Copyright © 1970, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

TARGET FUNCTION APPROACH TO LINEAR PURSUIT PROBLEMS

by

Wladimir Borgest and Pravin Varaiya

Memorandum No. ERL-M282

2 September 1970

Electronics Research Laboratory

College of Engineering
University of California, Berkeley
94720

Target Function Approach to Linear Pursuit Problems*

Wladimir Borges† and Pravin Varaiya†

Abstract

The notion of target function is presented, and is applied for problems of pursuit with linear dynamics. It is shown that this approach unifies and extends the work of Pontriagin, Pshenichnyi and many others. The approach is geometric in nature, and its application relies heavily on the geometric difference of convex sets, and support functions of convex sets.

†Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California 94720. W. Borgest is on leave from Instituto Technologico de Aeronautica, Saõ José dos Campos, Brazil.

*Research sponsored by the Joint Services Electronics Program, Grant AFOSR-68-1488.

I. INTRODUCTION AND CONTENTS

Consider the linear system

$$\dot{z}(t) = Az(t) + u(t) - v(t) \quad , \quad t \geq 0$$

$$z(0) = z_0 \quad ,$$

where at time t, z(t) is the state, u(t)[v(t)] is the control selected by player P[E] subject to the constraint $u(t) \in U[v(t) \in V]$, where U[V] is a fixed subset. A is a constant matrix. There is also specified a fixed subset M, called the target set. It is the objective of P to choose u(t) for each t, based on the information available to P at time t, in such a way as to steer z(t) to M. We say that capture has occurred if $z(t) \in M$. On the other hand, E selects v(t) based on the information available to E at time t, in such a way as to prevent capture.

Various situations must be distinguished, depending upon the information available to each player. We shall distinguish two main types, open loop and closed loop. We are interested in the latter case only, but it is sometimes helpful to analyze the former case. A different classification is obtained depending on whether we require the capture to occur at a fixed time T independent of the control employed by E, or whether we permit the capture time to vary according to E's control. These situations are precisely defined in the next section.

Target functions are defined, and some of their fundamental properties are studied in Section III. The basic idea involved consists in substituting for the eventual fixed target M an immediate moving target

$F(t)$ for each time t. The function $F(t)$ should be such that if $z(t) \in F(t)$ for all t, then eventually capture must occur. The attractiveness of this approach depends upon the ease with which one can discover target functions and derive closed loop controls from them. In this sense the approach is analogous to Lyapunov's second method. In Section IV we present some "natural" candidates for target functions. There we also show that many of the solutions proposed in the literature correspond to selecting one of these candidates as a target function. The last section consists of some critical comments.

II. PROBLEM FORMULATION

Consider the linear system

$$\dot{z}(t) = Az(t) + u(t) - v(t)$$

where $z(t) \in R^n$, $u(\cdot)$ and $v(\cdot)$ are input functions under the control of players P, E respectively and constrained by $u(t) \in U$, $v(t) \in V$ where U and V are fixed compact, convex subsets of R^n. A is a constant $n \times n$ matrix.

Let M be a fixed closed, convex subset of R^n. M is called the <u>target</u>.

<u>Definition 1</u>: a. A control $u(\cdot)$ $[v(\cdot)]$ defined over some interval $[a,b]$ is said to be <u>admissible</u> if it is piecewise continuous, continuous from the right, and $u(t) \in U$ $[v(t) \in V]$ for all $t \in [a,b]$. We sometimes denote this function as $u_{[a,b]}$ $[v_{[a,b]}]$.

b. If $u_{[a,b]}$, $v_{[a,b]}$ are admissible, then $z(b,z_a,u_{[a,b]},v_{[a,b]})$ is the state of (1) at time t, corresponding to the inputs $u_{[a,b]}$, $v_{[a,b]}$ and initial state $z(a) = z_a$.

<u>Definition 2</u>: a. Starting at $z(0) = z_0$, <u>P can capture E in time T</u> if for all admissible $v_{[0,T]}$ there is an admissible $u_{[0,T]}$, with $u(\tau) = \psi(z(\tau),v(\tau))$ depending only on $z(\tau)$ and $v(\tau)$, such that $z(t^*) \in M$ for some $t^* \in [0,T]$. (Note that t^* depends on z_0 and $v_{[0,T]}$.)

b. Let $T^*(z_0) = \inf \{t^* | \text{starting at } z_0, \text{ P can capture E in time } t^*\}$.
If P cannot capture E starting at z_0 in any time $T < \infty$, we let

$$T^*(z_0) = \infty.$$

c. For each t, let

$$Z^*(t) = \{z_0 | T^*(z_0) = t\}.$$

Let

$$Z^* = \{z_0 | T^*(z_0) < \infty\}$$

<u>Remark 1</u>

The assumption that at time τ, P knows the current control $v(\tau)$ is less realistic than a closed loop formulation where P only knows the past. However, it is very helpful technically and can be rationalized as follows. Let $z(0) = z_0$ with $T^*(z_0) < \infty$. Let $u(\tau) = \psi(z(\tau),v(\tau))$ be a closed-loop control using which P can capture E in time T, in the sense of Definition 2a. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that with the closed-loop control $u^\delta(\tau) = \psi(z(\tau),v(\tau-\delta))$†, P can ε-capture E in time T i.e., for all admissible $v_{[0,T]}$, there exists $t^* \in [0,T]$ such that $z(t^*,z_0,u^\delta{}_{[0,t^*]},v_{[0,t^*]}) \in M + S_\varepsilon$, where $S_\varepsilon = \{z | z \in R^n, |z| \leq \varepsilon\}$.†† Note that the

---

†We set $u_\delta(\tau) = u_0$, $0 \leq t < \delta$ where $u_0 \in U$ is arbitrary.

††Here and throughout $|z|$ is the Euclidean norm of $z \in R^n$.

control u^δ is based on the present state and past control of E. The proof of the above statement follows easily from the boundedness of V and the fact that $T^*(z_0)$ is finite. It is even possible to let ε approach zero. However various delicate technical problems arise which are not of importance here. The interested reader may consult [1] - [3].

Now let z_0 be such that $T^*(z_0) < \infty$. In some cases the time t^* in Definition 2a, can be chosen to be independent of the control $v_{[0,T]}$ of E. Such situations are of interest because some important cases fall into the category, and the problem is greatly simplified when this property is taken into account. We therefore propose the following definition.

Definition 3: a. Starting at $z(0) = z_0$, P can capture E (in the absorbtion sense) at time T if for all controls† $v_{[0,T]}$ there is a control $u_{[0,T]}$, with $u(\tau) = \psi(z(\tau),v(\tau))$ depending only on $z(\tau)$ and $v(\tau)$ such that

$$z(T,z_0,u_{[0,T]},v_{[0,T]}) \in M.$$

b. Let

$$T(z_0) = \inf\{T \mid \text{starting at } z_0\text{, P can capture E at time T}\}$$

If P cannot capture E starting at z_0 at any time $T < \infty$, we let

$$T(z_0) = \infty.$$

c. For each t, let

†From now on whenever we say control we mean admissible control.

$$Z(t) = \{z_0 | T(z_0) = t\}.$$

Let

$$Z = \{z_0 | T(z_0) < \infty\}$$

Remark 2

(i) The condition that the capture time be independent of the control of E is highly restrictive. First of all it is clear that $T^*(z) \leq T(z)$ for all z. In fact as shown in Example 1 below it may easily be the case that $T^*(z) < \infty$ but $T(z) = \infty$.

(ii) The set Z(t) is convex. To see this let $z_i \in Z(t)$, and let $u_i(\tau) = \psi_i(z(\tau),v(\tau))$ be the closed-loop control which captures E at time t starting in z_i, for i = 1, 2. Let $0 \leq \lambda \leq 1$. Then from the linearity of (1), for the closed-loop control $u(\tau) = \lambda\psi_1(z(\tau),v(\tau)) + (1-\lambda)\,\psi_2(z(\tau),v(\tau))$, we can see that $(\lambda z_1 + (1-\lambda)z_2) \in Z(t)$. The convexity of Z(t) makes it very convenient to characterize, and therefore much of the work reported in the literature [4],[6]-[9],[13],[19] deals with the problem of capture in the absorbtion sense. Example 2 below shows that $Z^*(t)$ need not be convex.

(iii) Definitions 2 and 3 are both concerned with closed-loop capture as opposed to open-loop capture where P knows at time 0 the open-loop control $v_{[0,\infty)}$ chosen by E. This problem is not very interesting from our point of view, however it has received some attention [7], [10],[11]. We might add that these papers are concerned with open-loop capture in the absorbtion sense.

The problem of pursuit can now be defined as characterizing the sets

Z^* and Z, and finding the closed-loop pursuit control $u(\tau) = \psi(z(\tau),v(\tau))$ which achieves capture. We are not restricting ourselves to finding closed-loop controls which achieve capture in minimum time i.e., controls which guarantee capture in [at] time $T^*(z_0)$ $[T(z_0)]$. Indeed in this case the only general technique is to solve Isaac's main equation [12] (see [13] for an application of this technique for a particular case of capture in the absorbtion sense). Rather if $T^*(z_0) < \infty$ $[T(z_0) < \infty]$ we want to find <u>some</u> closed-loop capture which achieves capture in [at] some finite time.

<u>Example 1</u>: $(T(z_0) = \infty, T^*(z_0) < \infty)$. Consider the 3-dimensional system $\dot{z}_1 = u$, $\dot{z}_2 = v_1$, $\dot{z}_3 = v_2$ where $u \in [-2,2]$ and $v' = (v_1,v_2) \in \{\lambda(1,1/2) + (1-\lambda)(1/2,1) | 0 \leq \lambda \leq 1\}$ are the constraints on the control for P and E respectively. Let $M = \{(z_1,z_2,z_3)' | z_1 = z_2\}$. By inspection we can see that for the initial condition $z_0' = (0,-1,0)$, $T^*(z_0) = 2$ but $T(z_0) = \infty$.

<u>Example 2</u>: ($Z(t)$ convex, $Z^*(t)$ not convex). Consider the 1-dimensional system $\dot{z} = u - v$ with $v(t) \in [-a,a]$ and $u(t) \in [-2a,2a]$ where $a > 0$. Let $M = \{0\}$. Then $Z^*(t) = \{-ta,ta\}$ whereas $Z(t) = [-ta,ta]$.

It might be conjectured that $\bigcup_{\tau \leq t} Z^*(t)$ is convex. The next example shows that this is not true in general.

<u>Example 3</u>: Consider the 2-dimensional system,

$$\dot{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z + u - v$$

where $u \in U$, $v \in V$ and $U = V$ is an arbitrary set.

Let $M = \{(z_1,z_2)' | z_1 \in [a,b],\ z_2 = 0\}$. Then, $\bigcup_{\tau \leq t} Z^*(\tau) = \bigcup_{\tau \leq t} Z(\tau)$

$= \{(r \cos \theta, r \sin \theta)' | a \leq r \leq b,\ 0 \leq \theta \leq 2\pi t\}$ which is not convex for $t > 0$, unless $a = 0$.

III. TARGET FUNCTIONS: BASIC PROPERTIES

If $F \subset R^n$, then ∂F denotes the boundary of F. If $G \subset R^n$ and $z \in R^n$, then $d(z,G) = \inf \{|z-x|\ x \in G\}$.

From now on we consider the fixed system (1), with constraint sets U, V and target M.

Definition 4: A target function is any set-valued† function $F : [0,\infty) \to R^n$ such that $F(s)$ is closed for each s, and $F(0) \subset M$.

The basic idea behind the target function approach is given by the next elementary result.

Fact 1: Let $F(s)$ be a target function and define $\lambda(z,s) = +d(z,\partial F(s))$ if $z \in F(s)$ and $\lambda(z,s) = -d(z, F(s))$ if $z \notin F(s)$. Let $\lambda(z_0,s_0) > 0$ i.e., $z_0 \notin F(s_0)$. Suppose there exists $\alpha_0 > 0$ and a closed-loop control $u = \psi(z,v)$ such that for every control $v_{[0,T]}$, (i) a solution to $\dot{z}(t) = Az(t) + \psi(z(t), v(t))$ with $z(0) = z_0$ exists,

(ii) the corresponding function $u(\tau) = \psi(z(\tau), v(\tau))$ is an admissible control,

and (iii) along the trajectory

† i.e., $F(s)$ is a subset of R^n.

$$\lambda(z(t),\ s_0 - t\mu(t)) \geq 0 \quad , \quad 0 \leq t \leq T,$$

for some function $\mu(t) \geq \alpha_0$. Then starting at z_0, P can capture E in time $\frac{s_0}{\alpha_0}$, using the closed-loop control ψ.

The next (and more important) step is to find target functions F and closed-loop controls ψ which satisfy the above condition. We shall restrict our search to target functions F which are smooth, and for which F(s) is convex for all s.† We need to develop some elementary properties of convex sets.

Definition 5: Let $F \subset R^n$ be a closed and convex set. The support function of F is the function $W_F : R^n \to R \cup \{\infty\}$ defined by

$$W_F(\phi) = \sup \{\phi' x | x \in F\}.$$

We also define,

$$K_F = \{\phi | W_F(\phi) < \infty\} ,$$

$$\lambda_F = \inf \{W_F(\phi) \mid |\phi| = 1\},$$

$$\Gamma_F = \{\phi \mid |\phi| = 1,\ W_F(\phi) = \lambda_F\}.$$

The main properties of the support function are stated in the Appendix. It will be useful to note that $\lambda_F \geq 0$ if and only if $0 \in F$.

Definition 6 [4]: For closed convex sets A and B, their geometric

---

†Of course for the function $\mathcal{Z}^*(s) = \{z | T^*(z) \leq s\}$, a closed-loop control which satisfies the conditions of Fact 1 exists by definition (it is easy to show that $\mathcal{Z}^*(s)$ is closed). The situation is completely analogous to converse Lyapunov stability results. The difficulty is that in general we don't know $\mathcal{Z}^*(s)$.

different $A \stackrel{*}{-} B$, is given by

$$C = A \stackrel{*}{-} B = \{z | z + B \subset A\}.$$

It is shown in the Appendix that if B is bounded, then

$$W_C = \text{co } [W_A - W_B].$$

where for any function $W : R^n \to R$, co [W] is the largest convex function bounded from above by W.

Definition 7: If A, B are closed convex subsets of R^n define $W^*_{A,B}$ by

$$W^*_{A,B}(\phi) = W_A(\phi) - W_B(\phi).$$

(If $A \stackrel{*}{-} B = C$, we will sometimes write $W^*_C = W^*_{A,B}$ when there is no ambiguity). Also define,

$$\lambda^*_{A,B} = \inf \{W^*_{A,B}(\phi) \mid |\phi| = 1\},$$

$$\Gamma^*_{A,B} = \{\phi \mid |\phi| = 1, W^*_{A,B}(\phi) = \lambda^*_{A,B}\}.$$

Some properties of these functions are given in the Appendix. Note $\lambda^*_{A,B} \geq 0$ if and only if $B \subset A$.

Let F(s), $s \geq 0$ be a target function satisfying conditions C1 - C4.

C1. F(s) is convex for $s \geq 0$.

C2. F(s) is a continuous function of s for $s > 0$, in the Hausdorff metric.†

C3. $K_{F(s)}$ is closed for each $s \geq 0$.

C4. For all $\phi \in K_{F(s)}$, $s \geq 0$, the partial derivative of $W_{F(s)}(\phi)$ with respect to s exists and is continuous in ϕ and s.

---

†i.e., For each s and $\varepsilon > 0$, there is $\delta > 0$ such that $F(s) \subset F(s') + S_\varepsilon$ and $F(s') \subset F(s) + S_\varepsilon$ whenever $|s - s'| \leq \delta$.

Let $\lambda(z,s) \triangleq \lambda_{F(s)-z}$ and $\Gamma(z,s) \triangleq \Gamma_{F(s)-z}$ (see Definition 5).

Lemma 1

Let F be a target function satisfying C1 - C4. Then

a) $\Gamma(z,s)$ is upper[†] semi-continuous for $(z,s) \in R^n \times R^+$.[††]

b) $\Gamma(z,s)$ is differentiable along any direction in R^{n+1} for $(z,s) \in R^n \times R^+$. Furthermore, if $f = (e,\mu)'$ with $e \in R^n$, $\mu \in R$ then

$$\frac{\partial\lambda(z,s)}{\partial f} \triangleq \lim_{\tau\to 0+} \frac{\lambda(z + \tau e, s + \tau\mu) - \lambda(z,s)}{\tau}$$

$$= \min_{\phi \in \Gamma(z,s)} \left[\mu \frac{\partial}{\partial s} W_{F(s)-z}(\phi) + \left(\frac{\partial}{\partial z} W_{F(s)-z}(\phi)\right)'e\right]$$

$$= \min_{\phi \in \Gamma(z,s)} \left[\mu \frac{\partial}{\partial s} W_{F(s)}(\phi) - \phi' e\right].$$

c) If $e(\tau)$, $\mu(\tau)$ are such that $\lim_{\tau\to 0+} e(\tau) = e$ and $\lim_{\tau\to 0+} \mu(\tau) = \mu$, then

$$\lim_{\tau\to 0+} \frac{\lambda(z + \tau e(\tau), s + \tau\mu(\tau)) - \lambda(z,s)}{\tau} = \frac{\partial\lambda}{\partial f}(z,s)$$

where $f = (e,\mu)'$.

Proof: See Appendix.

From now on, unless explicitly stated otherwise, we only consider target functions which satisfy C1 - C4.

[†] i.e., For each s, z and $\varepsilon > 0$ there is $\delta > 0$ such that $\Gamma(s',z') \subset \Gamma(s,z) + S_\varepsilon$ whenever $|s - s'| + |z - z'| \leq \delta$.

[††] $R^+ = \{s | s > 0\}$.

Let F be a target function. For each $z \in R^n$ let $s(z) = \min \{s | z \in F(s)\}$ i.e., $s(z)$ is the smallest root of the equation $\lambda(z,s) = 0$. If $\lambda(z,s) = 0$ has no root then let $s(z) = \infty$.

Lemma 2

$s(z)$ is lower semicontinuous on the set $\{z | s(z) < \infty\}$.

Proof: Let z_i be a sequence converging to z such that $s(z_i)$ converges to $\bar{s} < \infty$. By definition, $\lambda(z_i, s(z_i)) = 0$. Since λ is continuous it follows that $\lambda(z,\bar{s}) = 0$ and hence $s(z) \leq \bar{s}$.

Let $u(\cdot)$, $v(\cdot)$ be any controls, then along any trajectory of (1) we must have

$$z(t+\tau) = z(t) + \tau e(\tau) \qquad \tau \geq 0$$

where,

$$e(\tau) = Az(t) + u(t) - v(t) + \frac{o(\tau)}{\tau}$$

and so

$$\lim_{\tau \to 0+} e(\tau) = Az(t) + u(t) - v(t).$$

Let s be a function such that $s(t+\tau) = s(t) - \tau\mu(\tau)$ with $\lim_{\tau \to 0+} \mu(\tau) = \mu$.

Then along this trajectory, we see from Lemma 1 that,

$$\left(\frac{d\lambda(z(t),s(t))}{dt}\right)^+ \triangleq \lim_{\tau \to 0+} \frac{\lambda(z(t) + \tau e(\tau), s(t) - \tau\mu(\tau)) - \lambda(z(t),s(t))}{\tau}$$

$$= \min_{\phi \in \Gamma(z(t),s(t))} [-\mu \frac{\partial}{\partial s} W_{F(s)}(\phi) - \phi'(Az(t) + u(t) - v(t))]. \quad (2)$$

Definition 8: For $z \in R^n$, $u \in U$, $v \in V$, $s > 0$, $\mu \in R$ let

$$\beta(z,s,u,v,\mu) = \min_{\phi \in \Gamma(z,s)} [- \mu \frac{\partial}{\partial s} W_{F(s)}(\phi) - \phi'(Az + u - v)] \qquad (3)$$

Lemma 3

$\beta(z,s,u,v,\mu)$ is continuous in u, v, μ and lower semicontinuous in z, s.

Proof: The continuity in u, v, μ is clear from (3). The lower semicontinuity in (z,s) follows from the fact that $\Gamma(z,s)$ is upper semicintinuous in (z,s) (Lemma 1a).

Definition 9: Let $\Omega(F) = \{z | 0 < s(z) < \infty\}$.

Theorem 1

Suppose there is $\alpha_0 > 0$ such that for all $z \in \Omega(F)$, for all $v \in V$, there exists $\mu \geq \alpha_0$ and $u \in U$ such that

$\beta(z,s(z),u,v,\mu) > 0.$

Then starting at $z_0 \in \Omega(F)$, P can capture E in time $\frac{s(z_0)}{\alpha_0}$.

Proof: Let $v(\cdot)$ be any admissible control, and let $\mu_0 \geq \alpha_0$, $u_0 \in U$ be such that $\beta(z_0,s(z_0),u_0,v_0,\mu_0) > 0$. Let $u(t) = u_0$ for $0 \leq t < t_1$, where $t_1 \geq 0$ is the largest $t_1 \geq 0$ such that $\beta(z(t),s(z_0) - \mu_0 t,u_0,v(t),\mu_0) \geq 0$ for $t \leq t_1$. Since v(t) is continuous from the right it follows from Lemma 3 that $t_1 > 0$.

Therefore at $t = t_1$, $\lambda(z(t_1),s(z_0) - \mu_0 t_1) \geq 0$, because $\lambda(z_0,s(z_0)) = 0$ and from (2), $(\frac{d\lambda}{dt}(z(t),s(z_0) - \mu_0 t))^+ \geq 0$ for $t \leq t_1$. Hence,

$s(z(t_1)) \leq s(z_0) - \mu_0 t_1 \leq s(z_0) - \alpha_0 t_1$, and in particular $z(t_1) \in \Omega(F)$.

Next we start at $z_1 = z(t_1)$ and let $\mu_1 \geq \alpha_0$, $u_1 \in U$ be such that $\beta(z_1, s(z_1), u_1, v_1, \mu_1) > 0$ where $v_1 = v(t_1)$. Let $u(t) = u_1$ for $t_1 \leq t < t_2$, where $t_2 \geq 0$ is the largest $t_2 \geq 0$ such that $\beta(z(t), s(z_1) - \mu_1(t-t_1), u_1, v(t), \mu_1) \geq 0$ for $t_1 \leq t \leq t_2$. Once again $t_2 - t_1 > 0$, and

$$s(z(t_2)) \leq s(z_1) - \mu_1(t_2 - t_1) \leq s(z_0) - \alpha_0\, t_2.$$

Continuing in this way we define $u(t) = u_i$, for $t_i \leq t < t_{i+1}$ with

$$s(z(t_{i+1})) \leq s(z_0) - \alpha_0\, t_{i+1} \quad , \quad i = 0, 1, 2, \ldots \tag{5}$$

Either in a finite number of steps $s(z(t)) = 0$ and $z(t) \in M$ (with $t \leq \frac{s(z_0)}{\alpha_0}$) or t_i converges to t^* and $z(t_i)$ converges to $z(t^*)$, and from (5) and Lemma 2, $0 \leq s(z(t^*)) \leq s(z_0) - \alpha_0\, t^*$. If $s(z(t^*)) = 0$, then $z(t^*) \in M$. Otherwise $0 < s(z(t^*)) \leq s(z_0) - \alpha_0\, t^*$. But then $z(t^*) \in \Omega(F)$. Starting at $z(t^*)$ we can repeat the procedure. Eventually for some t, $s(z(t)) = 0$ with $t \leq \frac{s(z_0)}{\alpha_0}$.

Remark 3

(i) Theorem 1 and its proof were suggested by [4]. In the light of Theorem 1, [4] can be considered as a proof of Theorem 1 for a particular F (see Section IV.1), and $\mu_i = 1$, for all i.

(ii) If we have a target function F, then (4) automatically gives us a closed-loop control $u = \psi(z,v)$ which achieves capture in time $\frac{s(z_0)}{\alpha_0}$. Thus the pursuit problem reduces to finding target functions which satisfy the hypothesis of the Theorem.

(iii) Sometimes it is convenient to relax the strict inequality in (4). But then we must assume that the closed-loop control is well behaved as follows.

Theorem 2

Suppose there is $\alpha_0 > 0$ and a closed-loop control $u = \psi(z,v)$ such that for all $z \in \Omega(F)$, $v \in V$, there exists $\mu \geq \alpha_0$ satisfying

$$\beta(z,s(z),\psi(z,v),v,\mu) \geq 0. \tag{6}$$

Suppose that for every admissible control $v(\cdot)$, and $z_0 \in \Omega(F)$, the solution to $\dot{z} = Az + \psi(z,v) - v$ with $z(0) = z_0$ exists, and the corresponding control $u(\tau) = \psi(z(\tau),v(\tau))$ is admissible. Then starting at $z_0 \in \Omega(F)$, P can capture E in time $\frac{s(z_0)}{\alpha_0}$ using the closed-loop control ψ.

Proof: Let $z(t)$ be a trajectory starting at $z_0 \in \Omega(F)$ and using admissible control $v(\cdot)$ and $u(\tau) = \psi(z(\tau),v(\tau))$. By hypothesis, there is a function $\mu(\tau) \geq \alpha_0$ such that

$$\beta(z(t),s(z(t)),u(t),v(t),\mu(t)) \geq 0.$$

It follows from (2) that

$$\left(\frac{d\lambda}{dt}(z(t),s(t))\right)^+ \geq 0$$

for

$$s(t) = s(z_0) - \int_0^t \mu(\tau)d\tau \leq s(z_0) - \alpha_0\, t.$$

Since $\lambda(z(0),s(z_0)) = 0$, we conclude that $\lambda(z(t),s(t)) \geq 0$ so that

$s(z(t)) \leq s(t) \leq s(z_0) - \alpha_0 t$. Hence $z(t) \in M$ for some $t \leq \dfrac{s(z_0)}{\alpha_0}$.

In some cases (6) implies (4) with α_0 replaced by $\alpha_0 - \delta$, as follows.

Definition 10 [5]: $z \in \Omega(F)$ is said to be regular if

$$\frac{\partial}{\partial s} W_{F(s)-z}(\phi)\bigg|_{s=s(z)} = \frac{\partial}{\partial s} W_{F(s)}(\phi)\bigg|_{s=s(z)} > 0 \text{ for all } \phi \in \Gamma(z,s(z)).$$

If every $z \in \Omega(F)$ is regular then $\Omega(F)$ is said to be regular.

Theorem 3

Suppose there is $\alpha_0 > 0$ such that for all $z \in \Omega(F)$, $v \in V$ there exists $u \in U$, $\mu \geq \alpha_0$ such that

$$\beta(z,s(z),u,v,\mu) \geq 0. \tag{7}$$

Suppose that $\Omega(F)$ is regular. Then starting at $z_0 \in \Omega(F)$, P can capture E in time $\dfrac{s(z_0)}{\alpha_0-\varepsilon}$ where $\varepsilon > 0$ is arbitrary.

Proof: Since $\Omega(F)$ is regular, it follows from (2), (7) that if $\delta > 0$,

$$\beta(z,s(z),u,v,\mu-\delta) > 0$$

for $z \in \Omega(F)$. Also $\mu-\delta \geq \alpha_0-\delta$ so the result follows from Theorem 1.

Remark 4

It may be helpful to note that $\Omega(F)$ is regular if and only if for all $z \in F(s(z))$, $z \in$ interior $(F(s(z) + \delta))$ for $\delta > 0$ sufficiently small.

In turn this implies $F(s) \cap M^C \subset F(s') \cap M^C$ and $\partial F(s) \cap \partial F(s') \subset M$ when $0 < s < s'$. (Here $M^C = \{z | z \notin M\}$).

Sometimes a target function F may be obtained as the geometric difference of two target functions† F_1, F_2 i.e., $F(s) \triangleq F_1(s) \underset{-}{*} F_2(s)$ (as for example in Section IV.1). Then instead of using the functions $W_{F(s)-z}, \lambda, \Gamma, s, \beta$ it is more convenient to use the functions $W^*_{F(s)-z}, \lambda^*, \Gamma^*, s^*, \beta^*$ which are defined as follows.

$$W^*_{F(s)-z}(\phi) = W_{F_1(s)-z}(\phi) - W_{F_2(s)}(\phi),\ \lambda^*(z,s) = \min \{W^*_{F(s)-z}(\phi)\ \ |\phi| = 1\},$$

$$\Gamma^*(z,s) = \{\phi\ \ |\phi| = 1,\ W^*_{F(s)-z}(\phi) = \lambda^*(z,s)\}, s^*(z) = \min \{s | s \geq 0, \lambda^*(z,s) = 0\}$$

and

$s^*(z) = \infty$ if $\lambda(z,s) > 0$ for all $s \geq 0$, and

$$\beta^*(z,s,u,v,\mu) = \min_{\phi \in \Gamma^*(z,s)} [-\mu \frac{\partial}{\partial s} W^*_{F(s)-z}(\phi) + (\frac{\partial}{\partial z} W^*_{F(s)-z}(\phi))'(Az+u-v)].$$

Let $\Omega^*(F) = \{z | 0 < s^*(z) < \infty\}$. The analogous version of Definition 10 is the following.

<u>Definition 11</u>: $z \in \Omega^*(F)$ is said to be <u>regular</u> if

$$\frac{\partial}{\partial s} W^*_{F(s)-z}(\phi)\Big|_{s=s^*(z)} = \frac{\partial}{\partial s} W_{F_1(s)}(\phi) - \frac{\partial}{\partial s} W_{F_2(s)}(\phi)\Big|_{s=s^*(z)} > 0 \text{ for all}$$

$\phi \in \Gamma^*(z, s^*(z))$.

$\Omega^*(F)$ is regular if every $z \in \Omega^*(F)$ is regular.

---

†In this case it is enough that F_1, F_2 satisfy C1 - C4, provided we use the functions W^*, λ^* defined below.

We can now obtain a development completely analogous to the development from Lemma 1 through Remark 4. Since there are no subleties involved we omit this. The interested reader can consult [14].

One further observation will prove useful.

Remark 5

Let $F(s)$ be a target function, or suppose F_1, F_2 are target functions and let $F = F_1 \overset{*}{-} F_2$. Let $\Phi(s)$ be a nonsingular $n \times n$ matrix function. It is sometimes more convenient to work with the set function $\Phi(s)\, F(s)$ instead of F directly. Since $\Phi(s)[F_1(s) \overset{*}{-} F_2(s)] = \Phi(s)\, F_1(s) \overset{*}{-} \Phi(s)\, F_2(s)$, and since $z \in F(s)$ if and only if $\Phi(s)\, z \in \Phi(s)\, F(s)$, we can equally well use the functions $W_{\Phi(s)F(s)-\Phi(s)z}(\phi)$ instead of $W_{F(s)-z}(\phi)$, and $W^*_{\Phi(s)F(s)-\Phi(s)z}(\phi)$ instead of $W^*_{F(s)-z}$; the functions λ^*, Γ^* etc., will also be changed appropriately. Such a nonsingular transformation will be employed often in the succeeding sections.

IV. SPECIFIC TARGET FUNCTIONS AND APPLICATIONS

IV.1. Open-loop capture in absorbtion sense. We again consider the system (1), together with the sets U, V, M.

Definition 12: Starting at $z(0) = z_0$ P can capture E (in the absorbtion sense) at time T, open-loop, if for all controls $v_{[0,T]}$, there exists a control $u_{[0,T]}$ (depending on $v_{[0,T]}$) such that $z(T, z_0, u_{[0,T]}, v_{[0,T]}) \in M$. Let $T^0(z_0)$ be the minimum such time; if none exists set $T^0(z_0) = \infty$. Let $Z^0(t) = \{z | T^0(z) = t\}$.

It is easy to see that $T^0(z_0) \leq T(z)$ for all z. In general there is no ordering between $T^0(z)$ and $T^*(z)$.

Definition 13:

a. Let $\mathcal{U}(t) = \int_0^t e^{A\tau}\, U d\tau \triangleq \{\int_0^t e^{A\tau}\, u(\tau)d\tau \,|\, u_{[0,t]} \text{ admissible}\}$

b. Let $\mathcal{V}(t) = \int_0^t e^{A\tau}\, V d\tau \triangleq \{\int_0^t e^{A\tau}\, v(\tau)d\tau \,|\, v_{[0,t]} \text{ admissible}\}$.

c. Let $\mathcal{a}^0(t) = [-M + \mathcal{U}(t)] \overset{*}{-} \mathcal{V}(t)$.

The next result is immediate.

Lemma 4

a. $W_{\mathcal{U}(T)}(\phi) = \int_0^T \max_{u \in U} (\phi' e^{A\tau} u) d\tau, W_{\mathcal{V}(T)}(\phi) = \int_0^T \max_{v \in V} (\phi' e^{A\tau} v) d\tau.$

b. $W_{\mathcal{a}^0(T)} = co\, [W^*_{\mathcal{a}^0(T)}]$ † where,

$$W^*_{\mathcal{a}^0(T)}(\phi) \triangleq W_{-M+\mathcal{U}(T)}(\phi) - W_{\mathcal{V}(T)}(\phi) = W_{-M}(\phi) + W_{\mathcal{U}(T)}(\phi) - W_{\mathcal{V}(T)}(\phi)$$

c. $z \in Z^0(T)$ if and only if $-e^{AT}z \in \mathcal{a}^0(T)$.

We can use Z^0 as a target function. Equivalently, from Lemma 4c and Remark 5 we may use the function $\mathcal{a}^0(s) = -e^{As}\, Z^0(s)$. Because of the simple characterization obtained in Lemma 4a, 4b we shall use $W^*_{\mathcal{a}^0} = W_{-M+\mathcal{U}} \overset{*}{-} W_{\mathcal{V}}$ †† instead of $W_{\mathcal{a}^0}$. Thus let,

---

† Recall co[W] = largest convex function bounded from above by W.

†† $W_{\mathcal{V}}$ satisfies conditions C1 - C4 since $\mathcal{V}$ is compact. In order that $W_{-M+\mathcal{U}} = W_{-M} + W_{\mathcal{U}}$ satisfy C1 - C4 we must assume that K_{-M} is closed. This is true if M is described by linear inequalities.

$$W^*(z,s,\phi) = W^*_{\mathcal{A}^0(s)+e^{As}z}(\phi) = \phi' e^{As} z + W_M(-\phi) + W_{\mathcal{U}(s)}(\phi) - W_{\mathcal{V}(s)}(\phi). \tag{8}$$

$$\lambda^*(z,s) = \mathrm{Min}\ \{W^*(z,s,\phi) \mid |\phi| = 1\} \tag{9}$$

$$\Gamma^*(z,s) = \{\phi \mid |\phi| = 1,\ W^*(z,s,\phi) = \lambda^*(z,s)\} \tag{10}$$

$$s^*(z) = \min\ \{s \mid s \geq 0,\ \lambda^*(z,s) = 0\},\ s^*(z) = \infty, \text{ if } \lambda^*(z,s) \geq 0 \text{ for all } s \geq 0. \tag{11}$$

$$\beta^*(z,s,u,v,\mu) = \min_{\phi \in \Gamma^*(z,s)} \left[-\mu \frac{\partial}{\partial s} W^*(z,s,\phi) + \left(\frac{\partial}{\partial z} W^*(z,s,\phi)\right)(Az+u-v)\right]$$

$$= \min_{\phi \in \Gamma^*(z,s)} \left[-\mu\left(\phi' A e^{As} z + \max_{u \in U} \phi' e^{As} u - \max_{v \in V} \phi' e^{As} v\right) + \phi' e^{As}(Az+u-v)\right]. \tag{12}$$

We can now specialize Theorems 1, 2, 3 in terms of the particular functions introduced above. As example, we specialize Theorem 3. Let $\Omega^*(\mathcal{A}^0) = \{z \mid 0 < s^*(z) < \infty\}$. Following Definition 11, we say that $\Omega^*(\mathcal{A}^0)$ is regular if for all $z \in \Omega^*(\mathcal{A}^0)$, and all $\phi \in \Gamma^*(z,s^*(z))$,

$$\phi' e^{As^*(z)} Az + \max_{u \in U} \phi' e^{As^*(z)} u - \max_{v \in V} \phi' e^{As^*(z)} v > 0.$$

<u>Theorem 4</u>

Suppose that $\Omega^*(\mathcal{A}^0)$ is regular. Suppose that there is $\alpha_0 > 0$ such that for all $z \in \Omega^*(\mathcal{A}^0)$ and $v \in V$, there exist $\mu \geq \alpha_0$ and $u \in U$ such that

$$\phi' e^{As^*(z)}(u-v) + (1-\mu)\phi' e^{As^*(z)} Az \geq \left[\max_{u \in U} \phi' e^{As^*(z)} u - \max_{v \in V} \phi' e^{As^*(z)} v\right]. \tag{13}$$

for all $\phi \in \Gamma^*(z,s^*(z))$. Then starting at $z_0 \in \Omega^*(\mathcal{Q}^0)$, P can capture E in time $\dfrac{s^*(z_0)}{\alpha_0 - \varepsilon}$ where $\varepsilon > 0$ is arbitrary.

Proof: Because of the representation (12), (13) is equivalent to (7).

Remark 6

The results of [4] correspond to Theorem 4 with the additional condition that (10) holds for $\mu = 1$, so that capture is guaranteed in time $s^*(z_0) + \varepsilon$ where $\varepsilon > 0$ is arbitrary.

IV.2. Extremal Aiming. By imposing additional conditions on the dynamics, and the sets U, V, M we can use the target functions defined in Section IV.1 to simplify and make rigorous (for the linear case) the extremal aiming pursuit strategy studied by Krasovskii [9], [15]-[18].[†] Thus consider system (1) again, and the sets U, V, M. We define the functions W^*, λ^*, Γ^*, s^*, β^* by the equations (8)-(12). Let $z_0 \notin M$ be a fixed initial state such that $s^* \triangleq s^*(z_0) < \infty$. Then s^* is the smallest t such that $[e^{At}z_0 + \mathcal{U}(t) - M] \supset \mathcal{V}(t)$. Consider Assumptions 1 and 2.

Assumption 1: For every $t \in [0,s^*]$ and every $z \in e^{At}z_0 + \mathcal{U}(t) - \mathcal{V}(t)$, $s^*(z) < \infty$, and furthermore

$$\partial[e^{As^*(z)} + \mathcal{U}(s^*(z)) - M] \cap \partial\mathcal{V}(s^*(z)) = \{p^*(z)\} \tag{14}$$

[†][9] is concerned with the linear case only. [15]-[18] consider nonlinear dynamics with applications to the linear case.

[††]Krasovskii considers dynamics which separate as $\dot{x} = A_1x + u$, $\dot{y} = A_2y + v$. He also requires M to be a subspace.

consists of a single point $p^*(z)$. Here ∂K denotes the relative boundary of a closed convex set K.

Assumption 2: The set $(\mathcal{U}(t) - M)$ has smooth boundaries i.e., at every point on the boundary of $\mathcal{U}(t) - M$ there is a unique outward normal to the set $\mathcal{U}(t) - M$.

From Assumptions 1 and 2 we immediately obtain the next fact.

Fact 2: Suppose Assumptions 1 and 2 are satisfied. Then,

$$\Gamma^*(z,s^*(z)) = \{\phi(z)\} \tag{15}$$

consists of a single vector, for all $z \in e^{At}z_0 + \mathcal{U}(t) - \mathcal{V}(t)$ and for all $0 \leq t \leq s^*(z_0)$.

Theorem 5

Suppose Assumption 1, 2 are satisfied. Let $\psi : R^n \to U$ be any function such that

$$\phi'(z)e^{As^*(z)}\psi(z) = \max_{u \in U} \phi'(z)e^{As^*(z)}u \tag{16}$$

for all $z \in e^{At}z_0 + \mathcal{U}(t) - \mathcal{V}(t)$, and all $0 \leq t \leq s^*(z_0)$. (In (16) $\phi(z)$ is given by (15).) Suppose that for every admissible control $v_{[0,s^*(z_0)]}$, a solution to

$$\dot{z}(t) = Az(t) + \psi(z(t)) - v(t),\ z(0) = z_0, \tag{17}$$

exists. Then, starting at $z(0) = z_0$, and using the closed-loop control $u = \psi(z)$, P can capture E in time $s^*(z_0)$.

Proof: From Fact 2, (12), and (16),

$$\beta^*(z,s^*(z),\psi(z),v,1) = -\left[\max_{u \in U} \phi'(z)e^{As^*(z)}u - \min_{v \in V} \phi'(z)e^{As^*(z)}v\right]$$

$$+ \phi'(z)e^{As^*(z)}(\psi(z)-v)$$

$$= \max_{v \in V} \phi'(z)e^{As^*(z)}v - \phi'(z)e^{As^*(z)}v \geq 0.$$

The result now follows from Theorem 2 (for the function β^* instead of β), and $\alpha_0 = \mu = 1$.

We shall now show why the control $u = \psi(z)$ is called the extremal aiming strategy.

Definition 14: Given a point $p^* \in [e^{At}z_0 + \mathcal{U}(t) - M]$, $u^* \in U$ is said to be aimed at p^* if there is an admissible control $u_{[0,t]}$ such that $u(0) = u^*$ and,

$$e^{At}z_0 + \int_0^t e^{A(t-\tau)}u(\tau)d\tau \in p^* + M.$$

Lemma 5

Suppose Assumptions 1, 2 hold. For every $z \in e^{At}z_0 + \mathcal{U}(t) - \mathcal{V}(t)$, and $0 \leq t \leq s^*(z_0)$, the control $u = \psi(z)$ is aimed at $p^*(z) \in e^{As^*(z)} + \mathcal{U}(s^*(z)) - M$, where $p^*(z)$ is given in (14).

Proof: By Assumption 1, there is a vector $q \in M$ and $r \in \partial\mathcal{U}(s^*(z))$ such that

$$e^{As^*(z)}z + r - q = p^*(z).$$

Also since $\{\phi(z)\} = \Gamma^*(z,s^*(z))$, it follows that $\phi(z)$ is the outward normal to $e^{As^*(z)}z + \mathcal{R}(s^*(z)) - M$ at the point $p^*(z)$. Hence $\phi(z)$ is also an outward normal to the set $\mathcal{R}(s^*(z))$ at the point r. From the Pontriagin maximum principle, we conclude that any open-loop control $u^*(t)$ satisfying

$$\phi'(t)u^*(t) = \max_{u \in U} \phi'(t)u \tag{18}$$

where

$$\phi(t) = e^{A'(s^*(z_0)-t)}\phi(z) \quad , \quad 0 \leq t \leq s^*(z)$$

also satisfies

$$r = \int_0^{s^*(z)} e^{A(s^*(z)-\tau)}u^*(\tau)d\tau .$$

But then from (16), (18) we can assume that $u^*(0) = \psi(z)$ and the lemma is proved.

Remark 7

(i) In [9], Krasovskii does not explicitly state Assumption 1. He appears to require it to hold only along every trajectory which results from the extremal-aiming control. But then it is not very clear how to define the function ψ precisely.

(ii) The two assumptions are extremely difficult to verify in general. Some simplification obtains if the dynamics separate and if M is a subspace as in the previous footnote. Theorem 2 of [6] states a relatively simple condition which presumably implies Assumptions 1 and 2.

Unfortunately the proof given there is erroneous, and in fact [14] contains a counter-example. In [7], Sakawa is concerned mainly with open-loop capture in the absorbtion sense. In his concluding remarks he does give the extremal aiming strategy. He states Assumption 1 explicitly, and assumes that Assumption 2 holds in general which is incorrect. Finally there is a serious mistake in [7] which limits its applicability to target sets $M = S_\varepsilon$.

(iii) In Theorem 5 we have assumed that ψ is well-behaved i.e., a solution to (17) always exists. This can be guaranteed if Assumption 1 is required to hold in an open neighborhood of every $z \in e^{At}z_0 + \mathcal{U}(t) - \mathcal{V}(t)$, $0 \leq t \leq s^*(z_0)$, and if in addition (16) uniquely determines $\psi(z)$. Then one can show that $\psi(z)$ is differentiable in z so that a solution to (17) is guaranteed. (See for instance Theorem 5.1 of [3]).

(iv) Note that the control ψ does not depend upon the control of E.

<u>IV.3. $Z(\cdot)$ as a Target Function</u>. Following Pontriagin [4], we can use the function $Z(\cdot)$ introduced in Definition 3, as a target function. It follows almost immediately from the definition that for all $\delta > 0$, $v_{[0,\delta]}$ admissible, and $z_0 \in Z(t)$, there exists a closed-loop control $u(\tau) = \psi(z(\tau),v(\tau))$ such that

$$z(\delta, z_0, u_{[0,\delta]}, v_{[0,\delta]}) \in Z(t-\delta).$$

Since $Z(0) = M$, we can see that the control $u(\tau) = \psi(z(\tau), v(\tau))$ and the function $Z(\cdot)$ satisfy the hypothesis of Fact 1[†] with $\mu(t) \equiv \alpha_0 = 1$. Thus

[†] $Z(\cdot)$ may not satisfy <u>C2</u> - <u>C4</u>, however.

$Z(\cdot)$ appears to be an attractive target function. In fact it can be appropriately modified to apply for the time-varying case also [19]. However two points must be kept in mind. First of all, if we are interested in capture in the sense of Definition 2, then the set Z may be a highly conservative estimate of Z^* (witness Example 1). Secondly, in general it is extremely difficult to compute $Z(\cdot)$. Under some special cases these drawbacks may be overcome, as follows.

Again consider system (1) and sets U, V, M.

<u>Definition 15</u>: The target set M is said to be <u>absorbing</u> if for all $z_0 \in M$, for all admissible $v_{[0,T]}$, there is a $u_{[0,T]}$ such that

$$z(T, z_0, u_{[0,T]}, v_{[0,T]}) \in M.$$

(Evidently M is absorbing if the above condition holds for all $0 < T \leq \delta$ where $\delta > 0$ is arbitrarily small.)

The next lemma is straightforward.

<u>Lemma 6</u>

If M is absorbing, then $Z(t) = Z^*(t)$ for $t \geq 0$.

<u>Remark 8</u>

(i) If M is a subspace, L is its orthogonal complement and π the orthogonal projection of R^n onto L, then M is absorbing if for all $z \in M$, $v \in V$, there is $u \in U$ such that $\pi(Az + u - v) = 0$.

(ii) The requirement that M be absorbing is highly restrictive. However it appears to be crucial if one wishes to prove existence of an optimal strategy for a zero-sum differential game where the pay-off to

E is $T^*(z_0)$ (see the assumptions in [1],[2],[20]).

We now turn to the second drawback noted above. Following [4], for each positive integer k we inductively define set function $Z^k(\cdot)$ as follows

$$Z^k(0) = M$$

for $0 < \tau \leq 2^{-k}$, $\quad Z^k(\tau) = \{z \mid -e^{A\tau}z \in [-Z^k(0) + \mathcal{U}(\tau) \overset{*}{-} \mathcal{V}(\tau)]\}$

$\vdots$

for $i2^{-k} < \tau \leq (i+1)2^{-k}$, $Z^k(\tau) = \{z \mid -e^{As}z \in [-Z^k(i2^{-k}) + \mathcal{U}(s) \overset{*}{-} \mathcal{V}(s)]\}$

where $s = \tau - i2^{-k}$.

In other words if $i2^{-k} < \tau \leq (i+1)2^{-k}$ and $s = \tau - i2^{-k}$, then $z \in Z^k(\tau)$ if and only if for every control $v_{[0,s]}$ there is a control $u_{[0,s]}$ such that

$$e^{As}z + \int_0^s e^{A(s-\sigma)}[u(\sigma) - v(\sigma)]d\sigma \in Z^k(i2^{-k}).$$

With this characterization we can obtain the first part of the next lemma, and the second part follows by a limiting argument (see [4] or [19] for details).

<u>Lemma 7</u>

(i) $Z^{k+1}(\tau) \subset Z^k(\tau)$ for all τ.

(ii) $Z(\tau) = \bigcap_{k=1}^{\infty} Z^k(\tau)$ for all τ.

Thus for sufficiently large k, $Z^k(\tau)$ is a good approximation for

$Z(\tau)$. However the "construction" of $Z^k(\tau)$ described above is completely impractical in general. There is one special case where we can directly obtain $Z(\cdot)$ in terms of its support function. This is the case where all the functions $Z^k(\cdot)$ coincide with the function $Z^0(\cdot)$ introduced in Definition 12 above. But first let us define

$$\mathcal{A}^k(\tau) = -e^{A\tau} Z^k(\tau) \triangleq \{-e^{A\tau}z \mid z \in Z^k(\tau)\},$$

$$\mathcal{A}(\tau) = -e^{A\tau} Z(\tau) \triangleq \{-e^{A\tau}z \mid z \in Z(\tau)\}.$$

<u>Lemma 8</u>

Statements (i), (ii) are equivalent, (iii), (iv) are equivalent, and (iii) implies (ii).

(i) $\mathcal{a}^0(\tau) = \mathcal{A}(\tau)$ for $0 \le \tau \le t$

(ii) $\mathcal{a}^0(\tau) = \mathcal{A}^k(\tau)$ for $0 \le \tau \le t$, $k = 1, 2, 3, \ldots.$

(iii) $W_{\mathcal{a}^0(\tau)} = W^*_{\mathcal{a}^0(\tau)} \triangleq W_{-M} + W_{\mathcal{N}(\tau)} - W_{\mathcal{V}(\tau)}$ for $0 \le \tau \le t$.

(iv) $W^*_{\mathcal{a}^0(\tau)}(\phi)$ is a convex function of ϕ for all $0 \le \tau \le t$.

<u>Proof:</u> Since $\mathcal{a}^0(\tau) \supset \mathcal{A}^1(\tau) \supset \mathcal{A}^2(\tau) \supset \ldots. \supset \mathcal{A}(\tau)$, (i) and (ii) are equivalent by Lemma 7 (ii). Since $W_{\mathcal{a}^0(\tau)} = \mathrm{co}[W^*_{\mathcal{a}^0(\tau)}]$, $W_{\mathcal{a}^0(\tau)} = W^*_{\mathcal{a}^0(\tau)}$ if and only if $W^*_{\mathcal{a}^0(\tau)}(\phi)$ is a convex function of ϕ, so that (iii) and (iv) are equivalent. The final assertion is proved in the Appendix.

From this lemma, we can see that if $W_{-M} + W_{\mathcal{N}(\tau)} - W_{\mathcal{V}(\tau)}$ is a convex function for each $\tau \in [0,t]$, then it is also the support function of $\mathcal{A}(\tau)$, for $\tau \in [0,t]$. The lemma also shows that this is a quite restrictive condition since it implies that the notions of open-loop capture in the absorbtion sense and closed-loop capture in the absorbtion sense coincide.

The above lemma can also be used to derive the results of Kimura [8]. He considers system (1) with control sets U, V. Instead of the target set M, suppose that the players P, E wish to minimize, maximize respectively the payoff function

$$J(u_{[0,T]},v_{[0,T]}) = |Dz(T,z_0,u_{[0,T]},v_{[0,T]})|$$

where z_0 is a fixed initial condition, T is a fixed time, and D is a fixed $m \times n$ matrix. The information available to P at time t is z(t) and v(t), so that the game must be considered in the closed-loop sense. Also since T is fixed, the situation is analogous to absorbtion type capture. Kimura proceeds by first considering the open-loop situation. Thus let

$$\varepsilon = \min_{v_{[0,T]}} \max_{u_{[0,T]}} |Dz(T,z_0,u_{[0,T]},v_{[0,T]})| \tag{19}$$

where of course the functions $v(\cdot)$, $u(\cdot)$ range over the set of admissible controls. Let

$$M_\varepsilon = \{z \mid |Dz| \leq \varepsilon\}.$$

Then from (19) and Definition 12 it follows that T is the smallest time t such that $z_0 \in Z^0_\varepsilon(t)$ i.e., $T^0_\varepsilon(z_0) = T$. (Here Z^0_ε, T^0_ε correspond to the target M_ε.) Then, Kimura derives conditions which guarantee that

$\varepsilon(z_0)$ is the optimum payoff even in the closed-loop case. The condition turns out to be exactly the same as Lemma 8 (iii). The closed-loop pursuit strategy can then be obtained from (13) for $\mu = 1$. Further sim- simplification can be achieved if the support function $W^*_{\mathcal{A}^0_\varepsilon(\tau)}$ is strictly convex for $\tau \in [0,T]$ because then the set Γ consists of only one point. For details see [14].[†]

IV.4. An Example. In this section we present a pursuit problem with trivial dynamics, namely where the matrix A in (1) is the zero matrix. The problem can be solved by inspection. We shall solve it by using a target function. The purpose of the example is to note that none of the target functions presented so far is applicable to this problem, if the set V contains at least two points and if it is not contained in U. (V, U are defined below.) Consider the system

$$\dot{\tilde{z}} = \tilde{u} + \tilde{v} \tag{20}$$

where $\tilde{u} \in \tilde{U}$, $\tilde{v} \in \tilde{V}$ with $\tilde{U}$, $\tilde{V}$ compact and convex subsets of R^n. The target $\tilde{M}$ is a subspace of R^n. Let π denote the orthogonal projection of R^n onto L, the orthogonal complement of $\tilde{M}$. Then the pursuit problem (20) is equivalent to (21),

$$\dot{z} = u + v \tag{21}$$

where $z = \pi\tilde{z}$, $u \in U = \pi(\tilde{U})$, $v \in V = \pi(\tilde{V})$ and the target is $M = \{0\}$.[††]

[†] The following notational correspondence may help the reader. $-G(t,x,\xi)$ in [8] $\leftrightarrow W^*_{\mathcal{A}^0_0(T)+e^{AT}z}$, $\bar{V}(t,x)$ in [8] $\leftrightarrow \varepsilon$, $-F_\alpha(t,\xi)$ in [8] $\leftrightarrow W^*_{\mathcal{A}^0_\alpha(T)}(\phi)$ and $R_t(\alpha)$ in [8] $\leftrightarrow \mathcal{A}^0_\alpha(T)$.

[††] This projection onto a lower-dimensional problem with target $\{0\}$ is possible only because π commutes with the transition matrix of (20), which is I. This fact was overlooked in [7], thereby drastically reducing its contribution.

Fact 3: Starting at $z_0 \neq 0$, P can capture E in a finite time if and only if for all $v \in V$, there is a $u \in U$ and $\alpha_0 > 0$ such that $u + v = -\mu z_0$ for some $\mu \geq \alpha_0$ (and then capture is possible in time $\frac{1}{\alpha_0}$).

Proof: Let $\bar{v} \in V$ be such that $u + \bar{v} \neq -\mu z_0$ for all $u \in U$, $\mu > 0$. Then for $v(t) \equiv \bar{v}$, $z(t) \neq 0$ for all $t \geq 0$ and all admissible $u(\cdot)$. Conversely consider the target function $F(s) = \{sz_0\}$. Then $\Omega(F) = \{rz_0 | 0 < \Omega < \infty\}$ and condition (6) becomes $-\phi'(\mu z_0 + \psi(z,v)+v) \geq 0$ for all $|\phi| = 1$, $z \in \Omega(F)$, so that $\mu z_0 + \psi(z,v) + v = 0$ for all $z \in \Omega(F)$. By hypothesis, this can be satisfied by a closed-loop control ψ for some $\mu \geq \alpha_0$. The sufficiency then follows from Theorem 2.

V. COMMENTS

The target function approach gives us an intuitive geometric framework which helps infinding strategies for linear pursuit problems. We hope that the examples of specific functions given have shown that the approach is also unifying and operational. The results of [22], [23] can also be derived using target functions. The approach can be extended to nonlinear dynamics as in [16]-[18]. We have not done this. In many ways the approach is similar to Lyapunov's second method. As in the latter theory, general results are easy to come by. Specific target functions are as difficult to find as specific Lyapunov functions. The analogy goes deeper. In a recent paper Krasovskii [21] has presented ideas quite similar to those presented here, but more along the lines of Lyapunov theory. However he does not develop them.

APPENDIX

Those statements below which are stated without proof, are elementary.

I. Support functions. Let F be a closed, convex set. Let W_F, λ_F, Γ_F be as in Definition 5.

Fact A1: $W_F : R^n \to R \cup \{\infty\}$ is a lower semicontinuous convex, positively homogeneous† function.

Fact A2: Let $W : R^n \to R \cup \{\infty\}$ be a lower semicontinuous, convex, positively homogenous function. Let

$$F = \{z \mid \phi' z \leq W(\phi) \text{ for all } \phi \in R^n\}$$

Then F is a closed, convex set and $W_F = W$.

Fact A3: Let G be a closed, convex set. Then

(i) $W_{F+G} = W_F + W_G$

(ii) $W_F = W_G$ if and only if $F = G$

(iii) $W_F \leq W_G$ if and only if $F \subset G$.

Corollary

The following statements are equivalent.

(i) $0 \in F$.

(ii) $W_F(\phi) \geq 0$ for all ϕ

(iii) $\lambda_F \geq 0$

Recall $d(z,G) \triangleq \min\{|z - x| \mid x \in G\}$ and $\partial F \triangleq$ boundary of F.

† i.e., $W_F(\alpha\phi) = \alpha W_F(\phi)$ for all $\alpha \geq 0$, $\phi \in R^n$.

Fact A4: (i) $\lambda_F = + d(0,\partial F)$ if $0 \in F$

$\lambda_F = - d(0,\partial F)$ if $0 \notin F$.

(ii) If $\lambda_F < 0$ then Γ_F consists of a single vector. If $\lambda_F = 0$, then $0 \in \partial F$, and Γ_F consists of all vectors $\phi \in R^n$ such that $|\phi| = 1$, and ϕ is an outward normal to a hyperplane supporting F at 0.

II. Let A, B be closed convex sets with B bounded. Let $C = A \underline{*} B$ and $W_C^* \triangleq W_{A,B}^* \triangleq W_A^* - W_B^*$.

Fact A5: C is a closed, convex (possibly empty) set.

Fact A6: $W_C = co[W_A - W_B] = co[W_C^*]$.

Proof: Since B is bounded, W_B is continuous so that $W_C^* = W_A - W_B$ is lower semicontinuous. Since W_A, W_B are positively homogeneous, so is W_C^*. Hence $co[W_C^*]$ is lower semicontinuous, convex, and positively homogeneous so that by Fact A2, there is a closed convex set C^* such that $W_{C^*} = co[W_C^*]$. Since $W_{C^*} \leq W_A - W_B$, it follows from Fact A3 that $C^* + B \subset A$, so that $C^* \subset C$. Next, since $C = A \underline{*} B$, $B + C \subset A$, so that $W_C \leq W_A - W_B$. Since W_C is convex, $W_C \leq co[W_A - W_B] = W_{C^*}$ so that $C \subset C^*$. But then $C = C^*$.

Fact A6: $\lambda_{A,B}^* \geq 0$ if an only if $B \subset A$.

Let F(s), $s \geq 0$ be a target function satisfying conditions C1-C4. Let $W(z,s,\phi) = W_{F(s)-z}(\phi)$, $\lambda(z,s) = \lambda_{F(s)-z}$, $\Gamma(z,s) = \Gamma_{F(s)-z}$.

Fact A7: λ is a continuous function of $(z,s) \in R^n \times [0, \infty)$.

Proof: Since F(s) varies continuously in s, $d(z,\partial F(s))$ is a continuous function of (z,s). By Fact A4 (i), it follows that λ is continuous.

III. Proof of Lemma 1.

a) Let $z_i \in R^n$, $s_i \geq 0$, $\phi_i \in \Gamma(z_i,s_i)$, $i = 1, 2, \ldots$ be sequences converging respectively to z, s, ϕ. By definition $|\phi_i| = 1$ for all i, so that $|\phi| = 1$. Next, $W(z_i,s_i,\phi_i) = \lambda(z_i,s_i)$ for all i, and since W, λ are continuous, it follows that $W(z,s,\phi) = \lambda(z,s)$ so that $\phi \in \Gamma(z,s)$. But this means that Γ is upper semicontinuous.

b) Set $z_\tau = z + \tau e$, $s_\tau = s + \tau\mu$. For all $\phi_\tau \in \Gamma(z_\tau,s_\tau)$, and $\phi \in \Gamma(z,s)$,

$$W(z_\tau,s_\tau,\phi_\tau) = \lambda(z_\tau,s_\tau) \leq W(z_\tau,s_\tau,\phi)$$

and

$$W(z,s,\phi_\tau) \geq \lambda(z,s) = W(z,s,\phi)$$

so that

$$\begin{aligned} W(z_\tau,s_\tau,\phi_\tau) - W(z,s,\phi_\tau) &\leq \lambda(z_\tau,s_\tau) - \lambda(z,s) \\ &\leq W(z_\tau,s_\tau,\phi) - W(z,s,\phi). \end{aligned}$$

From the mean-value theorem there exists $s^* = s + \tau^*\mu$, $z^* = z + \tau^* e$ with $0 \leq \tau^* \leq \tau$ such that $W(z_\tau,s_\tau,\phi_\tau) - W(z,s,\phi_\tau) = \tau[(\frac{\partial W}{\partial z}(z^*,s^*,\phi_{\tau^*}))' e + \frac{\partial W}{\partial s}(z^*,s^*,\phi_{\tau^*})\mu]$. Combining this with the left-hand inequality we see that (since Γ is upper semicontinuous),

$$\lim_{\tau \to 0+} \frac{\lambda(z_\tau,s_\tau) - \lambda(z,s)}{\tau} \geq \min_{\phi \in \Gamma(z,s)} [(\frac{\partial W}{\partial z}(z,s,\phi))' e + \mu \frac{\partial W}{\partial s}(z,s,\phi)]$$

Similarly, from the right-hand inequality,

$$\lim_{\tau \to 0+} \frac{\lambda(z_\tau, s_\tau) - \lambda(z,s)}{\tau} \leq \min_{\phi \in \Gamma(z,s)} [(\frac{\partial W}{\partial z}(z,s,\phi))'e + \mu \frac{\partial W}{\partial s}(z,s,\phi)]$$

c) The proof of this part is essentially a duplication of the above argument.

IV. Proof of Lemma 8.

We must show that statement (iii) of Lemma 8 implies statement (ii). This follows immediately from Fact A8 below.

Let M, U_1, U_2, ... and V_1, V_2, ... be closed convex sets with V_1, V_2, ... bounded. Define, $\mathcal{A}_\ell = (-M + \sum_{i=1}^{\ell} U_i) \overset{*}{-} (\sum_{i=1}^{\ell} V_i)$. Let $A_1 = \mathcal{A}_1 = (-M + U_1) \overset{*}{-} V_1$ and for $\ell \geq 1$, let $A_{\ell+1} = (A_\ell + U_{\ell+1}) \overset{*}{-} V_{\ell+1}$.

Fact A8: Statement (a) below, implies (b).

(a) $W_{\mathcal{A}_\ell} = W_{-M} + \sum_{i=1}^{\ell} W_{U_i} - \sum_{i=1}^{\ell} W_{V_i}$, $\ell = 1, \ldots, L$.

(b) $\mathcal{A}_\ell = A_\ell$, $\ell = 1, \ldots, L$.

Proof: For L = 1, the assertion is true by definition. Suppose it is true for L. Then,

$$W_{\mathcal{A}_{L+1}} = W_{-M} + \sum_{i=1}^{L+1} W_{U_i} - \sum_{i=1}^{L+1} W_{V_i}$$

$$= W_{\mathcal{A}_L} + W_{U_{L+1}} - W_{V_{L+1}} \qquad \text{by hypothesis}$$

$= W_{A_L} + W_{U_{L+1}} - W_{V_{L+1}}$ by induction hypothesis

$= \mathrm{co}[W_{A_L} + W_{U_{L+1}} - W_{V_{L+1}}],$ since $W_{\mathcal{A}_{L+1}}$ is convex,

$= W_{A_{L+1}}$, by definition and Fact A6

so that $\mathcal{A}_{L+1} = A_{L+1}$ by Fact A3 (ii).

REFERENCES

[1] A. Friedman, "Existence of Value and Saddle Points for Differential Games of Pursuit and Evasion," J. Diff. Eq. Vol. 7, No. 1, (1970), 92-110.

[2] P. Varaiya and J. Lin, "Existence of Saddle Points in Differential Games", SIAM J. Control Vol. 7, No. 1 (1969), 141-157.

[3] B. N. Pshenichnyi, "Structure of Differential Games," Dokl. Akad. Nauk SSSR Vol. 184, No. 2 (1969), 285-287.

[4] L. S. Pontriagin, "Linear Differential Games I and II," Soviet Math. Dokl. Vol. 8, Nos. 2 and 3 (1967), 769-771 and 910-912.

[5] B. N. Pshenichnyi, "Linear Differential Games," Automation and Remote Control Vol. 29, (1968), 65-78.

[6] N. Sugino, "Sufficient Strategies in Pursuit-Evasion Games," Proc. First Int. Conf. on Theory and Appl. of Differential Games, Univ. of Massachusetts, Amherst, Mass. (1969).

[7] Y. Sakawa, "Solution of Linear Pursuit-Evasion Games," SIAM J. Control, Vol. 8, No. 1, (1970), 100-112.

[8] H. Kimura, "Linear Differential Games with Terminal Payoff," IEEE Trans. Automatic Control, Vol. AC-15, No. 1, (1970), 58-66.

[9] N. N. Krasovskii, Theory of Control of Movement, Isd. Nauka, Moscow, (1969) (in Russian).

[10] D. L. Kelendzheridge, "On the Theory of Optimal Pursuit," Soviet Math., Vol. 2, (1961) 654-656.

[11] R. Chattopadhyay, "Functional Analytic Analysis of a Pursuit Problem," J. Optimization Theory and Appl., Vol. 2, No. 4, (1968), 230-239.

[12] R. Isaacs, Differential Games, Wiley, New York, (1965).

[13] G. Lietmann, "A Simple Differential Game," J. Optimaization Theory and Appl., Vol. 2, No. 4, (1968), 270-275.

[14] W. Borgest, "Linear Pursuit-Evasion Games," Ph.D. Dissertation, Univ. of California at Berkeley, California (1970).

[15] N. N. Krasovskii, "On a Problem of Pursuit," J. Appl. Math. Mech., Vol. 27, No. 2, (1963), 363-369.

[16] N. N. Krasovskii, "On a Game of Interception," Dokl. Akad. Nauk SSSR, Vol. 181, No. 5, (1968), 1062-1064 (in Russian).

[17] N. N. Krasovskii and A. I. Subbotin, "A Problem of Interception of Controlled Objects," Prikl, Mat. Mekh., Vol. 32, No. 4, (1968), 575-586, (in Russian).

[18] N. N. Krasovskii, Yu. M. Repin, and V. I. Tretiakor, "Theory of Games in Control System," Technical Cybernetics, No. 4, (1965), 7-11.

[19] M. S. Nikolsky, "Direct L. S. Pontrjagin Methods in the Linear Nonstationary Differential Games of Pursuit," Proc. First Int'l Conf. on Theory and Appl. of Differential Games, Univ. of Massachusetts, Amherst, Mass., (1969).

[20] W. H. Fleming, "The Convergence Problem for Differential Games," J. Math. Anal. Appl., Vol. 3, (1961), 102-116.

[21] N. N. Krasovskii, "On the Pursuit Problem," Dokl. Akad. Nauk SSSR, Vol. 191, No. 2, (1970) 523-526 (in Russian).

[22] B. N. Pshennichnyi, "On Linear Differential Games," Kibernetica, (Jan.-Feb. 1968) (in Russian).

[23] B. N. Pshennichnyi, and Yu. N. Onopchnk, "Linear Differential Games with Integral Restrictions," Teknicheskaya Kibernetika, (1), (1968), 13-22 (in Russian) (English Trans. Available from Joint Publications Research Service JPRS 45881 (9July 1968), Department of Commerce, USA).