SOME RESULTS CONCERNING TIME-VARYING NETWORKS HAVING A TIME-INVARIANT TERMINAL BEHAVIOR

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Ronald A. Skoog

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Electronics Research Laboratory

College of Engineering
University of California, Berkeley
94720

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Ronald A. Skoog

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

Necessary conditions are obtained for a time-varying network to non-trivially realize a time-invariant terminal behavior. Networks containing one type of time-varying element and more than one type of time-varying element are considered. Some implications of the results are that with one type of time-varying element present there must be at least two time-varying components, and with time-varying RC or RL networks there must be at least two time-varying components of each type in order to realize a time-invariant terminal behavior. There are in addition certain constraints on the derivatives of the component values.

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1. Introduction

It has been demonstrated[1-3] that time-varying RLC networks can realize a time-invariant terminal behavior in a nontrivial way (i.e., the time-varying components contribute to the terminal behavior). In particular, examples are known of a time-varying R fixed C network having a driving point impedance with inductive reactance[1] [2], a time-varying C or L and fixed R network having a driving point impedance with negative real part[2], and non-reciprocal two ports can be realized with time-varying resistors, capacitors, or inductors[2].

In this paper certain necessary conditions are obtained for a network containing time-varying components to nontrivially realize a time-invariant terminal behavior. These results provide partial answers to the questions regarding how many time-varying elements are required to realize a time-invariant terminal behavior, and what the relationship between these elements must be. In particular, it follows from these results that there must be at least two time-varying components of the same type, and the derivatives of their component values must never have the same sign. It is not known, however, if two time-varying components are sufficient.

2. Preliminaries

It has been shown[1-4] that except for certain degenerate cases, any linear time-invarient RLCT network of the form shown in Fig. 1 can be described by the mathematical representation

$$\dot{x}(t) = Ax(t) + B_1u(t) + B_2i(t) \tag{1a}$$

$$y(t) = C_1x(t) + D_1u(t) + D_2i(t) \tag{1b}$$

$$v(t) = C_2x(t) + D_3u(t) + D_4i(t) \ . \tag{1c}$$

The p-vector u consists of voltage and current inputs, and the output p-vector y consists of the corresponding conjugate variables at those ports. The input current vector i(t) and output voltage vector v(t) are both q-vectors. The state vector x(t) is assumed to be an n-vector. The matrices in (2.1) are of orders compatible with the vector dimensions given above.

Associated with the representation in (1) are two signature matrices which will be denoted by Σ_1 and Σ_2. The matrix Σ_1 is called an external signature matrix[3], and is a p x p diagonal matrix with a + 1 in those positions corresponding to a current and a - 1 in those positions corresponding to a voltage in the vector u. The matrix Σ_2 is called an internal signature matrix, and is an n x n diagonal matrix with $\pm$ 1's on the diagonal. This matrix is determined by the capacitors and inductors in the network (see [1-3] for details, and [8] for related ideas).

The fact that the representation (1) arises from an RLCT network imposes certain symmetry conditions on the matrices. These can be simply stated in terms of the signature matrices Σ_1 and Σ_2 as follows:

i)	$\Sigma_2 A \Sigma_2 = A'$	iv)	$\Sigma_1 D_1 \Sigma_1 = D_1'$
ii)	$-\Sigma_2 B_1 \Sigma_1 = C_1'$	v)	$\Sigma_1 D_2 = D_3'$
iii)	$-\Sigma_2 B_2 = C_2'$	vi)	$D_4' = D_4$

where a prime denotes matrix transposition.

Passivity imposes further conditions on the matrices in (1). In particular, if the network of Fig. 1 consists of positive R, L, and C's then the symmetric part of the matrix

$$\begin{bmatrix} -A & -B_1 & -B_2 \\ C_1 & D_1 & D_2 \\ C_2 & D_3 & D_4 \end{bmatrix}$$

will be non-negative definite. Note in particular, that since D_4 is symmetric it must be non-negative definite.

Consider now the dynamical system

$$\dot{x}(t) = A(t)\, x\,(t) + B(t)u(t) \tag{2a}$$

$$y(t) = C(t)\, x\,(t) + D(t)u(t), \tag{2b}$$

which will be denoted by (A,B,C,D). The response y to the input u with zero initial state at time t_0 is given by

$$y(t) = \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \tag{3}$$

where Φ is the transition matrix associated with A. Following [5] we call the function $w(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$ the <u>weighting pattern</u>, and a weighting pattern w is called <u>stationary</u> if $w(t,\tau) = w(t-\tau,0)$.

<u>Definition 1</u>: Two systems (A,B,C,D) and (F,G,H,J) are called <u>zero-state equivalent</u> if $D(t) = J(t)$ and they have the same weighting pattern.

Our main concern will be with systems which are zero-state equivalent to a constant coefficient system. For (A,B,C,D) to be zero-state equivalent to a constant coefficient system, it is necessary and sufficient that

D be constant and its weighting pattern be stationary, (see [5] and [6]). In order to characterize these systems completely in terms of A,B,C, and D we must define the operators δ and Δ associated with the system (A,B,C,D). These operators are defined as follows.

$$(\delta C)(t) = \frac{d}{dt} C(t) + C(t)A(t) \tag{4}$$

and

$$(\Delta B)(t) = - \frac{d}{dt} B(t) + A(t)B(t) \tag{5}$$

Powers of δ and Δ are defined in the obvious way;

$$(\delta^k C)(t) = \frac{d}{dt} (\delta^{k-1} C)(t) + (\delta^{k-1} C)(t)A(t) \text{ , } (\delta^0 C)(t) = C(t) \tag{6}$$

$$(\Delta^k B)(t) = - \frac{d}{dt} (\Delta^{k-1} B)(t) + A(t)(\Delta^{k-1} B)(t) \text{ , } (\Delta^0 B)(t) = B(t) \tag{7}$$

The following result is proved in [7] (see also [6] for similar results), and provides the desired characterization of systems which are zero-state equivalent to a constant coefficient system.

Theorem 1: Let the system (A,B,C,D) be such that A is 2n-2 times continuously differentiable (where n is the order of A), B, and C are 2n-1 times continuously differentiable. Then a necessary and sufficient condition for (A,B,C,D) to be zero-state equivalent to a constant coefficient system is that D and $(\delta^k C)B$ be constant for $k = 0,1,\ldots, 2n-1$. Also, if (A,B,C,D) is zero-state equivalent to a constant coefficient system, then $(\delta^j C)(\Delta^k B)$ is is constant for all $j, k \leq 2n-1$.

3. Networks with Time-Varying Resistors

Consider the network shown in Fig. 2 in which the linear, passive, time-invariant RLCT (p+q) port is described by (1), and $R(\cdot)$ is the resistance matrix of the time-varying resistance q-port. It will be assumed throughout that $R(t)$ is a symmetric positive definite matrix for all t.

The q-vectors i and v in (1) are now related by

$$v(t) = - R(t)i(t), \tag{8}$$

which, when used in (1), yields the following state equations for the network of Fig. 2:

$$\dot{x}(t) = [A - B_2[D_4 + R(t)]^{-1}C_2]x(t) + [B_1 - B_2[D_4 + R(t)]^{-1}D_3]u(t) \tag{9a}$$

$$y(t) = [C_1 - D_2[D_4 + R(t)]^{-1}C_2]x(t) + [D_1 - D_2[D_4 + R(t)]^{-1}D_3]u(t) \cdot \tag{9b}$$

Note that since D_4 is non-negative definite and $R(t)$ is positive definite, $D_4 + R(t)$ is invertable for all t.

The main result of this section is given by the following theorem.

Theorem 2: Let $R(\cdot)$ be 2n-1 times continuously differentiable, and let there exist some time t at which $\dot{R}(t)$ is either positive or negative definite. Then, if the system in (9) is zero-state equivalent to a constant coefficient system, it is zero-state equivalent to the system obtained from (1a,b) by setting $i(t) = 0$(i.e., the system obtained from Fig. 2 by replacing R(t) by an open circuit).

The following implications of Theorem 2 are immediate

a.) If a linear RLCT network containing one linear time-varying resistor has a time-invariant terminal behavior, then the same terminal behavior would be obtained after the time varying resistor was replaced by an open circuit.

b.) Suppose the resistance matrix R(t) is of the form $R(t) = R_0 + r(t)R_1$ where r(t) is a scalar, $\frac{dr(t)}{dt} \not\equiv 0$, and R_1 is positive or negative definite. Then, if the network of Fig. 2 has a time-invariant terminal behavior, the same terminal behavior would be obtained after removing the time-varying resistance network.

c.) In order for a network consisting of linear, fixed R,L,C's and linear time-varying resistors* to realize a time-invariant terminal behavior nontrivially (i.e., the time-varying resistors contribute to the terminal behavior) it is necessary that there be at least two time-varying resistors, say $r_1(t)$ and $r_2(t)$, with $\dot{r}_1(t)\dot{r}_2(t) \leq 0$ for all t.

Proof of Theorem 2: From Theorem 1 it is seen first of all that for (9) to be zero-state equivalent to a constant coefficient system it is necessary that $D_1 - D_2[D_4 + R(t)]^{-1}D_3$ be constant (see (9b)). Differentiating,

*The term linear time-varying resistor should be understood as a linear resistor whose resistance r(t) is such that $\dot{r}(t) \not\equiv 0$.

this gives

$$D_2[D_4 + R(t)]^{-1} \dot{R}(t)[D_4 + R(t)]^{-1}D_3 \equiv 0 \tag{10}$$

From the symmetry condition v) in Section 2 we have that $D_2 = \Sigma_1 D_3'$ (note Σ_1 and Σ_2 are their own inverses). Thus (10) requires

$$\Sigma_1 D_3'[D_4 + R(t)]^{-1} \dot{R}(t) \; [D_4 + R(t)]^{-1}D_3 \equiv 0 \tag{11}$$

Since $\dot{R}(t)$ is positive or negative definite at some time, say t_1, it follows from (11) that

$$[D_4 + R(t_1)]^{-1}D_3 = 0 \tag{12}$$

and hence that $D_3 = 0$. Since $D_2 = \Sigma_1 D_3' = 0$, (9) becomes

$$\dot{x}(t) = [A - B_2[D_4 - R(t)]^{-1}C_2] \; x(t) + B_1 u(t) \tag{13a}$$

$$y(t) = C_1 \; x(t) + D_1 u(t) \tag{13b}$$

Again using Theorem 1, for (13) to be zero-state equivalent to a constant coefficient system it is necessary that $(\delta C_1)(t)B_1$ be constant. From (4) it is seen that

$$(\delta C_1)(t) = C_1[A - B_2[D_4 - R(t)]^{-1}C_2] \; , \tag{14}$$

and therefore we require

$$C_1[A - B_2[D_4 - R(t)]^{-1}C_2]B_1 = \text{const.} \tag{15}$$

Differentiating (15) there results

$$C_1B_2[D_4 - R(t)]^{-1}\,\dot{R}(t)[D_4 - R(t)]^{-1}C_2B_1 \equiv 0 \qquad (16)$$

Using symmetry conditions ii) and iii) of Section 2 it is seen that

$$C_2B_1 = B_2'C_1'\Sigma_1 \qquad (17)$$

and therefore (16) gives

$$C_1B_2[D_4 - R(t)]^{-1}\,\dot{R}(t)[D_4 - R(t)]^{-1}\,B_2'C_1'\Sigma_1 \equiv 0\,. \qquad (18)$$

Since $\dot{R}(t_1)$ is either positive or negative definite, it follows from (18) that $C_1B_2 = 0$.

It will now be shown by induction that $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n-1$. For this purpose, we need the following lemma.

<u>Lemma 1</u>: In the system of (13), if $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n$, then $(\delta^kC_1)(t) = C_1A^k$ and $(\Delta^kB_1)(t) = A^kB_1$ for $k = 0,1,\ldots, n+1$.

Proof: Suppose $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n$. The proof that $(\delta^jC_1)(t) = C_1A^j$ for $j = 0,1,\ldots, n+1$ will proceed by induction on j. First of all, $\delta^0C_1 = C_1$ so the result is true for $j = 0$. Suppose $(\delta^jC_1)(t) = C_1A^j$ for some $j \leq n$. Then

$$(\delta^{j+1}C_1)(t) = \frac{d}{dt}(\delta^jC_1)(t) + (\delta^jC_1)(t)[A - B_2[D_4 - R(t)]^{-1}C_2]$$

$$= C_1A^{j+1}$$

since $C_1A^jB_2 = 0$.

The result that $(\Delta^kB_1)(t) = A^kB_1$ follows similarly by first noting from the symmetry conditions i) - iii) that $C_1A^kB_2 = (C_2A^kB_1\Sigma_1)'$ and thus

$C_2A^kB_1 = 0$ for $k = 0,1,\ldots, n$. Q.E.D.

Returning now to the induction proof that $C_1A^kB_2 = 0$ for $k = 0, 1,\ldots, n-1$, note first of all that it was shown above that $C_1B_2 = 0$ and thus the assertion is true for $k = 0$. Suppose now it is true for $k = 0,1,\ldots, j$ where $j < n-1$. From Theorem 1 it is known that $(\delta^{j+2}C_1)(\Delta^{j+1}B_1)$ is constant, and from Lemma 1 that $(\delta^{j+1}C_1) = C_1A^{j+1}$ and $(\Delta^{j+1}B_1) = A^{j+1}B_1$. Thus

$$(\delta^{j+2}C_1)(t) = C_1A^{j+1}(A - B_2[D_4 - R(t)]^{-1}C_2) \tag{19}$$

and hence

$$(\delta^{j+2}C_1)(t)(\Delta^{j+1}B_1)(t) = C_1A^{2j+2}B_1 - C_1A^{j+1}B_2[D_4 - R(t)]^{-1}C_2A^{j+1}B_1 \tag{20}$$

Since the right side of (20) must be constant, its derivative is zero, and thus

$$C_1A^{j+1}B_2[D_4 - R(t)]^{-1}\dot{R}(t)[D_4 - R(t)]^{-1}(C_1A^{j+1}B_2)^1\Sigma_1 \equiv 0 \tag{21}$$

where the symmetry conditions have been used to obtain $C_2A^{j+1}B_1 = (C_1A^{j+1}B_2)^1\Sigma_1$. Using the fact that $\dot{R}(t_1)$ is either positive or negative definite, it follows from (21) that $C_1A^{j+1}B_2 = 0$. Thus, by induction, $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n-1$.

To complete the proof, observe from (13) that if $x(t_0) = 0$ then $x(t)$ is a solution of

$$x(t) = \int_{t_0}^{t} e^{A(t - \tau)}[B_1u(\tau) - B_2[D_4 - R(\tau)]^{-1}C_2x(\tau)]d\tau \tag{22}$$

Since $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n-1$, it follows that $C_1e^{At}B_2 \equiv 0$. Therefore, $y(t)$ is given by

$$y(t) = \int_{t_0}^{t} C_1e^{A(t - \tau)}B_1u(\tau)d\tau + D_1u(t) \tag{23}$$

Hence the zero-state response of (9) given in (23) is the zero-state response of the constant coefficient system

$$\dot{x}(t) = Ax(t) + B_1u(t) \tag{24a}$$

$$y(t) = C_1x(t) + D_1u(t) \tag{24b}$$

and this is the system obtained from (1a,b) with $i(t) \equiv 0$. Q.E.D.

4. Networks with Time-Varying Capacitors

Consider the network shown in Fig. 3 in which the linear, passive, time-invariant RLCT (p+q) port is described by

$$\dot{x}(t) = Ax(t) + B_1u(t) + B_2v(t) \tag{25a}$$

$$y(t) = C_1x(t) + D_1u(t) + D_2v(t) \tag{25b}$$

$$i(t) = C_2x(t) + D_3u(t) + D_4v(t) \tag{25c}$$

and the linear time-varying capacitance network relates v and i by

$$i(t) = -\dot{q}(t) \tag{26}$$

$$S(t)q(t) = v(t) \tag{27}$$

where S is the elastance matrix for the capacitance network, and is assumed throughout to be symmetric and positive definite. The only difference between (25) and (1) is simply that the voltage vector v is considered as the input and the current vector i as an output. The symmetry conditions for (25) are obtained from those for (1) by replacing Σ_1 by $-\Sigma_1$ in i) - vi).

Using (26) and (27) in (25) there results

$$\dot{x}(t) = Ax(t) + B_2S(t)q(t) + B_1u(t) \tag{28a}$$

$$\dot{q}(t) = -C_2x(t) - D_4S(t)q(t) - D_3u(t) \tag{28b}$$

$$y(t) = C_1x(t) + D_2S(t)q(t) + D_1u(t) \tag{28c}$$

In regards to the network of Fig. 3 having a time-invariant terminal behavior, we have the following result.

<u>Theorem 3</u>: Let $S(\cdot)$ be 2n-1 times continuously differentiable, and let there exist some time t such that $\dot{S}(t)$ is either positive or negative definite. Then, if the system of (28) is zero-state equivalent to a constant coefficient system, it is zero-state equivalent to the system obtained from (25a,b) by setting $v(t) \equiv 0$ (i.e., the system obtained from Fig. 3 by shorting the capacitive ports).

As in the case of Theorem 2, we have the following immediate implications of Theorem 3:

a.) If a linear RLCT network containing one linear time-varying capacitor has a time-invariant terminal behavior, then the same terminal behavior would be obtained after replacing the time-varying capacitor by a short circuit.

b.) Suppose $S(t)$ is of the form $S(t) = S_0 + s(t)S_1$ with S_1 positive or negative definite. Then if the network of Fig. 3 has a time-invariant terminal behavior, the same terminal behavior would be obtained after shorting the capacitor ports.

c.) In order for a network consisting of linear, fixed, R,L,C's and linear time-varying capacitors to realize a time-invariant terminal behavior nontrivially, it is necessary that there be at least two time-varying capacitors, say with elastances $s_1(t)$ and $s_2(t)$, with $\dot{s}_1(t)\dot{s}_2(t) \leq 0$ for all t.

Proof of Theorem 3: Applying Theorem 1 to (28), the condition that $(\delta^0 C)B =$ constant gives for (28).

$$[C_1,\ D_2S(t)] \begin{bmatrix} B_1 \\ -D_3 \end{bmatrix} = C_1B_1 - D_2S(t)D_3 = \text{const.} \tag{29}$$

Differentiating (26) there results

$$D_2\dot{S}(t)D_3 \equiv 0\ . \tag{30}$$

From the symmetry conditions it follows that $D_2 = -\Sigma_1 D_3'$, and hence (30) gives

$$\Sigma_1 D_3'\ \dot{S}(t)D_3 \equiv 0\ . \tag{31}$$

However, since $\dot{S}(t)$ is positive or negative definite for some t it follows from (31) that $D_3 = 0$, and thus also $D_2 = -\Sigma_1 D_3' = 0$. As a result, (28) becomes

$$\begin{bmatrix} \dot{x}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} A & B_2 S(t) \\ -C_2 & -D_4 S(t) \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \tag{32a}$$

$$y(t) = C_1 x(t) + D_1 u(t) \tag{32b}$$

The condition $(\delta^2 C_1)B$ = const. from Theorem 1 applied to (32) gives

$$C_1 A^2 B_1 - C_1 B_2 S(t) C_2 B_1 = \text{const.} \tag{33}$$

which requires

$$C_1 B_2 \dot{S}(t) C_2 B_1 \equiv 0 \tag{34}$$

From the symmetry conditions it follows that $C_2 B_1 = - B_2' C'_1 \Sigma_1$, and thus (31) requires

$$C_1 B_2 \dot{S}(t) B_2' C_1' \Sigma_1 \equiv 0 \ . \tag{35}$$

Since $\dot{S}(t)$ is either negative or positive definite for some t, it then follows from (35) that $C_1 B_2 = 0$.

It will now be shown by induction that $C_1 A^k B_2 = 0$ for $k = 0,1,\ldots,$ $n-1$. In this case, we need the following lemma.

<u>Lemma 2</u>: In the system of (32), if $C_1 A^k B_2 = 0$ for $k = 0,1,\ldots, n$, then

$(\delta^k C)(t) = (C_1 A^k, 0)$ and $(\Delta^k B)(t) = \begin{bmatrix} A^k B_1 \\ 0 \end{bmatrix}$ for $k = 0,1,\ldots, n+1$, where $C = (C_1, 0)$ and $B = \binom{B_1}{0}$.

The proof of this lemma is by induction along lines similar to the proof of Lemma 1, and will be omitted.

Suppose now that $C_1 A^k B_2 = 0$ for $k = 0,1,\ldots, j$ where $j < n-1$. From Theorem 1 it is known that $(\delta^{j+2}C)$ $(\Delta^{j+2}B)$ is constant, and from Lemma 2 that $(\delta^{j+1}C) = (C_1 A^{j+1}, 0)$ and $(\Delta^{j+1}B) = \begin{bmatrix} A^{j+1}B_1 \\ 0 \end{bmatrix}$. Thus

$$(\delta^{j+2}C) = (C_1 A^{j+2},\ C_1 A^{j+1} B_2 S(t)) \tag{36}$$

$$(\Delta^{j+2}B) = \begin{bmatrix} A^{j+2}B_1 \\ -C_2 A^{j+1} B_1 \end{bmatrix}, \tag{37}$$

so

$$(\delta^{j+2}C)(\Delta^{j+2}B) = C_1 A^{2j+4} B_1 - C_1 A^{j+1} B_2 S(t) C_2 A^{j+1} B_1 \tag{38}$$

Since the right side of (38) is constant, and from the symmetry conditions it follows that $C_2 A^{j+1} B_1 = -(C_1 A^{j+1} B_2)^1 \Sigma_1$, there results from (38)

$$C_1 A^{j+1} B_2 \dot{S}(t) (C_1 A^{j+1} B_2)^1 \Sigma_1 \equiv 0 \tag{39}$$

From the fact that $\dot{S}(t)$ is positive or negative definite at some time t, (39) implies that $C_1 A^{j+1} B_2 = 0$. Hence by induction $C_1 A^k B_2 = 0$ for $k = 0,1,\ldots, n-1$.

From (32a) if $x(t_0) = 0$ then $x(t)$ is given by

$$x(t) = \int_{t_0}^{t} e^{A(t-\tau)} [B_1 u(\tau) + B_2 S(\tau) q(\tau)] d\tau. \tag{40}$$

Since $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n-1$, it follows that $C_1e^{At}B_2 \equiv 0$ and thus from (40) it follows that

$$y(t) = \int_{t_0}^{t} C_1e^{A(t - \tau)} B_1u(\tau)d\tau + D_1u(t) \qquad (41)$$

Thus the zero-state response of (28) is the same as that for the system of (25a,b) with $v(t) \equiv 0$. Q.E.D.

It is remarked that a similar result to Theorem 3 holds for time-varying inductor networks. This result would be obtained by simply changing the voltage drive $v(\cdot)$ in (25) into a current drive $i(\cdot)$, and exchanging the inverse inductance matrix $\Gamma(\cdot)$ for $S(\cdot)$ in the theorem statement.

5. Networks With More Than One Kind of Time-varying Element

We will begin by considering the network of Fig. 4 containing time-varying resistors and capacitors. The linear time-invariant portion of the network is described by

$$\dot{x}(t) = Ax(t) + B_1u(t) + B_2v_C(t) + B_3i_R(t) \qquad (42a)$$

$$y(t) = C_1x(t) + D_1u(t) + D_2v_C(t) + D_3i_R(t) \qquad (42b)$$

$$i_C(t) = C_2x(t) + D_4u(t) + D_5v_C(t) + D_6i_R(t) \qquad (42c)$$

$$v_R(t) = C_3x(t) + D_7u(t) + D_8v_C(t) + D_9i_R(t) \qquad (42d)$$

and the vectors v_C, i_C and v_R, i_R are related by

$$\dot{q}(t) = i_C(t) \tag{43}$$

$$v_C(t) = S(t)q(t) \tag{44}$$

$$v_R(t) = R(t)i_R(t) \tag{45}$$

Using these relations in (42) gives

$$\begin{bmatrix} \dot{x}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} A - B_3[D_9 + R(t)]^{-1}C_3 & B_2S(t) \\ C_2 - D_6[D_9 + R(t)]^{-1}C_3 & D_5S(t) \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} B_1 - B_3[D_9 + R(t)]^{-1}D_7 \\ D_4 - D_6[D_9 + R(t)]^{-1}D_7 \end{bmatrix} u(t) \tag{46a}$$

$$y(t) = (C_1 - D_3[D_9 + R(t)]^{-1}C_3,\ (D_2 - D_3[D_9 + R(t)]^{-1}D_8)S(t)) \begin{pmatrix} x(t) \\ q(t) \end{pmatrix} + (D_1 - D_3[D_9 + R(t)]^{-1}D_7)u(t) \tag{46b}$$

The symmetry conditions are stated most compactly in the requirement that the matrix

$$\begin{bmatrix} \Sigma_2 & 0 & 0 & 0 \\ 0 & \Sigma_1 & 0 & 0 \\ 0 & 0 & -I_{q_C} & 0 \\ 0 & 0 & 0 & I_{q_R} \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & D_1 & D_2 & D_3 \\ C_2 & D_4 & D_5 & D_6 \\ C_3 & D_7 & D_8 & D_9 \end{bmatrix} \tag{47}$$

be symmetric.

$\underline{\text{Theorem 4}}$: Let $S(\cdot)$ and $R(\cdot)$ be 2n-1 times continuously differentiable and let there exist some time t_1 such that $\dot{S}(t_1)$ is either positive or negative definite, and a time t_2 such that $\dot{R}(t_2)$ is either positive or negative definite. Then, if the system (46) is zero-state equivalent to a constant coefficient system, it is zero-state equivalent to the system (42a,b) with $i_R(t) \equiv 0$ and $v_C(t) \equiv 0$.

Observe that this theorem implies that with both time-varying resistors and capacitors present one must have at least two time-varying resistors and two time-varying capacitors in order to realize a time-invariant terminal behavior nontrivially. If for example there were two time-varying resistors and one time-varying capacitor, then the time-varying capacitor could be removed without changing the terminal behavior.

$\underline{\text{Proof of Theorem 4}}$: Assuming (46) to be zero-state equivalent to a constant coefficient system, we have from Theorem 1 and (46b) that

$$D_3[D_9 + R(t)]^{-1}D_7 = \text{constant} \qquad (48)$$

Using identical arguments as those in the proof of Theorem 2 it is found that (48) implies that $D_3 = 0$ and $D_7 = 0$.

The condition from Theorem 1 that $(\delta^0 C)B$ = constant now gives for (46)

$$C_1B_1 + D_2S(t)D_4 = \text{constant} \qquad (49)$$

or equivalently

$$D_2\dot{S}(t)D_4 \equiv 0 \qquad (50)$$

Invoking the symmetry conditions and our assumptions on S there results from (50) that $D_2 = 0$ and $D_4 = 0$.

At this point (46) has been reduced to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} A - B_3[D_9 + R(t)]^{-1}C_3 & B_2S(t) \\ C_2 - D_6[D_9 + R(t)]^{-1}C_3 & D_5S(t) \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \tag{51a}$$

$$y(t) = C_1x(t) + D_1u(t) \tag{51b}$$

Now, the condition $(\delta^1 C)B$ = const. from Theorem 1 gives

$$C_1AB_1 - C_1B_3[D_9 + R(t)]^{-1}C_3B_1 = \text{const.} \tag{52}$$

Again as in the case of Theorem 2 (see (15) - (18)), (51) will require $C_1B_3 = 0$. Next, the condition $(\delta^2 C)B$ = const. leads to (note $C_1B_3 = 0$ implies that $C_3B_1 = 0$ due to symmetry)

$$C_1A^2B_1 - C_1B_2S(t)C_2B_1 = \text{constant} \tag{53}$$

and again from symmetry and the conditions on S it follows from (53) that $C_1B_2 = 0$.

It will now be shown by induction that $C_1A^kB_3 = 0$ and $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n-1$. We need the following lemma.

<u>Lemma 3</u>: For the system of (51), if $C_1A^kB_3 = 0$ and $C_1A^kB_2 = 0$ for $k = 0,1,2,\ldots, n$, then $(\delta^k C)(t) = (C_1A^k, 0)$ and $(\Delta^k B)(t) = \begin{pmatrix} A^kB_1 \\ 0 \end{pmatrix}$ for $k = 0, 1,2,\ldots, n+1$; where $C = (C_1, 0)$ and $B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$.

Proof: Suppose $C_1A^kB_3 = 0$ and $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, n$. By induction, suppose $(\delta^j C) = (C_1A^j, 0)$ for some $j \leq n$. Then

$$
\begin{aligned}
(\delta^{j+1}C)(t) &= (C_1A^{j+1} - C_1A^jB_3[D_9 + R(t)]^{-1}C_3,\ C_1A^jB_2S(t)) \\
&= (C_1A^{j+1}, 0)
\end{aligned}
\tag{54}
$$

since $C_1A^jB_3 = 0$ and $C_1A^jB_2 = 0$.

Similarly suppose $(\Delta^jB) = \begin{pmatrix} A^jB_1 \\ 0 \end{pmatrix}$. Then

$$
(\Delta^{j+1}B)(t) = \begin{bmatrix} A^{j+1}B_1 - B_3[D_9 + R(t)]^{-1}C_3A^jB_1 \\ C_2A^jB_1 - D_6[D_9 + R(t)]^{-1}C_3A^jB_1 \end{bmatrix}
\tag{55}
$$

From the symmetry conditions it follows that $C_3A^jB_1 = (C_1A^jB_3)^1\Sigma_1 = 0$ and $C_2A^jB_1 = -(C_1A^jB_2)^1\Sigma_1 = 0$. Hence (55) gives

$$
(\Delta^{j+1}B)(t) = \begin{bmatrix} A^{j+1}B_1 \\ 0 \end{bmatrix} .
\tag{56}
$$

Since $(\delta^0C) = (C_1, 0)$ and $(\Delta^0B) = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ the conclusion of the lemma follows by induction. Q.E.D.

Assume that $C_1A^kB_3 = 0$ and $C_1A^kB_2 = 0$ for $k = 0,1,\ldots, j$ with $j < n-1$. Then $(\delta^{j+1}C)(t) = (C_1A^{j+1}, 0)$ and $(\Delta^{j+1}B) = \begin{pmatrix} A^{j+1}B_1 \\ 0 \end{pmatrix}$ from Lemma 3. Thus

$$
\begin{aligned}
(\delta^{j+2}C)(t) = (C_1A^{j+2} - C_1A^{j+1}B_3(D_9 + R(t))^{-1}C_3, \\
C_1A^{j+1}B_2S(t))
\end{aligned}
\tag{57}
$$

and

$$(\Delta^{j+2}B)(t) = \begin{bmatrix} A^{j+2}B_1 - B_3[D_9 + R(t)]^{-1}C_3A^{j+1}B_1 \\ \\ C_2A^{j+1}B_1 - D_6[D_9 + R(t)]^{-1}C_3A^{j+1}B_1 \end{bmatrix} . \tag{58}$$

Now, from Theorem 1 $(\delta^{j+2}C)(\Delta^{j+1}B) =$ const., and using (57) we have

$$(\delta^{j+2}C)(\Delta^{j+1}B) = C_1A^{2j+3} - C_1A^{j+1}B_3[D_9 + R(t)]^{-1}C_3A^{j+1}B_1 \tag{59}$$

From the symmetry conditions it follows that $C_3A^{j+1}B_1 = (C_1A^{j+1}B_3)'\Sigma_1$. Thus, from the right side of (59) being constant there follows

$$C_1A^{j+1}B_3[D_9 + R(t)]^{-1}\,\dot{R}(t)[D_9 + R(t)]^{-1}(C_1A^{j+1}B_3)' \equiv 0, \tag{60}$$

and since $\dot{R}(t_2)$ is positive or negative definite, (60) implies that $C_1A^{j+1}B_3 = 0$.

Again using Theorem 1, $(\delta^{j+2}C)(\Delta^{j+2}B) =$ constant, and from (57) and (58) there follows

$$(\delta^{j+2}C)(\Delta^{j+2}B)(t) = C_1A^{2j+4}B_1 + C_1A^{j+1}B_2S(t)C_2A^{j+1}B_1 \tag{61}$$

since $C_1A^{j+1}B_3 = 0$. From symmetry, $C_2A^{j+1}B_1 = -(C_1A^{j+1}B_2)'\Sigma_1$. Thus from the right side of (61) being constant we have

$$(C_1A^{j+1}B_2)\,\dot{S}(t)(C_1A^{j+1}B_2)' \equiv 0 . \tag{62}$$

Since $\dot{S}(t_1)$ is either positive or negative definite, we have finally that $C_1A^{j+1}B_2 = 0$. Thus by induction it follows that $C_1A^kB_2 = 0$ and $C_1A^kB_3 = 0$ for $k = 0, 1, 2, \ldots, n-1$.

To complete the proof of Theorem 4, observe from (51) that if $x(t_o) = 0$ then x is a solution of

$$x(t) = \int_{t_0}^{t} e^{A(t - \tau)} [B_1u(\tau) - B_3[D_9 + R(t)]^{-1}C_3x(\tau) + B_2S(\tau)q(\tau)]d\tau \qquad (63)$$

Since from the above it follows that $C_1e^{At}B_2 \equiv 0$ and $C_1e^{At}B_3 \equiv 0$, $y(t)$ is seen to be

$$y(t) = \int_{t_0}^{t} C_1e^{A(t - \tau)}B_1u(\tau)d\tau + D_1u(t) \; . \qquad (64)$$

Thus the zero state response of (46) is the same as the zero-state response of (42a,b) with $v_C(t) \equiv 0$. Q.E.D.

Finally, consider the network of Fig. 5 in which there are time-varying resistors, capacitors, and inductors. Let $R(t)$ be the resistance network, $S(t)$ the elastance matrix of the time-varying capacitance network, and $\Gamma(t)$ the inverse inductance matrix of the time-varying inductance network. It will be assumed that each of these matrices is positive definite and symmetric for all t. The following result is proven along the same lines as the previous results.

Theorem 5: Let $R(\cdot)$, $S(\cdot)$, and $\Gamma(\cdot)$ be 2n-1 times continuously differentiable (where n is the order of the linear time-invariant portion of the network of Fig. 5), and let there exist some time t_1 such that $\dot{R}(t_1)$ is either positive or negative definite, and a time t_2 such that the matrix

$$Q(t_2) = \begin{bmatrix} \dot{S}(t_2) & 0 \\ 0 & -\dot{\Gamma}(t_2) \end{bmatrix}$$

is either positive or negative definite. Then if the network of Fig. 5 is zero-state equivalent to a constant coefficient system, it is zero-state equivalent to the network obtained by shorting the time-varying capacitive ports and open circuiting the time-varying resistive and inductive ports.

6. Conclusions

It has been shown that if a network containing fixed R,L,C's and one type of time varying element realizes a time-invariant terminal behavior, then the time-varying components will contribute to the terminal behavior only if the corresponding time-varying resistance, capacitance, or inductance matrix has a derivative which is never positive or negative definite. Thus, it follows that there must be at least a two-port time-varying network which is embedded in the time-invariant RLC network in order that a time-invariant terminal behavior be realized nontrivially. It has been shown by examples [1] [2] that it is possible to obtain time-invariant terminal behavior with an embedded three-port time-varying network (resistive, capacitive, or inductive). These networks require eight time-varying elements for a transformerless synthesis. As yet it is not known whether a two-port is sufficient for the realization of a time-invariant terminal behavior.

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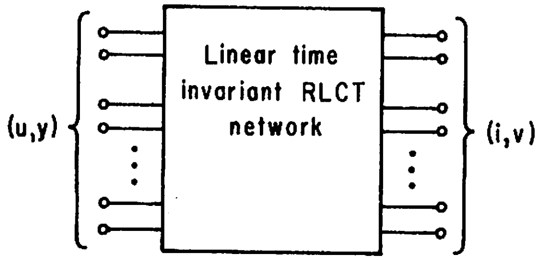

Fig. 1. Linear time invariant RLCT network.

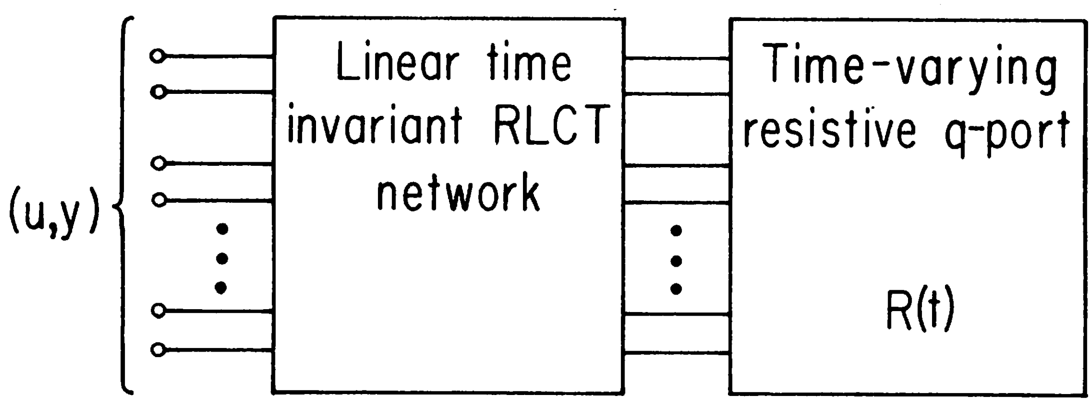

Fig. 2. Network with time-varying resistors.

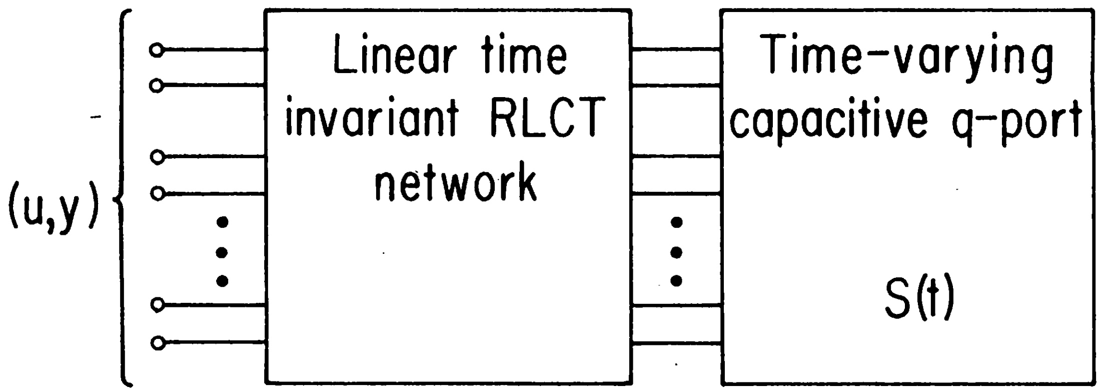

Fig. 3. Network with time varying capacitors

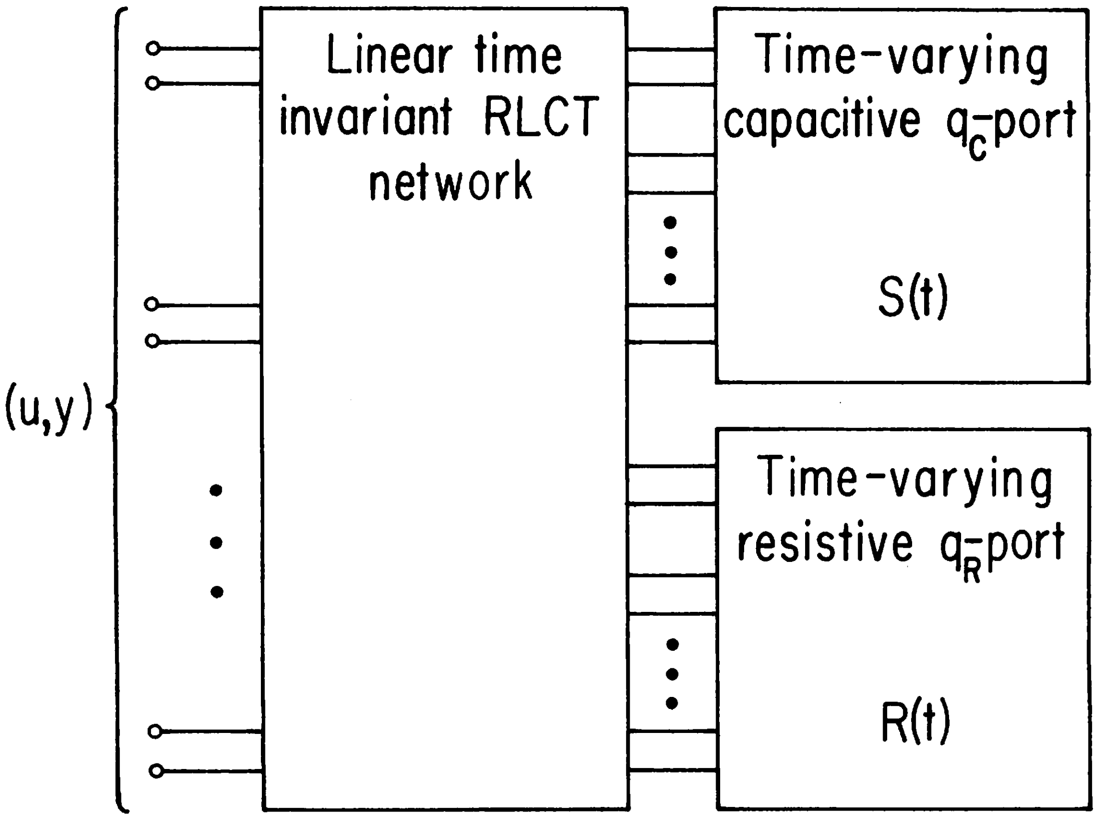

Fig. 4. Network with time-varying resistors and capacitors.

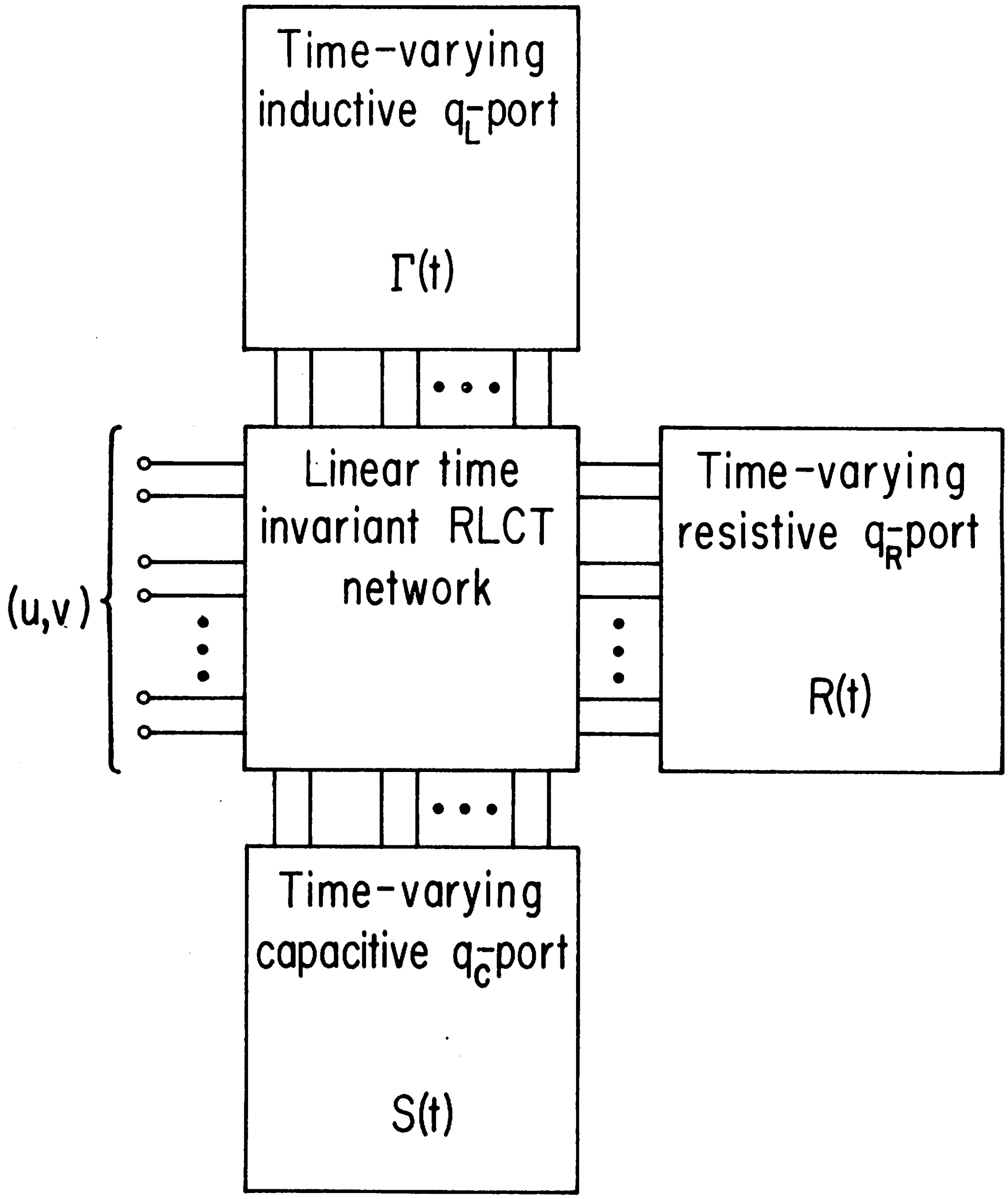

Fig. 5. Network with time-varying resistors.