ON DYNAMICAL SYSTEMS REALIZING STATIONARY WEIGHTING PATTERNS AND TIME-VARYING FEEDBACK SYSTEMS

by

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1. INTRODUCTION. In this paper necessary and sufficient conditions are obtained in terms of the matrices A, B, and C under which the linear time-varying dynamical system (A,B,C) realizes a stationary weighting pattern (i.e., its input-output mapping is time invariant). In many situations the possible system functions that can be realized is considerably increased by introducing time-varying components in such a way as to keep the input-output behavior time-invariant. Examples of this in network theory are given in Refs. [1] - [3]. The importance of these results lies in the fact that the synthesis of time-varying systems is in general most easily accomplished by using the state variable formulation. Thus, in attempting to realize stationary weighting patterns, the need for conditions on the matrices A, B, and C arrises naturally.

In the area of stability theory, it would be possible to obtain stability criteria for time varying feedback systems if one could "split off" a part of the time varying gain in such a way that the resulting feedback system had a time-invariant input-output behavior. It will be shown in Sec. 4 that for certain linear time-varying feedback systems, a time-invariant input-output behavior can only be obtained in trivial cases.

2. PRELIMINARIES. The systems to be considered here are those having a representation in the form

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) \tag{2.1}$$
$$y(t) = C(t)\, x(t)$$

where the state x(t) is a real n-vector, the input u(t) is a real m-vector, and the output y(t) is a real p-vector. The real matrices A(t), B(t), and C(t) are respectively $n \times n$, $n \times m$, and $p \times n$. A system in the form of (2.1) will be denoted by (A,B,C).

As is well known, the output y of system (2.1) is given by

$$(2.2) \qquad y(t) = C(t)\ \Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\ \Phi(t,\tau)\ B(\tau)\ u(\tau)$$

where x_0 is the initial state at time t_0 and Φ is the transition matrix associated with A. The matrix $W(t,\tau) = C(t)\ \Phi(t,\tau)\ B(\tau)$ will be called the <u>weighting pattern</u> of (2.1) [4], and a weighting pattern will be called <u>stationary</u> if $W(t,\tau) = W(t - \tau, 0)$. A weighting pattern W is called <u>realizable</u> if it can be realized by a finite dimensional system (A,B,C), and the system (A,B,C) is called a <u>realization</u> of W. The system (2.1) is called <u>minimal</u> if there are no other realizations of W having a lower order.

<u>Definition 2.1</u>: Two systems (A,B,C) and $(\hat{A},\hat{B},\hat{C})$ with corresponding state vectors x and $\hat{x}$ respectively are called <u>algebraically equivalent</u> whenever $\hat{x}(t) = T(t)\ x(t)$ for some absolutely continuous matrix function T possessing an absolutely continuous inverse. This equivalence will be denoted by $(A,B,C) \overset{T}{\rightarrow} (\hat{A},\hat{B},\hat{C})$.

If $(A,B,C) \overset{T}{\rightarrow} (\hat{A},\hat{B},\hat{C})$, then it is easily seen that $\hat{A}(t) = T(t)\ A(t)\ T^{-1}(t) + \dot{T}(t)\ T^{-1}(t)$, $\hat{B}(t) = T(t)\ B(t)$, and $\hat{C}(t) = C(t)\ T^{-1}(t)$. Also, if $\Phi(\cdot,\cdot)$ and $\hat{\Phi}(\cdot,\cdot)$ denote the transition matrices for A and $\hat{A}$ respectively, then $\hat{\Phi}(t,\tau) = T(t)\ \Phi(t,\tau)\ T^{-1}(\tau)$. Therefore it is seen that two algebraically equivalent systems have the same weighting pattern.

With regards to stationary weighting patterns, the following important result will be needed.

Theorem 2.1 (Youla [4]): All minimal realizations of a stationary weighting pattern are algebraically equivalent to a constant coefficient realization (i.e., a realization (A,B,C) in which the matrices A, B, and C are constant).

For any given system (A,B,C), the operator δ is defined by (in any expression it will be tacitly assumed that A, B, and C have the required number of derivatives)

$$(2.3) \qquad (\delta C)(t) = \frac{d}{dt} C(t) + C(t)\, A(t)$$

and the operator Δ by

$$(2.4) \qquad (\Delta B)(t) = - \frac{d}{dt} B(t) + A(t)\, B(t).$$

The powers δ^n and Δ^n are defined in the obvious way;

$$(2.5) \qquad (\delta^n C)(t) = \frac{d}{dt} (\delta^{n-1} C)(t) + (\delta^{n-1} C)(t)\, A(t)$$

$$(2.6) \qquad (\Delta^n B)(t) = - \frac{d}{dt} (\Delta^{n-1} B)(t) + A(t) (\Delta^{n-1} B)(t)$$

$$(2.7) \qquad (\delta^0 C)(t) = C(t)\ , \ (\Delta^0 B)(t) = B(t).$$

From these definitions, the "controllability" and "observability" matrices are defined respectively as

$$(2.8) \qquad Q_n(t) = \begin{bmatrix} (\delta^0 C)(t) \\ (\delta^1 C)(t) \\ \vdots \\ (\delta^{n-1} C)(t) \end{bmatrix}$$

and

$$(2.9) \qquad P_n(t) = [(\Delta^0 B)(t) | (\Delta^1 B)(t) | \ldots | (\Delta^{n-1} B)(t)]$$

These matrices play a predominant role in specifying the conditions for controllability and observability of the system (A,B,C) (see [5], [6], [7]).

For constant coefficient systems (A,B,C) the system function H(s) is given by $H(s) = C^1(Is - A)^{-1}B$. The system function H(s) has a Laurent series expansion about $s = \infty$ of the form

$$(2.10) \qquad H(s) = \sum_{k=1}^{\infty} H_{k-1} s^{-k}$$

and associated with this expansion is the Hankel matrix

$$(2.11) \qquad \begin{bmatrix} H_0 & H_1 & H_2 & \cdots \\ H_1 & H_2 & H_3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

A necessary and sufficient condition for a constant coefficient system (A,B,C) to be minimal is that the truncated Hankel matrix

$$(2.12) \qquad S_n = \begin{bmatrix} H_0 & H_1 & \cdots & H_{n-1} \\ H_1 & H_2 & \cdots & H_n \\ \cdots & \cdots & \cdots & \cdots \\ H_{n-1} & h_n & \cdots & H_{2n-2} \end{bmatrix}$$

have rank n[8]. Also for constant systems (A,B,C) the matrices Q_n and

P_n of (2.8) and (2.9) are given by

$$Q_n = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} \tag{2.13}$$

$$P_n = [B, AB, \ldots, A^{n-1} B] \tag{2.14}$$

and it is easily seen that

$$S_n = Q_n P_n \tag{2.15}$$

Thus it follows that (A,B,C) is minimal if and only if Q_n and P_n are of full rank.

For time-varying systems (A,B,C) the analog of the truncated Hankel matrix is:

$$S_n(t) = Q_n(t) P_n(t) \tag{2.16}$$

with Q_n and P_n given by (2.8) and (2.9). The matrix S_n of (2.16) plays an important role in realization theory [5].

3. A CONDITION FOR THE REALIZATION OF A STATIONARY WEIGHTING PATTERN. The main result of this section is given in the following theorem.

Theorem 3.1: Suppose a system (A,B,C) is such that A is 2n-2 times continuously differentiable (where n is the order of A(t)), B and C are

2n-1 times continuously differentiable. Then a necessary and sufficient condition for (A,B,C) to be a minimal realization of a stationary weighting pattern is that $(\delta^k C)(t)\ B(t)$ be constant for k = 0, 1, 2, ..., 2n-1, and the constant matrix

$$S = \begin{bmatrix} (\delta^0 C)B & (\delta^1 C)B & \cdots & (\delta^{n-1} C)B \\ (\delta^1 C)B & (\delta^2 C)B & \cdots & (\delta^n C)B \\ \cdots & \cdots & \cdots & \cdots \\ (\delta^{n-1} C)B & (\delta^n C)B & \cdots & (\delta^{2n-2} C)B \end{bmatrix} \tag{3.1}$$

have rank n.

Before giving a proof of Theorem 3.1, some preliminary lemmas will be needed. First of all, with regards to algebraically equivalent systems we have the following.

<u>Lemma 3.1</u> [5]: Suppose $(A,B,C) \overset{T}{\to} (\hat{A},\hat{B},\hat{C})$. Then

$$\delta^k \hat{C} = (\delta^k C)T^{-1} \qquad k = 0, 1, 2, \ldots \tag{3.2}$$

$$\Delta^k \hat{B} = T\ \Delta^k B \qquad k = 0, 1, 2, \ldots \tag{3.3}$$

and thus the matrix δ_n given by (2.16) is the same for all algebraically equivalent systems.

This lemma follows from a simple induction argument; see Ref. [5] for a proof. From this Lemma it is clear that if (A,B,C) is algebraically equivalent to a constant coefficient system, say $(\hat{A},\hat{B},\hat{C})$, then $(\delta^i C)(\Delta^j B)(t)$ must be constant for all i and j. Indeed, for some T we have

$$(3.4)\qquad (\delta^i C)(\Delta^j B) = (\delta^i C)T^{-1}\, T(\Delta^j \hat{B}) = (\delta^i \hat{C})(\Delta^j \hat{B})$$

and since $\hat{A}$, $\hat{B}$, and $\hat{C}$ are constant, $\delta^i \hat{C} = \hat{C}\hat{A}^i$ and $\Delta^j \hat{B} = \hat{A}^j \hat{B}$. Thus it follows that $(\delta^i C)(\Delta^j B)$ is constant.

Lemma 3.2: If for a given system (A,B,C), $(\delta^j C)(\Delta^k B)(t)$ is constant, then $(\delta^{j+1} C)(\Delta^k B) = (\delta^j C)(\Delta^{k+1} B)(t)$.

Proof. Since $(\delta^i C)(\Delta^k B)$ is constant, its derivative is zero, which implies that

$$(3.5)\qquad \frac{d}{dt}(\delta^j C)\ \Delta^k B = -\ \delta^j C\ \frac{d}{dt}(\Delta^k B).$$

But

$$(3.6)\qquad \begin{aligned}(\delta^{j+1} C)(\Delta^k B) &= [\frac{d}{dt}(\delta^j C) + (\delta^j C)A](\Delta^k B)\\ &= \frac{d}{dt}(\delta^j C)\ \Delta^k B + (\delta^j C)\ A(\Delta^k B),\end{aligned}$$

and using (3.5) in (3.6) gives

$$(3.7)\qquad \begin{aligned}(\delta^{j+1} C)(\Delta^k B) &= -\ (\delta^j C)\ \frac{d}{dt}(\Delta^k B) + (\delta^j C)\ A(\Delta^k B)\\ &= (\delta^j C)(\Delta^{k+1} B).\end{aligned}$$

Q.E.D.

Corollary 3.1: Suppose $(\delta^k C)(\Delta^0 B)$ is constant for $k = 0, 1, 2, \ldots, N$. Then $(\delta^i C)(\Delta^j B)$ is constant for all i and j satisfying $i + j \leq N$. Furthermore, if $i_1+j_1 = i_2+j_2 \leq N$, then $(\delta^{i_1} C)(\Delta^{j_1} B) = (\delta^{i_2} C)(\Delta^{j_2} B)$.

Proof. The proof proceeds by a repeated use of Lemma 3.2 and induction on $i + j$. First of all, by hypothesis $(\delta^0 C)(\Delta^0 B)$ is constant, so the conclusion is true for $i + j = 0$. Suppose that $(\delta^i C)(\Delta^j B)$ is constant for all i and j satisfying $i + j = n < N$. Then by Lemma 3.2

$$\begin{aligned}
(\delta^n C)(\Delta^0 B) \quad &\text{constant} \Rightarrow (\delta^{n+1} C)(\Delta^0 B) &&= (\delta^n C)(\Delta^1 B) \\
(\delta^{n-1})(\Delta^0 B) \quad &\text{constant} \Rightarrow (\delta^n C)(\Delta^1 B) &&= (\delta^{n-1})(\Delta^2 B) \\
&\vdots \\
(\delta^1 C)(\Delta^{n-1} B) \quad &\text{constant} \Rightarrow (\delta^2 C)(\Delta^{n-1} B) &&= (\delta^1 C)(\Delta^n B) \\
(\delta^0 C)(\Delta^n B) \quad &\text{constant} \Rightarrow (\delta^1 C)(\Delta^{n+1} B) &&= (\delta^0 C)(\Delta^{n+1} B)
\end{aligned}$$

Hence, it is seen that

$$(\delta^{n+1} C)(\Delta^0 B) = (\delta^n C)(\Delta^1 B) = \ldots = (\delta^0 C)(\Delta^{n+1} B). \tag{3.8}$$

Since $n < N$, $n + 1 \leq N$ and thus by hypothesis $(\delta^{n+1} C)(\Delta^0 B)$ is constant. Thus from (3.8) it is seen that the conclusion is true for $i + j = n + 1$ if it is true for $i + j = n$. Q.E.D.

By completely similar arguments it is easily seen that the following is also true.

Corollary 3.2: Suppose $(\delta^0 C)(\Delta^k B)$ is constant for $k = 0, 1, \ldots, N$. Then the conclusion of Corollary 3.1 is true.

We are now in a position to give the

Proof of Theorem 3.1: Necessity: If (A,B,C) is minimal and realizes a stationary weighting pattern, then from Theorem 2.1 it is known that it is algebraically equivalent to a minimal constant coefficient system $(\hat{A},\hat{B},\hat{C})$. Hence, from Lemma 3.1, and in particular (3.4), it is seen that

$(\delta^k C)B$ must be constant. Further, the matrix S is the truncated Hankel matrix S_n of (2.15) for $(\hat{A},\hat{B},\hat{C})$, and thus has rank n since $(\hat{A},\hat{B},\hat{C})$ is minimal.

Sufficiency: Suppose $(\delta^k C)(t)\, B(t)$ is constant for $k = 0, 1, \ldots 2n-1$, and that the matrix S of (3.1) has rank n. Let Q_n and P_n be given by (2.8) and (2.9) respectively. Then from Corollary 3.1 of Lemma 3.2 it follows that

$$S_n(t) = Q_n(t)\, P_n(t) \tag{3.9}$$

and

$$\hat{S}_n(t) = (\delta Q_n)(t)\, P_n(t) \tag{3.10}$$

are both constant, and also $S_n(t) = S$.

Since S has rank n, there exists an $n \times n$ submatrix $\hat{S}$ of S which is nonsingular. From (3.9) it is seen therefore that there exist $n \times n$ submatrices $\tilde{Q}_n$ and $\tilde{P}_n$ of Q_n and P_n respectively such that

$$\tilde{S} = \tilde{Q}_n(t)\, \tilde{P}_n(t) \tag{3.11}$$

Now consider the system $((\delta\tilde{Q}_n)\, \tilde{Q}_n^{-1},\ \tilde{Q}_n B,\ C\tilde{Q}_n^{-1})$. Note that $\tilde{Q}_n(t)$ is nonsingular due to (3.11) and $\tilde{S}$ being nonsingular. It will now be shown that this is a constant coefficient system. First of all, from (3.10) and $\hat{S}_n(t)$ being constant, it follows that $(\delta\tilde{Q}_n)(t)\, \tilde{P}_n(t)$ is constant since it is a submatrix of $\hat{S}_n(t)$. Also, $\tilde{S}$ is constant and nonsingular, which implies that

$$(\delta\tilde{Q}_n)\, \tilde{P}_n\, \tilde{S}^{-1} = (\delta\tilde{Q}_n)\, \tilde{Q}_n^{-1} \tag{3.12}$$

is constant. The matrix $(\delta\tilde{Q}_n)(t)\,B(t)$ is constant by hypothesis. Finally,

$$(3.13)\qquad C\tilde{Q}_n^{-1} = C\tilde{P}_n\,\tilde{P}_n^{-1}\,\tilde{Q}_n^{-1} = (C\tilde{P}_n)\,\tilde{S}^{-1},$$

and since $C\tilde{P}_n$ and $\tilde{S}$ are both submatrices of S, it is seen that $C\tilde{Q}_n^{-1}$ is constant. Thus $((\delta\tilde{Q}_n)\,\tilde{Q}_n^{-1},\ \tilde{Q}_nB,\ C\tilde{Q}_n^{-1})$ is constant.

Consider the transformation $((\delta\tilde{Q}_n)\,\tilde{Q}_n^{-1},\ \tilde{Q}_nB,\ C\tilde{Q}_n^{-1}) \overset{\tilde{Q}_n^{-1}}{\rightarrow} (\hat{A},\hat{B},\hat{C})$. Then

$$(3.14a)\qquad \hat{A} = \tilde{Q}_n^{-1}(\delta\tilde{Q}_n) - \tilde{Q}_n^{-1}\,\dot{\tilde{Q}}_n$$

$$(3.14b)\qquad \hat{B} = B$$

$$(3.14c)\qquad \hat{C} = C$$

However,

$$(3.15)\qquad \delta\tilde{Q}_n = \dot{\tilde{Q}}_n + \tilde{Q}_nA,$$

So (3.14a) becomes

$$(3.16)\qquad \hat{A} = (\tilde{Q}_n^{-1}\,\dot{\tilde{Q}}_n + A) - \tilde{Q}_n^{-1}\,\dot{\tilde{Q}}_n = A.$$

Therefore, (A,B,C) is algebraically equivalent to the minimal* constant system $((\delta\tilde{Q}_n)\,\tilde{Q}_n^{-1},\ \tilde{Q}_nB,\ C\tilde{Q}_n^{-1})$, and hence is a minimal realization of a stationary weighting pattern. Q.E.D.

By using Corollary 3.2, a theorem dual to Theorem 3.1 can be proved along analogous lines. This fact is recorded in the following theorem.

<u>Theorem 3.2</u>: Under the conditions of Theorem 2.1 a necessary and

*Minimality follows from S having rank n.

sufficient condition for (A,B,C) to be a minimal realization of a stationary weighting pattern is that $C(t)(\Delta^k B)(t)$ be constant for $k = 0, 1, \ldots, 2n-1$, and the constant matrix

$$S = \begin{bmatrix} C\Delta^0 B & C\Delta^1 B & \cdots & C\Delta^{n-1} B \\ C\Delta^1 B & C\Delta^2 B & \cdots & C\Delta^n B \\ \cdots & \cdots & \cdots & \cdots \\ C\Delta^{n-1} B & C\Delta^{n-2} B & \cdots & C\Delta^{2n-2} B \end{bmatrix}$$

have rank n.

A result similar to Theorems 3.1 and 3.2 has been obtained by Silverman and Meadows [5] which would require that the matrix $Q_{n+1}(t)\ P_{n+1}(t)$ be constant (see (2.8-9)). However, this condition requires the calculation of $(\delta^i C)(\Delta^j B)$ for all $0 \leq i, j \leq n+1$; whereas the conditions of Theorem 3.1 (or Theorem 3.2) requires only the calculation of $(\delta^i C)$ for $i = 0, 1, \ldots, 2n-1$. This reduction in the required number of calculations is made possible by the use of Lemma 3.2 and its corollaries.

4. AN APPLICATION TO FEEDBACK SYSTEMS. Consider the feedback system shown in Fig. 1, in which L is a linear time-invariant system with the representation

(4.1a) $\dot{x}(t) = Ax(t) + Bu_1(t) + d\tilde{u}_2(t)$

(4.1b) $y(t) = Cx(t)$

(4.1c) $y_2(t) = fx(t)$.

With the input $\tilde{u}_2(t) = u_2(t) - k(t)\ y_2(t)$, the dynamic equations for the feedback system are easily seen to be

$$\dot{x}(t) = (A - k(t)df)\ x(t) + Bu_1(t) + du_2(t) \tag{4.2a}$$

$$y_1(t) = Cx(t) \tag{4.2b}$$

$$y_2(t) = fx(t) \tag{4.2c}$$

The question is now raised as to whether the system of (4.2) can be a realization of a stationary weighting pattern if $\frac{dk}{dt} \not\equiv 0$. It will be shown that this can only happen in the trivial case when either d or f is the zero vector. This result is based on the following theorem.

Theorem 4.1: Consider the system $(A + k(\cdot)df,\ b,\ c)$ where A is a real constant $n \times n$ matrix, d, f', b, and c' are real constant n-vectors, and $k(\cdot)$ is a 2n-2 times continuously differentiable real valued function of t with $\frac{dk}{dt} \not\equiv 0$. Also assume that (A,b,c) is minimal. Then $(A + k(\cdot)df,b,c)$ is a minimal realization of a stationary weighting pattern if and only if $d = 0$ or $f = 0$.

Proof: Sufficiency. This is immediate since $d = 0$ or $f = 0$ results in the system (A,b,c).

Necessity. This will be proven by showing that the assumption that $(A + k(\cdot)df,b,c)$ is a minimal realization of a stationary weighting pattern implies that one of the following two conditions must be satisfied:

i) $cA^i d = 0, \quad i = 0, 1, 2, \ldots, n-1$

ii) $fA^i b = 0, \quad i = 0, 1, 2, \ldots, n-1$

Then since (A,b,c) is minimal, the matrices

$$Q = \begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix} \quad \text{and } P = [b, Ab, \ldots, A^{n-1}b]$$

are nonsingular. Since i) implies that $Qd = 0$, it follows that i) thereby implies that $d = 0$. Similarly ii) implies that $Pf = 0$ and thus that $f = 0$.

Before proving i) and ii) it will be convenient to first prove the following lemma.

<u>Lemma 4.1</u>: Consider the system $(A + k(\cdot)df, b, c)$. If $cA^k d = 0$ for $k = 0, \ldots, N$, then $(\delta^k C)(t) = cA^k$ for $k = 0, 1, \ldots, N+1$. If $fA^k b = 0$ for $k = 0, 1, \ldots, N$, then $(\Delta^k b)(t) = A^k b$ for $k = 0, 1, \ldots, N+1$.

<u>Proof</u>: Suppose $cA^k d = 0$ for $k = 0, 1, \ldots, N$. The proof that $(\delta^i c)(t) = cA^j$ for $j = 0, 1, \ldots, N+1$, will proceed by induction on j. First of all, $\delta^0 c = c$ so the result is true for $j = 0$. Suppose $(\delta^j c)(t) = cA^j$ for some $j \leq N$. Then

$$(4.3) \qquad (\delta^{j+1}c)(t) = \frac{d}{dt}(\delta^j c)(t) + (\delta^j c)(t)(A + k(t)df)$$

and since $(\delta^j c)(t) = cA^j$ and $cA^j d = 0$ (4.3) becomes

$$(4.4) \qquad (\delta^{j+1}c)(t) = cA^{j+1} + k(t)\, cA^j df = cA^{j+1}$$

which proves the result. A similar proof is used to show $\Delta^k b = A^k b$ for $k = 0, \ldots, N+1$ if $fA^k b = 0$ for $k = 0, 1, \ldots, N$.

Returning now to the necessity proof of Theorem 4.1, suppose

$(A + k(\cdot)df,b,c)$ is a minimal realization of a stationary weighting pattern. From Theorems 3.1 and 3.2 it is known that $(\delta^i c)b$ and $c(\Delta^i b)$ must be constant for $i = 0, 1, \ldots, 2n-1$. Since b and c are constant, these conditions are satisfied for $i = 0$. For $i = 1$ they give

$$[cA + k(t)cdf]b = \text{const.} \tag{4.5}$$

and

$$c[Ab + k(t)dfb] = \text{const.} \tag{4.6}$$

Since $\frac{dk}{dt} \neq 0$ it is seen from (4.5) and (4.6) that $cdfb = 0$, which leads to three possibilities:

a) $cd = 0$ and $fb \neq 0$

b) $cd \neq 0$ and $fb = 0$

c) $cd = fb = 0$

Consider a) first, then i) will be obtained by induction. Therefore, suppose a) holds, and $cA^k d = 0$ for $k = 0, 1, \ldots, N-1$ $(N < 2n-2)$. Then from Lemma 4.1, $(\delta^k c) = cA^k$ for $k = 0, 1, \ldots, N$. Hence

$$(\delta^{N+1}c)b = cA^{N+1}b + k(t)cA^N dfb \tag{4.7}$$

Using the fact that $(\delta^i c)b$ is constant for $i = 0, 1, \ldots, 2n-1$, it follows from (4.7) that $cA^N dfb = 0$. But $fb \neq 0$, so $cA^N d = 0$. Consequently, by induction on k, it is seen that $cA^k d = 0$ for $k = 0, 1, \ldots, n-1$ establishing i).

In a similar manner b) yields ii). For case c) let j be the first integer such that $cA^i d$ and $fA^i b$ do not both vanish. If there is no such integer $j \leq n-1$, then both i) and ii) hold and we are done. Therefore suppose $j < n-1$. From Corollary 3.1 of Lemma 3.2 and Theorem 3.1 it follows that $(\delta^m c)(\Delta^k b)$ is constant for $m+k \leq 2n-1$. Hence $(\delta^m c)(\Delta^j b)$ is constant for $m \leq 2n-1-j$. Also, from Lemma 4.1 it is known that $cA^i d = fA^i b = 0$ for all $i < j$ implies that

$$(4.8) \qquad \delta^i c = cA^j$$

$$(4.9) \qquad \Delta^j b = A^j b.$$

Hence,

$$(4.10) \qquad (\delta^{j+1} c)(t)(\Delta^j b)(t) = [cA^{j+1} + k(t)\, cA^j df]A^j b$$

and since $j+1 < 2n-1-j$ (recall $j < n-1$), the left hand side of (4.10) is constant. Thus there follows from (4.10)

$$(4.11) \qquad (cA^j d)(fA^j b) = 0$$

However, one of the terms in (4.11) is nonzero by the choice of j. Therefore, either

a') $cA^j d = 0$ and $fA^j b \neq 0$, or

b') $cA^j d \neq 0$ and $fA^j b = 0$.

It is easily seen that a') leads to i) and b') leads to ii) in the same way as a) and b) led to i) and ii) respectively. Q.E.D.

The result of Theorem 4.1 can be extended to multiple input-multiple output systems, and this is done in the following theorem.

Theorem 4.2: Consider the system $(A + k(\cdot)df, B, C)$ where A, d, f, and

$k(\cdot)$ are as in Theorem 4.1, B is a real constant $n \times m$ matrix, and C is a real constant $p \times n$ matrix, and (A,B,C) is minimal. Then $(A + k(\cdot)df,B,C)$ is a minimal realization of a stationary weighting pattern if and only if $d = 0$ or $f = 0$.

Proof: Sufficiency is obvious. To prove necessity, write B and C as follows:

$$B = (b_1, b_2, \ldots, b_m)$$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

where b_i and c_i' are constant n-vectors. Since $(A + kdf,B,C)$ realizes a stationary weighting pattern, clearly every system $(A + k(\cdot)df,b_i,C_j)$ must also. Theorem 4.1 cannot be used immediately to conclude that either $d = 0$ or $f = 0$ since (A,b_i,c_j) may not be minimal for any i and j. However, $(A + k(\cdot)df,b_i,c_j)$ is algebraically equivalent to a constant system since $(A + k(\cdot)df,B,C)$ is. Therefore, using the same arguments as in the necessity proof of Theorem 4.1 it is seen that, for each i and j, one of the following must hold:

a) $c_iA^id = 0 \qquad k = 0, 1, \ldots, n-1$

b) $fA^kb_j = 0 \qquad k = 0, 1, \ldots, n-1$

Suppose b) does not hold for some j, then a) must hold for all i. Thus

$$CA^k d = 0 \qquad k = 0, 1, \ldots, n\text{-}1.$$

Similarly, if a) does not hold for all i, then

$$fA^k B = 0 \qquad k = 0, 1, \ldots, n\text{-}1$$

Hence the requirement that either a) or b) hold for all i and j is equivalent to the requirement that one of the following hold:

a') $Qd = 0$

b') $fP = 0$

where $Q' = (C', A'C', \ldots, A'^{n-1}C')$ and $P = (B, AB, \ldots, A^{n-1}B)$. Since (A,B,C) is minimal, P and Q are of full rank. Thus a') implies that $d = 0$ and b') implies that $f = 0$. Q.E.D.

Returning now to the feedback system of Fig. 1, it is seen that the representation of (4.2) is in the form considered in theorem 4.2. Since we are only concerned with input-output mappings it is always possible to select a representation such that (4.2) satisfies the minimality assumptions of the theorem. Thus it follows from Theorem 4.2 that the feedback system of Fig. 1 with $\dot{k} \not\equiv 0$ will realize a stationary weighting pattern only in the trivial cases of $d = 0$ or $f = 0$.

A less restrictive requirement on the system of Fig. 1 would be to set $u_2 \equiv 0$ and ask for a stationary weighting pattern for the input-output pair u_1 and y_1. For this case the dynamic equations would be

(4.12a) $\dot{x}(t) = (A - k(t)df)\, x(t) + Bu_1(t)$

(4.12b) $y_1(t) = Cx(t)$

We can no longer assume that (A,B,C) is minimal, but from the proof of Theorem 4.2 it is seen that for (4.12) to be a realization of a stationary weighting pattern it is necessary that either $CA^k d = 0$ or $fA^k B = 0$ for $k = 0, 1, \ldots, n-1$.

Suppose first that $CA^k d = 0$ for $k = 0, 1, \ldots, n-1$. It is easily seen that with zero initial conditions at $t = t_0$

$$(4.13) \qquad y_1(t) = C \int_{t_0}^{t} e^{A(t-\tau)}[Bu_1(\tau) - k(\tau)dfx(\tau)]d\tau,$$

and since $CA^k d = 0$ for $k = 0,1, \ldots, n-1$ implies that $Ce^{At}d \equiv 0$ this becomes

$$(4.14) \qquad y_1(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bu_1(\tau)d\tau.$$

Thus, in this case the weighting pattern for u_1 and y_1 is not changed by the feedback.

Now, suppose that $fA^k B = 0$ for $k = 0, 1, \ldots, n-1$. With $x(t_0) = 0$, it is seen that

$$(4.15) \qquad fx(t) = f \int_{t_0}^{t} e^{A(t-\tau)}[Bu_1(\tau) - k(\tau)dfx(\tau)]d\tau.$$

Since $fA^k B = 0$ for $k = 0, 1, \ldots, n-1$ implies that $fe^{At}B \equiv 0$, (4.15) becomes

$$(4.16) \qquad fx(t) = - \int_{t_0}^{t} k(\tau)fdfx(\tau)d\tau$$

Therefore, fx(t) satisfies the differential equation

$$\frac{d}{dt} fx(t) = - k(t)fdfx(t), \tag{4.17}$$

and since $fx(t_0) = 0$ it follows that $fx(t) \equiv 0$. From (4.13) it is then seen that y_1 is given by (4.14), and the weighting pattern for u_1 and y_1 is unchanged by the feedback.

These results are sumarized in the following Theorem.

Theorem 4.3: If the weighting pattern of any input-output pair in Fig. 1 is stationary, then it is independent of the feedback $k(\cdot)$.

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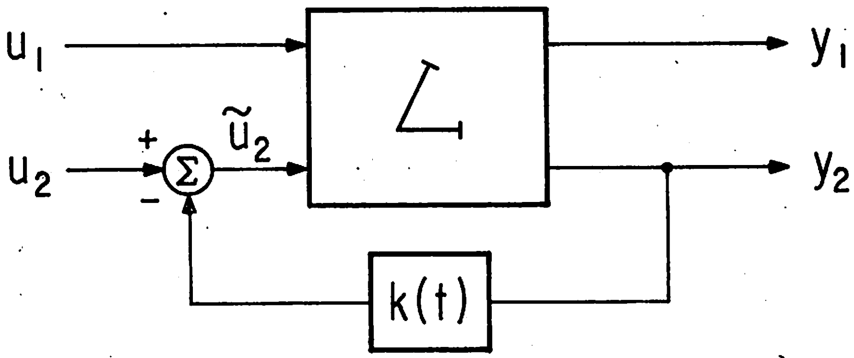

Fig. 1. Time-Varying Feedback System.