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T. Fujisawa† and E. S. Kuh††

Abstract

This paper deals with nonlinear networks which can be characterized by the equation $f(x) = y$ where $f(\cdot)$ is a piecewise-linear mapping of R^n into itself. x is a point in R^n and represents a set of chosen network variables, and y is an arbitrary point in R^n and represents the input to the network. Lipschitz condition and global homomorphism are studied in detail. Two theorems on sufficient conditions for the existence of a unique solution of the equation for all $y \in R^n$ in terms of the constant Jacobian matrices are derived. The theorems turn out to be of pertinent importance in the numerical computation of general nonlinear resistive networks based on the piecewise-linear analysis. A comprehensive study of the Katzenelson's algorithm applied to general networks is carried out, and conditions under which the method converges are obtained. Special attention is given to the problem of boundary crossing of a solution curve.

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PIECEWISE - LINEAR THEORY OF NONLINEAR NETWORKS

1. Introduction

In this paper we present the general theory of nonlinear resistive networks based on the piecewise-linear analysis. As far as we can determine, the first work on the use of piecewise linear analysis to fairly general nonlinear resistive networks was done by Katzenelson in 1965[1]. Specifically, Katzenelson developed a method for solving nonlinear resistive networks which contain uncoupled resistors of the strictly monotonically increasing type. For this special class of nonlinear networks, it is not difficult to see that his algorithm always converges, and it represents an efficient method of analysis. Recently Kuh and Hajj extended the applicability of Katzenelson's method in two directions[2]. First, they applied the Katzenelson's algorithm to analyze arbitrary nonlinear resistive networks which possess a unique solution under all possible inputs. Second, they developed a method of calculating input-output characteristics of nonlinear resistive networks which possess single or multiple solutions. In both cases, the method works well, yet proof was not given of the convergence of the method. Furthermore, it became obvious in the process of the development that there remain many theoretical problems to be investigated. Typically, what are the conditions for the existence of a unique solution of the equation $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ where $\underset{\sim}{f}$ maps R^n into itself and $\underset{\sim}{f}$ is continuous but not differentiable? What actually happens when a "solution curve" hits the boundaries of regions which are defined by the piecewise linear mapping? These questions are investigated in detail in this paper. The conclusion provides a theoretical basis for the piecewise-linear analysis of nonlinear networks, and furthermore, will lead to better algorithms in the numerical computation procedure.

In order to make this paper reasonably self-contained, we include in our presentation a quick review of the method of mixed analysis[3, 4] in nonlinear networks and a brief discussion of the Katzenelson's method of solution. The principal results of the paper are in Sections 3 and 4, where the problem of existence and uniqueness of solution in piecewise-linear mapping is treated and a detailed study of the problem of boundary crossing is given. We found that the sign of the network determinant in each linear region is of crucial significance in the piecewise-linear analysis.

2. Network equations and modeling

In dealing with general nonlinear networks we must include voltage-controlled resistors, current-controlled resistors, and coupled nonlinear elements. Thus it is most convenient to use the method of mixed analysis. In mixed analysis, we partition the network into two subnetworks such that we depend on the loop analysis for one and the cut-set analysis for the other. Let $\mathcal{N}$ denote the complete network and its graph; let $\mathcal{N}_a$ denote the subnetwork and its graph for which the cut-set analysis is used; and let $\mathcal{N}_b$ denote the subnetwork and its graph for which the loop analysis is used. The branch voltage vector $\underset{\sim}{v}$ can be partitioned into $\underset{\sim}{v}_a$ and $\underset{\sim}{v}_b$, and similarly the branch current $\underset{\sim}{i}$ can be partitioned into $\underset{\sim}{i}_a$ and $\underset{\sim}{i}_b$ to designate the voltages and currents pertaining to the subnetworks. The element characteristics are specified in the hybrid form by

$$\begin{bmatrix} \underset{\sim}{i}_a \\ \\ \underset{\sim}{v}_b \end{bmatrix} = \underset{\sim}{h} \begin{bmatrix} \underset{\sim}{v}_a \\ \\ \underset{\sim}{i}_b \end{bmatrix} \tag{1}$$

It is seen that electric couplings among elements in the subnetwork $\mathcal{N}_a$ and the subnetwork $\mathcal{N}_b$ are allowed in this representation. As far as the two subgraphs are concerned, $\mathcal{N}_a$ is obtained from $\mathcal{N}$ by opening all branches which belong to $\mathcal{N}_b$. Similaraly, $\mathcal{N}_b$ is obtained from $\mathcal{N}$ by shorting all branches which belong to $\mathcal{N}_a$. For simplicity in derivation, we assume that all current sources belong to $\mathcal{N}_a$ and all voltage sources belong to $\mathcal{N}_b$*.

Let us further partition the elements in $\mathcal{N}_a$ and $\mathcal{N}_b$ into tree branches and links of the two subgraphs. If the subgraphs are not connected graphs, then tree branches are actually branches which belong to a forest. For simplicity we will use the terms tree and cotree to signify the two-way partition; and we use subcripts t(tree) and ℓ(loop) in the voltage and current vectors to designate those which belong to the tree and those which belong to the co-tree, respectively. Thus, the following variables are introduced.

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*With this assumption, the resulting network equation is of the form $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ where $\underset{\sim}{x}$ represents the network variables to be defined and $\underset{\sim}{y}$ represents the input. Without this assumption, the equation is of the form $\underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}) = \underset{\sim}{y}$, where $\underset{\sim}{u}$ represents additional inputs.

$$\underset{\sim}{v} = \begin{bmatrix} \underset{\sim}{v}_{a\ell} \\ \underset{\sim}{v}_{b\ell} \\ \underset{\sim}{v}_{at} \\ \underset{\sim}{v}_{bt} \end{bmatrix} \qquad \underset{\sim}{i} = \begin{bmatrix} \underset{\sim}{i}_{a\ell} \\ \underset{\sim}{i}_{b\ell} \\ \underset{\sim}{i}_{at} \\ \underset{\sim}{i}_{bt} \end{bmatrix} \tag{2}$$

It is also obvious from the way that the two subgraphs are introduced that the collection of tree branches of $\mathcal{N}_a$ and $\mathcal{N}_b$ forms the tree of $\mathcal{N}$ and the collection of links of $\mathcal{N}_a$ and $\mathcal{N}_b$ forms the co-tree of $\mathcal{N}$. Let $\mathcal{T}$ and $\mathcal{L}$ designate the set of all tree branches and links, respectively, of $\mathcal{N}$; let $\mathcal{T}_a$ and $\mathcal{L}_a$ designate the set of all tree branches and links, of $\mathcal{N}_a$, respectively; and let $\mathcal{T}_b$ and $\mathcal{L}_b$ designate the set of all tree branches and links, of $\mathcal{N}_b$, respectively. Then it follows that $\mathcal{T} = \mathcal{T}_a \cup \mathcal{T}_b$, $\mathcal{L} = \mathcal{L}_a \cup \mathcal{L}_b$. Furthermore, links in $\mathcal{L}_a$ form fundamental loops in $\mathcal{N}$ with only tree branches in $\mathcal{T}_a$. Because of the last fact, the fundamental loop matrix $\underset{\sim}{B}$ of $\mathcal{N}$ is of the following special form:

$$\underset{\sim}{B} = [\underset{\sim}{1} \quad \underset{\sim}{F}] = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{0} & \underset{\sim}{F}_{aa} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1} & \underset{\sim}{F}_{ab} & \underset{\sim}{F}_{bb} \end{bmatrix} \tag{3}$$

Thus, Kirchhoff voltage and current laws lead to the following equations:

$$\text{KVL:} \begin{cases} \underset{\sim}{v}_{a\ell} + \underset{\sim}{F}_{aa}\,\underset{\sim}{v}_{at} = \underset{\sim}{0} & (4) \\ \underset{\sim}{v}_{b\ell} + \underset{\sim}{F}_{ab}\,\underset{\sim}{v}_{at} + \underset{\sim}{F}_{bb}\,\underset{\sim}{v}_{bt} = \underset{\sim}{e}_b & (5) \end{cases}$$

$$\text{KCL:}\quad \begin{cases} -\underset{\sim}{F}^T_{aa}\,\underset{\sim}{i}_{a\ell} - \underset{\sim}{F}^T_{ab}\,\underset{\sim}{i}_{b\ell} + \underset{\sim}{i}_{at} = \underset{\sim}{j}_a & (6) \\ -\underset{\sim}{F}^T_{bb}\,\underset{\sim}{i}_{b\ell} - \underset{\sim}{i}_{bt} = \underset{\sim}{0} & (7) \end{cases}$$

where $\underset{\sim}{e}_b$ and $\underset{\sim}{j}_a$ represent, respectively, the fundamental loop voltage source in $\mathcal{N}_b$, and the fundamental cut-set current sources in $\mathcal{N}_a$. Let us denote the fundamental loop matrices of $\mathcal{N}_a$ and $\mathcal{N}_b$ by $\underset{\sim}{B}_a$ and $\underset{\sim}{B}_b$, respectively, and the fundamental cut-set matrices of $\mathcal{N}_a$ and $\mathcal{N}_b$ by $\underset{\sim}{Q}_a$ and $\underset{\sim}{Q}_b$, respectively. Then it is clear that

$$\underset{\sim}{B}_a = [\underset{\sim}{1} \quad \underset{\sim}{F}_{aa}] \tag{8}$$

$$\underset{\sim}{Q}_a = [-\underset{\sim}{F}^T_{aa} \quad \underset{\sim}{1}] \tag{9}$$

$$\underset{\sim}{B}_b = [\underset{\sim}{1} \quad \underset{\sim}{F}_{bb}] \tag{10}$$

$$\underset{\sim}{Q}_b = [-\underset{\sim}{F}^T_{bb} \quad \underset{\sim}{1}] \tag{11}$$

Note that the matrix $\underset{\sim}{F}_{ab}$ in (3) contribute to the topological coupling between $\mathcal{N}_a$ and $\mathcal{N}_b$. With the above notation, Eqs. (5) and (6) can be written as

$$\begin{bmatrix} \underset{\sim}{Q}_a & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_b \end{bmatrix} \begin{bmatrix} \underset{\sim}{i}_a \\ \underset{\sim}{v}_b \end{bmatrix} + \begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{F}^T_{ab} \\ \underset{\sim}{F}_{ab} & \underset{\sim}{0} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{at} \\ \underset{\sim}{i}_{b\ell} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{j}_a \\ \underset{\sim}{e}_b \end{bmatrix} \tag{12}$$

We shall use the tree-branch voltages $\underset{\sim}{v}_{at}$ of $\mathcal{N}_a$ and the link currents $\underset{\sim}{i}_{b\ell}$ of $\mathcal{N}_b$ as the set of independent network variables. From Eqs. (4) and (7), we express all the branch variables $\begin{bmatrix} \underset{\sim}{v}_a \\ \underset{\sim}{i}_b \end{bmatrix}$ in terms of chosen ones $\begin{bmatrix} \underset{\sim}{v}_{at} \\ \underset{\sim}{i}_{b\ell} \end{bmatrix}$ as:

$$\begin{bmatrix} \underset{\sim}{v}_a \\ \underset{\sim}{i}_b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{v}_{a\ell} \\ \underset{\sim}{v}_{at} \\ \underset{\sim}{i}_{b\ell} \\ \underset{\sim}{i}_{bt} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Q}_a^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_b^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{at} \\ \underset{\sim}{i}_{b\ell} \end{bmatrix} \tag{13}$$

Next, the element characteristics in Eq. (1) are combined with the Kirchhoff formulas in (12) and (13) to yield the basic set of network equations:

$$\begin{bmatrix} \underset{\sim}{Q}_a & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_b \end{bmatrix} \underset{\sim}{h} \left\{ \begin{bmatrix} \underset{\sim}{Q}_a^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_b^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{at} \\ \underset{\sim}{i}_{b\ell} \end{bmatrix} \right\} + \begin{bmatrix} \underset{\sim}{0} & -\underset{\sim}{F}_{ab}^T \\ \underset{\sim}{F}_{ab} & \underset{\sim}{0} \end{bmatrix} \begin{bmatrix} \underset{\sim}{v}_{at} \\ \underset{\sim}{i}_{b\ell} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{j}_a \\ \underset{\sim}{e}_b \end{bmatrix} \tag{14}$$

Let $\underset{\sim}{x}$ denote the set of independent network variables:

$$\underset{\sim}{x} \stackrel{\Delta}{=} \begin{bmatrix} \underset{\sim}{v}_{at} \\ \underset{\sim}{i}_{b\ell} \end{bmatrix} \tag{15}$$

and $\underset{\sim}{y}$ denotes the set of inputs

$$\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{j}_a \\ \underset{\sim}{e}_b \end{bmatrix} \tag{16}$$

The resulting network equations in (14) can be written as:

$$f(x) = y \tag{17}$$

where x and y are assumed to have a dimension n.

$$f(x) = C\, h(C^T x) + D\, x \tag{18}$$

where

$$C \triangleq \begin{bmatrix} Q_a & 0 \\ 0 & B_b \end{bmatrix} \tag{19}$$

characterizes the subgraphs $\mathcal{N}_a$ and $\mathcal{N}_b$, and

$$D \triangleq \begin{bmatrix} 0 & -F_{ab}^T \\ F_{ab} & 0 \end{bmatrix} \tag{20}$$

specifies the topological relation between $\mathcal{N}_a$ and $\mathcal{N}_b$.

The Jacobian matrix of the mapping is then

$$J(x) = \frac{\partial f}{\partial x} = C\, H(x)\, C^T + D \tag{21}$$

where

$$H(x) \triangleq \frac{\partial h}{\partial x} \tag{22}$$

is the incremental hybrix matrix representing the branch characteristies.

Let p denote the total number of branches in $\mathcal{N}$, then $\underset{\sim}{H}$ is of order p and $p \geq n$.

In the piecewise linear analysis, the Jacobian matrix $\underset{\sim}{J}$ is a constant matrix in each region of the n-dimensional $\underset{\sim}{x}$-space. Eq. (17) becomes

$$\underset{\sim}{J}^{(k)}\underset{\sim}{x} + \underset{\sim}{w}^{(k)} = \underset{\sim}{y} \quad , \qquad k=1,2,\cdots r \tag{23}$$

where $\underset{\sim}{J}^{(k)}$ is the constant Jacobian matrix and $\underset{\sim}{w}^{(k)}$ is a constant vector defined in a given region k, and r denotes the total number of regions in the $\underset{\sim}{x}$-space. Both $\underset{\sim}{J}^{(k)}$ and $\underset{\sim}{w}^{(k)}$ are obtained from the piecewise-linear approximation of the function $\underset{\sim}{f}(\cdot)$ in the n-dimension $\underset{\sim}{x}$-space.

The general multi-dimensional piecewise-linear approximation is far from a trivial problem and will not be treated in this paper. Fortunately, all practical devices encountered in circuits have nonlinear characteristics of a special "separable" form, that is, each dependent variable is represented by the sum of nonlinear functions of single variables. For example, the Ebers-Moll model of a transistor is given by

$$\begin{aligned} i_1 &= a_{11}(e^{v_1/v_T} -1) + a_{12}(e^{v_2/v_T} -1) \\ i_2 &= a_{21}(e^{v_1/v_T} -1) + a_{22}(e^{v_2/v_T} -1) \end{aligned} \tag{24}$$

The piecewise-linear approximation for this model is reduced to the approximation of a scalar function, in this case, the diode characteristics

$(e^{v/v_T}-1)$; and the familiar "line-segment" method is used. Thus, the piecewise-linear approximation is needed only to be undertaken for the branch characteristics rather than for the network variable $\underset{\sim}{x}$ in Eq. (17). In the p-dimensional branch-variable space, the boundaries of the piecewise-linear regions are defined by p equations:

$$\begin{bmatrix} \underset{\sim}{v}_a \\ \underset{\sim}{i}_b \end{bmatrix} = \underset{\sim}{g}_i \qquad i = 1, 2, \cdots \tag{25}$$

where $\underset{\sim}{g}_i$ is a p-dimensional constant vector which is picked by the circuit analysists in obtaining the piecewise-linear approximation according to nonlinear characteristics of the elements. In terms of the network variable $\underset{\sim}{x}$, we have from Eq. (13)

$$\underset{\sim}{C}^T \underset{\sim}{x} = \underset{\sim}{g}_i \qquad i = 1, 2, \ldots \tag{26}$$

This equation which specifies the boundaries in the $\underset{\sim}{x}$-space will be used in Section 4 where we study in detail the boundary crossing problem.

3. Existence and Uniqueness of Solution

In this section we shall confine ourselves to the problem of existence and uniqueness of solution of Eq. (17), $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$, where $\underset{\sim}{f}$ is a continuous piecewise-linear function. A piecewise-linear function is defined by a finite number of hyperplanes which divide the whole space into a finite number of convex regions surrounded by boundary hyperplanes, and by a set of constant Jacobian matrices $J^{(k)}$, k = 1, 2, ... r, each of which describes the linear behavior in each region as in Eq. (23). Only regions containing non-empty open sets need be considered, for any region of dimension less than n is contained in a boundary hyperplane. Throughout this section, the constant vectors $\tilde{w}^{(1)}, \ldots, \tilde{w}^{(r)}$ in (23) do not appear explicitly, and hence nothing is mentioned about them. Continuous, piecewise-linear functions are not differentiable on the boundaries of regions; thus usual analytical techniques can not be applied to study the behavior of functions in the whole space. Let $R_1, \ldots, R_r$ be the regions, and let $\tilde{J}^{(1)}, \ldots, \tilde{J}^{(r)}$ be the constant Jacobian matrices of regions $R_1, \ldots R_r$, respectively. If some of the Jacobian matrices are singular, then the uniqueness of solution does not follow because each region contains a non-empty open set. Therefore, we assume at the outset

$$\det \tilde{J}^{(k)} \neq 0 \qquad \text{for } k = 1, 2, \cdots, r \tag{27}$$

throughout this section.

The principal purpose of this section is to derive useful theorems which give conditions for the existence of a unique solution of continuous piecewise-linear mappings. The results will be used in Section 4 to present the solution method. However, first we need to give some necessary theoretical backgrounds. Especially, we will discuss thoroughly the Lipschitz condition pertaining to continuous piecewise-linear mappings and some results on global homeomorphism.

Lipschitz conditions

It is easy to show that $\tilde{f}$ satisfies a Lipschitz condition. Let

$\underset{\sim}{x}^{(1)}$ and $\underset{\sim}{x}^{(2)}$ be two distinct points in the $\underset{\sim}{x}$-space. The line-segment joining these two points, denoted by $L_{\underset{\sim}{x}}$, is parametically represented by:

$$L_{\underset{\sim}{x}} = \{\underset{\sim}{x} : \underset{\sim}{x} = \underset{\sim}{x}(\lambda) = (1 - \lambda)\, \underset{\sim}{x}^{(1)} + \lambda \underset{\sim}{x}^{(2)},\ 0 \leq \lambda \leq 1\}, \tag{28}$$

$L_{\underset{\sim}{x}}$ is split up into a finite number of portions $L_i = \{\underset{\sim}{x} : \underset{\sim}{x} = \underset{\sim}{x}(\lambda), \lambda_{i-1} \leq \lambda \leq \lambda_i\}$ $(i = 1, \ldots, m)$, where $\lambda_0 = 0$ and $\lambda_m = 1$, and each portion L_i belongs to a region R_i as shown in Fig. 1. Then, it follows from Eq. (23) and the definition of L_i:

$$\begin{aligned} \underset{\sim}{f}(\underset{\sim}{x}^{(2)}) - \underset{\sim}{f}(\underset{\sim}{x}^{(1)}) &= \sum_{i=1}^{m} \{\underset{\sim}{f}(\underset{\sim}{x}(\lambda_i)) - \underset{\sim}{f}(\underset{\sim}{x}(\lambda_{i-1}))\} \\ &= \left[\sum_{i=1}^{m} (\lambda_i - \lambda_{i-1})\, \underset{\sim}{J}^{(i)}\right] \{\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)}\} \end{aligned} \tag{29}$$

Let K_1 be the maximum of the matrix norms of $\underset{\sim}{J}^{(1)}, \ldots, \underset{\sim}{J}^{(r)}$, then we obtain

$$\|\underset{\sim}{f}(\underset{\sim}{x}^{(2)}) - \underset{\sim}{f}(\underset{\sim}{x}^{(1)})\| \leq K_1 \|\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)}\| \tag{30}$$

which is a Lipschitz condition of $\underset{\sim}{f}$. We will see below that if Eq. (17) has a unique solution for all $\underset{\sim}{y} \in R^n$, then the uniquely determined inverse $\underset{\sim}{f}^{-1}$ also satisfies a Lipschitz condition. For this, we first present the following lemma.

<u>Lemma 1</u>

Let $\underset{\sim}{f}$ be a continuous piecewise-linear mapping of R^n into itself. If Eq. (17) has a unique solution for all $\underset{\sim}{y} \in R^n$, then the inverse image of any straight line has the following property: If two distinct points of the inverse image belong to one region, then the intersection of the region with the straight line joining the two points is a portion of the inverse image; and furthermore, no other points of the inverse image belong to the region.

Proof: Let

$$L_{\underset{\sim}{y}} = \{\underset{\sim}{y} : \underset{\sim}{y} = \underset{\sim}{y}(\lambda) = (1-\lambda)\, \underset{\sim}{y}^{(1)} + \lambda \underset{\sim}{y}^{(2)}, \quad -\infty < \lambda < \infty\} \tag{31}$$

be a straight line, and assume that $\underset{\sim}{x}^{(1)} = \underset{\sim}{f}^{-1}(\underset{\sim}{y}^{(1)})$ and $\underset{\sim}{x}^{(2)} = \underset{\sim}{f}^{-1}(\underset{\sim}{y}^{(2)})$ are the two distinct points in a region, say R_1. Then, any point of the intersection of R_1 and the straight line joining $\underset{\sim}{x}^{(1)}$ ans $\underset{\sim}{x}^{(2)}$ may be represented by $\underset{\sim}{x} = (1-\lambda)\, \underset{\sim}{x}^{(1)} + \lambda x^{(2)}$, or equivalently $\underset{\sim}{x} - \underset{\sim}{x}^{(1)} = \lambda(\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)})$. This is shown in Fig. 2. Then it is seen that

$$\underset{\sim}{f}(\underset{\sim}{x}) - \underset{\sim}{f}(\underset{\sim}{x}^{(1)}) = \underset{\sim}{J}^{(1)}(\underset{\sim}{x} - \underset{\sim}{x}^{(1)}) = \lambda \underset{\sim}{J}^{(1)}(\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)})$$

$$= \lambda\, \{\underset{\sim}{f}(\underset{\sim}{x}^{(2)}) - \underset{\sim}{f}(\underset{\sim}{x}^{(1)})\}$$

Therefore, the following relation is immediate

$$\underset{\sim}{f}(\underset{\sim}{x}) = (1-\lambda)\, \underset{\sim}{f}(\underset{\sim}{x}^{(1)}) + \lambda \underset{\sim}{f}(\underset{\sim}{x}^{(2)}) \tag{32}$$

from which $\underset{\sim}{f}(\underset{\sim}{x}) \in L_{\underset{\sim}{y}}$ and hence $\underset{\sim}{x}$ is in the inverse image of $L_{\underset{\sim}{y}}$.

To show the second half of the lemma, let $\underset{\sim}{x}^{(3)}$ be the inverse image of a point $\underset{\sim}{y}^{(3)} = (1-\mu)\, \underset{\sim}{y}^{(1)} + \mu \underset{\sim}{y}^{(2)} \in L_{\underset{\sim}{y}}$, which lies in R_1 and is not on the straight line joining $\underset{\sim}{x}^{(1)}$ and $\underset{\sim}{x}^{(2)}$. It is clear that the two vectors $\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)}$ and $\underset{\sim}{x}^{(3)} - \underset{\sim}{x}^{(1)}$ are linearly independent. Now it is readily seen that

$$\underset{\sim}{J}^{(1)}(\underset{\sim}{x}^{(3)} - \underset{\sim}{x}^{(1)}) = \underset{\sim}{f}(\underset{\sim}{x}^{(3)}) - \underset{\sim}{f}(\underset{\sim}{x}^{(1)})$$
$$= \underset{\sim}{y}^{(3)} - \underset{\sim}{y}^{(1)} = \mu(\underset{\sim}{y}^{(2)} - \underset{\sim}{y}^{(1)})$$

This implies that the simultaneous linear equations

$$\underset{\sim}{J}^{(1)}\underset{\sim}{z} = \underset{\sim}{y}^{(2)} - \underset{\sim}{y}^{(1)} \tag{33}$$

has two linearly independent solutions $\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)}$ and $(\underset{\sim}{x}^{(3)} - \underset{\sim}{x}^{(1)})/\mu$, which is impossible due to the non-singularity of $\underset{\sim}{J}^{(1)}$. This completes the proof of Lemma 1.

Assuming that Eq. (17) has a unique solution for all $\underset{\sim}{y} \varepsilon R^n$, the above lemma implies the following: There exist real numbers $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_t \leq 1$ such that the inverse image of the portion of $L_{\underset{\sim}{y}}$ for $\lambda_{i-1} \leq \lambda \leq \lambda_i$ lies in region R_i for $i = 1, 2, \ldots, t$ as shown in Fig. 3. Then

$$\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)} = \underset{\sim}{f}^{-1}(\underset{\sim}{y}^{(2)}) - \underset{\sim}{f}^{-1}(\underset{\sim}{y}^{(1)})$$

$$= \left[\sum_{i=1}^{t} (\lambda_i - \lambda_{i-1}) \underset{\sim}{J}^{(i)}\right]^{-1} (\underset{\sim}{y}^{(2)} - \underset{\sim}{y}^{(1)}) \tag{34}$$

If we denote the maximum of the matrix norms of $\underset{\sim}{J}^{(1)^{-1}}, \ldots, \underset{\sim}{J}^{(r)^{-1}}$ by K_2, it is seen from (34) that

$$\|\underset{\sim}{x}^{(2)} - \underset{\sim}{x}^{(1)}\| \leq K_2 \|\underset{\sim}{y}^{(2)} - \underset{\sim}{y}^{(1)}\| \tag{35}$$

This proves the following lemma.

Lemma 2

Let $\underset{\sim}{f}$ be a continuous piecewise-linear mapping of R^n into itself. If Eq. (17), has a unique solution for all $\underset{\sim}{y} \in R^n$, then the uniquely determined inverse mapping $\underset{\sim}{f}^{-1}$ satisfies a Lipschitz condition.

Theorems on homeomorphism

Our primary concern has been to find conditions for Eq. (17) to have a unique solution for all $\underset{\sim}{y} \in R^n$. The foregoing discussion has made it clear that $\underset{\sim}{f}^{-1}$ is continuous if Eq. (17) has a unique solution for all $\underset{\sim}{y} \in R^n$. In mathematical terminology, $\underset{\sim}{f}$ is said to be a homeomorphism of R^n onto itself. Hence, what we are trying to find are conditions for $\underset{\sim}{f}$ to be a homeomorphism.

Holtzmann and Liu gave necessary and sufficient conditions for a continuous mapping $\underset{\sim}{f}$ to be a homeomorphism as a corollary of Palais theorem[5,6]. Their conditions are: (i) $\underset{\sim}{f}$ is a local homeomorphism, and (ii) $\lim \|\underset{\sim}{f}(\underset{\sim}{x})\| = \infty$ as $\|\underset{\sim}{x}\| \to \infty$. The local homeomorphism is defined as follows: For each $\underset{\sim}{x}$ there exists a neighborhood of $\underset{\sim}{x}$ which is mapped homeomorphically onto a neighborhood of $\underset{\sim}{f}(\underset{\sim}{x})$. Let ∇ be a neighborhood, then $\underset{\sim}{f} : \nabla \to \underset{\sim}{f}(\nabla)$ is required to be one-to-one, and furthermore it is required that $\underset{\sim}{f}^{-1}$: $\underset{\sim}{f}(\nabla) \to \nabla$ is continuous.

We will give a corollary to the above result which is suitable to continuous piecewise-linear mappings that form a special subclass of general continuous mappings. For this purpose we need the following lemma.

Lemma 3

Let $\underset{\sim}{f}$ be continuous piecewise-linear mapping which is a homeomorphism of R^n onto itself. Then, for any unit vector $\underset{\sim}{\alpha}$ and for any point $\underset{\sim}{x} \in R^n$ there exists one and only one non-zero vector $\underset{\sim}{\beta}(\underset{\sim}{\alpha}, \underset{\sim}{x})$ such that

$$\underset{\sim}{f}(\underset{\sim}{x} + \nu\underset{\sim}{\beta}) = \underset{\sim}{f}(\underset{\sim}{x}) + \nu\underset{\sim}{\alpha} \tag{36}$$

for sufficiently small positive ν.

Proof: Set $\underset{\sim}{y}^{(1)} = \underset{\sim}{f}(\underset{\sim}{x})$ and $\underset{\sim}{y}^{(2)} = \underset{\sim}{f}(\underset{\sim}{x}) + \underset{\sim}{\alpha}$. Let $L_{\underset{\sim}{y}}$ be the straight line joining two points $\underset{\sim}{y}^{(1)}$ and $\underset{\sim}{y}^{(2)}$, as defined in Eq. (31). Since $\underset{\sim}{f}^{-1}$ is continuous, $\underset{\sim}{f}^{-1}(\underset{\sim}{y}\ (\lambda)) \to \underset{\sim}{f}^{-1}(\underset{\sim}{y}^{(1)}) = \underset{\sim}{x}$ as $\lambda \to 0$. Hence, there exists a positive constant $\lambda_0 \to 0$ such that x and $f^{-1}(y(\lambda_0))$ both lie in a same region, say R_1. Let $\underset{\sim}{\beta}(\underset{\sim}{\alpha}, \underset{\sim}{x})$ be the vector defined by

$$\underset{\sim}{\beta}(\underset{\sim}{\alpha}, \underset{\sim}{x}) = \frac{1}{\lambda_0} \{\underset{\sim}{f}^{-1}(\underset{\sim}{y}(\lambda_0)) - \underset{\sim}{x}\}. \tag{37}$$

If $0 \leq \nu \leq \lambda_0$, then the point $\underset{\sim}{x} + \nu\underset{\sim}{\beta}$ lies on the line segment joining $\underset{\sim}{x}$ and $\underset{\sim}{f}^{-1}(\underset{\sim}{y}(\lambda_0))$, which is wholly contained in region R_1. Therefore, the following relation follows immediately

$$\underset{\sim}{f}(\underset{\sim}{x} + \nu\underset{\sim}{\beta}) - \underset{\sim}{f}(\underset{\sim}{x}) = \nu\underset{\sim}{J}^{(1)}\underset{\sim}{\beta} = \frac{\nu}{\lambda_0}\underset{\sim}{J}^{(1)}\{\underset{\sim}{f}^{-1}(\underset{\sim}{y}(\lambda_0) - \underset{\sim}{x}\}$$

$$= \frac{\nu}{\lambda_0}\{\underset{\sim}{y}(\lambda_0) - \underset{\sim}{f}(\underset{\sim}{x})\} = \nu\underset{\sim}{\alpha} \tag{38}$$

Thus, the existence of one such $\underset{\sim}{\beta}$ has been demonstrated. If more than one $\underset{\sim}{\beta}$ exist, then the one-to-one correspondence of $\underset{\sim}{f}$ does not hold; hence, the uniqueness of $\underset{\sim}{\beta}$ follows. This concludes the proof of Lemma 3.

Lemma 4

Let $\underset{\sim}{f}$ be a continuous piecewise-linear mapping of R^n into itself. A necessary and sufficient condition for f to be homeomorphism of R^n onto itself is that for any unit vector $\underset{\sim}{\alpha}$ and for any $\underset{\sim}{x} \in R^n$ there exists one and only one nonzero vector $\underset{\sim}{\beta}(\underset{\sim}{\alpha}, \underset{\sim}{x})$ such that

$$\underset{\sim}{f}(\underset{\sim}{x} + \nu\underset{\sim}{\beta}) = \underset{\sim}{f}(\underset{\sim}{x}) + \nu\underset{\sim}{\alpha}$$

for sufficiently small positive ν.

Proof: The necessity follows from Lemma 3, and hence we need to prove sufficiency only. It is interesting to see that condition (ii) of Holtzman and Liu's statement is automatically satisfied for continuous piecewise-linear mapping provided that the relation in (27) is valid. Let us assume the contrary, that is, $\|\underset{\sim}{x}^{(k)}\| \to \infty$ while $\|\underset{\sim}{f}(\underset{\sim}{x}^{(k)})\| \leq B$, where B is a positive constant. Since the number of regions is finite, there should be at least one region, say R_1, which contains infinitely many distinct points of the sequence $\{\underset{\sim}{x}^{(k)}\}$. Therefore, without loss of generality, we can assume that $\{\underset{\sim}{x}^{(k)}\}$ belongs to region R_1. Then, we obtain

$$\underset{\sim}{f}(\underset{\sim}{x}^{(k)}) - \underset{\sim}{f}(\underset{\sim}{x}^{(1)}) = \underset{\sim}{J}^{(1)} (\underset{\sim}{x}^{(k)} - \underset{\sim}{x}^{(1)})$$

The left-hand side of the equation is bounded and the right-hand side has to diverge for $\|x^{(k)} - x^{(1)}\| \to \infty$ as $k \to \infty$. This is the desired contradiction.

It remains to be shown that the condition of the lemma implies local homeomorphism. Let x^* be an arbitrary chosen point of R^n, and let y^* be the corresponding point, that is, $y^* = f(x^*)$. We are concerned with a local property of f at a neighborhood of x^*, and therefore we may throw away all boundary hyperplanes except those on which x^* lies. Such a modified continuous piecewise-linear function will be denoted by $\hat{f}$. In this case, the whole space is divided into a finite number of cones with the common vector x^*, each of which is a region as shown in Fig. 4. Then, for any point $x' \in R^n$, the half-line

$$HL_x = \{x : x = x(\lambda) = \lambda(x' - x^*), \lambda \geq 0\} \tag{39}$$

is wholly contained in a region. Therefore, we obtain the following statement: For any unit vector α there exists one and only one nonzero vector $\beta(\alpha)$, such that

$$\hat{f}(x^* + \nu\beta) = \hat{f}(x^*) + \nu\alpha \tag{40}$$

for $\nu \geq 0$. This statement indicates that the mapping $\hat{f}$ of R^n into itself is a one-to-one onto mapping. Therefore, $\hat{f}^{-1}$ is continuous due to Lemma 2. Hence, $\hat{f}$ is actually a homeomorphism of R^n onto itself. Since $\hat{f}$ coincides with f in a neighborhood ∇ of x^*, the mapping f of ∇ onto $f(\nabla) = \hat{f}(\nabla)$, which is a neighborhood of y^*, is a homeomorphism. This implies that f is a local homeomorphism. Thus the proof of Lemma 4 has been completed.

Sufficient conditions. Holtzmann and Liu's corollary as well as Lemma 4 are highly useful to discuss theoretically the properties of $\tilde{f}$. However, they are not of much practical value to judge whether or not a given function $\tilde{f}$ is a homeomorphism. In this respect it is important to have various sufficient conditions under which homeomorphism of $\tilde{f}$ is guaranteed. Two theorems will be given, they represent extensions of theorems for continuously differentiable mappings[7].

Theorem 1.

Let $\tilde{f}$ be a continuous piecewise-linear mapping of R^n into itself, and let $\tilde{J}_k^{(j)}$ denote the matrix composed of the first k-rows and the first k-columns of the constant Jacobian matrix $\tilde{J}^{(j)}$ of region R_j. The mapping is a homeomorphism of R^n onto itself if, for each $k = 1, 2, \ldots, n$ the determinants

$$\det \tilde{J}_k^{(1)}, \cdots, \det \tilde{J}_k^{(r)}$$

do not vanish and have the same sign.

Remark.

If the constant Jacobian matrices $\tilde{J}^{(1)}, \ldots, \tilde{J}^{(r)}$ all belong to class P introduced by Fiedler and Ptak[7-10], then the sign condition in the above theorem is automatically satisfied. Since class P matrices are generalizations of positive-definite matrices, positive-definiteness of Jacobians also implies the sign condition. It is clear that matrices which satisfy the sign condition are not restricted to class P matrices[7].

Proof: It is easy to see that the theorem is valid for $n = 1$, because in this case y_1 is a continuous piecewise-linear function of x_1, which has positive slopes or negative slopes everywhere, as shown in Fig. 5. Therefore, the theorem can be proven by induction.

Let us assume that the statement of the theorem is valid for $n = k - 1$ and the case of $n = k$ is considered. In this case,

$$y_i = \underset{\sim}{f}(x_1, \cdots, x_{k-1}, x_k) \qquad (i=1, \cdots, k) \tag{41}$$

or in the form of vector function, we have

$$\underset{\sim}{y} = \underset{\sim}{f}(\underset{\sim}{x}) \tag{42}$$

For the purpose of notational simplicity we introduce the following conventions:

$$\begin{aligned} \underset{\sim}{x}_{-k} &= [x_1, \cdots, x_{k-1}]^T \\ \underset{\sim}{y}_{-k} &= [y_1, \cdots, y_{k-1}]^T \\ \underset{\sim}{f}_{-k} &= [f_1, \cdots, f_{k-1}]^T \end{aligned} \tag{43}$$

and

$$\underset{\sim}{y}_{-k} = \underset{\sim}{f}_{-k} (\underset{\sim}{x}_{-k}, x_k) \tag{44}$$

If the value of x_k is fixed, the mapping $\underset{\sim}{f}_{-k}$ of R^{k-1} into itself is still a continuous piecewise-linear mapping. Furthermore it is clear that the mapping $\underset{\sim}{f}_{-k}$ satisfies the condition of the theorem for $n = k - 1$, and hence $\underset{\sim}{f}_{-k}$ is a homeomorphism of R^{k-1} onto itself. From Lemma 2 the inverse mapping $\underset{\sim}{f}_{-k}^{-1}$ defined by

$$\underset{\sim}{x}_{-k} = \underset{\sim}{f}_{-k}^{-1} (\underset{\sim}{y}_{-k}, x_k) \tag{45}$$

satisfies a Lipschitz condition. As seen from the proof of Lemma 2, the Lipschitz constant solely depends on $\underset{\sim}{J}_{k-1}^{(1)}, \ldots, \underset{\sim}{J}_{k-1}^{(r)}$, and hence the constant can be chosen independently of the value of x_k.

Let $\underset{\sim}{x}^{(0)} = \{x_1^{(0)}, \cdots, x_{k-1}^{(0)}, x_k^{(0)}]^T$ be an arbitrary point and let $\underset{\sim}{y}^{(0)} = \{y_1^{(0)}, \cdots, y_{k-1}^{(0)}, y_k^{(0)}]^T$ be the corresponding point. That is, $\underset{\sim}{y}^{(0)} = \underset{\sim}{f}(x^{(0)})$. We define a half-line starting with $\underset{\sim}{x}^{(0)}$ as follows:

$$HL_\Delta = \{\underset{\sim}{x} : \underset{\sim}{x} = \underset{\sim}{x}(\Delta x_k) = [x_1^{(0)}, \cdots, x_{k-1}^{(0)}, x_k^{(0)} + \Delta x_k]^T, \Delta x_k \geq 0\} \tag{46}$$

Then it is clear that

$$\|\underset{\sim}{x}^{(0)} - \underset{\sim}{x}(\Delta x_k)\| = \Delta x_k \tag{47}$$

Since $\underset{\sim}{f}$ satisfies a Lipschitz condition with Lipschitz constant K_1, it follows that

$$\begin{aligned} &\|\underset{\sim}{y}_{-k}^{(0)} - \underset{\sim}{f}_{-k}(\underset{\sim}{x}_{-k}(\Delta x_k), x_k^{(0)} + \Delta x_k)\| \\ &= \|\underset{\sim}{y}_{-k}^{(0)} - \underset{\sim}{f}_{-k}(\underset{\sim}{x}_{-k}^{(0)}, x_k^{(0)} + \Delta x_k)\| \leq \|\underset{\sim}{y}^{(0)} - \underset{\sim}{f}(\underset{\sim}{x}(\Delta x_k))\| \\ &\leq K_1 \Delta x_k \end{aligned} \tag{48}$$

Since $\underset{\sim}{f}_{-k}$ is a homeomorphism, there eixsts one and only one point

$$\underset{\sim}{x}'(\Delta x_k) = [x_1'(\Delta x_k), \cdots, x_{k-1}'(\Delta x_k), x_k^{(0)} + \Delta x_k]^T \tag{49}$$

such that

$$\underset{\sim}{y}_{-k}^{(0)} = \underset{\sim}{f}_{-k}(\underset{\sim}{x}'_{-k}(\Delta x_k), x_k^{(0)} + \Delta x_k) \tag{50}$$

for any Δx_k. From (48), (50) and from the fact that $\underset{\sim}{f}_k^{-1}$ is Lipschitzian, it is seen that

$$\|\underset{\sim}{x}(\Delta x_k) - \underset{\sim}{x}'(\Delta x_k)\| = \|\underset{\sim}{x}_{-k}(\Delta x_k) - \underset{\sim}{x}'_{-k}(\Delta x_k)\|$$
$$\leq K_1 K_2 \Delta x_k. \tag{51}$$

This together with (47) implies

$$\|\underset{\sim}{x}^{(0)} - \underset{\sim}{x}'(\Delta x_k)\| \leq (1 + K_1 K_2)\, \Delta x_k \tag{52}$$

Then, there exists a positive constant $\delta > 0$ such that the points $\underset{\sim}{x}^{(0)}$ and $\underset{\sim}{x}'(\delta)$ belong to the same region, say R_1, regardless of whether or not the point $\underset{\sim}{x}^{(0)}$ is an inner point of a region as shown in Fig. 6. It is to be noted that f is linear in region R_1. Therefore, for $0 \leq \Delta x_k \leq \delta$

$$\underset{\sim}{x}'(\Delta x_k) = \underset{\sim}{x}^{(0)} + \frac{\Delta x_k}{\delta} \{\underset{\sim}{x}'(\delta) - \underset{\sim}{x}^{(0)}\}, \tag{53}$$

which is equivalent of saying that the line-segment joining $\underset{\sim}{x}^{(0)}$ and $\underset{\sim}{x}'(\delta)$ is the set $\{\underset{\sim}{x} : \underset{\sim}{x} = \underset{\sim}{x}'(\Delta x_k),\ 0 \leq \Delta x_k \leq \delta\}$. In region R_1, the behavior of the function f is described by the Jacobian matrix

$$\underset{\sim}{J}^{(1)} = \left[\begin{array}{c|c} \underset{\sim}{J}^{(1)}_{k-1} & \underset{\sim}{b}^{(1)} \\ \hline \underset{\sim}{a}^{(1)} & c^{(1)} \end{array}\right] \tag{54}$$

Therefore, for $0 \leq \Delta x_k \leq \delta$

$$
\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \\ \Delta y_k \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{J}^{(1)}_{k-1} & \underset{\sim}{b}^{(1)} \\ \hline \underset{\sim}{a}^{(1)} & c^{(1)} \end{array}\right] \begin{bmatrix} x'_1(\Delta x_k) - x_1^{(0)} \\ \cdot \\ \cdot \\ x'_{k-1}(\Delta x_k) - x_{k-1}^{(0)} \\ \\ \Delta x_k \end{bmatrix} \tag{55}
$$

The application of Cramer's rule yields

$$
\frac{\Delta y_k}{\Delta x_k} = \frac{\det \underset{\sim}{J}^{(1)}}{\det \underset{\sim}{J}^{(1)}_{k-1}} = \frac{\det \underset{\sim}{J}^{(1)}_{k}}{\det \underset{\sim}{J}^{(1)}_{k-1}} \tag{56}
$$

which has the same sign throughout the space according to the assumptions made in the theorem.

The arguments above make the following statement legitimate: For any point $\underset{\sim}{x}$ there exists one and only one unit vector $\underset{\sim}{\alpha}^{+}(\underset{\sim}{x})$ such that a sufficiently small change along this direction vector makes no change in the values of $y_1, \ldots, y_{k-1}$ but causes an increase of the value of x_k. Similarly, there exists one and only one unit vector $\underset{\sim}{\alpha}^{-}(\underset{\sim}{x})$ such that a sufficiently small change along this direction vector makes no change in the values of $y_1, \ldots, y_{k-1}$ but causes a decrease of the value of x_k.

Let $\underset{\sim}{y}^* = \{y_1^*, \ldots, y_{k-1}^*, y_k^*]^T$ be an arbitrarily given point, then the homeomorphism of $\underset{\sim}{f}_{-k}$ implies that there exists one and only one point $\underset{\sim}{x}^{(1)} = \{x_1^{(1)}, \ldots, x_{k-1}^{(1)}, 0]^T$ for which

$$
\underset{\sim}{y}^*_{-k} = \underset{\sim}{f}_{-k}(\underset{\sim}{x}^{(1)}_{-k}, 0) \tag{57}
$$

Starting with the initial point $\tilde{x}^{(1)}$ and following the way described in the preceding paragraphs, we can either increase or decrease the value of x_k while preserving the values $y_1 = y_1^*, \ldots, y_{k-1} = y_{k-1}^*$ constant. As described in the preceding paragraphs, the direction vector along which the point tranverses can be determined at any point. The linear movement along the direction vector can thus be described as long as the point lies in the same region. At the critical point where a boundary is crossed, a new direction vector must be determined. In this manner, a continuous piecewise-linear curve can be traced, and the curve so obtained is called a "trajectory", which is parameterically represented by the value of x_k. It is evident that the trajectory does not intersect with itself. Furthermore, following the line of reasoning which was used to prove Lemma 1, we conclude that the trajectory is a continuous curve consisting of a finite number of line-segments and half-lines for $-\infty < x_k < \infty$, each of which lies in a region. The derivative dy_k/dx_k in each region is given by one of the following ratios:

$$\frac{\det \tilde{J}^{(1)}}{\det \tilde{J}_{k-1}^{(1)}}, \ldots, \frac{\det \tilde{J}^{(1)}}{\det \tilde{J}_{k-1}^{(1)}} \tag{58}$$

which are all positive or negative. Therefore, the function y_k of variable x_k is strictly monotone increasing or strictly monotone decreasing; and furthermore the range of y_k covers the whole real line $-\infty < y_k < \infty$. This implies that there exists one and only one value $x_k = x_k^*$ which gives

$y_k = y_k^*$. If we denote the point of the trajectory which corresponds to the parameter value $x_k = x_k^*$ by $\underset{\sim}{x}^*$, it is clear that

$$\underset{\sim}{y}^* = \underset{\sim}{f}(\underset{\sim}{x}^*) \tag{59}$$

and the point $\underset{\sim}{x}^*$ is the only point which satisfies Eq. (59). These imply that f is a one-to-one onto mapping, thus the proof of Theorem 1 is completed.

By applying interchange of rows and of columns the following theorem may be easily derived from the first theorem

Theorem 2

The mapping $\underset{\sim}{f}$ is a homeomorphism of R^n onto itself if there exist two permutations $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$ such that the determinants

$$\det \underset{\sim}{J}^{(1)} \begin{pmatrix} i_1, & \cdots, & i_k \\ j_1, & \cdots, & j_k \end{pmatrix}, \cdots, \det \underset{\sim}{J}^{(r)} \begin{pmatrix} i_1, & \cdots, & i_k \\ j_1, & \cdots, & j_k \end{pmatrix} \tag{60}$$

do not vanish and have the same sign for each $k = 1, 2, \cdots, n$, where the (α, β) - element of the square matrix $\underset{\sim}{J}^{(\ell)} \begin{pmatrix} i_1, & \cdots, & i_k \\ j_1, & \cdots, & j_k \end{pmatrix}$ of order k is the (i_α, j_β) - element of the matrix $\underset{\sim}{J}^{(\ell)}$.

This theorem further extends the applicability of Theorem 1, since non-principal cofactors may be used to ascertain the global homeomorphism of continuous piecewise linear mappings. It should be pointed out that the theorem only gives sufficient conditions. The following example illustrates that the conditions are not necessary.

Example 1

Consider the piecewise linear mapping described by the following two equations:

$$\text{In } R_1: \begin{cases} y_1 = \frac{1}{2} x_1 + x_2 \\ y_2 = \frac{1}{2} x_2 \end{cases} \qquad \underset{\sim}{J}^{(1)} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\text{In } R_2: \begin{cases} y_1 = -\frac{1}{2} x_1 \\ y_2 = -x_1 - \frac{1}{2} x_2 \end{cases} \qquad \underset{\sim}{J}^{(2)} = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & -\frac{1}{2} \end{bmatrix}$$

The two regions R_1 and R_2 are shown in Fig. 7. Note that the Jacobian determinants have the same sign but the matrices do not satisfy Theorem 2 since there exist no corresponding cofactors which are both nonzero and have the same sign. Yet, the mapping is a global homeomorphism of R^2 onto itself.

4. Method of solution

Algorithm

We first describe briefly the Katzenelson algorithm for obtaining the solution of Eq. (17):

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y} \tag{61}$$

where $\underset{\sim}{f}$ is a continuous piecewise linear mapping of R^n onto itself. To begin, we choose arbitrarily a point $\underset{\sim}{x}^{(1)}$ in the $\underset{\sim}{x}$-space and denote the region where $\underset{\sim}{x}^{(1)}$ lies R_1. In R_1 the equations in (61) are linear and are represented by

$$\underset{\sim}{J}^{(1)} \underset{\sim}{x} + \underset{\sim}{w}^{(1)} = \underset{\sim}{y} \tag{62}$$

Substituting $\underset{\sim}{x}^{(1)}$ into the above equation, we obtain

$$\underset{\sim}{J}^{(1)} \underset{\sim}{x}^{(1)} + \underset{\sim}{w}^{(1)} = \underset{\sim}{y}^{(1)} \tag{63}$$

where $\underset{\sim}{y}^{(1)}$ is the corresponding point in the y-space, i.e. $\underset{\sim}{f}(\underset{\sim}{x}^{(1)}) = \underset{\sim}{y}^{(1)}$. Let the actual input to the network be $\underset{\sim}{y}^{(f)}$, and denote the line segment joining $\underset{\sim}{y}^{(1)}$ and $\underset{\sim}{y}^{(f)}$ by $L_{\underset{\sim}{y}}$. The problem is then to determine the inverse image of the line segment $L_{\underset{\sim}{y}}$. The inverse image of $\underset{\sim}{y}^{(f)}$ is the actual solution $\underset{\sim}{x}^{(f)}$, and the inverse image of $L_{\underset{\sim}{y}}$ is called the solution curve in the $\underset{\sim}{x}$-space. Combining (62) and (63) and letting $\underset{\sim}{y} = \underset{\sim}{y}^{(f)}$, we obtain

$$\underset{\sim}{x} = \underset{\sim}{x}^{(1)} + \underset{\sim}{J}^{(1)^{-1}} (\underset{\sim}{y}^{(f)} - \underset{\sim}{y}^{(1)}) \tag{64}$$

If it happens that the $\underset{\sim}{x}$ obtained above is in R_1, it is then the actual solution; and the solution curve is the line segment $L_{\underset{\sim}{x}^{(1)}}$ connecting $\underset{\sim}{x}^{(1)}$ and $\underset{\sim}{x}$ and it is in R_1. Usually it is not, and we proceed with the following iteration. The next starting point, $\underset{\sim}{x}^{(2)}$, is the intersection of the line segment $L_{\underset{\sim}{x}^{(1)}}$ and the hyperplane which represents the boundary of R_1. Thus

$$\underset{\sim}{x}^{(2)} = \underset{\sim}{x}^{(1)} + \lambda^{(1)} \underset{\sim}{J}^{(1)^{-1}} (\underset{\sim}{y}^{(f)} - \underset{\sim}{y}^{(1)}) \tag{65}$$

where $\lambda^{(1)}$ is a parameter and

$$0 < \lambda^{(1)} \leq 1 \tag{66}$$

It is chosen to force $\underset{\sim}{x}^{(2)}$ to lie on the boundary of R_1. Let $\underset{\sim}{y}$ be the corresponding point in the $\underset{\sim}{y}$-space. Clearly $\underset{\sim}{y}^{(2)} \in L_{\underset{\sim}{y}}$ and the first portion of the inverse image of $L_{\underset{\sim}{y}}$ is thus determined. Next, we move the input from $\underset{\sim}{y}^{(2)}$ along $L_{\underset{\sim}{y}}$ toward $\underset{\sim}{y}^{(f)}$, the inverse image will move out of R_1 in the $\underset{\sim}{x}$-space and enter a new region R_2. This fact has been demonstrated in the previous section.

In the next iteration, we start with $\underset{\sim}{x}^{(2)}$ in R_2. We repeat what was done before but use $\underset{\sim}{J}^{(2)}$ to compute the next point $\underset{\sim}{x}^{(3)}$. Thus

$$\underset{\sim}{x}^{(3)} = \underset{\sim}{x}^{(2)} + \lambda^{(2)} \underset{\sim}{J}^{(2)} (\underset{\sim}{y}^{(f)} - \underset{\sim}{y}^{(2)}) \tag{67}$$

As in the previous case $\lambda^{(2)}$ is chosen so that $\underset{\sim}{x}^{(3)}$ lies on the common boundary of R_2 and the succeeding region R_3, and

$$0 < \lambda^{(2)} \leq 1 \tag{68}$$

In Fig. 8, the first two iterations are illustrated. The process continues until the actual solution $\underset{\sim}{x}^{(f)}$ is reached.

Boundary crossing

The purpose of this section is to present a detailed analysis of the problem of boundary crossing in the iteration without recourse to the theory of Section 3.

In Section 2 we discussed briefly the problem of piecewise linear approximation. Because of the nature of the circuit elements usually encountered, it is feasible to make the piecewise linear approximation directly to the network branch characteristics in terms of the branch $\begin{bmatrix} \underset{\sim}{v}_n \\ \underset{\sim}{i}_b \end{bmatrix}$. Let $\underset{\sim}{H}^{(1)}$ and $\underset{\sim}{H}^{(2)}$ be the constant branch hybrid matrices corresponding to regions R_1 and R_2, respectively. Recall that the common boundary is characterized by the branch variable, say v_k, the k-th equation of (25). The branch hybrid matrices for the two regions are related by

$$\underset{\sim}{H}^{(2)} = \underset{\sim}{H}^{(1)} + \delta\underset{\sim}{H} = \underset{\sim}{H}^{(1)} + \begin{bmatrix} 0 & 0 \dots & \delta_{1k} & \dots 0 \\ \hline 0 & 0 \dots & \delta_{jk} & \dots 0 \\ \hline 0 & 0 \dots & \delta_{pk} & \dots 0 \end{bmatrix} \tag{69}$$

where $\underset{\sim}{H}^{(2)}$ and $\underset{\sim}{H}^{(1)}$ are of order p. Note that $\delta\underset{\sim}{H}$ which represents the difference between $\underset{\sim}{H}^{(2)}$ and $\underset{\sim}{H}^{(1)}$ is nonzero only in the k-th column since the boundary is specified by v_k = constant. In the n-dimensional $\underset{\sim}{x}$-space, the corresponding common boundary is given by (26)

$$\underset{\sim}{c}_k^T \underset{\sim}{x} = \text{constant} \tag{70}$$

where $\underset{\sim}{c}_k$ is the k-th column of the matrix $\underset{\sim}{C}$ which specifies the two subgraphs. The Jacobian matrices $\underset{\sim}{J}^{(1)}$ and $\underset{\sim}{J}^{(2)}$ in the two regions are related according to Eqs. (69) and (19):

$$\underset{\sim}{J}^{(2)} = \underset{\sim}{C} \; (\underset{\sim}{H}^{(1)} + \delta\underset{\sim}{H}) \; \underset{\sim}{C}^T + D \; = \underset{\sim}{J}^{(1)} + \underset{\sim}{C} \; \delta\underset{\sim}{H} \; \underset{\sim}{C}^T$$

$$= \underset{\sim}{J}^{(1)} + \sum_{j=1}^{p} \delta_{jk} \, \underset{\sim}{c}_j \, \underset{\sim}{c}_k^T = \underset{\sim}{J}^{(1)} + \underset{\sim}{\sigma} \, \underset{\sim}{c}_k^T \tag{71}$$

where

$$\underset{\sim}{\sigma} \triangleq \sum_{j=1}^{p} \delta_{jk} \, \underset{\sim}{c}_j \tag{72}$$

Thus the difference of the two Jacobian matrices is a dyad, which is of rank one. The above gives a clear relation among the n-dimensional network variable $\underset{\sim}{x}$, the p-dimensional branch characteristics and the network topology.

In the $\underset{\sim}{x}$-space, the common boundary between R_1 and R_2 can be represented in terms of the normal vector of the hyperplane, $\underset{\sim}{n}$

$$(\underset{\sim}{x} \; , \; \underset{\sim}{n}) = \text{constant} \tag{73}$$

or

$$(\Delta\underset{\sim}{x} \; , \; \underset{\sim}{n}) = 0 \tag{74}$$

Comparing (73) and (70), we note that the normal vector $\underset{\sim}{n}$ can be identified with the vector $\underset{\sim}{c}_k$ which describes the graph. Eq. (74) implies the following:

$$\underset{\sim}{J}^{(2)} \; \Delta\underset{\sim}{x} = \underset{\sim}{J}^{(1)} \; \Delta\underset{\sim}{x} + \underset{\sim}{\sigma}\underset{\sim}{n}^T \; \Delta\underset{\sim}{x} = \underset{\sim}{J}^{(1)} \; \Delta\underset{\sim}{x}$$

Hence, the piecewise-linear representations of $f(x)$, $J^{(1)}x + w^{(1)}$ and $J^{(2)}x + w^{(2)}$, coincide with each other on the boundary hyperplane in (73) if $w^{(1)}$ and $w^{(2)}$ are appropriately chosen. This guarantees the continuity of the piecewise-linear function $f(x)$.

Let us now consider the solution curve in R_1 and R_2 as shown in Fig. 8. The normal vector n is shown. It is seen that the scalar product

$$(x^{(2)} - x^{(1)}, n) > 0 \tag{75}$$

We wish to derive the conditions on $J^{(2)}$ and $J^{(1)}$, which insures

$$(x^{(3)} - x^{(2)}, n) > 0 \tag{76}$$

From Eq. (65), we have

$$x^{(2)} - x^{(1)} = J^{(1)^{-1}} \lambda^{(1)} (y^{(f)} - y^{(1)}) = J^{(1)^{-1}} \lambda^{(1)} \Delta y \tag{77}$$

Substituting (77) into (75), we obtain

$$(J^{(1)^{-1}} \lambda^{(1)} \Delta y, \ n) > 0$$

or

$$(\lambda^{(1)} \Delta y, J^{(1)^{-1T}} n) > 0 \tag{78}$$

Note that the first term in the scalar product of (78) is a direction vector in the y-space, the second term contains the information of the network in R_1 and the boundary. In order to determine the nature of the solution curve beyond R_1, we need to compute the corresponding vector $\underset{\sim}{J}^{(2)^{-1T}} \underset{\sim}{n}$. Using (71) and identifying $\underset{\sim}{c}_k$ with $\underset{\sim}{n}$, we obtain

$$\underset{\sim}{J}^{(2)^{-1T}} \underset{\sim}{n} = (\underset{\sim}{J}^{(1)} + \underset{\sim}{\sigma}\, \underset{\sim}{n}^T)^{-1T} \underset{\sim}{n} = (\underset{\sim}{J}^{(1)^T} + \underset{\sim}{n}\, \underset{\sim}{\sigma}^T)^{-1} \underset{\sim}{n}$$

$$= \underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n} - \underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n}\, (\underset{\sim}{\sigma}^T \underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n} + 1)^{-1} \underset{\sim}{\sigma}^T\, \underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n}$$

$$= K\, \underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n} \tag{79}$$

where

$$K = (1 + \underset{\sim}{\sigma}^T\, \underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n})^{-1} \tag{80}$$

is a scaler. Thus the relation between the two vectors $\underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n}$ and $\underset{\sim}{J}^{(2)^{-1T}} \underset{\sim}{n}$ is specified by the scalar K in Eq. (80). Using the familiar determinant identity:

$$\det\, (\underset{\sim}{1} + \underset{\sim}{P}\, \underset{\sim}{Q}) = \det\, (\underset{\sim}{1} + \underset{\sim}{Q}\, \underset{\sim}{P}) \tag{81}$$

it is not difficult to show that

$$\det\, \underset{\sim}{J}^{(2)} = K \det\, \underset{\sim}{J}^{(1)} \tag{82}$$

Therefore, we obtain the following important relation:

$$\underset{\sim}{J}^{(2)^{-1T}} \underset{\sim}{n} = \frac{\det\, \underset{\sim}{J}^{(1)}}{\det\, \underset{\sim}{J}^{(2)}}\, \underset{\sim}{J}^{(1)^{-1T}} \underset{\sim}{n} \tag{83}$$

With the above relation, it is easily shown that if the determinants of Jacobian matrices $\underset{\sim}{J}^{(2)}$ and $\underset{\sim}{J}^{(1)}$ have the same sign, Eq. (76) follows. Thus the sign condition on the determinants alone ensures that the solution curve will enter a new region when a common boundary is reached.

In view of the above study, and Theorem 2 and Example 1 of Section 3, it might be conjectured that the sign condition on the determinant $\underset{\sim}{J}^{(k)}$ alone is necessary and sufficient for global homeomorphism of a continuous piecewise linear mappings. The following example indicates that this is not the case.

Example 2

In Fig. 9, we indicate the six regions of a 2-dimensional $\underset{\sim}{x}$-space. The continuous piecewise linear mapping, and its linear Jacobian matrices are given as follows:

$$\text{In } R_1: \begin{cases} y_1 = x_1 - \frac{2}{\sqrt{3}} x_2 \\ y_2 = x_2 \end{cases} \qquad \underset{\sim}{J}^{(1)} = \begin{bmatrix} 1 & -\frac{2}{\sqrt{3}} \\ 0 & 1 \end{bmatrix}$$

$$\text{In } R_2: \begin{cases} y_1 = -\frac{1}{\sqrt{3}} x_2 \\ y_2 = \sqrt{3}\, x_1 \end{cases} \qquad \underset{\sim}{J}^{(2)} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} \\ \sqrt{3} & 0 \end{bmatrix}$$

$$= \underset{\sim}{J}^{(1)} + \begin{bmatrix} \frac{2}{\sqrt{3}} \\ -2 \end{bmatrix} [-\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}]$$

In R_3: $\begin{cases} y_1 = -x_1 - \frac{2}{\sqrt{3}} x_2 \\ y_2 = \qquad\quad -x_2 \end{cases}$

$$\underset{\sim}{J}^{(3)} = \begin{bmatrix} -1 & -\frac{2}{\sqrt{3}} \\ 0 & -1 \end{bmatrix}$$

$$= \underset{\sim}{J}^{(2)} + \begin{bmatrix} \frac{2}{\sqrt{3}} \\ 2 \end{bmatrix} [-\frac{\sqrt{3}}{2}, -\frac{1}{2}]$$

In R_4: $\begin{cases} y_1 = -x_1 + \frac{2}{\sqrt{3}} x_2 \\ y_2 = \qquad\quad -x_2 \end{cases}$

$$\underset{\sim}{J}^{(4)} = \begin{bmatrix} -1 & \frac{2}{\sqrt{3}} \\ 0 & -1 \end{bmatrix}$$

$$= \underset{\sim}{J}^{(3)} + \begin{bmatrix} -\frac{4}{\sqrt{3}} \\ 0 \end{bmatrix} [0, -1]$$

In R_5: $\begin{cases} y_1 = \qquad\quad \frac{1}{\sqrt{3}} x_2 \\ y_2 = -\sqrt{3}\, x_1 \end{cases}$

$$\underset{\sim}{J}^{(5)} = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} \\ -\ 3 & 0 \end{bmatrix}$$

$$= \underset{\sim}{J}^{(4)} + \begin{bmatrix} \frac{2}{\sqrt{3}} \\ -2 \end{bmatrix} [\frac{\sqrt{3}}{2}, -\frac{1}{2}]$$

In R_6: $\begin{cases} y_1 = x_1 + \frac{2}{\sqrt{3}} x_2 \\ y_2 = x_2 \end{cases}$ $\qquad \underset{\sim}{J}^{(6)} = \begin{bmatrix} 1 & \frac{2}{\sqrt{3}} \\ 0 & 1 \end{bmatrix}$

$$= \underset{\sim}{J}^{(5)} + \begin{bmatrix} \frac{2}{\sqrt{3}} \\ 2 \end{bmatrix} [\frac{\sqrt{3}}{2}, \ \frac{1}{2}]$$

In each case, we indicate the relation of the Jacobian matrices of the neighboring regions in terms of the normal vectors. To complete the picture, we need to check the relations between $\underset{\sim}{J}^{(1)}$ and $\underset{\sim}{J}^{(6)}$. We note that

$$\underset{\sim}{J}^{(1)} = \begin{bmatrix} 1 & -\frac{2}{\sqrt{3}} \\ 0 & 1 \end{bmatrix} = \underset{\sim}{J}^{(6)} + \begin{bmatrix} -\frac{4}{\sqrt{3}} \\ 0 \end{bmatrix} [0, 1]$$

In this example det $\underset{\sim}{J}^{(i)} = 1$, for $i = 1, 2, \ldots. 6$.

Let us next examine the mappings in some detail. Let $\underset{\sim}{x}^{(1)} = [1, 0]$, $\underset{\sim}{x}^{(2)} = [\frac{1}{2}, \frac{\sqrt{3}}{2}]$, $\underset{\sim}{x}^{(3)} = [-\frac{1}{2}, \frac{\sqrt{3}}{2}]$, $\underset{\sim}{x}^{(4)} = [-1, 0]$, $\underset{\sim}{x}^{(5)} = [-\frac{1}{2}, -\frac{\sqrt{3}}{2}]$ and $\underset{\sim}{x}^{(6)} = [\frac{1}{2}, -\frac{\sqrt{3}}{2}]$ as shown in the figure. It is easy to see that

$$f(x^{(1)}) = x^{(1)} \ , \quad f(x^{(2)}) = x^{(3)} \text{ and } f(R_1) = R_1 \cup R_2$$

$$f(x^{(2)}) = x^{(3)} \ , \quad f(x^{(3)}) = x^{(5)} \text{ and } f(R_2) = R_3 \cup R_4$$

$$f(x^{(3)}) = x^{(5)} \ , \quad f(x^{(4)}) = x^{(1)} \text{ and } f(R_3) = R_5 \cup R_6$$

$$f(x^{(4)}) = x^{(1)} \ , \quad f(x^{(5)}) = x^{(3)} \text{ and } f(R_4) = R_1 \cup R_2$$

$$f(x^{(5)}) = x^{(3)} \ , \quad f(x^{(6)}) = x^{(5)} \text{ and } f(R_5) = R_3 \cup R_4$$

$$f(x^{(6)}) = x^{(5)} \ , \quad f(x^{(1)}) = x^{(1)} \text{ and } f(R_6) = R_5 \cup R_6$$

Therefore, for $y \neq 0$, there exists two distinct solutions; for $y = 0$, there is a unique solution $x = 0$. Thus, even though the determinants of the Jacobian matrices in all six regions have the same sign, the mapping is not homeomorphic.

Corner problem

Up to now we have not considered the possibility that the solution curve might hit a corner, a common boundary for three or more regions. In the following we will study this possibility and indicate what will be done in such a case.

Let us assume that the solution curve hits a corner point $x^{(k)}$ which is on mutually distinct s-boundary hyperplanes

$$H_i = \{x : (x - x^{(k)}, n^{(i)}) = 0\} \qquad (i=1,2,\cdots,s) \tag{84}$$

where $\underset{\sim}{n}^{(1)}, \ldots, \underset{\sim}{n}^{(s)}$ are notmal unit vectors. The corner problem under consideration is to find a region, in which the extension of the solution curve lies. Since the problem is a local one at $\underset{\sim}{x}^{(k)}$, we may forget about all boundaries other than $H_1, \ldots, H_s$.

The solution curve is a portion of the inverse image of the line segment $L_{\underset{\sim}{y}}$ joining $\underset{\sim}{y}^{(1)}$ and $\underset{\sim}{y}$ with direction vector

$$\underset{\sim}{\alpha} = \underset{\sim}{y} - \underset{\sim}{y}^{(1)} \tag{85}$$

Let L be any straight-line with direction vector $\underset{\sim}{\alpha}$, then the inverse image of L has two half-lines, one of which is in parallel with $\underset{\sim}{x}^{(k+1)} - \underset{\sim}{x}^{(k)}$ and lies in the same region as $\underset{\sim}{x}^{(k+1)}$, and the other one is in in parallel with $\underset{\sim}{x}^{(k)} - \underset{\sim}{x}^{(k-1)}$ and lies in the same region as $\underset{\sim}{x}^{(k)}$, as shown in Fig. 10. Therefore, by constructing a straight-line L, whose inverse image crosses one and only one boundary at a time and by tracing the inverse image, we can determine the outgoing vector $\underset{\sim}{x}^{(k+1)} - \underset{\sim}{x}^{(k)}$.

We are going to use a perturbation method to construct a desirable line L. The underlying idea and the process of construction can be seen from the following description:

(i) First choose an arbitrary inner point, $\underset{\sim}{z}^{(1)}$, of a region R_1, in which $\underset{\sim}{x}^{(k-1)}$ lies. At this step, the straight-line L is the one passing through $\underset{\sim}{f}(\underset{\sim}{Z}^{(1)})$ and having a direction vector $\underset{\sim}{\alpha}$. We trace the inverse image of L, in Region R_1, starting with $\underset{\sim}{Z}^{(1)}$. A small perturbation applied to L can be found so that the inverse image hits one and only one boundary. Let $\underset{\sim}{Z}^{(1)}(\Sigma_1)$ and $L(\Sigma_1)$ be the modified $\underset{\sim}{Z}^{(1)}$ and L, respectively. Assume that the inverse image of $L(\Sigma_1)$ hits only one boundary, say $\underset{\sim}{H}_1$, at $\underset{\sim}{z}^{(2)}(\Sigma_1)$, and enters into the neighbouring region, say R_2.

(ii) We trace the inverse image of $L(\Sigma_1)$, in region R_2, starting with $\underset{\sim}{z}^{(2)}(\Sigma_1)$. If the inverse image never hits any boundaries, then the region R_2 is the one to be found, in which the extension of the solution curve lies. If otherwise, a small perturbation applied to $L(\Sigma_1)$ can be found so that the inverse image hits one and only one boundary. Let $\underset{\sim}{z}^{(2)}(\Sigma_1, \Sigma_2)$ and $L(\Sigma_1, \Sigma_2)$ be the modified $\underset{\sim}{z}^{(2)}(\Sigma_1)$ and $L(\Sigma_1)$, respectively. Assume the inverse image of $L(\Sigma_1, \Sigma_2)$ hits one and only one boundary, say H_2, at $\underset{\sim}{z}^{(3)}(\Sigma_1, \Sigma_2)$, and enters into a neighbouring region, say R_3. It is important to notice that the second modification is small enough so that the inverse image of $L(\Sigma_1, \Sigma_2)$ crosses H_1 as in the case of $L(\Sigma_1)$. Thus, the inverse image of $L(\Sigma_1, \Sigma_2)$ crosses H_1 and H_2.

(iii) In general, we have $\underset{\sim}{z}^{(\ell)}(\Sigma_1, \ldots, \Sigma_{\ell-1})$ and $L(\Sigma_1, \ldots, \Sigma_{\ell-1})$, of which inverse image crosses $H_1, \ldots, H_\ell$. We trace the image, in region R_ℓ, starting with $\underset{\sim}{z}(\Sigma_1, \ldots \Sigma_{\ell-1})$. If the image never hits any boundaries, then the region R_ℓ is the one to be found , and the process terminates here. If otherwise, a small perturbation can be found so that the inverse image of $L(\Sigma_1, \ldots, \Sigma_{\ell-1}, \Sigma_\ell)$ hits one and only one boundary and crosses $H_1, \ldots, H_\ell$ as the inverse image of $L(\Sigma_1, \ldots, \Sigma_{\ell-1})$ does. This process cannot continue indefinitely because there are only a finite number of regions. Therefore, after a finite number of steps, we can find a region where the extension of the solution curve lies. The computational details of the process will not be treated here.

5. Conclusion

In this paper we presented the theory of piecewise linear analysis of nonlinear resistive networks. In order to allow the most general class of network elements, the mixed analysis is used in the network formulation. The resulting network equation is of the form $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{y}$ where $\underset{\sim}{x}$ represents the chosen network variables, $\underset{\sim}{y}$ is the set of input, and $\underset{\sim}{f}$ is a continuous piecewise linear function which maps R into itself.

The main problem of interest is to determine the conditions on $\underset{\sim}{f}$ for which the equation has a unique solution for all $\underset{\sim}{y}$. Detailed studies are given with respect to Lipschitz conditions, Holtzmann and Liu's Lemma on homeomorphism and new necessary and sufficient conditions pertinent to the piecewise linear mapping. Two useful sufficient conditions in terms of the constant Jacobian matrices in all regions are derived.

Special attention is given to the problem of boundary crossing in obtaining the solution curve. It is shown that the sign of the Jacobian determinants of all the regions is of crucial importance in tracing the solution curve. Finally, a brief discussion is given on the corner problem where more than two regions intersect.

It seems that there still exist two important unsolved problems in piecewise linear network analysis. The first is the approximation in the multi-dimensinal space. In this connection, the approach used by Iri based on the Barycentric coordinates seems to be promising[11]. The second is the problem of developing an efficient computational method for equations with multiple solutions[2].

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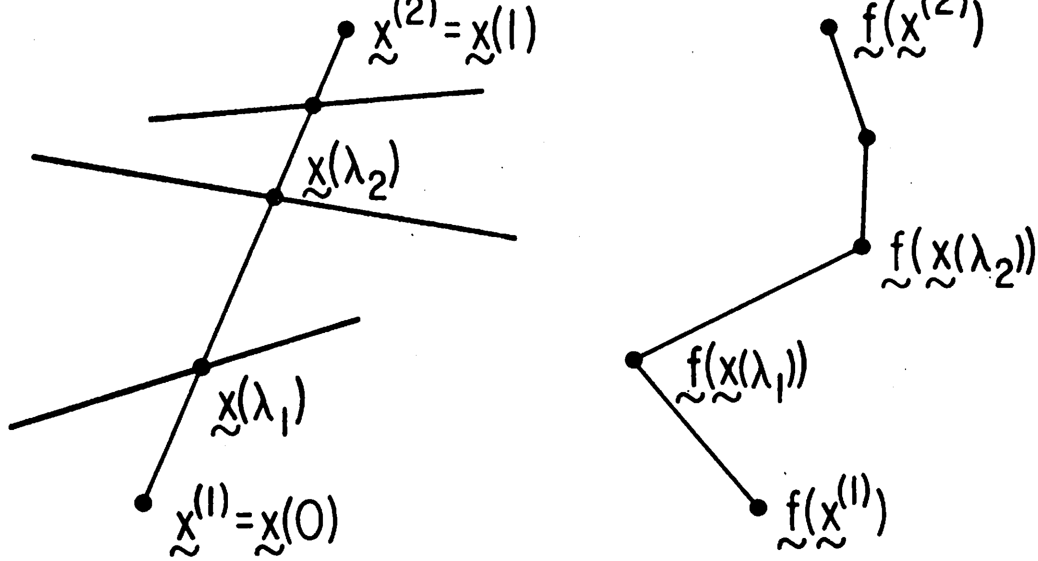


Fig. 1. Illustration of the piecewise-linear mapping of a straight line joining $\underset{\sim}{x}^{(1)}$ and $\underset{\sim}{x}^{(2)}$

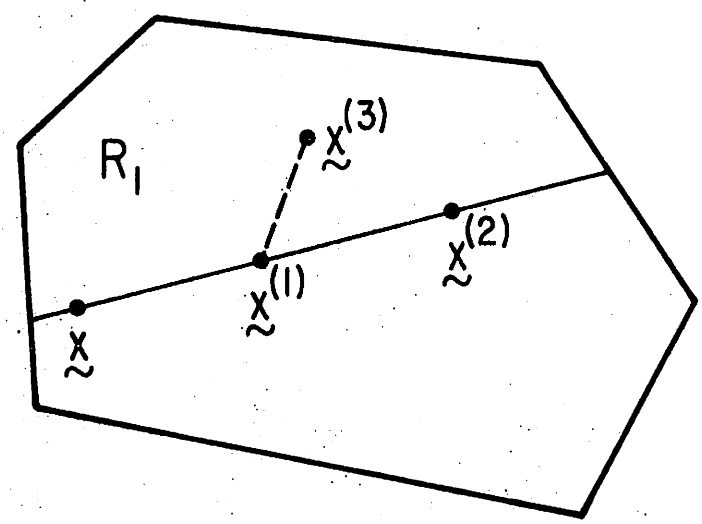


Fig. 2. Line segment in R_1 as represented by $\underset{\sim}{x} = (1 - \lambda)\, \underset{\sim}{x}^{(1)} + \lambda \underset{\sim}{x}^{(2)}$

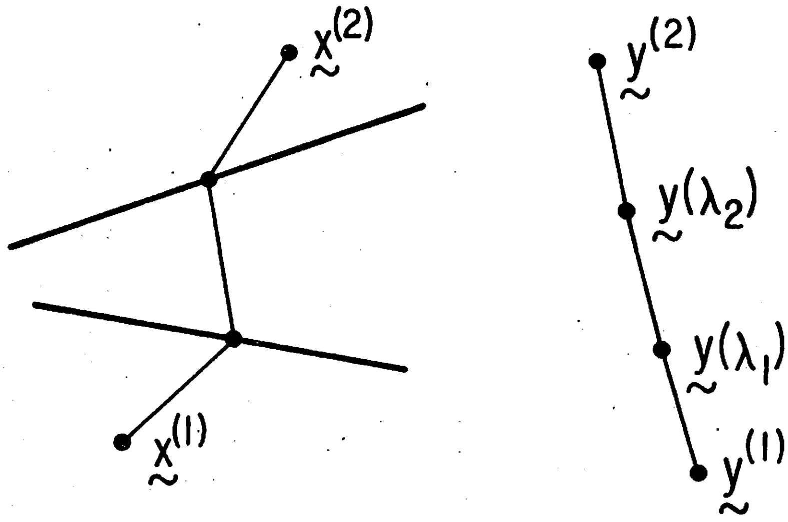


Fig. 3. Illustration of the inverse mapping of a straight line joining $y^{(1)}$ and $y^{(2)}$

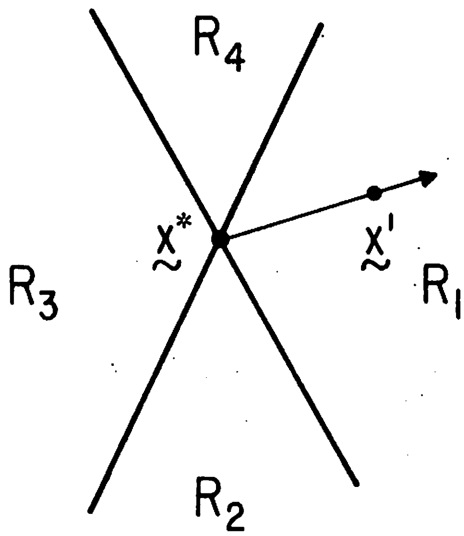


Fig. 4. Study of local homeomorphism

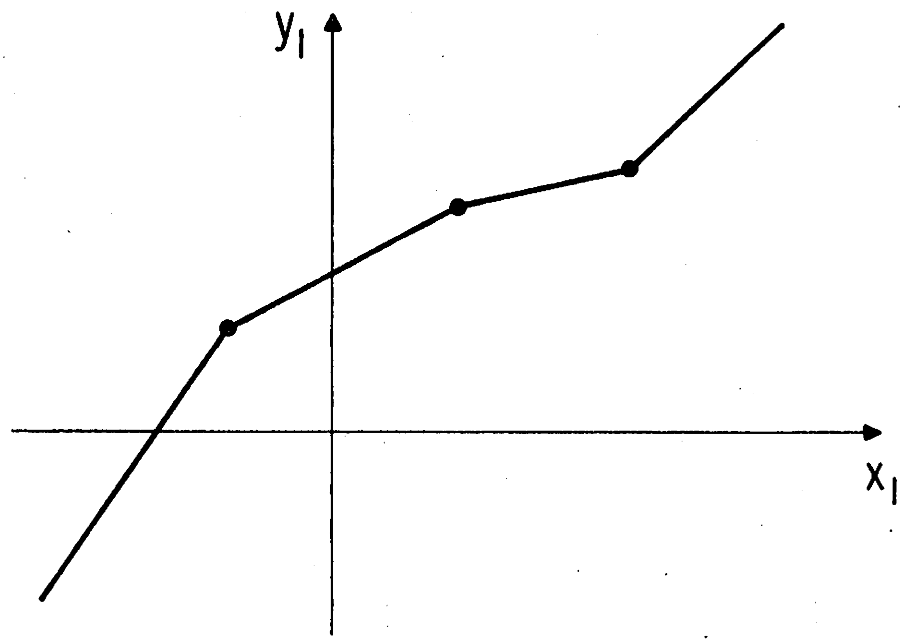


Fig. 5. One-dimensional monotonically-increasing piecewise-linear mapping

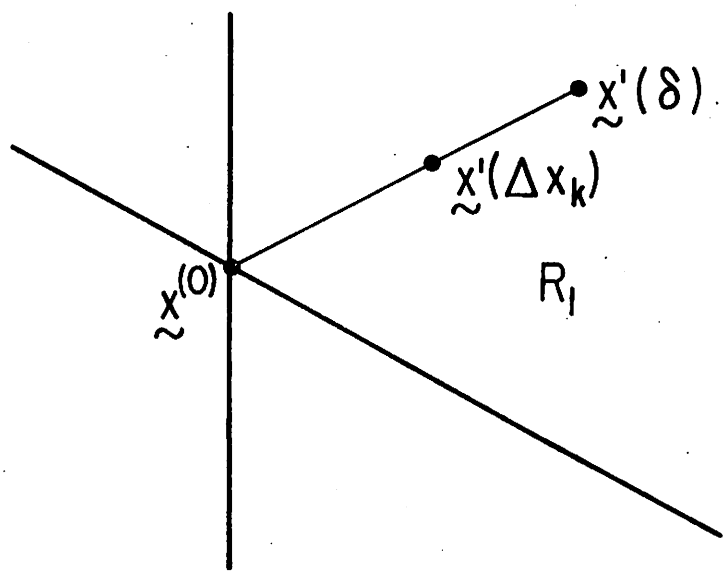


Fig. 6. Illustration of the proof of Theorem 1

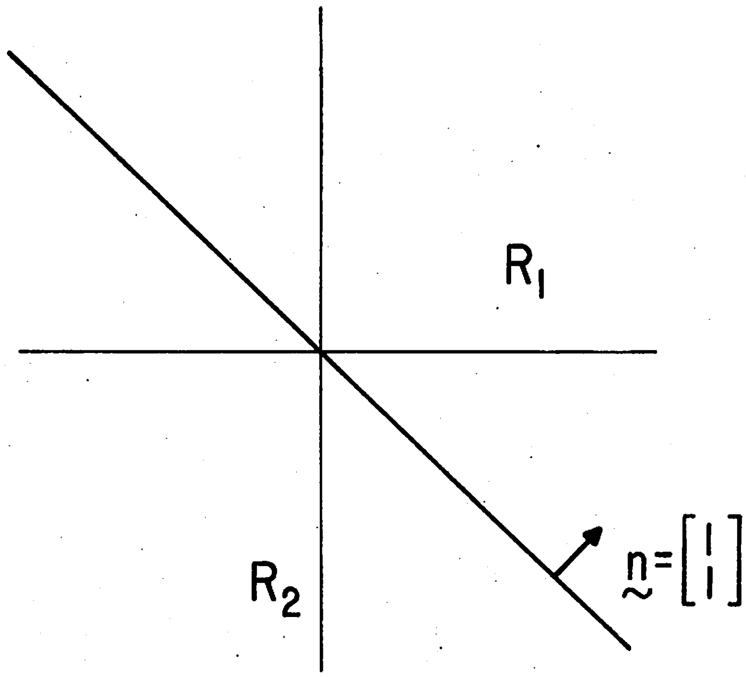


Fig. 7. Example 1

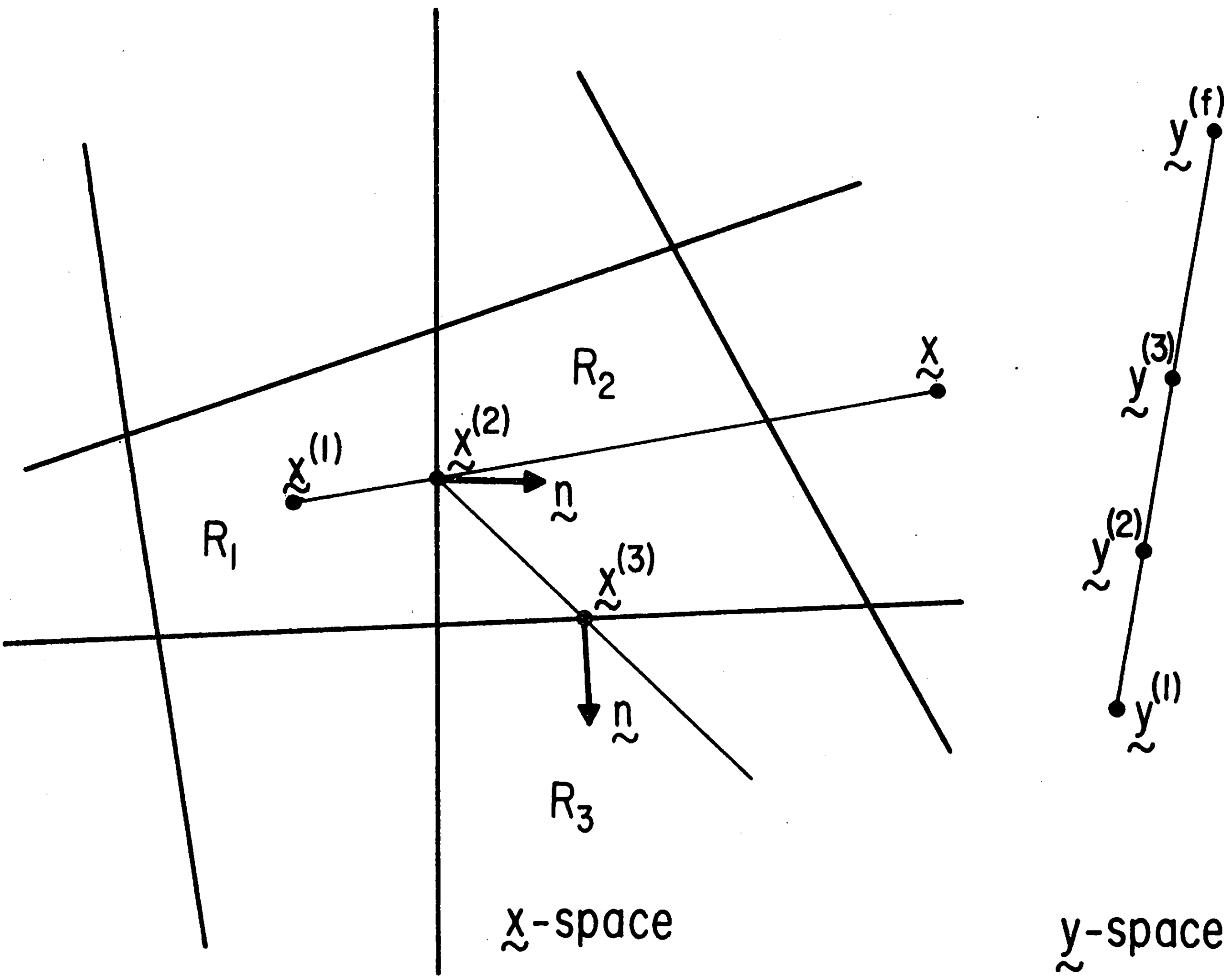
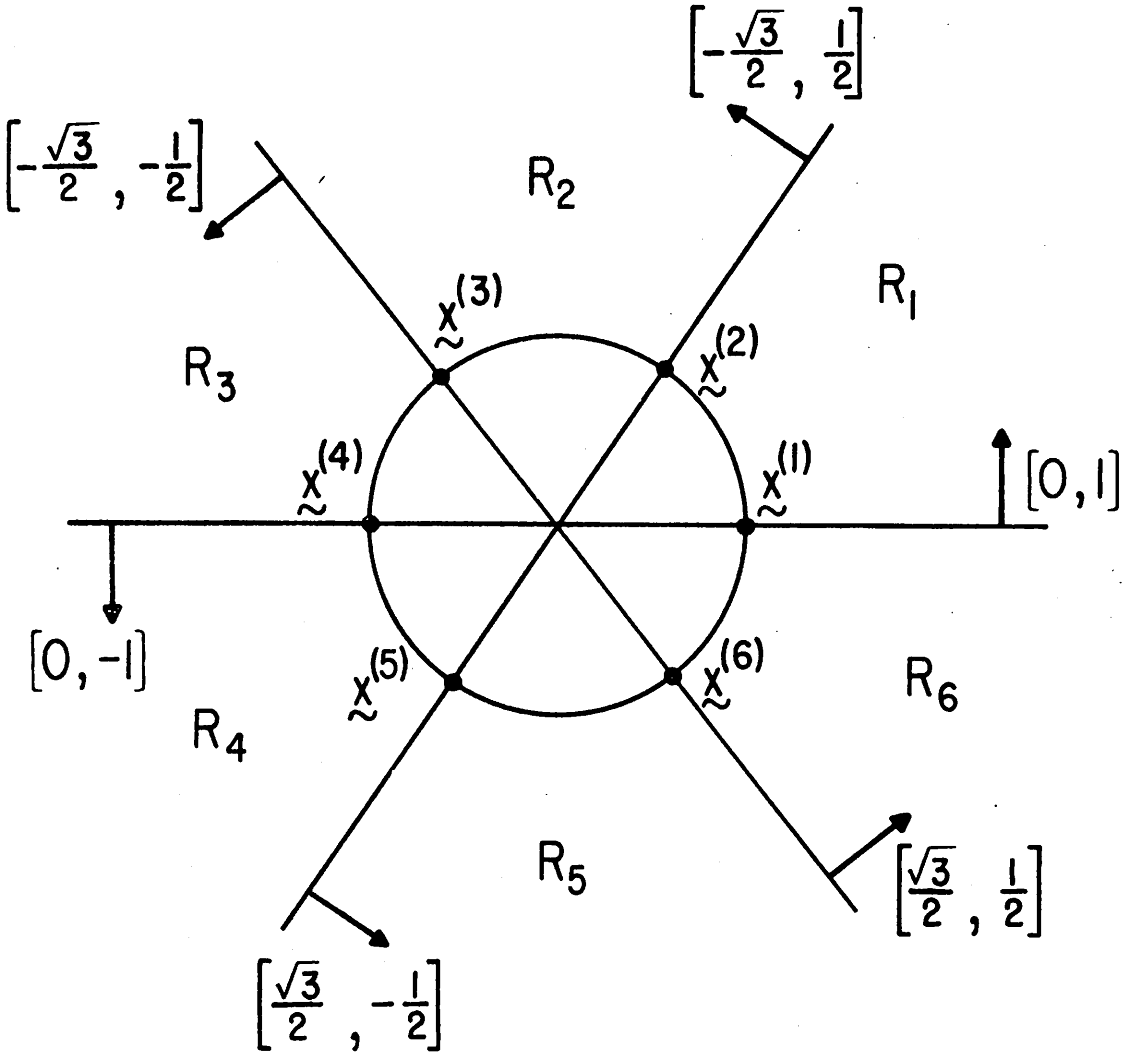


Fig. 8. Solution curve to illustrate the Katzenelson algorithm



$$\underset{\sim}{x}^{(1)} = [1, 0] \qquad \underset{\sim}{x}^{(2)} = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$$

$$\underset{\sim}{x}^{(3)} = \left[-\frac{1}{2}, \frac{\sqrt{3}}{2}\right] \qquad \underset{\sim}{x}^{(4)} = [-1, 0]$$

$$\underset{\sim}{x}^{(5)} = \left[-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right] \qquad \underset{\sim}{x}^{(6)} = \left[\frac{1}{2}, -\frac{\sqrt{3}}{2}\right]$$

Fig. 9. Example 2

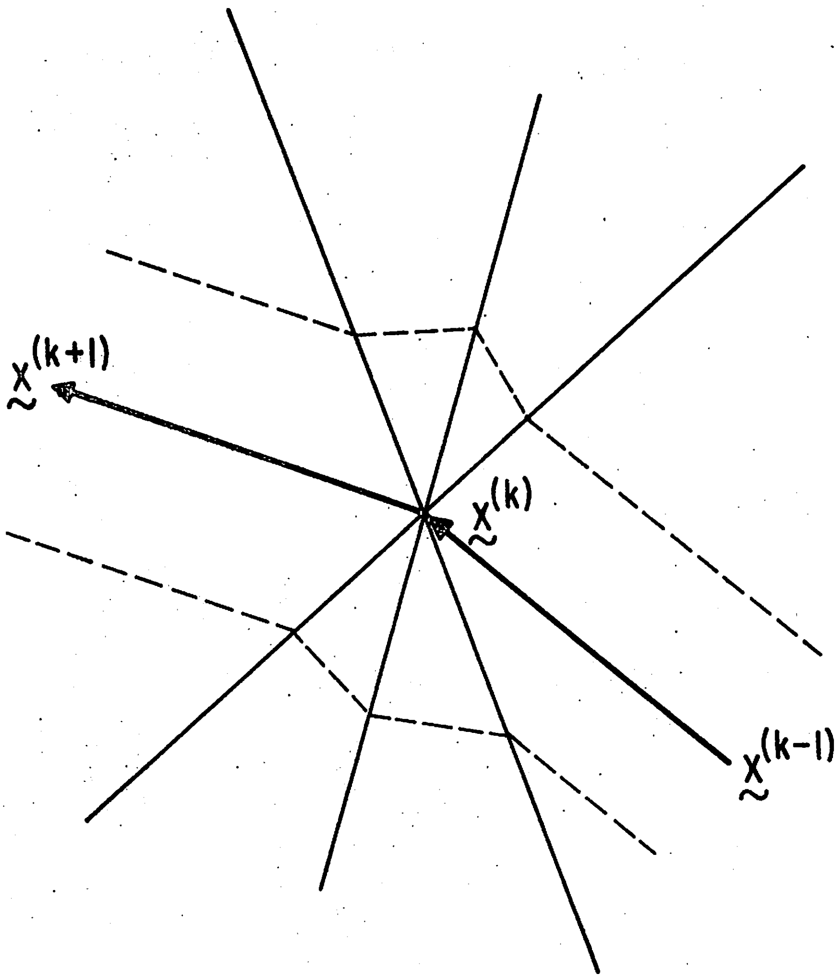


Fig. 10. Illustration of the corner problem.