$$

which is independent of Y.

3. Representation of Martingales

Not every sample-continuous second-order martingale can be represented as a stochastic integral in the form of (10). It is clear from (13) that for such a representation to be possible the increasing process $\langle X \rangle(\omega, t)$ must be an absolutely continuous function: of t (ω.r.t. the Lebesgue measure) for almost all ω . As Fisk has observed[5], this condition is also sufficient by virtue of a theorem of Doob[3, p. 449], but it may be necessary to enlarge the underlying probability space by the adjunction of a Brownian motion. Specifically, Doob proved the following:

Theorem 3.1 (Doob) Let $\{X_t, \mathcal{A}_t, 0 \leq t \leq T\}$ be a sample-continuous second-order martingale. Suppose that there exists a non-negative measurable process $\{\psi_t, 0 \leq t \leq T\}$ adopted to $\{\mathcal{A}_t\}$ such that for $t > s$

$$E^{\mathcal{A}_s}(X_t - X_s)^2 = \int_s^t E^{\mathcal{A}_s}\psi_\tau \, d\tau \tag{14}$$

If the set $\{(\omega,t): \psi(\omega,t) = 0\}$ has zero $d\mathcal{P}dt$ measure then there exists a Brownian motion. $\{W_t, \mathcal{A}_t, 0 \leq t \leq T\}$ such that

$$X_t = X_0 + \int_0^t \psi_s^{1/2} \, dW_s \tag{15}$$

with probability 1. Without the hypothesis that ψ vanishes almost nowhere, representation (15) is still valid with the adjunction of a Brownian motion to the probability space.

The condition that $\langle X \rangle$ be almost surely continuous with respect to the Lebesgue measure is both somewhat stringent and difficult to verify. Perhaps, it is more natural to consider representations of the form

$$X_t(\omega) = X_0(\omega) + \int_0^t \phi_s(\omega)\, dW_{F(s)}(\omega) \tag{16}$$

where W is a Brownian motion and F is an increasing function defined by

$$\begin{aligned} F(t) &= E(X_t - X_0)^2 \\ &= E\langle X \rangle_t \end{aligned} \tag{17}$$

If a representation of the form (10) exists then F is necessairly absolutely continuous with respect to the Lebesgue measure and a change of variable puts (10) into the form of (16). Of course, (16) may exist even when (10) does not. For example, if F is a continuous increasing function singular with respect to the Lebesgue measure and W is a Brownian motion then $W_{F(t)}$ has no representation of the form (10).

<u>Theorem 3.2</u> Let $\{X_t, \mathcal{A}_t, 0 \leq t \leq T\}$ be a sample-continuous second-order martingale. We assume that $\{\mathcal{A}_t\}$ is right continuous (i.e. $\bigcap_{s>\tau} \mathcal{A}_s = \mathcal{A}_t$) and each $\mathcal{A}_t$ is completed. A representation of the form (16) exists if and only if $\langle X \rangle$ is absolutely continuous with respect to F with probability 1.

proof: We only need to prove the theorem for the case $F(t) = t$, because $\{X_t, \mathcal{A}_t\}$ can be transformed into a sample-continuous second-order martingale $\{\tilde{X}_t, \tilde{\mathcal{A}}_t\}$ with $E\,\tilde{X}_t^2 = t$ by defining

$$F^{-1}(t) = \inf\{s: \ F(s) = t\}$$

$$\tilde{X}_t = X_{F^{-1}(t)}$$

$$\tilde{\mathcal{A}}_t = \mathcal{A}_{F^{-1}(t)}$$

Even though F^{-1} may be discontinuous $\{\tilde{\mathcal{A}}_t\}$ is right continuous and $\tilde{X}$ is still sample-continuous, because $F(t) = F(s)$ emplies $X_t = X_s$ almost surely. Since $X_t = \tilde{X}_{F(t)}$ with probability 1, a representation

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \phi_s \, dW_s$$

where $\{W_t, \tilde{\mathcal{A}}_t\}$ is a Brownian motion implies a representation.

$$X_t = \tilde{X}_{F(t)} = X_0 + \int_0^{F(t)} \tilde{\phi}_s \, dW_s$$

$$= X_0 + \int_0^t \phi_s \, dW_{F(s)}$$

which is just (16).

To prove theorem 3.2 for the case $F(t) = t$, we first note that necessity follows from (13). To prove sufficiency, we assume that $\langle X \rangle$ is absolutely continuous with probability 1 and write

$$\langle X \rangle_t(\omega) = \int_0^t \psi_s(\omega) \, ds$$

where ψ can always be chosen to be a measurable process because $\langle X \rangle$ is a measurable process. For each t ψ_t is measurable with respect to $\bigcap_{s>t} \mathcal{A}_s$ which is equal to $\mathcal{A}_t$ by assumption.

Because X is a martingale

$$E^{\mathcal{A}_s}(X_t - X_s)^2 = E^{\mathcal{A}_s}(X_t^2 - X_s^2)$$

It follows from the definition of $\langle X \rangle$ and (18) that

$$\begin{aligned} E^{\mathcal{A}_s}(X_t - X_s)^2 &= E^{\mathcal{A}_s}[\langle X \rangle_t - \langle X \rangle_s] \\ &= \int_s^t E^{\mathcal{A}_s} \psi_t \, d\tau \end{aligned}$$

so that condition (14) is satisfied. Applying theorem 3.1 completes the proof.

Theorem 3.2 is basically the same as theorem 2.1 of[5], except for the introduction of the increasing function F. We now come to the main result of the section, namely, a sufficient condition for the representation (16) that can be verified in terms of two-dimensional distributions of the martingale X.

<u>Theorem 3.3</u> Let $\{X_t, \mathcal{A}_t, 0 \leq t \leq T\}$ be a sample-continuous second-order martingale and let $F(t) = E(X_t - X_0)^2 = E\langle X \rangle_t$. Suppose that there exist finite positive constants α and β such that

$$\sup_{0 < F(t) - F(s) \leq \beta} \frac{E\,|X_t - X_s|^{2+2\alpha}}{[F(t) - F(s)]^{1+\alpha}} < \infty \tag{19}$$

then $\langle X \rangle$ is almost surely absolutely continuous with respect to the Lebesgue measure, and X has a representation of the form of (16).

proof: By virtue of the Lebesgue decomposition, we can always write

$$\langle X \rangle_t(\omega) = \int_0^t \psi_s(\omega)\, dF(s) + \mu_t(\omega) \tag{20}$$

where μ is almost surely singular with respect to F. Now, E ψ_s = 1 implies

$$E\,\mu_t = - \int_0^t dF(t) + E\,\langle X \rangle_t = 0$$

which in turn implies that $\mu \equiv 0$ almost surely since μ is non-negative and sample-continuous. Therefore, we only need to prove that (19) implies $E\,\psi_s = 1,\ 0 \le s \le T$.

Let $T_n = \{t_\nu^{(n)},\ \nu = 0, 1, \ldots, n\}$ be a sequence of nested (i.e., $T_{n+1} \supset T_n$) partitions of the interval [0, T] such that

$$\max_\nu \left[F\left(t_{\nu+1}^{(n)}\right) - F\left(t_\nu^{(n)}\right)\right] \xrightarrow[n \to \infty]{} 0$$

Define ψ_{nt}, $0 \le t \le T$, as follows:

$$\psi_{nt} = \frac{\langle X \rangle_{t_{\nu+1}^{(n)}} - \langle X \rangle_{t_\nu^{(n)}}}{F(t_{\nu+1}^{(n)}) - F(t_\nu^{(n)})}, \qquad t_\nu^{(n)} \le t < t_{\nu+1}^{(n)} \tag{21}$$

It is well known [see e.g., 3, pp. 346-347] that for each ω ψ_{nt} converges for almost all t (F-measure) to the Radon-Nikodym derivative of the absolutely continuous component of $\langle X \rangle$ with respect to F. That is , $\psi_{nt} \to \psi_t$

for almost all (ω, t). Since it is obvious that $E\psi_{nt} = 1$, the desired result $E\psi_t = 1$ will follow if for each t $\{\psi_{nt}\}$ is a uniformly integrable family of random variables.

Now, it is known[6] that for any $p > \frac{1}{2}$ there exists a constant κ_p such that

$$E\left|\langle X\rangle_t - \langle X\rangle_s\right|^p \leq \kappa_p E\left|X_t - X_s\right|^{2p}$$

Therefore, if we let N be the smallest n such that

$$\max_\nu [F(t_{\nu+1}^{(n)}) - F(t_\nu^{(n)})] \leq \beta$$

then

$$\sup_{n \geq N} E\psi_{nt}^{1+\alpha} \leq \kappa_{1+\alpha} \sup_{0 < F(t) - F(s) \leq \beta} \left\{\frac{E|X_t - X_s|^{2+2\alpha}}{[F(t) - F(s)]^{1+\alpha}}\right\} < \infty$$

so that $\{\psi_{nt}\}$ is a uniformly integrable family of random variables. This, together with theorem 3.2, complete the proof.

Theorem 3.2 is reminiscent of Kolmogorov's condition for sample continuity and has similar advantages, the primary one being that it can be verified in terms of the two-dimensional distributions of X.

4. Quadratic Variation

Let $T_n = \{t_\nu^{(n)}\}$ be a nested sequence of partitions of $[0, T]$ such that $\max_\nu (t_\nu^{(n)} - t_\nu^{(n)}) \xrightarrow[n \to \infty]{} 0$. Let $t \wedge s$ denote $\min(t, s)$. Fisk[5] has shown that the sequence of sums

(22) $$Q_n(t) = \sum_\nu \left[X_{t\,t_{\nu+1}^{(n)}} - X_{t\,t_\nu^{(n)}}\right]^2$$

converges to $\langle X \rangle_t$ in L^1 - mean, i.e.,

(23) $$E\left|Q_n(t) - \langle X \rangle_t\right| \xrightarrow[n \to \infty]{} 0$$

For this reason $\langle X \rangle_t$ is said to be the quadratic variation of X on [0, t]. Now, suppose that $\{Z_t,\ 0 \leq t \leq T\}$ is a sample-continuous process the sample functions of which are almost surely of bounded variation. Then there exists an almost surely finite random variable A such that

$$\sup_n \sum_\nu \left|Z_{t_{\nu+1}^{(n)}} - Z_{t_\nu^{(n)}}\right| \leq A$$

Therefore,

$$\sum_\nu \left|Z_{t_{\nu+1}^{(n)}} - Z_{t_\nu^{(n)}}\right|^2 \leq A \max_\nu \left|Z_{t_{\nu+1}^{(n)}} - Z_{t_\nu^{(n)}}\right| \xrightarrow[n \to \infty]{a.s.} 0$$

so that Z has zero quadratic variation on [0, T].

Theorem 4.1 Let $\{\mathcal{A}_t\}$ and $\{\tilde{\mathcal{A}}_t\}$ be two increasing families of σ-algebras and let $\{X_t, \mathcal{A}_t\}$ and $\{\tilde{X}_t, \tilde{\mathcal{A}}_t\}$ be sample-continuous second-order martingales. If $X_t + \tilde{X}_t$ or $X_t - \tilde{X}_t$ is of bounded variation then $\langle X \rangle_t = \langle \tilde{X} \rangle_t$ almost surely.

Proof: We can make use of the fact that a sample continuous process of bounded variation has zero quadratic variation and the inequality

$$\left(\sqrt{\sum_k a_k^2} - \sqrt{\sum_k b_k^2}\right)^2 \leq \sum_k \left(a_k \pm b_k\right)^2 \leq \left(\sqrt{\sum_k a_k^2} + \sqrt{\sum_k b_k^2}\right)^2$$

to show that the quadratic variation of X must be equal to that of $\tilde{X}$ Since $\langle X \rangle$ is the quadratic variation of X, this proves the theorem.

We should note that in theorem 4.1 we do not assume that X and $\tilde{X}$ are martingales with respect to the same family of σ-algebras. If they are then $X + \tilde{X}$ and $X - \tilde{X}$ are both martingales. If one of them is also of bounded variation, say $X + X$, then $X_t + \tilde{X}_t = X_0 + \tilde{X}_0$ with probability 1 for all $t \in [0, T]$. In which case the result of theorem 4.1 trivially follows. The more interesting cases arise when neither $X + \tilde{X}$ nor $X - \tilde{X}$ is a martingale.

An interesting application of theorem 4.1 is in connection with quasi-martingales[7]. A process $\{X_t, 0 \leq t \leq T\}$ is said to be a quasi-martingale with respect to $\{\mathcal{A}_t\}$ if there exist $\{B_t, 0 \leq t \leq T\}$ and $\{M_t, 0 \leq t \leq T\}$ both adapted to $\{\mathcal{A}_t\}$ such that $X = M + B$, B is of bounded variation, and $\{M_t, \mathcal{A}_t\}$ is a martingale. We shall be interested only in those cases where both M and B are sample continuous and where the total variation of B has a finite expectation. Under these assumptions if $\{X_t, \mathcal{A}_t\}$ is a quasi-martingale then $\{X_t, \mathcal{A}_{xt}\}$ is always a quasi-martingale, where $\mathcal{A}_{xt}$ denotes the σ-algebra generated by $\{X_s, 0 \leq s \leq t\}$. Of course, this statement is vacuous if $\mathcal{A}_t = \mathcal{A}_{xt}$. If not, then there exist $\tilde{B}$ and $\tilde{M}$, distinct from B and M, adapted to $\{\mathcal{A}_{xt}\}$ such that $X = \tilde{B} + \tilde{M}$, $\tilde{B}$ is of bounded variation, $\{\tilde{M}_t, \mathcal{A}_{xt}\}$ is a martingale, both $\tilde{M}$ and $\tilde{B}$ are sample-continuous, and the total variation of $\tilde{B}$ has finite expectations.

It follows from theorem 4.1 that if M and $\tilde{M}$ are second order then $\langle M \rangle_t =$ $\langle \tilde{M} \rangle_t$ for every t.

An important class of quasi-martingales is made up of Ito processes, which are processes having the representation

$$X_t = X_0 + \int_0^t \psi_s \, ds + \int_0^t \phi_s \, dW_s \quad , \ 0 \leq t \leq T \tag{24}$$

where ψ and ϕ are measurable processes adapted to an increasing family of σ-algebras $\{\mathcal{A}_t\}$, $\{W_t, \mathcal{A}_t\}$ is a Brownian motion, and X_0 is $\mathcal{A}_0$-measurable. In addition, we assume

$$\int_0^T E|\psi_s| \, ds \quad < \infty \tag{25}$$

$$\int_0^T E \, \phi_s^2 \, ds \quad < \infty \tag{26}$$

It is clear that $\{X_t, \mathcal{A}_t\}$ is a quasi-martingale. Thus, $\{X_t, \mathcal{A}_{xt}\}$ is also a quasi-martingale and the representation of X as a quasi-martingale with respect to $\{\mathcal{A}_t\}$ has the form

$$X_t = X_0 + \int_0^t (E^{\mathcal{A}_{xs}} \psi_s) \, ds + \tilde{M}_t \tag{27}$$

where $\{\tilde{M}_t, \mathcal{A}_{xt}\}$ is a martingale. Since the quadratic variation of the martingale term in (24) is $\int_0^t \phi_s^2 \, ds$, $\tilde{M}_t$ has a representation

$$\tilde{M}_t = \int_0^t |\phi_s| \, d\tilde{W}_s \tag{28}$$

where $\tilde{W}$ is a Brownian motion. Since

$$\langle \tilde{M} \rangle_t = \int_0^t \phi_s^2 \, ds \tag{29}$$

and $\tilde{M}$ is adapted to $\{\mathcal{A}_{xt}\}$, ϕ_t^2 is $\mathcal{A}_{xt}$-measurable for almost all t. These results can be summarized as follows:

<u>Theorem 4.2</u> Let $\{X_t, 0 \leq t \leq T\}$ be an Ito process satisfying (24) - (26). Then there exists a representation of X in the form

$$X_t = X_o + \int_0^t \tilde{\psi}_s(X_0^s) \, ds + \int_0^t \tilde{\phi}_s(X_0^s) \, d\tilde{W}_s \tag{30}$$

where X_0^s denotes $\{X_\tau, 0 \leq \tau \leq s\}$, $\tilde{W}$ is a Brownian motion, and for each t $\tilde{\psi}_t$ and $\tilde{\phi}_t$ are functionals on C[0, t] defined by

$$\tilde{\psi}_t(X_0^t(\omega)) = (E^{\mathcal{A}_{xt}} \psi_t) \; (\omega) \tag{31}$$

and

$$\tilde{\phi}_t(X_0^t(\omega)) = [\frac{d}{dt} \langle X \rangle_t]^{\frac{1}{2}} = |\phi_t(\omega)| \tag{32}$$

In (32) $\frac{d}{dt}$ stands for Radon-Nikodym derivative and $\langle X \rangle_t$ is defined by

(33) $$\langle X \rangle_t = X_t^2 - X_0^2 - 2 \int_0^t X_s \, dX_s$$

Theorem 4.2 can be viewed as a generalization of a result, due originally to Wonham[8] and termed the innovation theorem by Kailath[9], which corresponds to the case $\phi \equiv 1$. For that case we can write

$$\int_0^t [\psi_s - E^{\mathcal{A}_{xs}}\psi_s] \, ds = W_t + \tilde{W}_t$$

where both W and $\tilde{W}$ are Brownian motions.

The fact that ϕ_t^2 (or $|\phi_t|$) is $\mathcal{A}_{xt}$-measurable for almost all t is a simple observation based on considerations of quadratic variations. Yet, it has surprising consequences. For example, (24) is often used as a model for a system disturbed by a Gaussian white noise, with $\frac{dW_t}{dt}$ formally playing the role of the white noise. An incidental result of theorem 4.2 is that

$$E^{\mathcal{A}_{xt}}|\phi_t| = |\phi_t|$$

for almost all (ω, t), so that if ϕ never changes sign then it is completely recoverable from observing $\{X_s, 0 \le s \le t\}$. This clearly has implications for such models. Implications of quadratic variation for singular detection have already been observed in a previous paper[10].

Finally, we strongly suspect that Theorem 4.1 is also true for local martingales[2]. However, for local martingales the quadratic variation interpretation of $\langle X \rangle$ needs modification, and the proof of Theorem 4.1 is no longer valid. Extension of Theorem 4.1 to local martingales would allow (25) and (26) to be removed from the hypotheses of Theorem 4.2.

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