RESONANT MODIFICATION AND DESTRUCTION OF ADIABATIC INVARIANTS

by

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Memorandum No. ERL-M293

17 September 1970

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This work is partly based on a Ph.D. thesis of E. F. Jaeger. A. J. Lichtenberg was a Miller Professor and E. F. Jaeger and AEC Fellow and an NSF Fellow during part of this research. The work was also partially supported by the National Science Foundation under Grant GK-2978, and Air Force Office of Scientific Research under Grant AF-AFOSR-69-1754.

ABSTRACT

For a harmonic oscillator with time varying coefficients a relation is obtained between the action integral and the first order adiabatic invariant. It is shown that a small resonant perturbation can modify the invariant. A canonical transformation generates a new action variable constant to first order in the perturbation with the old action-angle variables playing the role of oscillatory momentum and extension. There are two separate ways in which the new adiabatic constant can break down, leading either to a modified invariant or to a situation in which no invariant exists. A simple criterion for the destruction of adiabatic invariance is derived. A general theory is developed to use the procedure for multidimensional systems. It is shown that two distinct types of resonances are possible that lead to qualitatively different results. The general theory is applied to a coupled oscillator system in forms demonstrating both types of resonances. A detailed example of the more important resonance is made for single particle motion in a magnetic mirror with a perturbing r.f. field. Averages are performed first, over a time comparable to the r.f. period and second, over a time comparable to the longitudinal bounce period. Numerical integrations of both the averaged and exact Hamiltonian equations show the maximum value of the perturbation for which the averaging process is valid. The particle's energy is found to oscillate but, as long as the averaging process is valid, this oscillation is adiabatically separated from the longitudinal bounce by the existence of an adiabatic invariant. When averaging is not valid, a resonant coupling occurs between the bounce and energy oscillations. The

canonical transformation as described in the general theory is made near a particular resonance to obtain the oscillation in the adiabatic invariant. For oscillations about neighboring resonances which do not interact strongly then a new approximate invariant exists. For strong interaction the break-up of the invariant curves is demonstrated and the strength of the perturbation necessary for breakup is compared with the simple criterion.

I. INTRODUCTION

For a one dimensional oscillator in which the Hamiltonian is a slowly varying function of time, the action integral, $J = \oint p\, dq$ is an approximate constant or "adiabatic invariant" of the motion. Here p and q are canonical co-ordinates for the momentum and position and the integration is carried over a complete period of the oscillation. If the Hamiltonian is initially constant, then slowly varying with time, and finally constant again, the action integral is constant to all orders in an expansion parameter, ε.[1]-[3] These results do not mean that J is an exact invariant but only that the total time derivative of J approaches zero faster than any power of ε. An iterative technique has been used to calculate the change in the magnetic moment of a particle in a magnetic field.[4] Vandervoort[5] uses a similar iterative method to calculate the change in the action integral for the harmonic oscillator. An exponentially small change in the action is obtained which is not inconsistent with the asymptotic result. Changes in the action can also be calculated using the asymptotic method if discontinuities are assumed in higher order derivatives of the time varying parameter.[6] These discontinuities approximate changes in the derivatives which result from the finite rate at which the Hamiltonian varies. The unperturbed action is the first term of the asymptotic series which gives the total adiabatic invariant for the time varying problem. As we shall see in Section II, if the next term in the series is included, its time rate of change exactly cancels the change in the first term as calculated by Vandervoort. This cancellation occurs to each successive order in the series, and it is not until a term including a "discontinuous" derivative is reached that the asymptotic series breaks down. If the slow time dependence of the Hamiltonian is periodic, a rapidly varying derivative is equivalent to a resonance between a harmonic of the slow time dependence and the oscillator frequency, which destroys the adiabatic invariance of the action integral in that order. In practical cases only the lowest order resonances are strong enough to have significant effect.

For harmonic oscillators exact invariants also exist.[7] For nonlinear oscillators a similar invariant can be obtained by expansion in the nonlinearity,[8] but in the case of periodic coefficients the expansion may not converge. By properly choosing the constants of integration the exact invariant can be chosen equal to the adiabatic invariant at any initial time. Then, by comparing the two invariants at a later time, an estimate is made of the change in the adiabatic invariant. If the adiabatic invariant is destroyed due to resonances then we expect the "exact" invariant to be destroyed also.

For multidimensional problems such as nonlinearly coupled oscillators, the motion of charged particles in electric and magnetic fields, and the restricted three body problem, if the oscillation in one degree of freedom is very much faster than the motion in the remaining degrees of freedom, then the rapid oscillation can be isolated from the remaining motion by treating it as a one dimensional oscillator with slowly varying parameters. Bogoliubov and co-workers[9],[10] developed a "method of averaging," for the rapidly rotating phase associated with the fast oscillation, which gives the motion of the slow drift alone, dependent only on the adiabatic invariants of the fast motion. Kruskal[11] developed a method similar to that of Bogoliubov demonstrating that the adiabatic invariant is constant to all orders in the expansion parameter. McNamara and Whiteman[12],[13] use a simpler procedure to calculate higher order adiabatic invariants for multidimensional problems and they show their results are equivalent to those of Kruskal. Contopoulos[14] has pointed out that, for Hamiltonians which are periodic in time, two distinct forms of invariants can be constructed. The second form is known as a third integral because of its use in dynamical problems having two known integrals. Although both adiabatic invariants and third integrals are expansions in a small parameter, the small parameter in the third integral case refers to the small term in the Hamiltonian while in the adiabatic invariant case, it refers to a slow dependence of the Hamiltonian on a particular variable. For a one dimensional problem in which both expansions can be calculated, Contopoulos has shown the adiabatic invariant to be better conserved than the third integral when the Hamiltonian varies slowly with time. When the time variation is not slow but the perturbation term is small the third integral is better conserved. He also notes that if a resonance exists between the perturbation and the frequency of oscillation, both expansions fail due to secularities. It is the effect of resonances between harmonics of the slow oscillations and the fast oscillation in modifying and destroying adiabatic invariants that is the main subject of this paper.

In multidimensional systems, an invariant may be an "isolating integral" in that it separates two degrees of freedom. It is well known that, in multidimensional problems, there are transitions between regions in which isolating integrals exist and regions in which they do not[15]-[17]. The technique for observing these transitions is to look at the crossings of the trajectory in a surface of section corresponding to a particular phase of the most rapid oscillation. If the crossings lie on a smooth curve, the motion in that plane is isolated from the rest of the motion, and the isolating integral exists. If the crossings are random the integral does not exist. The computations indicate that transitions between smooth and ergodic trajectories often are accompanied by the breaking of a single curve into a number of discrete curves or islands. From the island structure we can identify the order of the resonance. The numerical results and their relationship to the destruction of invariants have been summarized by Lichtenberg.[17]

Chirikov[18] considers resonances between the Larmor rotation and the longitudinal bounce of a particle in a magnetic mirror and calculates the rate of change of the magnetic moment μ due to the resonance. Because of nonlinearities the system does not remain in resonance, and μ oscillates about its resonant value. A criterian for destruction of the invariant proposed by Chirikov is that the values of the action from the two adjacent resonances overlap. Similar overlap criteria have been used for the problem of the destruction of magnetic surfaces,[19],[20] and for a study of the stability of the Korteweg-deVries equation.[21] None of the above results of resonant destruction of invariants has been verified by numerical computations of the exact equations of motion. However, exact orbit calculations for electron-cyclotron heating in a magnetic mirror have demonstrated that the destruction of the invariants occurs near a resonance between two degrees of freedom of the system.[22] This problem will be reexamined in Section IV. Walker and Ford[23] have also predicted the onset of invariant destruction for some simple two dimensional oscillator systems. They calculate the value of the perturbation at which the separatrices of neighboring resonant oscillations overlap and obtain good agreement with numerical solutions. Their oscillator systems only contain the resonant terms, however, and therefore the effect of other harmonics of the nonlinear oscillators is not revealed by their calculation. In addition to the fact that the resonance width can only be calculated approximately, the concept of overlap is itself approximate. It is not obvious how close together resonances must be for breakdown to occur. In fact, we shall show with numerical examples that the interaction of the resonant terms with the nonresonant terms can lead to destruction of the invariant even if the resonances are well separated. Nevertheless, the overlap criterian is correct within an order of magnitude, and for simplicity we shall use the term "resonance overlap" to indicate invariant destruction, with the implicit notion that "overlap" means a strong interaction between neighboring resonances. Another complication, is that the primary resonance in the system is not always the cause for the breakdown of the invariant. Instead, the primary resonance may generate islands that have harmonic frequencies creating secondary resonances, and these secondary resonances may cause the breakdown. In fact, there is a hierarchy of resonances generated in the multiply periodic system and invariant destruction can, in principle, occur at any level of the hierarchy. The removal of a degeneracy in two degrees of freedom, with a low order resonance, does not generally lead to overlapping resonances, except near the separatrix of the oscillations. The destruction of the invariant in these cases is usually caused by secondary resonances. This has led to some confusion in the work on the breakup of magnetic surfaces in toruses. For the adiabatic problem, on the other hand, the resonance destruction is most likely caused by "overlap" of the resonances caused by two adjacent harmonics of the slow oscillation.

The destruction of adiabatic invariants due to resonances is related to the problem of small denominators in classical perturbation theory. This problem has occupied mathematicians since the time of Poincare[24], and there have been many methods devised to correct the perturbation technique so that terms with small denominators do not occur to destroy the convergence of the series. One such method which we use in subsequent sections is degenerate perturbation theory or the method of secular perturbations.[25] The resonant or degenerate variables are eliminated from the unperturbed Hamiltonian by a canonical transformation to a frame of reference that rotates with the resonant frequency. The new coordinates then measure the slow oscillation of the variables about their values at resonance. The procedure is similar to that used by Chirikov except that here canonical variables are preserved throughout the transformations. The adiabatic expansion procedures can be used to average over the rapidly rotating phase after the resonance has been removed. However, if ε is not small enough, higher order accidental degeneracies may occur to destroy the new adiabatic invariant. These higher order resonances can also be removed by transforming to new coordinates that rotate with the resonant frequency. Under certain conditions, the system may remain close enough to the higher order resonance that averaging can be performed a second time. In this case, modified adiabatic invariants are found which govern the oscillation in the previous adiabatic invariant about its value at resonance.

The existence of resonances is closely linked to the convergence of the asymptotic series in multidimensional systems. For almost periodic solutions represented as motion on an n-dimensional torus and for non-linear coupling that is sufficiently small with frequencies that are sufficiently incommensurable, the effect of the perturbation is only to slightly deform the toroidal surfaces. But for frequencies that are commensurable, the perturbed tori are deformed greatly and the particle orbits do not remain close to the unperturbed torus.[26]-[29] One application of this theorem is the proof of the eternal invariance of the action for the one dimensional oscillator with slow periodic variation of the Hamiltonian. However quantitative measures of sufficient incomensurability and sufficient slowness are difficult to obtain from these theorems. The radius of convergence of the series is related to the presence of the interacting resonant terms, and can at present only be estimated from numerical studies.

The procedure for determining the invariants of a multidimensional system is rather involved, and we here outline the general steps as they will be used in the following sections. 1. The Hamiltonian is divided into two parts, a zero order part which can be transformed to action angle variables and a first order part (in some small parameter, ε) which cannot be transformed. 2. (a) If one frequency of the unperturbed system is much faster than the others, the method of averaging is employed to obtain the first order invariant of the fast oscillation, including the perturbation. (b) If two frequencies have a low order commensurability a canonical transformation is made to a rotating frame in which there is only a single frequency for the unperturbed motion (intrinsic degeneracy) or two widely spaced frequencies (accidental degeneracy). The averaging can then be performed as in (a). 3. If after the transformation a resonance persists between a fairly low harmonic of the slow variable and the fast variable, the averaging process is not valid, and the additional resonance must be removed, leading to an "island oscillation". However, the transformed variables in the rotating frame are no longer in action-angle form. The Hamilton-Jacobi equation must be solved to reintroduce action variables. This is generally done by a four step process. (i) The Hamiltonian is averaged over the fast variable. (ii) The averaged Hamiltonian is expanded about an elliptic singular point and action angle variables are obtained to lowest order in the expansion. (iii) Perturbation theory is used to obtain the action-angle variables to higher order for the averaged Hamiltonian. (iv) The angle dependent terms are then reintroduced to obtain a Hamiltonian as in 1. 4. The process in 2.(b) is then repeated for the island oscillation to obtain the new invariant of the motion.

II. THE ONE DIMENSIONAL OSCILLATOR WITH TIME VARYING FREQUENCY

We consider here a one dimensional example to illustrate the physical effect of resonances on the adiabatic invariance of the action integral. A resonance can be observed in one dimension by allowing a slow periodic time dependence in the Hamiltonian. Commensurability between harmonics of the slowly varying parameter and the frequency of the one dimensional oscillation generates resonances similar to those in multidimensional systems. The problem is somewhat simpler in that the slowly varying frequencies are constants, not dependent on the value of an action.

To limit the amount of algebra in this section, we introduce certain canonical transformations intuitively rather than deriving them from generating functions. These transformations will be derived more rigorously in later sections but here we concentrate more on the physical meaning of the variables.

Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \tag{2.1}$$

where p, q, and m denote momentum, position, and mass, respectively, for an oscillating particle. The frequency of the oscillation, ω, is assumed to be a slowly varying function of time:

$$\omega = \omega(\varepsilon t). \tag{2.2}$$

We define new variables, P and w, which are action-angle variables for the unperturbed case ($\varepsilon = 0$). Physically, these variables represent a polar co-ordinate system in the q-p plane where w is an angle and $\sqrt{2P/m\omega}$ is a radial co-ordinate:

$$q = \sqrt{\frac{2P}{m\omega}} \sin w$$
$$p = \sqrt{2Pm\omega} \cos w \tag{2.3}$$

With a simple canonical transformation[5] the Hamiltonian can be written in terms of P and w as

$$H^* = \omega P + \varepsilon \frac{\omega'}{2\omega} P \sin 2w \tag{2.4}$$

where the prime denotes differentiation with respect to the variable $\tau = \varepsilon t$. Hamilton's equations for H^* are

$$\dot{w} = \frac{\partial H^*}{\partial P} = \omega + \varepsilon \frac{\omega'}{2\omega} \sin 2w \tag{2.5}$$

$$\dot{P} = \frac{-\partial H^*}{\partial w} = - \varepsilon \frac{\omega'}{\omega} P \cos 2w . \tag{2.6}$$

Equation (2.6) gives Vandervoort's result for the lowest order change in the action integral due to the slow time variation of the Hamiltonian.

According to Whiteman and McNamara[13] and Kruskal[11] the first order adiabatic invariant for this problem is

$$I = P + \varepsilon \frac{\omega'}{2\omega^2} P \sin 2w . \tag{2.7}$$

To verify the approximate constancy of I, we take a time derivative of (2.7). Substituting for $\dot{P}$ and $\dot{w}$ from (2.5) and (2.6) and keeping terms up to first order in ε we obtain

$$\dot{I} = \dot{P} + \varepsilon \left(\frac{\dot{\omega}'}{2\omega^2}\right) P \sin 2w + \varepsilon \left(\frac{\omega'}{\omega}\right) P \cos 2w + O(\varepsilon^2). \tag{2.8}$$

The first and third terms on the right cancel by virtue of Vandervoort's result (2.6) leaving to lowest order

$$\dot{I} = \varepsilon \left(\frac{\dot{\omega}'}{2\omega^2}\right) P \sin 2w \;. \tag{2.9}$$

From (2.9) we see that, if $\left(\frac{\dot{\omega}'}{2\omega^2}\right) = O(\varepsilon)$ then $\dot{I}$ is of order ε^2 and hence I is a first order invariant. This assumption is standard in asymptotic theories. It is equivalent to assuming a slow time dependence not only for the frequency ω but also for the first derivative of the frequency ω' so that $\omega' = \omega'(\varepsilon t)$. A second order asymptotic calculation would similarly require that $\omega'' = \omega''(\varepsilon t)$ and so on. From the above development we see that the asymptotic result that I is a first order adiabatic invariant is in no way inconsistent with Vandervoort's first order iterative calculation of the change in the unperturbed action integral, P. Rather, the two results are mutually dependent.

We now consider the possibility of secular changes in the adiabatic invariant I due to resonances between the oscillation and the time rate of change of the frequency ω. According to asymptotic theory, $\dot{I}$ in (2.9) is not identically zero but is second order. A secularity is said to occur if, after a long time, $\Delta t = \frac{1}{\varepsilon}$, the accumulated second order changes add up to destroy the first order invariance of I. To insure that this does not happen, we must show that the time average of $\dot{I}$ is of order ε^3 in which case, the accumulated change ΔI in a time $\frac{1}{\varepsilon}$ is of order ε^2. It will generally be the case for equation (2.9) that the time average of $\dot{I}$ will be nearly zero due to the rapid variation of the sin 2w term. Hence the condition that $\bar{\dot{I}} = O(\varepsilon^3)$ is easily satisfied and there will be no secularities. However, this is not the case if ω has a periodic time dependence and if one harmonic of that time dependence resonates with the frequency of oscillation of the term sin 2w. To see this we assume a slow periodicity in ω with a frequency ω_1 such that

$$\frac{\omega_1}{\omega} = O(\varepsilon). \tag{2.10}$$

We further assume that ω consists of one large constant term, a_0, with a small ripple superimposed on it. In this case, we can write ω as

$$\omega = a_0 + \varepsilon \sum_{n\neq 0} a_n \, e^{in\omega_1 t} \tag{2.11}$$

Using (2.11), $\dot{I}$ becomes to lowest order

$$\dot{I} = \frac{-\varepsilon}{4i} \left(\frac{\omega_1}{a_0}\right)^2 I \sum_{n\neq 0} n^2 \, a_n \left[e^{i(n\omega_1 t+2w)} - e^{i(n\omega_1 t-2w)}\right] \tag{2.12}$$

where we have substituted I for P to first order in ε. A time average of (2.12) gives zero unless there is a commensurability between the oscillations in t and w of the form

$$\frac{\omega}{\omega_1} = \frac{s}{2} \tag{2.13}$$

where s is an integer of order $\frac{1}{\varepsilon}$. For the length of time that the system remains close to the resonance, there will be slowly varying terms in the sum of (2.12) which do not average to zero. These will be the $n = \pm s$ terms. Assuming for simplicity that $a_{-n} = a_n$ so that ω is an even function of time, the average of (2.12) can be written as

$$\frac{\bar{\dot{I}}}{I} = \frac{\varepsilon}{2}\left(\frac{s\omega_1}{a_0}\right)^2 a_s \sin\phi \tag{2.14}$$

where ϕ is the slowly varying phase:

$$\phi = s\omega_1 t - 2w \ . \tag{2.15}$$

Equation (2.14) gives a time rate of change in I of the order of $\varepsilon\, a_s$ since from (2.11) and (2.13), $s\omega_1 \simeq 2\omega \simeq 2a_0$. Furthermore, for a low order resonance, a_s can be of order unity so that $\bar{\dot{I}}$ is of order ε. In that case, if the system remains in resonance for a time $\Delta t = \frac{1}{\varepsilon}$, a secular change $\Delta I = 0(1)$ is possible and the lowest order adiabatic invariant is destroyed. The particular choice (2.11) for the time dependence of ω allows ω to change at most by order ε in a time $\frac{1}{\varepsilon}$ so that the resonance condition (2.13) is still approximately satisfied. For the more general case in which $\omega = \sum_n a_n e^{in\omega_1 t}$, this is not necessarily true. The choice (2.11) for the time dependence of ω, however, is not overly restrictive since we will see in later chapters that multidimensional oscillations very often remain near resonance for long times. The a_n generally decrease with increasing n, such that a_n, itself may be of order $(\frac{1}{n})^m = \varepsilon^m$ where m depends on the smoothness of ω. In this case the asymptotic series would not be expected to fail until the (m+1)st order (see [6]). Since ϕ is slowly varying in the time $\frac{1}{\varepsilon}$, (2.14) represents a slow oscillation in I about its value at exact resonance. To find a new approximate invariant we employ the techniques of secular perturbation theory. To do this, we first transform from the action-angle variables for the unperturbed case ($\varepsilon = 0$), P, w, to new variables J and θ which are action-angle variables for the perturbed case ($\varepsilon \neq 0$). The new variable J is just the adiabatic invariant I defined in (2.7). The corresponding angle variable is

$$\theta = w + \varepsilon\left(\frac{\omega'}{4\omega^2}\right) \cos 2w \tag{2.16}$$

with Hamiltonian

$$H = \omega J + \varepsilon \left(\frac{\dot{\omega}'}{4\omega^2}\right) J \cos 2\theta. \tag{2.17}$$

We now assume a resonance condition of the form (2.13) and transform to new canonical variables $\hat{J}$ and $\hat{\theta}$ which put the observer in a frame of reference rotating with the resonance. $\hat{\theta}$ is the slowly varying difference phase $-\phi$ of (2.15), and $\hat{J}$ is $+\frac{1}{2}$ J. In terms of $\hat{J}$ and $\hat{\theta}$ (2.17) becomes

$$H = (2\omega - s\omega_1)\hat{J} + \varepsilon \left(\frac{\dot{\omega}'}{2\omega^2}\right) \hat{J} \cos (\hat{\theta} + s\omega_1 t). \tag{2.18}$$

If the system remains close to resonance, $\hat{\theta}$ will be slowly varying compared to $\omega_1 t$ and we expect that a time average of (2.18) would not appreciably effect the slow oscillation. Substituting $- 2s^2\omega_1^2 a_s \cos s\omega_1 t$ for

$\dot{\omega}'$, from (2.11), the average yields:

$$\overline{H} = (2a_0 - s\omega_1)\hat{J} - \frac{\varepsilon}{2}\left(\frac{s\omega_1}{a_0}\right)^2 a_s \hat{J} \cos\hat{\theta} \tag{2.19}$$

which is independent of time and hence a constant of the motion. The equation defines $\hat{J}$ (or J) as an explicit function of $\hat{\theta}$. In the presence of the resonance, the simple adiabatic invariant J is no longer a constant to first order in ε; instead there is a new approximate invariant defined by equation (2.19). For exact average resonance, such that the first term of (2.19) vanishes, there are infinities in J due to the independence of the average frequency from the action J. In actual systems nonlinearities limit the change in J. To see this, we assume that the average frequency a_0 depends nonlinearly on J in the following way:

$$a_0 = \alpha - \beta J \tag{2.20}$$

where $\beta = \frac{\partial a_0}{\partial J}$, the first term in the Taylor expansion about J = 0. Substituting (2.20) into (2.19) we obtain, after some rearranging of terms,

$$H = s\omega_1 \left\{ \left(-1 + \frac{2\alpha}{s\omega_1} - \frac{4\beta}{s\omega_1}\hat{J}\right)\hat{J} - \varepsilon \frac{\hat{J}}{\left(\frac{2\alpha}{s\omega_1} - \frac{4\beta}{s\omega_1}\hat{J}\right)\left(\frac{\alpha}{a_s} - \frac{2\beta}{a_s}\hat{J}\right)} \cos\hat{\theta} \right\} = \text{Constant} \tag{2.21}$$

We plot $\hat{J}$ as a function of $\hat{\theta}$ from (2.21)

$$\frac{2\alpha}{s\omega_1} = 1.1,\ \frac{\beta}{\alpha} = .091,\ \frac{a_s}{\alpha} = .91,\ \text{and } \varepsilon = .1.$$

For $\hat{J} = 1/2$, this corresponds to the case of average resonance in which $\frac{a_0}{\omega_1} = \frac{\alpha - 2\beta\hat{J}}{\omega_1} = \frac{s}{2}$ so that the first term in (2.21) vanishes. The results (solid lines in Fig. 1) show closed phase loops centered about $\hat{\theta} = \pm \pi$. The areas within the closed phase loops are the adiabatic invariants of the transformed system, which replace the previous invariants that no longer exist in consequence of the resonant interaction.

The results embodied in (2.21) can be made clearer by writing (2.21) in the functional form

$$\overline{H} = \omega_0 \hat{J} - \gamma \hat{J}^2 - \varepsilon\delta(\hat{J}) \cos \hat{\theta} = \text{const.} \tag{2.22}$$

We note there is an elliptic singular point at $\overline{J} \simeq 1/2$, $\overline{\theta} = \pm \pi$. This can be determined analytically by setting

$$\frac{\partial \overline{H}}{\partial \hat{J}} = \frac{\partial \overline{H}}{\partial \hat{\theta}} = 0 \ .$$

If we further expand $\hat{J}$ about the singularity, as

$$\hat{J} = \overline{J} + \Delta J$$

and assuming $\frac{\Delta J}{J} << 1$, (2.22) yields

$$\gamma(\Delta J)^2 + \varepsilon\delta(\overline{J}) \cos \hat{\theta} = \text{const.} \tag{2.23}$$

The separatrix for the oscillation is found from the Hamiltonian corresponding to the initial conditions $\Delta J = 0$ at $\hat{\theta} = 0$, and the maximum value of ΔJ is then found at $\hat{\theta} = \pi$ to be

$$(\Delta J)_{max} = \left(\frac{2\varepsilon\delta(\overline{J})}{\gamma}\right)^{1/2} \tag{2.24}$$

We see that the maximum excursion of J due to the resonance is proportional to $\varepsilon^{1/2}$ or, more completely, proportional to the square root of the ratio of the coefficient of the perturbation term divided by the coefficient of the first order nonlinearity in J. This result, found by Rosenbluth et al.[19] for the perturbation of magnetic surfaces, is a general property of a system in which the nonlinearity occurs independently of the perturbation. If the nonlinearity occurs only in the perturbation then the island amplitude is governed by a diffent law, which we describe in the next section. We plot the curve of constant Hamiltonian at the separatrix, from the approximate expression given in (2.23), to obtain the dashed line in Fig. 1, which although differing in detail, gives a value of ΔJ_{max} in good agreement with the results from (2.21).

The frequency of the island oscillation can also be found near the elliptic singularity by linearizing (2.23) to obtain

$$\gamma(\Delta J)^2 + \varepsilon\delta\frac{(\Delta\hat{\theta})^2}{2} = \text{const.} \tag{2.25}$$

with the frequency given by

$$\nu = (2\gamma\varepsilon\delta)^{1/2} \tag{2.26}$$

which is also proportional to $\varepsilon^{1/2}$.

The new invariant generated by removing resonances can break down in two ways. 1. The frequency can drift away from the resonance adiabatically. 2. Neighboring resonances can interact sufficiently strongly that a single resonance does not describe the motion. In the first case an adiabatic invariant usually exists, but the phase curves are distorted by the passage of frequencies near successive resonances.

In the case of strongly interacting resonances, values of ε can be chosen appropriately to give both successive island resonances and ergodic regions between them. For larger perturbations the islands can interact sufficiently strongly for the entire island structure to be absorbed by the ergodic region. The width of a resonance relative to the distance between resonances is the important parameter in determining breakdown of invariants. We have obtained an approximate formula for the width of the s-resonance in terms of the variation of the action. We obtain the variation of the frequency from

$$\Delta\omega = \frac{d\omega}{d\hat{J}} 2\Delta J$$

and compare it with the separation between the s and s + 1 resonance which is approximately ω/s, to obtain a criterian for resonance overlap

$$\frac{s}{\omega} \frac{d\omega}{d\hat{J}} 2\Delta J = O(1) \tag{2.27}$$

and since $d\omega/d\hat{J} \cong \frac{1}{2}\gamma$, we have, using (2.24) and (2.26), that for overlap

$$\frac{s\nu}{\omega} = O(1). \tag{2.28}$$

We explore the validity of this simple criterian in the detailed numerical example of section V.

III. GENERAL THEORY

A. Removal of Degeneracies

We assume the Hamiltonian consists of two parts in the form

$H = H_0 + \varepsilon H_1$, where H_0 is solvable in action angle variables such that

$$H = H_0(P_1,P_2) + \varepsilon H_1(P_1,P_2,w_1,w_2) \tag{3.1}$$

with ε a small number. If a resonance exists between the unperturbed frequencies

$$\frac{\nu_2^0(P_1,P_2)}{\nu_1^0(P_1,P_2)} = \frac{r}{s} \qquad \text{(r,s integers)} \tag{3.2}$$

then an attempt to solve the motion in action-angle variables by perturbation theory leads to a secularity in the solution.[25] We will take equation (3.2) to represent either a primary resonance in the system or a secondary resonance created by harmonic frequencies of islands generated by the primary resonance. In either case, the secularity can be removed by applying a transformation which eliminates one of the original actions, P_1 or P_2, from the unperturbed Hamiltonian H_0. We choose the generating function

$$F_2 = (rw_1 - sw_2)\hat{P}_1 + w_2\hat{P}_2 \tag{3.3}$$

which defines a canonical transformation from P_1,P_2,w_1,w_2 to $\hat{P}_1,\hat{P}_2,\hat{w}_1,\hat{w}_2$ such that

$$\begin{aligned} \hat{w}_1 &= \frac{\partial F_2}{\partial \hat{P}_1} = rw_1 - sw_2 \qquad & P_1 &= \frac{\partial F_2}{\partial w_1} = r\hat{P}_1 \\ \hat{w}_2 &= \frac{\partial F_2}{\partial \hat{P}_2} = w_2 \qquad & P_2 &= \frac{\partial F_2}{\partial w_2} = \hat{P}_2 - s\hat{P}_1. \end{aligned} \tag{3.4}$$

These coordinates put the observer in a rotating frame in which the rate of change of the new variable $\dot{\hat{w}}_1 = r\dot{w}_1 - s\dot{w}_2$ measures the slow deviation from resonance. There are two cases of importance: (1) If the resonance condition is met for the unperturbed frequencies for all P_1 and P_2, then the Hamiltonian is intrinsically degenerate. (2) If the resonance is satisfied only for a particular value of P_1 and P_2, then the Hamiltonian is accidentally degenerate. A primary resonance can be either accidental or intrinsic, but a secondary resonance is almost always accidental due to the complicated way in which the island frequency depends on the actions. Applying the transformation (3.4) to the Hamiltonian for the intrinsically degenerate case, we obtain a new Hamiltonian of the form

$$H = \psi(\hat{P}_2) + \varepsilon\Lambda(\hat{P}_1,\hat{P}_2,\hat{w}_1,\hat{w}_2). \tag{3.5}$$

For the accidentally degenerate case, we obtain

$$H = \psi(\hat{P}\ ,\hat{P}\ ) + \varepsilon\Lambda(\hat{P}_1,\hat{P}_2,\hat{w}_1,\hat{w}_2). \tag{3.6}$$

We note that if $r >> s$, then $\nu_2^0/\nu_1^0 = r/s = O(1/\varepsilon)$, and the Hamiltonian (3.1) is already in the form of (3.5).

In the case of intrinsic degeneracy, (Eq. (3.5)), Hamiltonian's equations are

$$\dot{\hat{w}}_1 = \frac{\partial H}{\partial \hat{P}_1} = \varepsilon \frac{\partial \Lambda}{\partial \hat{P}_1} = O(\varepsilon) \tag{3.7}$$

$$\dot{\hat{w}}_2 = \frac{\partial H}{\partial \hat{P}_2} = \frac{\partial \psi}{\partial \hat{P}_2} + \varepsilon \frac{\partial \Lambda}{\partial \hat{P}_2} = O(1). \tag{3.8}$$

We see that $\hat{w}_1$ is slowly varying compared to $\hat{w}_2$ and hence we can average (3.5) over $\hat{w}_2$ to give

$$\overline{H} = \psi(\hat{P}_2) + \varepsilon\overline{\Lambda}(\hat{P}_1,\hat{P}_2,\hat{w}_1) \tag{3.9}$$

where

$$\overline{\Lambda} = \frac{1}{2\pi}\int_0^{2\pi} \Lambda(\hat{P}_1,\hat{P}_2,\hat{w}_1,\hat{w}_2)d\hat{w}_2. \tag{3.10}$$

Since $\overline{H}$ is independent of $\hat{w}_2$, we have the approximate result that

$$\hat{P}_2 \simeq \text{constant.} \tag{3.11}$$

This is the first term of the series for the adiabatic invariant of Hamiltonian (3.5).

We see from (3.4) that $\hat{P}_2$ represents a combined invariant for the degenerate system, namely $\hat{P}_2 = P_2 + \frac{s}{r}P_1$. The effect of the rotating coordinates is to explicitly exhibit the single invariant of the resonant system. Notice, however, that for a high order resonance in which r >> s, the invariant for the resonant system, $\hat{P}_2$, reduces to that of the nonresonant system, P_2. Hence, the only resonances of importance are those with low harmonic numbers.

If an elliptic singularity exists in the $\hat{P}_1 - \hat{w}_1$ phase plane at $\overline{\hat{P}}_1,\overline{\hat{w}}_1$, then

$$\left.\frac{\partial\overline{H}}{\partial\hat{P}_1}\right|_{\overline{\hat{P}}_1,\overline{\hat{w}}_1} = 0 \qquad\qquad \left.\frac{\partial\overline{H}}{\partial\hat{w}_1}\right|_{\overline{\hat{P}}_1,\overline{\hat{w}}_1} = 0 \tag{3.12}$$

and Hamiltonian (3.9) can be expanded about the singularity as

$$\overline{H} \simeq \psi(\hat{P}_2) + \varepsilon\overline{\Lambda}(\overline{\hat{P}}_1,\overline{\hat{P}}_2,\overline{\hat{w}}_1) + g\,\frac{(\Delta P)^2}{2} + f\,\frac{(\Delta w)^2}{2} + \ldots \tag{3.13}$$

where

$$\Delta P = \hat{P}_1 - \overline{\hat{P}}_1$$

$$\Delta w = \hat{w}_1 - \overline{\hat{w}}_1$$

and

$$\left.\begin{aligned} g &= \varepsilon \left.\frac{\partial^2\overline{\Lambda}}{\partial\hat{P}_1^2}\right|_{\overline{\hat{P}}_1,\overline{\hat{w}}_1} = O(\varepsilon) \\ f &= \varepsilon \left.\frac{\partial^2\overline{\Lambda}}{\partial\hat{w}_1^2}\right|_{\overline{\hat{P}}_1,\overline{\hat{w}}_1} = O(\varepsilon). \end{aligned}\right. \tag{3.14}$$

Terms linear in ΔP and Δw are absent by virtue of (3.12). The coefficient of the $\Delta P\Delta w$ cross term has been assumed to vanish in the average since this is the observed behavior in our numerical examples. Hamiltonian (3.13) can be used to plot elliptic phase orbits in the ΔP-Δw phase plane. The frequency of oscillation around a phase loop in general depends on the area enclosed by the loop; however, for loops close to the singularity, this frequency approaches a constant value of [17]

$$\omega^0 = (fg)^{1/2} \propto \varepsilon,$$

i.e. the frequency of the $\hat{P}_1 - \hat{w}_1$ oscillation is a factor of ε slower than the frequency of the $\hat{P}_2 - \hat{w}_2$ oscillation. The ratio of the lengths of the semiaxes of the ellipse can be calculated approximately as[17]

$$\frac{(\Delta P)_{max}}{(\Delta w)_{max}} = \left(\frac{f}{g}\right)^{1/2} \quad ,$$

such that if the maximum excursion in Δw is of order unity, then the maximum excursion in $\Delta P/\overline{P}$ is also of order unity. To complete the solution, formally, we transform to action angle variables for the slow oscillation of ΔP and Δw. We postpone this calculation until we have considered the more general case of accidental degeneracy. The transformation to action angle variables is generally unnecessary unless an island resonance must also be removed.

In the case of accidental degeneracy, we consider the Hamiltonian of equation (3.6). If there is an elliptic singularity in the $\hat{P}_1 - \hat{w}_1$ phase plane, then an average can be performed similar to that for the intrinsic case. To see this, we assume an elliptic singularity at $\overline{\hat{P}}_1, \overline{\hat{w}}_1$. The Hamiltonian (3.6) is expanded as

$$H = \psi(\overline{\hat{P}}_1, \hat{P}_2) + \varepsilon\Lambda(\overline{\hat{P}}_1, \hat{P}_2, \overline{\hat{w}}_1, \hat{w}_2) + G\,\frac{(\Delta P)^2}{2} + L\,\Delta P \Delta w + F\,\frac{(\Delta w)^2}{2} + \ldots \tag{3.15}$$

where

$$G = \left.\frac{\partial^2 \psi}{\partial \hat{P}_1^2}\right|_{\overline{\hat{P}}_1, \overline{\hat{w}}_1} + \varepsilon \left.\frac{\partial^2 \Lambda}{\partial \hat{P}_1^2}\right|_{\overline{\hat{P}}_1, \overline{\hat{w}}_1} = 0(1)$$

$$L = \varepsilon \left.\frac{\partial^2 \Lambda}{\partial \hat{P}_1 \partial \hat{w}_1}\right|_{\overline{\hat{P}}_1, \overline{\hat{w}}_1} = 0(\varepsilon) \tag{3.16}$$

$$F = \varepsilon \left.\frac{\partial^2 \Lambda}{\partial w_1^2}\right|_{\overline{\hat{P}}_1, \overline{\hat{w}}_1} = 0(\varepsilon).$$

For the linear part of Hamiltonian (3.15) the cross product can be eliminated by a standard diagonalization procedure to obtain the eigenvalues

$$\lambda_1 = \frac{1}{2}(F + G) + \frac{1}{2}\sqrt{(F + G)^2 - 4(FG - L^2)}$$

$$\lambda_2 = \frac{1}{2}(F + G) - \frac{1}{2}\sqrt{(F + G)^2 - 4(FG - L^2)}\ , \tag{3.17}$$

so that Hamiltonian (3.15) becomes

$$H = \psi(\overline{P}_1,P_2) + \varepsilon\Lambda(\overline{P}_1,P_2,\overline{w}_1,w_2) + \lambda_1\frac{(\Delta P')^2}{2} + \lambda_2\frac{(\Delta w')^2}{2} + \dots\ . \tag{3.18}$$

The linear frequency of the $\Delta P' - \Delta w'$ motion is then

$$\Omega' = (\lambda_1\lambda_2)^{1/2} = (FG - L^2)^{1/2} = 0(\varepsilon^{1/2}) \tag{3.19}$$

which justifies averaging the Hamiltonian (3.6) over $\hat{w}_2$. The average gives

$$\overline{H} = \psi(\hat{P}_1,\hat{P}_2) + \varepsilon\overline{\Lambda}(\hat{P}_1,\hat{P}_2,\hat{w}_1) \tag{3.20}$$

with $\hat{P}_2$ an approximate constant of the motion. Equation (3.20) is analogous to (3.5) and hence can be used to isolate the $\hat{P}_1 - \hat{w}_1$ phase trajectories.

An accidental degeneracy in a multidimensional system is equivalent to a resonance of the one dimensional anharmonic oscillator with periodically varying frequency, which we treated in section II. In both cases the frequency is a function of the momentum in the absence of the resonant coupling.

B. Higher Order Resonances

If ε is not sufficiently small, higher order resonances are present in Hamiltonians (3.5) and (3.6) which contribute secular terms that modify the invariant $\hat{P}_2$. These higher order resonances can be removed in a manner closely analogous to that used in section A. However, the results have some additional features which will be seen by carrying through some of the steps explicitly. In order to apply the theory of section A, the part of the Hamiltonian not containing the resonances must be in action-angle form. We transform to action-angle variables J_1, θ_1 by solving the Hamilton-Jacobi equation,

$$\overline{H}(\hat{P}_1,\hat{P}_2,\hat{w}_1) = K(\hat{P}_2,J_1), \tag{3.21}$$

where for $\overline{H}$, we substitute equation (3.5) for an intrinsic degeneracy or equation (3.20) for an accidental degeneracy. Here, we outline the steps involved in the solution for an accidental degeneracy. Substituting equation (3.20) for $\overline{H}$, (3.21) becomes

$$\psi(\hat{P}_1,\hat{P}_2) + \varepsilon\overline{\Lambda}(\hat{P}_1,\hat{P}_2,\hat{w}_1) = K(\hat{P}_2,J_1). \tag{3.22}$$

To obtain a solution near the elliptic singularity we expand $\overline{H}$, and the generating function for the transformation, as power series in J_1. For $\overline{H}$ we obtain

$$\overline{H} = \psi(\overline{\hat{P}}_1,\hat{P}_2) + \varepsilon\overline{\Lambda}(\overline{\hat{P}}_1,\hat{P}_2,\overline{\hat{w}}_1) + \overline{G}\,\frac{(\Delta P)^2}{2} + \overline{F}\,\frac{(\Delta w)^2}{2} + \ldots \tag{3.23}$$

where we have assumed L averages to zero. The transformation to action-angle variables J_1^0, θ_1^0 for the harmonic oscillator is well known[5] yielding

$$K_0(\hat{P}_2, \varepsilon^{1/2} J_1^0) = \psi(\bar{\hat{P}}_1, \hat{P}_2) + \varepsilon\bar{\Lambda}(\bar{\hat{P}}_1, \hat{P}_2, \bar{\hat{w}}_1) + \Omega^0 J_1^0 \tag{3.24}$$

where $\Omega^0 = (\bar{F}\,\bar{G})^{1/2}$ and the dependence on ε is shown explicitly in K_0 since $\partial K_0/\partial J_1^0 = \Omega^0 = 0(\varepsilon^{1/2})$. The old and the new variables are related by the generating function $F_1 = \frac{1}{2}\left(\frac{\bar{F}}{\bar{G}}\right)^{1/2} (\Delta w)^2 \cot \theta^0$ which yields the transformation to polar coordinates

$$\begin{aligned} \Delta P &= (2J_1^0\, R)^{1/2} \cos \theta_1^0 \\ \Delta w &= \left(\frac{2J_1^0}{R}\right)^{1/2} \sin \theta_1^0 \end{aligned} \tag{3.25}$$

where $R = (\bar{F}/\bar{G})^{1/2} = 0(\varepsilon^{1/2})$. The nonlinearity is reintroduced by including higher order terms in the expansion of $\bar{H}$ and transforming to new action-angle variables J_1, θ_1 using perturbation theory. Substituting (3.25) into (3.23) and keeping terms up to fourth order in ΔP and Δw,

$$\bar{H} = \psi(\bar{\hat{P}}_1, \hat{P}_2) + \varepsilon\bar{\Omega}(\bar{\hat{P}}_1, \hat{P}_2, \bar{\hat{w}}_1) + \bar{H}_0 + \bar{H}_1 + \bar{H}_2 + \ldots$$

where

$$\bar{H}_0 = \Omega^0 J_1^0$$

and

$$\frac{\bar{H}_1}{\bar{H}_0} = 0(\lambda), \qquad \frac{\bar{H}_2}{\bar{H}_0} = 0(\lambda^2)$$

where

$$\lambda = \frac{(J_1^0 R)^{3/2}}{\Omega^0 J_1^0} .$$

Recalling that $J_{1\ max} = \frac{1}{2}(\Delta P)_{max}(\Delta w)_{max} = \frac{1}{2} R(\Delta w)^2_{max}$, in the accidental case, $J_{1\ max} = 0(\varepsilon^{1/2})$ and $\lambda_{max} = 0(\varepsilon^{1/2})$. Therefore $|\overline{H}_2| << |\overline{H}_1| << |\overline{H}_0|$ and we can write $\overline{H}$ as

$$\overline{H} = \psi(\hat{\overline{P}}_1,\hat{P}_2) + \varepsilon\overline{\Omega}(\hat{\overline{P}}_1,\hat{P}_2,\hat{\overline{w}}_1) + \overline{H}_0 + \delta\cdot\overline{H}_1 + \delta^2\cdot\overline{H}_2 \tag{3.26}$$

where δ is an artificial constant measure of smallness which is set equal to unity at the end of the calculation. It is interesting to note, however, that for the intrinsic case, $J_{1\ max} = 0(1)$ so that $\lambda_{max} = 0(\frac{1}{\varepsilon})$ and λ therefore is not necessarily small. Hence the expansion (3.26) will have a much greater range of validity in the accidental case than in the intrinsic case. To apply perturbation theory, we make the following expansions for the new Hamiltonian and the generating function

$$K = K_0 + \delta K_1 + \delta^2\ K_2$$

$$S = S_0 + \delta S_1 + \delta^2\ S_2$$

where S_0 is the identity transformation, $\theta_1^0 J_1$. We then solve for K and S to each order in δ (See [25]). The results are, to first order

$$K_1 = \overline{\overline{H}}_1(J_1)$$

and

$$\Omega_1^0(J_1)\ \frac{\partial S_1}{\partial\theta_1^0} = -\ [H_1(\theta_1^0,J_1) - \overline{\overline{H}}_1(J_1)] \tag{3.27}$$

and to second order

$$K_2 = \Omega^0(J_1)\,\frac{\partial S_2}{\partial \theta_1^0} + \frac{\partial \overline{H}_1}{\partial J_1}\,\frac{\partial S_1}{\partial \theta_1^0} + \overline{H}_2(J_1,\theta_1^0) \tag{3.28}$$

where $\overline{\overline{H}}_1$ is the average of $\overline{H}_1$ over θ_1^0. Since $\overline{H}_1$ has only odd powers of the sinusoids in θ_1^0, we have $\overline{\overline{H}}_1 = 0$. Thus there is no frequency shift to first order, and we must evaluate K_2 to obtain the lowest order effect of the nonlinearity on the frequency shift. This is accomplished by averaging K_2 over θ_1^0 giving

$$K_2 = \overline{\frac{\partial \overline{H}_1}{\partial J_1}\,\frac{\partial S_1}{\partial \theta_1^0}} + \overline{\overline{H}}_2(J_1). \tag{3.29}$$

If no first order terms appear in the Hamiltonian, $H_1 = 0$, and K_2 can be obtained from the simple average $\overline{\overline{H}}_2$.

Substituting for ΔP and Δw in terms of J_1 and θ_1 yields for the average part of the Hamiltonian

$$K(\hat{P}_2,\varepsilon^{1/2}J_1) = \psi(\overline{\hat{P}}_1,\hat{P}_2) + \varepsilon\overline{\Lambda}(\overline{\hat{P}}_1,\hat{P}_2,\overline{\hat{w}}_1) + \Omega^0 J_1(1 + \lambda^2(\hat{P}_2,J_1)m(\hat{P}_2) + \dots) \tag{3.30}$$

where m is in general a complicated function of $\hat{P}_2$. Eq. (3.30) is the formal solution in the case that the average over $\hat{w}_2$ is valid. It is independent of angles so that $\hat{P}_2$ and J_1 are the two constants of the motion.

To take into account the effect of the island resonance in modifying this solution we reintroduce the terms ignored in the simple average over $\hat{w}_2$. We denote these terms as $\varepsilon\tilde{\Lambda}(P_1,P_2,w_1,w_2)$ where for simplicity of notation we omit the hats from all variables. This is reasonable to do here, as we will perform a second transformation to new hat variables to remove the second order resonance. We have

$$\tilde{\Lambda}(P_1,P_2,w_1,w_2) = \Lambda(P_1,P_2,w_1,w_2) - \overline{\Lambda}(P_1,P_2,w_1). \tag{3.31}$$

For multiply periodic systems, Λ can be Fourier expanded in w_1 and w_2 giving for $\tilde{\Lambda}$:

$$\tilde{\Lambda} = \sum_{\substack{\ell,m \\ m\neq 0}} \Lambda_{\ell,m}(P_1,P_2)e^{i(\ell w_1 + mw_2)}. \tag{3.32}$$

Expanding about the elliptic singularity gives

$$\tilde{\Lambda} = \sum_{\substack{\ell,m \\ m\neq 0}} \Lambda_{\ell,m}(\overline{P}_1 + \Delta P,P_2)e^{i(\ell[\overline{w}_1 + \Delta w] + mw_2)}. \tag{3.33}$$

In the case of accidental degeneracy,

$$\frac{\Delta P_{max}}{\Delta w_{max}} = R = 0(\varepsilon^{1/2}) \tag{3.34}$$

so that to lowest order, we can ignore the variation in P_1. Transforming to action angle variables by (3.25)

$$\tilde{\Lambda} = \sum_{\substack{\ell,m \\ m\neq 0}} \Lambda_{\ell,m}(\overline{P}_1,P_2)e^{i(\ell\overline{w}_1 + mw_2)} \; e^{i\ell\left(\frac{2J_1}{R}\right)^{1/2} \sin\theta_1} \tag{3.35}$$

and, expanding the second exponential

$$\tilde{\Lambda} \cong \sum_{\substack{\ell,m,n \\ m\neq 0}} \Lambda_{\ell,m}(\bar{P}_1,P_2) J_n\left(\ell\sqrt{\frac{2\bar{J}_1}{R}}\right) e^{i(\ell\bar{w}_1 + mw_2 + n\theta_1)} \tag{3.36}$$

where J_n is the Bessel function of order n. From (3.36), it is evident that there can be higher order resonances between w_2 and θ_1 such that the average of (3.36) over w_2 is not zero. Considering the resonance

$$\frac{\nu_2}{\Omega} = \frac{p}{q} \qquad \text{(p, q integers)} \tag{3.37}$$

where

$$\nu_2 = \frac{\partial K}{\partial P_2} = 0(1)$$

$$\Omega = \frac{\partial K}{\partial J_1} = 0(\varepsilon^{1/2})$$

then the terms $m = \pm q$, $n = \mp p$ and their harmonics will remain after the average. The magnitude of the lowest harmonic depends on $J_p\left(\sqrt{\frac{2J_1}{R}}\right)$ where p is a large integer of order $\varepsilon^{-1/2}$. Since $J_{1\ max} \approx 0(\varepsilon^{1/2})$ and $R \approx 0(\varepsilon^{1/2})$, the maximum value of the argument $(2J_1/R)^{1/2}$ is of order unity which allows the Bessel function to be approximated by the first term in the expansion for small argument

$$J_p\left(\sqrt{\frac{2J_1}{R}}\right) \simeq \frac{\left(\frac{J_1}{2R}\right)^{p/2}}{p!} \simeq 0\left(\frac{1}{(q\varepsilon^{-1/2})!}\right) \tag{3.38}$$

Higher harmonics such as $m = -2q$, $n = 2p$ decrease rapidly and can be neglected. From (3.38) we see that the amplitude of the interaction term is also proportional to $J_1^{p/2}$ such that the island oscillations decrease in size rapidly with decreasing J_1. To obtain the new invariant, we write the total Hamiltonian as

$$H = K(P_2, \varepsilon^{1/2} J_1) + \varepsilon \tilde{\Lambda}(J_1, P_2, \theta_1, w_2). \tag{3.39}$$

This has a form similar to (3.1). Therefore, we can use the method of section A to remove the degeneracy given by (3.37). This involves a transformation to new variables which we denote as $\hat{P}_1$, $\hat{P}_2$, $\hat{w}_1$, $\hat{w}_2$ where one of the new angles $\hat{w}_1$ is the slow variable $p\Theta_1 - qw_2$. An average over the fast $\hat{w}_2$ is then equivalent to keeping only $m = q$, $n = -p$, and $m = -q$, $n = p$, together with harmonics in $\tilde{\Lambda}$. Letting $q = 0(1)$ the result will be of the form

$$\overline{\overline{H}} = K(\hat{P}_1, \hat{P}_2) + \left(\frac{\varepsilon}{\varepsilon^{-1/2}!}\right) \overline{\tilde{\Lambda}}'(\hat{P}_1, \hat{P}_2, \hat{w}_1) \tag{3.40}$$

where we have factored the ε dependence from $\overline{\tilde{\Lambda}}$, using (3.38), a procedure which can only be carried out after averaging. The double bar distinguishes this average from the simple average of section A which ignores all of the terms in $\overline{\tilde{\Lambda}}$. Since $\overline{\overline{H}}_1$ is cyclic in $\hat{w}_2$, we have also

$$\hat{P}_2 = P_2 + q\hat{P}_1 \simeq \text{constant}. \tag{3.41}$$

such that the $\hat{P}_1 - \hat{w}_1$ phase orbits are isolated. Comparing equations (3.40) and (3.20), we see that the amplitude and frequency of the oscillation in $\hat{P}_1$ will be a factor of $0(1/\varepsilon^{-1/2}!)^{1/2}$ smaller than the amplitude and frequency of the oscillation in P_1. We call the new $\hat{P}_1$ oscillation an island oscillation because it appears as a chain of islands in the $P_1 - w_1$ phase plane.

Although the procedure for exhibiting the invariant curves of the island oscillation, arising from the second order resonance, is the same as that used for obtaining the invariant curves of the primary resonance, the results have a somewhat different character. The strength of the island resonance is related explicitly to a particular form of phase nonlinearity, depending strongly on ε. The strength of the primary resonance is related weakly to ε, as $\varepsilon^{1/2}$. Thus, for relatively small ε, island oscillations rapidly become of negligible importance. For relatively large ε, on the other hand, the island oscillations may be more important than the primary resonance in determining the limits of adiabatic invariance. In principle the above procedure may be extended to third order resonances, but because of the factorial dependence on ε we find that the island oscillations either interact sufficiently strongly to terminate the series or the third order resonances have vanishingly small effect.

For intrinsic degeneracies the second order resonances do not have the same explicit form of ε dependence. However, the conclusion above, that the importance of the island oscillations is strongly dependent on the magnitude of ε, is still valid.

IV. NONLINEAR COUPLED OSCILLATOR

In this section, we present numerical results from an example which has an intrinsically degenerate fundamental resonance. By introducing a term to make the degeneracy accidental, we can compare the two problems and observe numerically the differences which we predicted analytically in the last section.

We consider a two dimensional oscillator in x and y with the Hamiltonian

$$H = H_0(p_x,p_y,x,y) + \varepsilon H_1(x,y) \tag{4.1}$$

where

$$H_0 = \frac{p_x^2}{2} + \frac{x^2}{2} + \frac{p_y^2}{2} + 2y^2 \tag{4.2}$$

and

$$H_1 = x^2y - \frac{4}{3}y^3. \tag{4.3}$$

ε is an artificial small parameter to be set equal to unity at the end of the calculation. The significance of ε is to remind us that for small amplitude oscillations in x and y, the terms in the perturbation, H_1, are smaller than the terms in H_0 by a factor of the square root of the unperturbed energy. The potential, U(x,y), is

$$U(x,y) = \frac{1}{2}(x^2 + 4y^2 + 2x^2y - \frac{8}{3}y^3).$$

As U increases, the curves of equipotential (U = constant) become increasingly nonlinear until a triangular separatrix at U = .6666 is reached, which separates bound from unbound trajectories.

We transform (4.1) to the appropriate form to apply our general theory by introducing in two dimensions, the action-angle variables for the harmonic oscillator as given in (3.25)

$$x = \sqrt{2P_1}\sin w_1 \qquad y = \sqrt{P_2}\sin w_2$$
$$p_x = \sqrt{2P_1}\cos w_1 \qquad p_y = \sqrt{4P_2}\cos w_2 \tag{4.4}$$

which gives

$$H = H_0(P_1,P_2) + \varepsilon H_1(P_1,P_2,w_1,w_2) \tag{4.5}$$

where

$$H_0 = P_1 + 2P_2 \tag{4.6}$$

and

$$H_1 = 2P_1(P_2)^{1/2}\ \sin^2 w_1 \sin w_2 - \tfrac{4}{3}(P_2)^{3/2}\sin^3 w_2. \tag{4.7}$$

The unperturbed frequencies in the two degrees of freedom are then

$$\omega_x^0 = \frac{\partial H_0}{\partial P_1} = 1, \qquad \omega_y^0 = \frac{\partial H_0}{\partial P_2} = 2$$

so that the resonance condition becomes

$$\frac{\omega_y^0}{\omega_x^0} = 2. \tag{4.8}$$

Note that this is intrinsic rather than accidental because ω_x^0 and ω_y^0 are independent of P_1 and P_2.

This problem bears close resemblence to the problem treated by Hénon and Heiles[15] and McNamara and Whiteman.[12] The only difference is that we have assumed a one to two resonance between the frequencies ω_x^0 and ω_y^0 while the above authors treat the one to one resonance. The advantage of the one to two case is that it provides a lowest order invariant which is zero order rather than first order in ε, and hence it is easier to calculate.

To remove the intrinsic degeneracy, we employ the generating function as in (3.3)

$$F_2 = (2w_1 - w_2)\hat{P}_1 + w_2\hat{P}_2, \tag{4.9}$$

giving a transformation to new variables $\hat{P}_1, \hat{P}_2, \hat{w}_1, \hat{w}_2$ such that the unperturbed part of the Hamiltonian H_0 is a function of $\hat{P}_2$ alone:

$$H_0 = 2\hat{P}_2. \tag{4.10}$$

The total Hamiltonian, in terms of the hat variables, is

$$H = 2\hat{P}_2 + 4\hat{P}_1(\hat{P}_2 - \hat{P}_1)^{1/2}\sin^2\left(\frac{\hat{w}_1 + \hat{w}_2}{2}\right) \sin\hat{w}_2 - \frac{4}{3}(\hat{P}_2 - \hat{P}_1)^{3/2}\sin^3\hat{w}_2, \tag{4.11}$$

and can be averaged over $\hat{w}_2$ to give

$$\overline{H} = 2\hat{P}_2 + \hat{P}_1(\hat{P}_2 - \hat{P}_1)^{1/2}\sin\hat{w}_1 \tag{4.12}$$

where we have set $\varepsilon = 1$. The lowest order adiabatic invariant is therefore,

$$\hat{P}_2 = P_2 + \frac{1}{2}P_1 \simeq \text{constant} \tag{4.13}$$

which, together with $\overline{H}$ = constant, isolates the $\hat{P}_1 - \hat{w}_1$ phase motion.
In removing a degeneracy by use of a generating function as in (4.9), there is an arbitrary choice of which of the original phase variables to keep. We have chosen here to keep $w_2 = \hat{w}_2$, such that the average in (4.11) is taken over the faster variable, w_2. Although this choice is convenient here, if 2^{nd} order island resonances are to be removed it is possible to lose the lowest order interaction unless the original transformation leads to an average over the slower of the original phase variables. We average over the slower oscillation

in the example of the next section in which the second order resonances are removed.

The light lines in Figure 2 show the $\hat{P}_1 - \hat{w}_1$ phase loops plotted from equation (4.12) for $\hat{P}_2 = .04166$ and a total energy $E = .08333$. As a check on the averaging procedure, we have also shown in Figure 2 (heavy lines) the trajectories obtained by numerically integrating Hamilton's equations before the average over $\hat{w}_2$. We differentiate equation (4.11) with respect to $\hat{P}_1$, $\hat{P}_2$, $\hat{w}_1$, $\hat{w}_2$ and solve the resulting four simultaneous differential equations by a standard subroutine on a CDC 6400 computer. The results are plotted in Figure 2 for a plane of section

$$\sin \hat{w}_2 = 1.0. \tag{4.14}$$

This particular plane has no special significance and similar results are obtained using any other plane. Comparing the analytic and numerical solutions in Fig. 2, we see that there is slightly more asymmetry in the numerical result. Presumably, if we carry the averaging procedure to higher orders in ε, the agreement will improve.

Also, from Fig. 2, we notice that for the outer-most phase loops, the fractional change in $\hat{w}_1$ is of order unity, and the fractional change in $\hat{P}_1$ is likewise of order unity. This result is a direct consequence of the intrinsic nature of the resonance as we showed in the last section.

Alternately, we can follow McNamara and Whiteman[12],[13] by writing the first order adiabatic invariant I in terms of x, y, p_x, p_y, eliminating p_x between I = constant and H = E, a constant, and evaluating the result in the x = 0 plane of section. We obtain,

$$\frac{E}{2} + \frac{1}{8}\, y(2E + \frac{8}{3}\, y^3 - p_y^2 - 4y^2) = \text{Constant} \tag{4.15}$$

which gives the isolated motion in the $y - p_y$ phase plane shown by the light lines in Fig. 3. Both McNamara and Whiteman[12] and Hénon and Heiles[15] show results in this plane rather than in the $\hat{P}_1 - \hat{w}_1$ plane of Fig. 2. The two methods of plotting are equivalent, however. The dark lines in Fig. 3 show numerical results obtained by differentiating the original Hamiltonian (4.1) with respect to x, p_x, y, p_y and integrating the four resulting equations with the aid of a computer. These results are plotted in the plane of section x = 0. Again there is more asymmetry in the numerical result, but as McNamara and Whiteman[12] have shown, if the adiabatic invariant is calculated to higher order, the agreement can be quite good. For the oscillator with $\omega_x^0 = \omega_y^0 = 1$, they have obtained reasonable agreement in fourth order for this same energy.

For the low value of energy considered up to this point, we can in principle obtain arbitrarily good agreement between theoretical and numerical solutions by calculating the invariant to higher orders in ε. This is not the case, however, for higher values of the energy because the invariant used in the theoretical calculations breaks down due to higher order resonances. This is shown by the island formation in Fig. 4 which was calculated numerically for an energy of E = .33333. The non-connected points correspond to a single particle orbit as do the five-island trajectories. A corresponding first order theoretical curve plotted from equation (4.12) would show niether the ergodic region nor the chain of islands. This is to be expected since the islands occur as a result of resonances which are ignored in the averaging leading to the simple invariants.

Note that the frequency of the phase oscillation in Fig. 4 is approximately twice that in Fig. 2. This is shown by the numbering of the points. In Fig. 2 there are about 10 oscillations in $\hat{w}_2$ for each one in $\hat{w}_1$ while there are only 5 in Fig. 4. Since the energy in Fig. 4 is 4 times that in Fig. 2, we see that the frequency of the $\hat{P}_1 - \hat{w}_1$ phase oscillation varies linearly with $\varepsilon \simeq \sqrt{E}$. As pointed out in the previous section, this is a general result for all phase oscillations that result from the removal of intrinsic degeneracies and it accounts for the relatively high value of energy ($E = .3333$) necessary to observe breakdown in Fig. 4.

In order to contrast this behavior with that resulting from the removal of an accidental degeneracy, we introduce another term in the unperturbed Hamiltonian (4.6) so that the unperturbed frequencies ω_1^0 and ω_2^0 depend on P_1 and P_2. We take

$$H_0 = P_1 + 2P_2 - P_1^{1/2} P_2^{1/2} \tag{4.16}$$

giving

$$\omega_x^0 = \frac{\partial H_0}{\partial P_1} = 1 - \frac{1}{2}\left(\frac{P_2}{P_1}\right)^{1/2} \tag{4.17}$$

$$\omega_y^0 = \frac{\partial H_0}{\partial P_2} = 2 - \frac{1}{2}\left(\frac{P_1}{P_2}\right)^{1/2}$$

so that the resonance condition, $\omega_y^0/\omega_x^0 = 2$, is satisfied only accidentally. Although the additional term in H_0 has been introduced somewhat artificially here, such terms do occur naturally in many problems, and we will treat such a case in the next section. With the nonlinear term present in the unperturbed Hamiltonian, the transformation (4.9) does not eliminate the

$\hat{P}_1$ dependence from H_0. But as we saw in Section 3, an average over $\hat{w}_2$ may still be justified when there is an elliptic singularity in the $\hat{P}_1 - \hat{w}_1$ phase plane. Assuming that this is the case, and performing the average we obtain

$$\overline{H} = 2\hat{P}_2 - (2\hat{P}_1)^{1/2}(\hat{P}_2 - \hat{P}_1)^{1/2} + \hat{P}_1(\hat{P}_2 - \hat{P}_1)^{1/2}\sin\hat{w}_1 \tag{4.18}$$

with a lowest order invariant $\hat{P}_2$ as defined in (4.13). In Fig. 5a, we show the $\hat{P}_1 - \hat{w}_1$ phase curves plotted from (4.18) for $\hat{P}_2 = .006443$ and a total energy of $E = .008333$. The corresponding numerical plot, shown in Fig. 5b, was plotted by integrating Hamilton's equations directly before the average over $\hat{w}_2$. There is an elliptic singularity evident in Fig. 5b so that we expect the averaging which led to Fig. 5a to be valid. Near the separatrix of the oscillation, however, the averaging breaks down as indicated by the island formation and ergodic region. Because the frequency of the $\hat{P}_1 - \hat{w}_1$ phase oscillation in the accidental case varies as $\varepsilon^{1/2} \simeq E^{1/4}$ whereas it varies as $\varepsilon \simeq E^{1/2}$ in the intrinsic case, a much smaller value of the energy is required for accidental degeneracy to obtain the same resonant harmonic interaction. The ratio of the frequency of the island oscillation to the frequency of main invariant curves is observed to be 1/7 which is in reasonable agreement with the predicted value of $(1/p!)^{1/2} \cong 1/10$ as predicted from Eq. (3.40). In both Figures 4 and 5 breakdown occurs at the fifth order resonance as indicated by the chain of five islands surrounding the adiabatic phase space curves.

The amplitude of the oscillation in $\hat{P}_1$ given in Fig. 5a is no longer of the same order of magnitude as the amplitude of the $\hat{w}_1$ oscillation. The outer most phase loop shows a maximum fractional change in $\hat{w}_1$ of order unity while the maximum fractional change in $\hat{P}_1$ is 0.3 which is of the order of the fourth root of the energy, consistent with the general results given in Section III for accidental degeneracies.

In Fig. 5c, we have lowered the energy from .0083 to .0068 (we have doubled the vertical scale for clarity). We note that this causes the five-island trajectory to move slightly inward toward the elliptic singularity in the $\hat{P}_1 - \hat{w}_1$ plane. This is to be expected since the frequency of the $\hat{P}_1 - \hat{w}_1$ phase loops varies as $\varepsilon^{1/2}$. A smaller ε, therefore, results in a smaller frequency, and to obtain the same resonant interaction it is necessary to reduce the nonlinear frequency shift by reducing the size of the phase loops. In Fig. 5c, break-up no longer occurs outside of the five-island orbit. Rather, there are smooth phase loops outside of the islands and the trajectories break up due to interaction between the chain of six islands and nonresonant terms. A stabilization of the five island trajectory as it moves inward is to be expected from the results of section III where we saw that the nonresonant terms have amplitudes depending on Bessel functions with arguments proportional to the area enclosed by $\hat{P}_1 - \hat{w}_1$ phase loops. Hence as the five islands move in, the area enclosed decreases and the secular terms become less important. The index of the Bessel function, on the other hand, depends on the resonance number so that the size of the islands decreases as the number of islands goes up. Thus, we might expect the chain of six islands to be smaller than the chain of five islands, but the competing effect of a larger area enclosed by the chain makes the size of the six islands in Fig. 5c comparable if not larger than the five islands. The higher harmonic island chains are also closer together, increasing the interaction of neighboring island chains. We note that an 11 island chain also appears, corresponding to the 11/2 resonance, such that the trajectory returns to a given island after two oscillations of the main invariant curve. The islands are of much smaller amplitude as the interaction strength depends on the amplitude of a Bessel function of 11th order, rather than the 5th and 6th order Bessel functions corresponding to the 5 and 6 island trajectories. Narrow bands of stochasticity may exist near the separatrices of the 5 and 11 island chain resonances, but these trajectories are enclosed by adiabatic trajectories which confine them to limited regions of the phase plane.

In the next section, we consider electron cyclotron resonance of magnetically confined particles as an example of a two dimensional coupled oscillator. The nonlinear term in the unperturbed part of the Hamiltonian occurrs naturally in this problem and the fundamental degeneracy is accidental, resulting in phase curves resembling those in Fig. 5 rather than those in Figures 3 and 4.

V. CYCLOTRON RESONANCE IN A MAGNETIC MIRROR FIELD

As an example of the effect of generation of invariants by removal of an accidental degeneracy, the modification of those invariants due to secondary resonances giving rise to island oscillations, and the destruction of the invariants for sufficiently large perturbation, we treat the problem of cyclotron resonance between a particle gyrating in a magnetic mirror field and an electromagnetic wave. The system is shown schematically in Fig. 6. All lengths are normalized to the length L where 2L is the distance between mirror points. The magnetic field B is approximated by $\frac{B}{B_0} = 1 + a\eta^2$, $a = \frac{B_{max}}{B_0} - 1$ where B_0 is the field strength at the midplane ($\eta=0$) and B_{max} is the field strength at $\eta = \frac{z}{L} = 1$. The Hamiltonian for this problem was expressed in action-angle variables by Seidl[30]. Following Seidl, we designate actions for the Larmor, azimuthal, and longitudinal motions, P_1, P_2 and P_3 respectively. The corresponding angle variables are w_1, w_2 and w_3. P_3 is equivalent to the longitudinal action integral

$$P_3 = \frac{1}{2\pi} \oint p_\eta \, d\eta \tag{5.1}$$

and w_3 is the phase of the longitudinal oscillation so that $\eta = \eta_m \sin w_3$ where η_m is the maximum longitudinal penetration which can be shown to be $\eta_m^2 = (2/a)^{1/2} P_3/P_1^{1/2}$. Similarly, w_2 is the phase angle for the azimuthal drift of the guiding center and P_2 is proportional to the flux through the drift orbit. w_1 is related to the Larmor angle Q_1 by the relation

$$Q_1 = w_1 - \frac{1}{4} \frac{P_3}{P_1} \sin 2w_3$$

such that the longitudinal variation of the Larmor motion has been subtracted out to make $\nu_1 = \dot{w}_1$ the average value of the Larmor frequency over a longitudinal bounce, rather than the instantaneous value. In the midplane where $w_3 = 0$, P_1 is proportional to the magnetic moment $\mu = \frac{E_\perp}{B}$. Hence P_1 is proportional to the perpendicular energy $E_\perp$ in the midplane.

In terms of P_1, P_2, P_3, w_1, w_2, w_3 an expansion in powers of P_2 yields

$$H_0 = \omega_0 P_1 \left[\gamma_0(P_2) + \gamma_1(P_2) \frac{P_3}{\sqrt{P_1}} + \gamma_2(P_2) \left(\frac{P_3}{\sqrt{P_1}} \right)^2 + \cdots \right] \quad (5.2)$$

where ω_0 is the Larmor frequency at the midplane and

$$\gamma_0(P_2) = 1 - a P_2 - \frac{1}{2} a^2 P_2^2$$

$$\gamma_1(P_2) = \sqrt{2a} \, (1 + a P_2 - 3a^2 P_2^2 + \cdots)$$

$$\gamma_2(P_2) = 10 a^2 P_2 (1 - 4a P_2 + \frac{9}{2} a^2 P_2^2 + \cdots) .$$

We note that (5.2) is independent of w_1, w_2, w_3, and time t, so that P_1, P_2, P_3, and H are constants of the motion.

If we add to the system a radio frequency electric field propagating parallel to the magnetic field; this R. F. field can be treated as a perturbation on the original Hamiltonian H_0 so that

$$H = H_0(P_1, P_2, P_3) - \varepsilon \, \omega_0 \, H_1 \, (w_1, w_3, P_1, P_3 \; \omega t) \quad (5.3)$$

where

$$H_1 = \sqrt{P_1} \cos \left(w_1 - \frac{1}{4} \frac{P_3}{P_1} \sin 2w_3 \right) \cos (kL\eta_m \sin 2w_3) \sin \omega t \quad (5.4)$$

$$\varepsilon = \sqrt{2} \, \frac{E_0}{\omega B_0 L} \quad (5.5)$$

and ω, E_0 and k are the frequency, amplitude, and wave number of the electric field, respectively. For $\varepsilon \neq 0$, P_1, P_3, and H are no longer constants of the motion. The problem is to determine if new constants exist in the presence of the perturbation. Seidl removed the primary resonance by transforming to a rotating frame as in (3.3) in which the new phase variable $w_1' = w_1 - \omega t$, is slowly varying near cyclotron resonance, and employed the method of secular perturbations to average over w_3 and ωt which are both assumed to be rapidly varying with respect to w_1'. Expanding the perturbation H_1 in a Fourier series, keeping only the slowly varying term,

$$H_1 = \frac{-\sqrt{P_1}}{4i} \sum_{n=-\infty}^{\infty} \left\{ \left(\sum_{m=-\infty}^{\infty} J_{2m}(\beta)\, J_{n-2m}(\alpha) \right) \left(- e^{-iw_1'} + (-1)^n e^{iw_1'} + e^{-i(w_1' + 2\omega t)} - (-1)^n e^{i(w_1' + 2\omega t)} \right) \right\} e^{i2nw_3} \tag{5.6}$$

where

$$\alpha = \frac{1}{4} \frac{P_3}{P_1}\,, \quad \beta = kL\eta_m$$

and averaging over w_3 and ωt, the total averaged Hamiltonian is

$$\bar{H} = \omega_0 \left[(\gamma_0 - \frac{\omega}{\omega_0}) P_1 + \gamma_1 \sqrt{P_1}\, P_3 + \gamma_2 P_3^2 \right] + \varepsilon\, \omega_0 \frac{\sqrt{P_1}}{2} f(P_1, P_3) \sin w_1' \tag{5.7}$$

where

$$f = \sum_{m=-\infty}^{\infty} J_{2m}(\alpha)\, J_{2m}(\beta) \quad .$$

Since $\overline{H}$ is independent of w_3, the lowest order adiabatic invariant is

$$P_3 \simeq \text{constant} \tag{5.8}$$

which is the main conclusion of Seidl's work. We note that the range of validity of the averaging is considerably more restricted here than in the previous section, as the average is not only over the fastest variable with frequency ω but also over the slower longitudinal variable with bounce frequency w_3. As in section IV, we check the accuracy of secular perturbation theory for averaging over w_3, by comparing the slow P_1, w_1' oscillation from (5.7) with numerical integration before the average over w_3 but after the average over ωt, as shown in Figures 3a and b. The numbers in Fig. 7b represent successive crossings of a plane of section in w_3,

$$\sin 2w_3 = 1.0. \tag{5.9}$$

There are approximately six longitudinal bounces for each oscillation in the perpendicular energy P_1, the particle crossing a resonance twice during each longitudinal period. The slight scatter in the points plotted is due to inaccuracies in satisfying (5.9) exactly. The relatively smooth curves traced out in the $P_1 - w_1'$ phase plane indicate the existence of an adiabatic invariant which in this case we know is P_3 to lowest order. The values of P_3, averaged over a longitudinal bounce are plotted in Fig. 8 for one of the phase loops of Fig. 7b and we see that P_3 is approximately constant and equal to it's initial value of 4×10^{-4}. As a further check on the theory, we have integrated the exact equations of motion numerically for parameters corresponding to Fig. 7b (see [31]). The results shown in Fig. 7c are similar to those in Fig. 7b indicating that the assumptions are valid.

Figures 9a, b and c are similar to Figures 7a, b and c except that the value of ε has been increased sufficiently for a chain of islands to appear surrounding the closed phase orbits, indicating a significant bounce-energy resonance. We include the exact numerical calculation in 9c, for the island chain only, to demonstrate that islands can be observed in the absence of any approximations. A plot of P_3 averaged over the longitudinal oscillation for the island trajectory is shown in Fig. 10. P_3 oscillates with the period of the island oscillation as is shown by the numbering of the points. Points 11 - 16 lie on the inside of the islands in Fig. 9b and hence enclose a small area inside their phase loop. We see from Fig. 10 that these same points have relatively low values of P_3. On the other hand, points 21 - 26 lie on the outside of the islands enclosing a larger area inside their phase orbit, and we see from Fig. 10 that these points have relatively high values of P_3. Although it is possible to obtain completely stable islands, the islands in Figs. 9b and 9c are not stable. The resonant terms which generate the island are perturbed by non-resonant terms that cause the trajectory to slowly drift away from the island.

To explain the behavior exhibited in these figures, it is necessary to consider the higher order resonance in the system which in this case is the bounce-energy resonance. To treat the bounce-energy resonance, we transform to the action-angle variables of the P_1 oscillation, i.e. solve the Hamilton-Jacobi equation

$$\bar{\bar{H}} \left(\frac{\partial S}{\partial w_1'}, w_1', P_2, P_3 \right) = E(J_1, P_2, P_3) \qquad (5.10)$$

where $S(w_1' J_1)$ is the partial generating function to transform to action-angles J_1, θ_1. As in Section III we can expand the average Hamiltonian, $\bar{H}$, about the elliptic singular point $\overline{P_1}$, $\overline{w_1'}$ The average part of the Hamiltonian in action-angle variables is then of the form

$$\bar{H} = \omega_0[(\gamma_0(P_2) - \frac{\omega_0}{\omega_0})\overline{P_1} + \gamma_1(P_2)\sqrt{\overline{P_1}}\,P_3 + \gamma_2(P_2)\,P_3^2]$$

$$+ F(P_3) - \Omega^0_{(P_3)}J_1\,[1 - \lambda^2(J_1, P_3)M(P_3)] \qquad (5.11)$$

where Ω^0 is the lowest order energy oscillation frequency, $\lambda = \dfrac{(2J_1R)^{3/2}}{2\Omega^0 J_1} \propto \varepsilon^{1/2}$, and we have written J_2 and J_3 as P_2 and P_3 respectively since these variables are unchanged by the transformation. F, R, Ω^0 and M are functions of P_3 resulting from the expansion which we derive in Appendix 1. The varying part of the Hamiltonian can be Fourier analyzed as

$$\tilde{H}_1 = \frac{-\sqrt{\overline{P_1}}}{4}\sum_{\substack{\ell,n\\ n\neq 0}} S_n\,(\overline{P_1}, P_3)((-1)^\ell + (-1)^n)J_\ell(\sqrt{\frac{2J_1}{R}})e^{i(\ell\,\theta_1 + 2nw_3)} \qquad (5.12)$$

where

$$S_n(\overline{P_1}, P_3) = \sum_{m=-\infty}^{\infty} J_{2m}\,(\bar{\beta})\,J_{n-2m}(\bar{\alpha})$$

and $\bar{\alpha}$ and $\bar{\beta}$ are defined as in (5.6) except with $\overline{P_1}$ replacing P_1. J_ℓ is the ℓ^{th} order Bessel function. The total Hamiltonian is

$$H = \bar{H}(J_1,P_2,P_3) - \varepsilon\omega_0\,\tilde{H}_1(J_1,P_2,P_3,\,\theta_1,\,w_3) \qquad (5.13)$$

where $\bar{H}$ and $\tilde{H}$ are given by (5.11) and (5.12) respectively.

If we now define the unperturbed frequencies:

$$\overline{\nu_3}(J_1, P_2, P_3) = \frac{\partial\bar{H}}{\partial P_3} = \text{longitudinal bounce frequency} \tag{5.14}$$

$$\bar{\Omega}(J_1, P_2, P_3) = \frac{\partial\bar{H}}{\partial J_1} = \text{energy oscillation frequency} \tag{5.15}$$

we see from the exponent in (5.12) that there can be resonances in (5.13) of the form

$$\frac{\bar{\nu}_3(J_1, P_2, P_3)}{\bar{\Omega}(J_1, P_2, P_3)} = \frac{r}{2s} \quad (r,\ s\ \text{integers})\ . \tag{5.16}$$

These resonances will introduce secularities in the time rate of change of the simple adiabatic invariant P_3. The resonances only occur for certain values of J_1 and P_3 which in general may vary due to the secularities. As we see in Fig. 9b, islands form around elliptic singularities in the J_1, θ_1 plane. Hence we can transform to coordinates $(\hat{\theta}_1, \hat{w}_2, \hat{w}_3, \hat{J}_1, \hat{P}_2, \hat{P}_3)$ in which $\hat{\theta}_1$ is the difference phase $2sw_3 + r\theta_1$ which is slowly varying near the elliptic singularity. The required generating function is

$$F_2 = (2sw_3 + r\theta_1)\hat{J}_1 + w_2\hat{P}_2 + w_3\hat{P}_3 \tag{5.17}$$

which defines the transformation to the hat variables in the rotating frame,

$$\hat{\theta}_1 = \frac{\partial F_2}{\partial \hat{J}_1} = 2sw_3 + r\theta_1 \qquad J_1 = \frac{\partial F_2}{\partial \theta_1} = r\hat{J}_1$$

$$\hat{w}_2 = \frac{\partial F_2}{\partial \hat{P}_2} = w_2 \qquad P_2 = \frac{\partial F_2}{\partial w_2} = \hat{P}_2 \tag{5.18}$$

$$\hat{w}_3 = \frac{\partial F_2}{\partial \hat{P}_3} = w_3 \qquad P_3 = \frac{\partial F_2}{\partial w_3} = \hat{P}_3 + 2s\hat{J}_1 \ .$$

Writing the perturbation (5.12) in terms of the hat variables:

$$\tilde{H}_1 = -\frac{\sqrt{\overline{P}_1}}{4}\sum_{\substack{\ell,n \\ n\neq 0}} S_n(\overline{P}_1,\ \hat{P}_3 + 2s\hat{J}_1)((-1)^\ell + (-1)^n)J_\ell\left(\sqrt{\frac{2r\hat{J}_1}{R}}\right)e^{\frac{i}{r}\{2(nr-\ell s)\hat{w}_3+\ell\hat{\theta}_1\}} \tag{5.19}$$

and averaging over $\hat{w}_3$, we get

$$\tilde{\overline{\overline{H}}}_1 = -\frac{\sqrt{\overline{P}_1}}{4}\sum_{\substack{\ell,n \\ n\neq 0 \\ nr-\ell s=0}} S_n(\overline{P}_1,\hat{P}_3 + 2s\hat{J}_1)((-1)^\ell + (-1)^n)J_\ell\left(\sqrt{\frac{2r\hat{J}_1}{R}}\right)e^{i\frac{\ell}{r}\hat{\theta}_1}. \tag{5.20}$$

The double bar average corresponds to keeping just the most slowly varying terms of the sum (5.19) which, for the $\frac{\overline{\nu}_3}{\overline{\Omega}} = \frac{r}{2s}$ resonance, are the $\ell = jr$ and $n = js$ terms where j runs over all positive integers.

The total average Hamiltonian taking into account the bounce-energy resonance can now be written as

$$
\begin{aligned}
\bar{\bar{H}} = {} & \omega_0 [(\gamma_0(\hat{P}_2) - \frac{\omega}{\omega_0}) \overline{P_1} + \gamma_1(\hat{P}_2)\sqrt{\overline{P_1}}(\hat{P}_3 + 2s\hat{J}_1) + \gamma_2(\hat{P}_2)(\hat{P}_3 + 2s\hat{J}_1)^2] \\
& + F(\hat{P}_3,\hat{J}_1) - \Omega^0(\hat{P}_3,\hat{J}_1)\, r\hat{J}_1 \,[1-\lambda^2 (\hat{P}_3,\hat{J}_1)\, M(\hat{P}_3,\hat{J}_1)] \\
& + \varepsilon\, \omega_0 \frac{\sqrt{\overline{P_1}}}{2} \sum_{j=1}^{\infty} S_{js}(\hat{P}_3,\hat{J}_1)((-1)^{jr} + (-1)^{js}) J_{jr}(\sqrt{\frac{2r\hat{J}_1}{R}}) \cos j\, \hat{\theta}_1
\end{aligned}
\tag{5.21}
$$

where all quantities are derived in the appendix. Since $\bar{\bar{H}}$ is independent of $\hat{w}_3$, the correct adiabatic invariant for the island case is

$$
\hat{P}_3 = P_3 - \frac{2s}{r} J_1 \simeq \text{constant} \tag{5.22}
$$

which reduces to the invariant P_3 for a very high order resonance, $r \gg s$. Since $\bar{\bar{H}}$ in (5.21) is independent of time, we can use $\bar{\bar{H}}$ = constant together with $\hat{P}_3$ = constant to plot $\hat{J}_1$ versus $\hat{\theta}_1$ for various values of the constant, $\bar{\bar{H}}$. Rather than do this directly we see, from (5.22), that if J_1 oscillates then P_3, the adiabatic invariant without resonances must also oscillate. This explains the sychronized oscillation between P_3 and J_1 observed in Figs. 9b and 10. We plot the oscillation in P_3, $\hat{\theta}_1$ for the five island resonance as solid lines in Fig. 11. The bold curve corresponds to the initial conditions of the five island trajectory in Fig. 9b. On the right hand axis we give the ratio ν_3/Ω. The oscillation centers about an average bounce-energy resonance number of $\nu_3/\Omega = 2.5$ as does the five island trajectory in the numerical integration. At the extremes of the oscillation, the ratio ν_3/Ω never moves very far from 2.5 going to 2.54 at the top of the phase loop and 2.46 at the bottom, thus justifying the fundamental assumption for keeping only the most slowly varying terms, that the system should remain close to a particular bounce-energy resonance.

In the numerical integrations of Fig. 9b, we also find relatively smooth phase loops near the elliptic singular point and ergodic trajectories beyond the islands. By plotting P_3, $\hat{\theta}_1$ phase diagrams for these other types of behavior, we can distinguish the physical mechanism that differentiates among them. The most slowly varying term for the islands ($\nu_3/\Omega = 2.5$) had ℓ=5, n=1 in the sum (5.12), plus higher harmonics. Taking r=7 and s=1 leads to the much smaller oscillation about $\nu_3/\Omega = 3.5$ shown at the top of Fig. 11. Also, r=3, s=1 contributes the drifting oscillations shown as dashed lines. (The integral resonances $\frac{\nu_3}{\Omega} = \frac{r}{2s} = 1,2,3, \cdots$ lead only to extremely small oscillations in the adiabatic invariant, P_3, and are not considered). Close to the 2.5 resonance, the nonresonant terms average nearly to zero over an island oscillation. To see this, we recall that the nonresonant terms average exactly to zero when the system is right at the 2.5 resonance due to the orthogonality of the exponentials making up the Fourier series (5.12). If the system is not exactly at resonance, the nonresonant terms do contribute, but in the approximation of a symmetric oscillation about resonance, the contributions above and below resonance cancel. The assumption of symmetry around the resonance is valid in the linear region close to the island singularity so that the total nonresonant contribution averages nearly to zero over the island oscillation. Thus the double bar average with r = 5 and s = 1 is a reasonable approximation for the island trajectory in Fig. 9b. For the ergodic trajectory in this figure, the nonresonant terms do not average to zero since the system is not close enough to the 2.5 resonance. The random trajectory, shown as the bold curve in Fig. 12, lies between the 2.5 and 3.5 resonances, and is not close enough to either resonance to justify keeping only one term in the sum. Furthermore, P_3 cannot remain constant in this case because of the large variations introduced by the $\frac{r}{2s} = \frac{3}{2}$ term. This term will also be rapidly varying with respect to the natural bounce-energy frequency of the system which is approximately 2.72; so that the net result is a random mixing of the three rapidly varying terms $\frac{r}{2s} = \frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$ leading to the ergodic behavior observed.

If we plot P_3, $\hat{\theta}_1$ phase diagrams for one of the relatively well behaved phase loops near the elliptic singularity of Fig. 9b, we find that although there is no single dominant slowly varying term, none of the terms introduces significant variation in P_3 as seen in Fig. 13. This result is to be expected, as the strength of the near resonant harmonic terms is proportional to Bessel functions depending on $J_1^{1/2}$. The amplitudes of the harmonics decrease rapidly with decreasing J_1.

If we compare the amplitude of the oscillation in P_3 as shown by Fig. 10 to the amplitude of the 2.5 island oscillation given by the bold curve in Fig. 11, we see that the observed variation in P_3 is a factor of 2 to 3 times larger than the value predicted by removing the higher order resonance. An underestimation of the variation of P_3 is to be expected since we have overestimated the nonlinearity by expanding the average Hamiltonian to only fourth order in Δp and Δw. The next term would be of opposite sign thus reducing the nonlinearity and making the amplitude of the predicted island oscillations larger. We note, however, that even if the phase loops in Fig. 11 were 2 to 3 times larger in amplitude, the variation in P_3 would still be only $\frac{1}{5}$ to $\frac{1}{3}$ of that necessary for marginal overlap between the 2.5 and 3.5 resonances. The actual overlap of neighboring island oscillations is not necessary for breakdown to occur. Rather, it is only necessary that the islands be close enough that a phase orbit between the two resonances is affected significantly by both terms as is the case in Fig. 12.

VI. CONCLUSIONS

For multidimensional oscillatory systems with widely differing periods, adiabatic invariants can be found that separate the degrees of freedom. If a low harmonic number resonance exists between two degrees of freedom a transformation may be employed to remove the resonance. Two cases must be distinguished: (1) Intrinsic degeneracy in which the frequencies of the oscillations are independent of the momenta in the absence of the terms which couple the two degrees of freedom. In this case the transformed degrees of freedom have frequencies that are separated by the strength of the coupling term ε. (2) Accidental degeneracy in which the oscillations depend on the mementa. In this case the transformed degrees of freedom have frequencies whose ratio is proportional to $\varepsilon^{1/2}$. In either case, if ε is small enough, adiabatic invariants exist in the transformed coordinates that separate the degrees of freedom. For increasingly large ε, harmonics of the slow oscillation resonate with the fast oscillation to perturb the invariants. The harmonic amplitudes are strongly dependent on the harmonic number, and therefore the lower harmonic resonances of an accidental degeneracy will become important at a lower value of ε than for an intrinsic degeneracy. The perturbation of the invariants leads to islands which have new invariants associated with them. The island invariant is obtained by removal of the island resonance. The procedure of removing resonances can be carried to higher order, by considering resonances between the island oscillations and the faster periods. The frequencies of the island oscillations are slower than the next faster frequency by $O(1/\varepsilon^{-1/2}!)^{1/2}$.

In practical cases the strength of the harmonic amplitudes contributing to the island formation decreases rapidly such that higher order resonances need not be considered. There is a rapid transition with increasing ε between the value at which significant islands appear and the somewhat larger value of ε at which the terms contributing to neighboring islands interact sufficiently strongly to destroy the invariants of the motion. Invariant destruction leads to apparent ergodic filling of the phase plane. From numerical computations the invariants are first destroyed in parts of the phase plane where the island interaction perturbs the invariants such that the amplitude of the oscillation in the invariant is comparable to but not necessarily as large as the amplitude necessary to have overlapping of harmonic resonances.

ACKNOWLEDGEMENT

The authors wish to acknowledge the many helpful conversations with Dr. Milos Seidl. The work is partly based on a Ph.D. thesis of E. F. Jaeger. A. J. Lichtenberg was a Miller Professor and E. F. Jaeger an

AEC Fellow and an NSF Fellow during part of this research. The work was also partially supported by NSF Grant GK-2978, AFOSR Grant AFOSR-69-1754, and by the University of California at Berkeley Computer Center.

REFERENCES

1. Kulsrud, R. M., Phys. Rev., 106, 205 (1957).
2. Gardner, G. S., Phys. Rev., 115, 791 (1959).
3. Lenard, A., Ann. Phys., 6, 261 (1959).
4. Chandrasekkar, S., Plasma Physics, Chap. 3, University of Chiacago Press, Chicago (1960).
5. Vandervoort, P. O., Ann. Phys., 12, 436 (1961).
6. Lichtenberg, A. J., Plasma Phys. (J.N.E. Pt. C), 6, 135 (1964).
7. Lewis, H. R. Jr., J. Math Phys., 9, 1976 (1968).
8. Symon, K. R., J. Math Phys., 11, 1320 (1970).
9. Krylov, N. and Bogoliubov, N. N., On Nonlinear Mechanics, Acad. Sci. USSR (1945).
10. Bogoliubov, N. N. and Mitropolsky, Y. A., Asymptotic Methods in the Theory of Nonlinear Oscillations (in translation), Gordon and Breach, New York (1961).
11. Kruskal, M., J. Math Phys., 3, 806 (1962).
12. McNamara, B., and Whiteman, K. J., J. Math Phys., 8, 2029 (1967).
13. Whiteman, K. J. and McNamara, B., J. Math Phys., 9, 1385 (1968).
14. Contopoulos, G., J. Math. Phys., 7, 788 (1966).
15. Hénon, M. and Heiles, C., Astr. J., 69, 73 (1964).
16. Dragt, A. J., Rev. Geophysics, 3, 255 (1965).
17. Lichtenberg, A. J., Phase Space Dynamics of Particles, John Wiley and Sons, Inc., New York (1969).
18. Chirikov, B. V., Plasma Physics (J.N.E. Pt. C), 1, 253 (1960).

19. Rosenbluth, M. N., Sagdeev, R. Z., Taylor, J. B., Zaslavsky, G. M., Nuclear Fusion, 6, 297 (1966).

20. Filonenko, N. N., Sagdeev, R. Z., and Zaslavskii, G. M. Nuclear Fusion, 7, 253 (1967).

21. Zaslavskii, B. M. and Filonenko, N. N., Soviet Physics JEPT, 29, 572 (1969).

22. Lichtenberg, A. J. and Jaeger, F., Phys. of Fluids, 13, 392 (1970).

23. Walker, G. H., and Ford, J., Phys. Rev. 188, 416 (1969).

24. Poincaré, H., Méthodes Nouvelles de la Méchanique Céleste, Dover Publications, Inc., New York (1892).

25. Born, M., The Mechanics of the Atom (in translation) G. Bell and Sons, Ltd., London (1927).

26. Kolmogorov, A. N., Dokl. Akad. Nauk. SSSR, 98, 527 (1954).

27. Arnol'd, V. I., Russian Math. Surveys, 18, 9 (1963).

28. Moser, J., Nachr. Acad. Wiss. Gottingen, II Math Physik Kl, 1 (1962).

29. Arnol'd, V. I., DAN, 142, 758 (1962).

30. Seidl, M., Plasma Physics (J.N.E. Pt. C), 6, 597 (1964).

31. Tuma, D. T. and Lichtenberg, A. J. , Plasma Physics (J.N.E. Pt. C), 9, 87 (1967).

Appendix

In this appendix, we expand the average Hamiltonian (5.7) about it's elliptic singularity and use 2nd order perturbation theory to transform to action-angle variables. Repeating (5.7)

$$\bar{H} = \omega_0 \left[(\gamma_0 - \frac{\omega}{\omega_0}) P_1 + \gamma_1 \sqrt{P_1}\, P_3 + \gamma_2 P_3^2 \right] + \varepsilon\, \omega_0 \frac{\sqrt{P_1}}{2} f(P_1, P_3) \sin w_1' \tag{A.1}$$

we write Hamilton's equations

$$\dot{P}_1 = -\frac{\partial \bar{H}}{\partial W_1'} = -\frac{1}{2}\, \varepsilon\, \omega_0 \sqrt{P_1}\, f(P_1, P_3) \cos w_1' \tag{A.2}$$

$$\dot{w}_1' = \frac{\partial \bar{H}}{\partial P_1} = \omega_0 \left[(\gamma_0 - \frac{\omega}{\omega_0}) + \frac{1}{2} \frac{\gamma_1 P_3}{\sqrt{P_1}} \right] + \frac{1}{2}\, \varepsilon\, \omega_0 (\sqrt{P_1} \frac{\partial f}{\partial P_1} + \frac{1}{2} \frac{f}{\sqrt{P_1}}) \sin w_1' . \tag{A.3}$$

Defining the elliptic singularity $\bar{P}_1$, $\bar{w}_1'$ as in (3.12) and using (A.2) and (A.3), we obtain

$$\dot{w}_1' = \frac{\pi}{2} \tag{A.4}$$

$$\left.\frac{\partial f}{\partial P_1}\right|_{\bar{P}_1} = \frac{-2}{\varepsilon \sqrt{\bar{P}_1}} \left[(\gamma_0 - \frac{\omega}{\omega_0}) + \frac{\gamma_1 P_3}{2\sqrt{\bar{P}_1}} \right] - \frac{f(\bar{P}_1, P_3)}{2\bar{P}_1} \tag{A.5}$$

where (A.5) must be solved numerically for $\bar{P}_1$. We expand the functions $f(P_1,P_3)$, $\sqrt{P_1}$, and $\sin w_1'$ about $\bar{P}_1,\bar{w}_1'$. Defining

$$\begin{aligned} \Delta P &= P_1 - \bar{P}_1 \\ \Delta w &= w_1' - \bar{w}_1' \end{aligned} \tag{A.6}$$

we obtain

$$\bar{H} = \omega_0 \left[(\gamma_0 - \frac{\omega}{\omega_0})\bar{P}_1 + \gamma_1 \sqrt{\bar{P}_1}\, P_3 + \gamma_2 P_3^{\,2} \right] + F(P_3)$$

$$- G(P_3)\,\frac{(\Delta P)^2}{2} - F(P_3)\,\frac{(\Delta w)^2}{2} + A(P_3)\,\frac{(\Delta P)^3}{3} + B(P_3)\,\frac{\Delta P(\Delta w)^2}{3} + I(P_3)\,\frac{(\Delta P)^4}{4}$$

$$+ D(P_3)\,\frac{(\Delta P)^2(\Delta w)^2}{4} + E(P_3)\,\frac{(\Delta w)^4}{4} \tag{A.7}$$

where

$$F = \frac{1}{2}\,\omega_0\, \varepsilon\, \sqrt{\bar{P}_1}\; f(\bar{P}_1,P_3)$$

$$G = \frac{\omega_0\, \varepsilon\, \sqrt{\bar{P}_1}}{2} \left(\frac{\dfrac{3\gamma_1 P_3}{\varepsilon} + \dfrac{3f(\bar{P}_1,P_3)}{2}}{2\bar{P}_1^{\,2}} + \frac{2(\gamma_0 - \frac{\omega}{\omega_0})}{\varepsilon\, \bar{P}_1^{\,3/2}} - \left.\frac{\partial^2 f}{\partial P_1^{\,2}}\right|_{\bar{P}_1} \right)$$

$$A = \frac{\omega_0\, \varepsilon}{4\bar{P}_1^{\,1/2}} \left(\frac{\dfrac{3\gamma_1 P_3}{\varepsilon} + \dfrac{3f(\bar{P}_1,P_3)}{2}}{2\bar{P}_1^{\,2}} + \bar{P}_1 \left.\frac{\partial^3 f}{\partial P_1^{\,3}}\right|_{\bar{P}_1} + \frac{3}{2} \left.\frac{\partial^2 f}{\partial P_1^{\,2}}\right|_{\bar{P}_1} + \frac{3}{2}\, \frac{(\gamma_0 - \frac{\omega}{\omega_0})}{\varepsilon\, \bar{P}_1^{\,3/2}} \right)$$

$$B = \frac{\omega_0 \; \varepsilon \; \sqrt{\bar{P}_1}}{2} \left(\frac{3(\gamma_0 - \frac{\omega}{\omega_0})}{\varepsilon \;\; \bar{P}_1} + \frac{3\gamma_1 P_3}{2 \; \varepsilon \; \bar{P}_1} \right)$$

$$I = \frac{-3\omega_0 \; \varepsilon}{16\bar{P}_1^{\;3/2}} \left(\frac{\frac{3\gamma_1 P_3}{\varepsilon} + \frac{3f(\bar{P}_1,P_3)}{2}}{2\bar{P}_1^{\;2}} + \frac{4(\gamma_0 - \frac{\omega}{\omega_0})}{3 \; \varepsilon \; \bar{P}_1^{\;3/2}} + \frac{2}{3} \frac{\partial^2 f}{\partial P_1^{\;2}}\bigg|_{\bar{P}_1} - \frac{8}{9} \bar{P}_1 \frac{\partial^3 f}{\partial P_1^{\;3}}\bigg|_{\bar{P}_1} - \frac{4}{9} \bar{P}_1^{\;2} \frac{\partial^4 f}{\partial P_1^{\;4}}\bigg|_{\bar{P}_1} \right)$$

$$D = \frac{\omega_0 \; \varepsilon \; \sqrt{\bar{P}_1}}{2} \left(\frac{\frac{2\gamma_1 P_3}{\varepsilon} + \frac{3f(\bar{P}_1,P_3)}{2}}{2\bar{P}_1^{\;2}} + \frac{2(\gamma_0 - \frac{\omega}{\omega_0})}{\varepsilon \; \bar{P}_1^{\;3/2}} - \frac{\partial^2 f}{\partial P_1^{\;2}}\bigg|_{\bar{P}_1} \right)$$

$$E = \frac{\omega_0 \; \varepsilon \; \sqrt{\bar{P}_1}}{2} \frac{f(\bar{P}_1,P_3)}{6} \quad .$$

Since the first two terms on the left of (A.7) do not depend on the variables P_1, w_1', they can be combined with H in determining the ΔP - Δw motion with P_2 and P_3 constant. The Hamilton-Jacobi equation for Hamiltonian (A.7) becomes

$$-G(P_3)\frac{(\Delta P)^2}{2} - F(P_3)\frac{(\Delta w)^2}{2} + A(P_3)\frac{(\Delta P)^3}{3} + B(P_3)\frac{\Delta P(\Delta w)^2}{3} + I(P_3)\frac{(\Delta P)^4}{4} + D(P_3)\frac{(\Delta P)^2(\Delta w)^2}{4} + E(P_3)\frac{(\Delta w)^4}{4} = K(J_1,P_2,P_3). \tag{A.8}$$

We can now transform to action-angle variables J_1^0, θ_1^0 for the linear problem by applying the transformation (3.25) so that equation (A.8) becomes

$$\overline{H}_0(J_1^0) + \delta\overline{H}_1(J_1^0,\theta_1^0) + \delta^2\overline{H}_2(J_1^0,\theta_1^0) = K(J_1,P_2,P_3) \tag{A.9}$$

where

$$\overline{H}_0(J_1^0) = -\Omega^0 J_1^0$$

$$\overline{H}_1(J_1^0,\theta_1^0) = \lambda\left\{\Omega^0 J_1{}^0\ \frac{2}{3}(A\cos^3\theta_1^0 + \frac{B}{R^2}\cos\theta_1^0\sin^2\theta_1^0)\right\}$$

$$\overline{H}_2(J_1^0,\theta_1^0) = \lambda^2\left\{\Omega^0 J_1^0\ \frac{G}{2}(I\cos^4\theta_1{}^0 + \frac{D}{R^2}\cos^2\theta_1^0\sin^2\theta_1^0 + \frac{E}{R^4}\sin^4\theta_1^0)\right\}$$

$$R = (F/G)^{1/2},\ \Omega^0 = (FG)^{1/2},\ \lambda = (2J_1R)^{3/2}/2\Omega^0 J_1$$

and δ is an artificial constant measure of smallness used because $\lambda_{max} \propto \varepsilon^{1/2}$ in the accidental case. Using (3.27), we solve for S_1, to obtain

$$S_1 = \frac{2}{9} J_1\lambda \sin\theta_1\left[A\ (\cos^2\theta_1^0 + 2) + \frac{B}{R^2}\ (\sin^2\theta_1^0)\right], \tag{A.10}$$

and to find the additional term in the energy, we use S_1 from (A.10) and (3.29) to obtain

$$K_2 = \Omega^0\ J_1\ \lambda^2\ M(P_3) \tag{A.11}$$

where

$$M(P_3) = \frac{3}{16} GI + \frac{1}{16}\frac{GD}{R^2} + \frac{3}{16}\frac{GE}{R^4} + \frac{5}{24} A^2 + \frac{1}{12}\frac{AB}{R^2} + \frac{1}{24}\frac{B^2}{R^4}.$$

The average Hamiltonian $\bar{H}$ in action-angle variables J_1, θ_1 valid to λ^2 is

$$\bar{H}(J_1,P_2,P_3) = \omega_0 \left[(\gamma_0 - \frac{\omega}{\omega_0})\bar{P}_1 + \gamma_1 \sqrt{\bar{P}_1}\, P_3 + \gamma_2 P_3^2 \right] + F(P_3)$$

$$- \Omega^0 J_1 (1 - \lambda^2 M) \tag{A.12}$$

and the frequency of the energy oscillation, $\dot{\theta}_1$ is

$$\dot{\theta}_1 = \frac{\partial \bar{H}}{\partial J_1} = - \Omega^0(P_3) \left[1 - 2\, \lambda^2 (J_1,P_3) M(P_3) \right] . \tag{A.13}$$

The nonlinearity in the energy oscillation prevents a secular increase in the action.

Finally, substituting (A.11) into (3.28) gives the second order term in $S(J_1,\theta_1^0)$:

$$S_2 = J_1 \lambda^2 \Big\{ \frac{G}{8} \sin\theta_1^0 \cos\theta_1^0 \Big[I(\cos^2\theta_1^0 + \frac{3}{2}) + \frac{D}{R^2}(-\cos^2\theta_1^0 + \frac{1}{2})$$

$$+ \frac{E}{R^4}(-\sin^2\theta_1^0 - \frac{3}{2}) \Big] + \frac{2}{3} \sin\theta_1^0 \cos\theta_1^0 \Big[A^2 (\frac{1}{6}\cos^4\theta_1^0$$

$$+ \frac{5}{24}\cos^2\theta_1^0 + \frac{5}{16}) + \frac{AB}{R^2}(\frac{1}{3}\cos^2\theta_1^0 \sin^2\theta_1^0$$

$$- \frac{1}{4}\cos^2\theta_1^0 + \frac{1}{8}) + \frac{B^2}{R^4}(\frac{1}{6}\sin^4\theta_1^0 - \frac{1}{24}\sin^2\theta_1^0 - \frac{1}{16}) \Big] \Big\} \tag{A.14}$$

from which we obtain the transformation from variables ΔP, Δw to action-angle variables J_1, θ_1 correct to second order in δ. As a check of this transformation, it can be substituted into equation (A.8) to verify that this equation is satisfied.

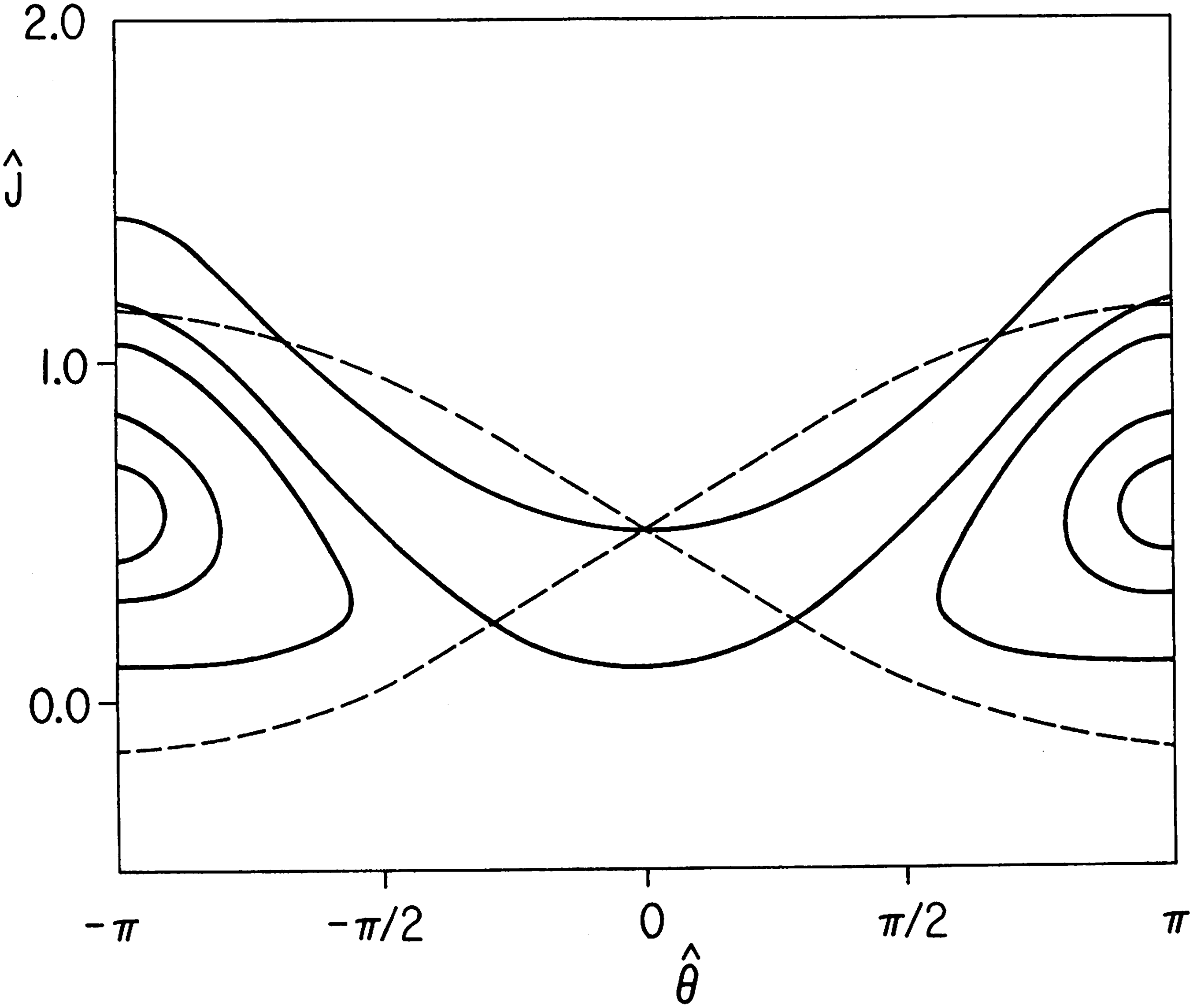

Fig. 1. Phase space of the oscillation in the adiabatic invariant due to resonance between the frequency of a one dimensional oscillator and one harmonic of the time varying frequency. The solid lines are plotted from (2.21) and the dashed lines from (2.23) with the constant chosen at the separatrix.

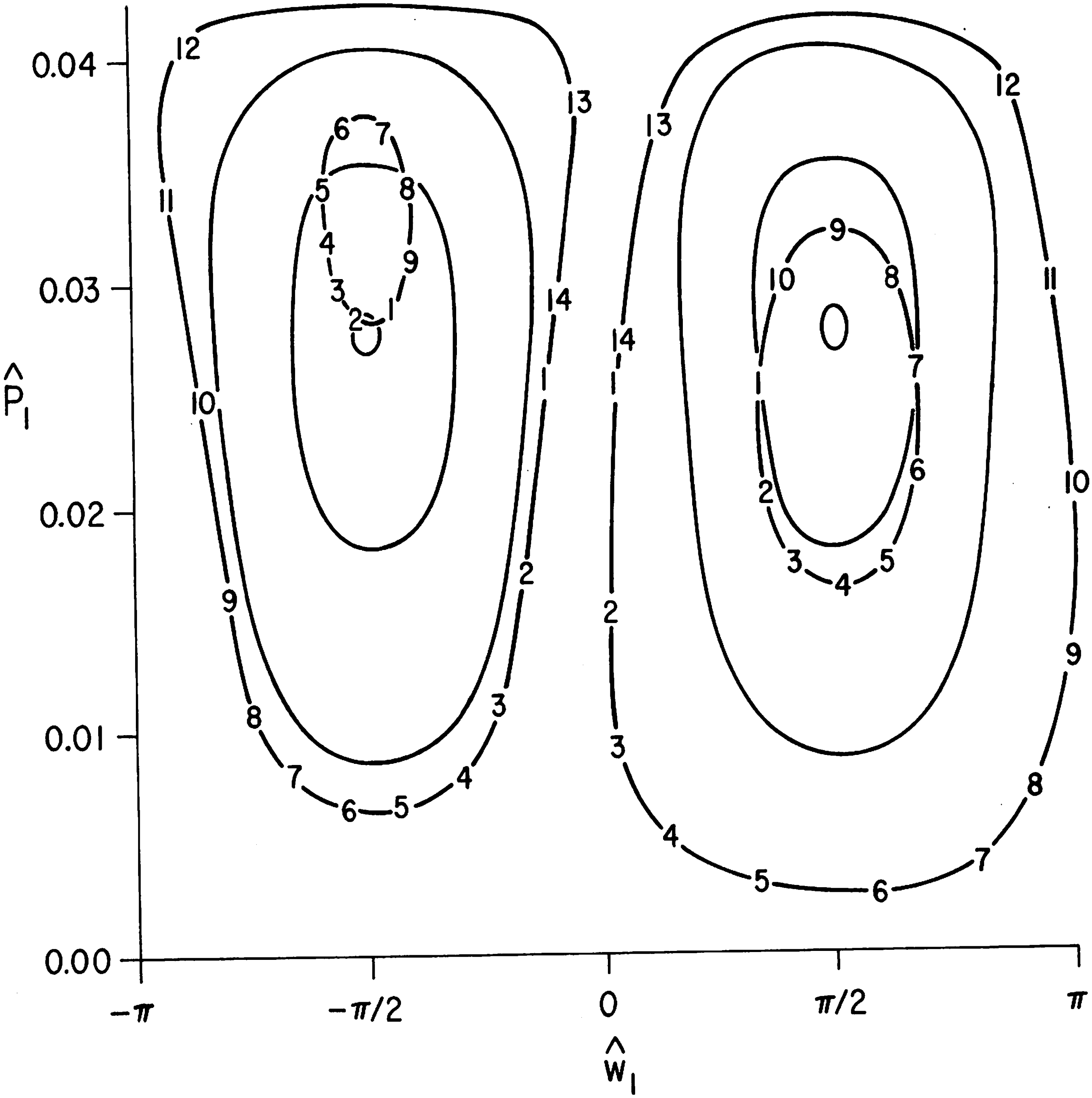

Fig. 2. Phase trajectories resulting from removal of an intrinsic degeneracy in a two dimensional coupled oscillator. The dark lines show the Hamiltonian curves before averaging over the faster of the two oscillations and the light lines show the Hamiltonian curves after averaging.

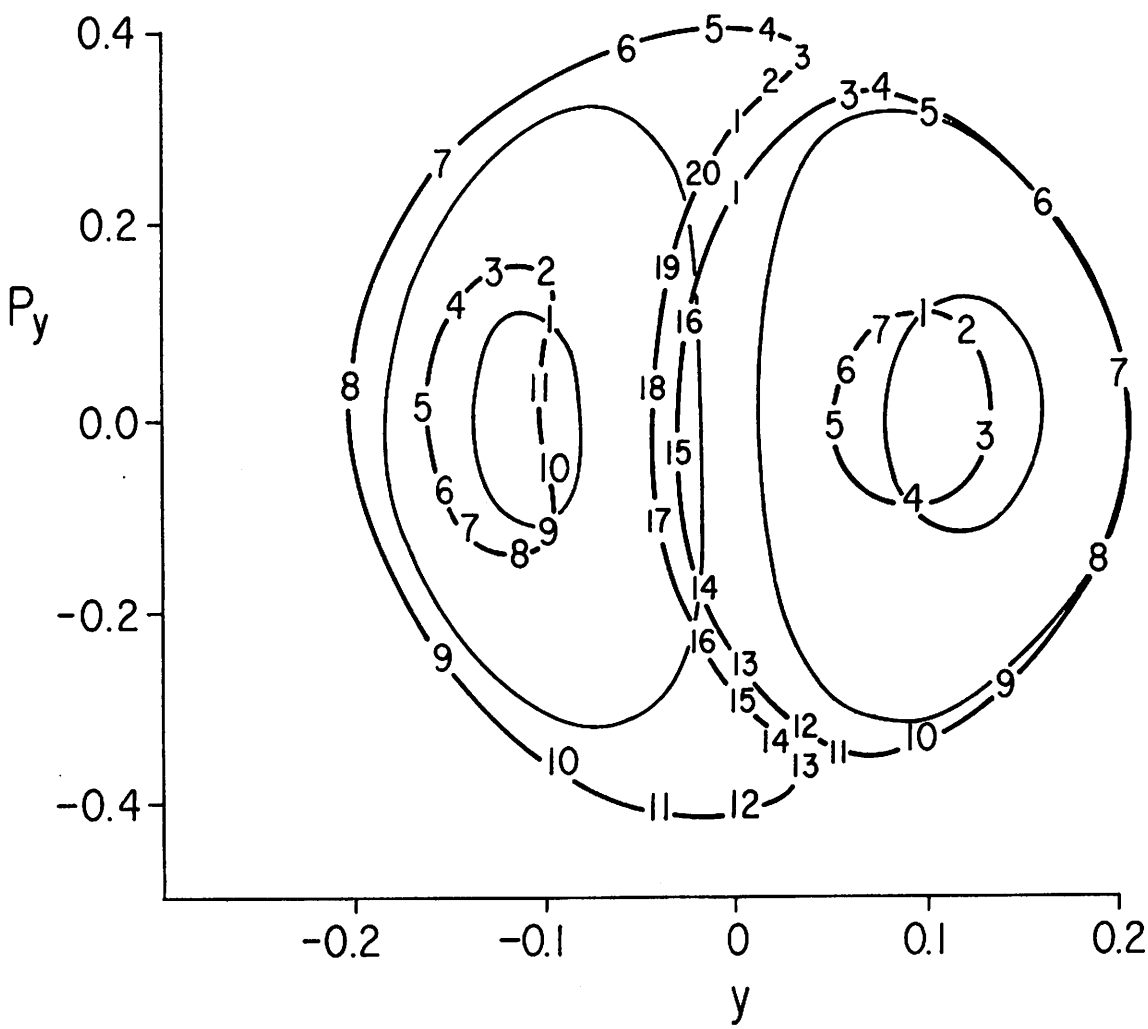

Fig. 3. Phase trajectories corresponding to those in Fig. 2, but plotted in the phase plane of the y oscillation.

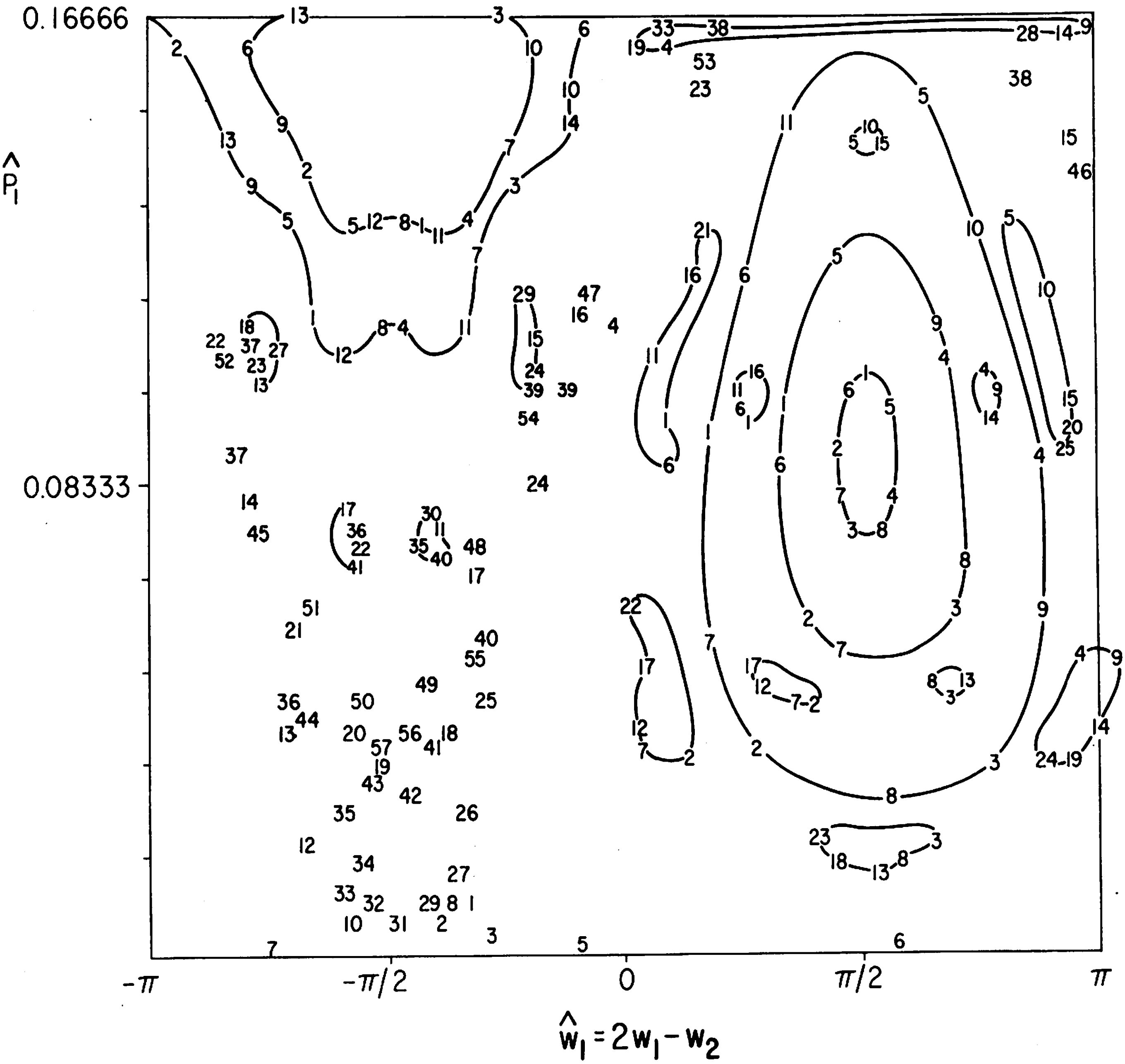

Fig. 4. Breakup of the Hamiltonian curves before averaging due to higher order resonances when the energy is increased.

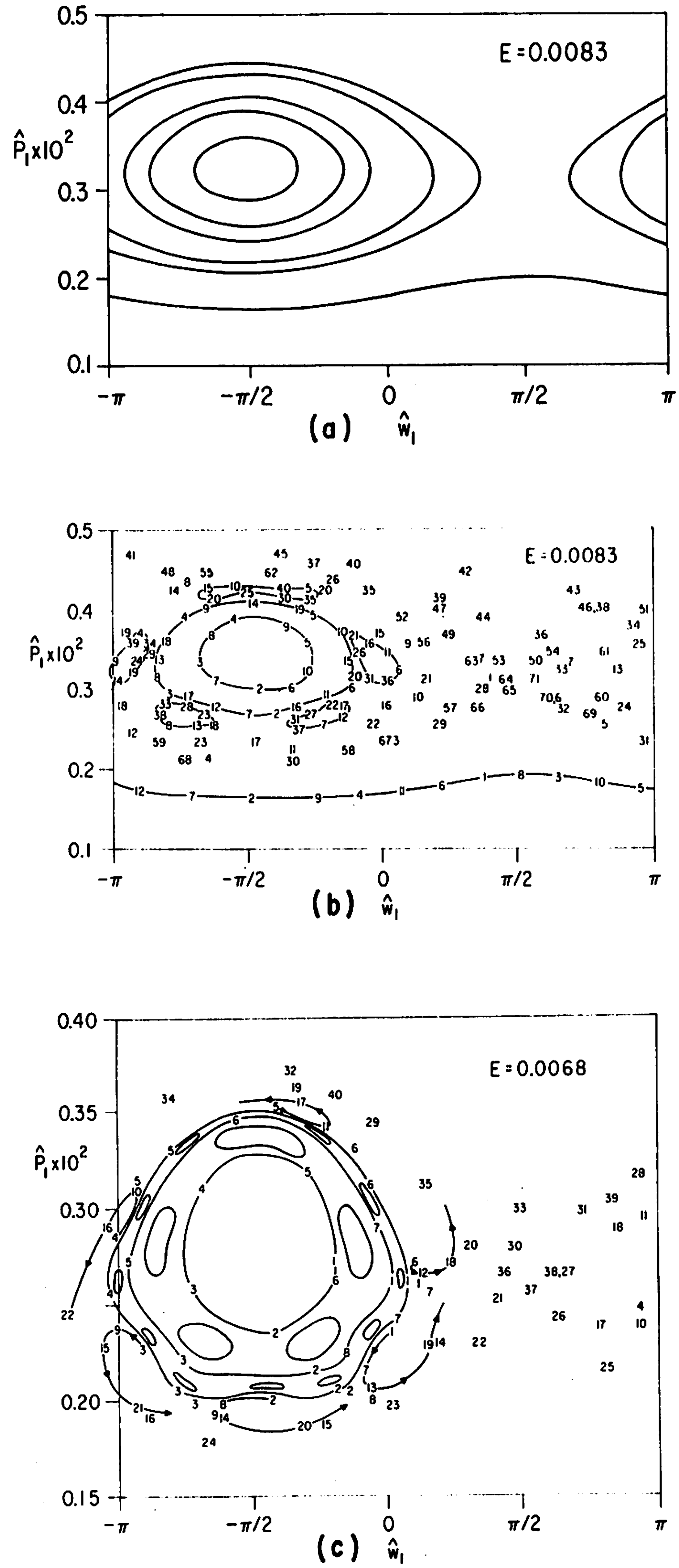

Fig. 5. (a) Averaged Hamiltonian curves resulting from removal of an accidental degeneracy in a two dimensional coupled oscillator. (b) Hamiltonian curves before averaging for parameters corresponding to those in (a). (c) Hamiltonian curves before averaging for an energy slightly lower than that in (a) and (b).

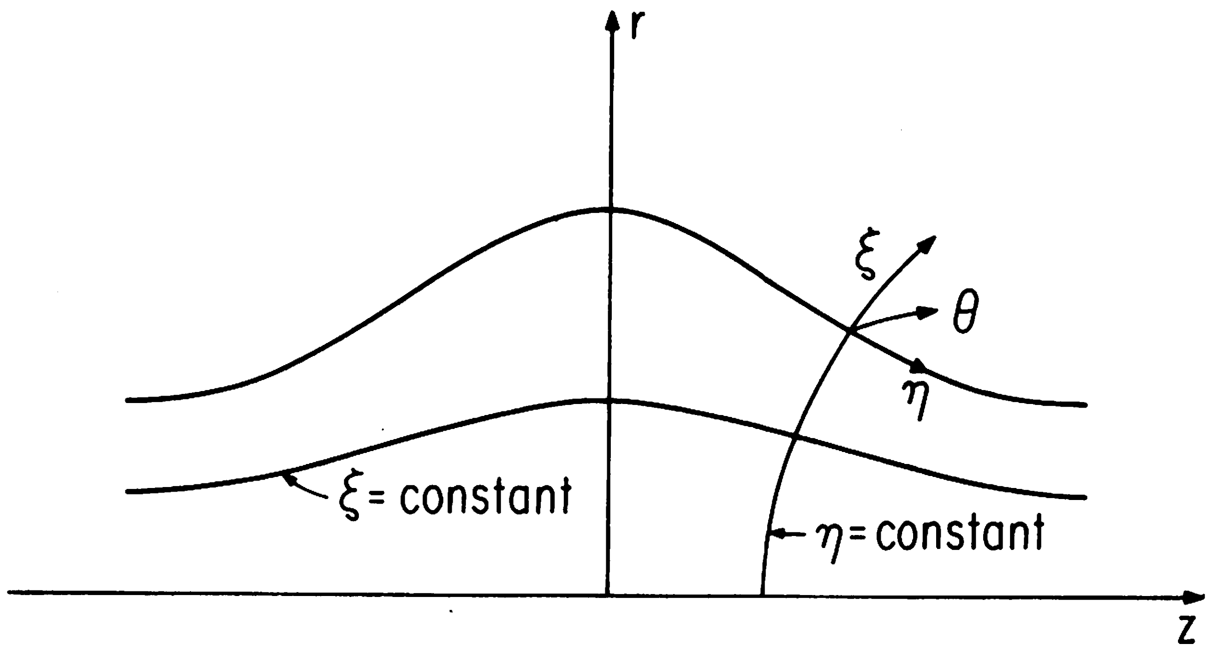

Fig. 6. The ξ, η, θ coordinates.

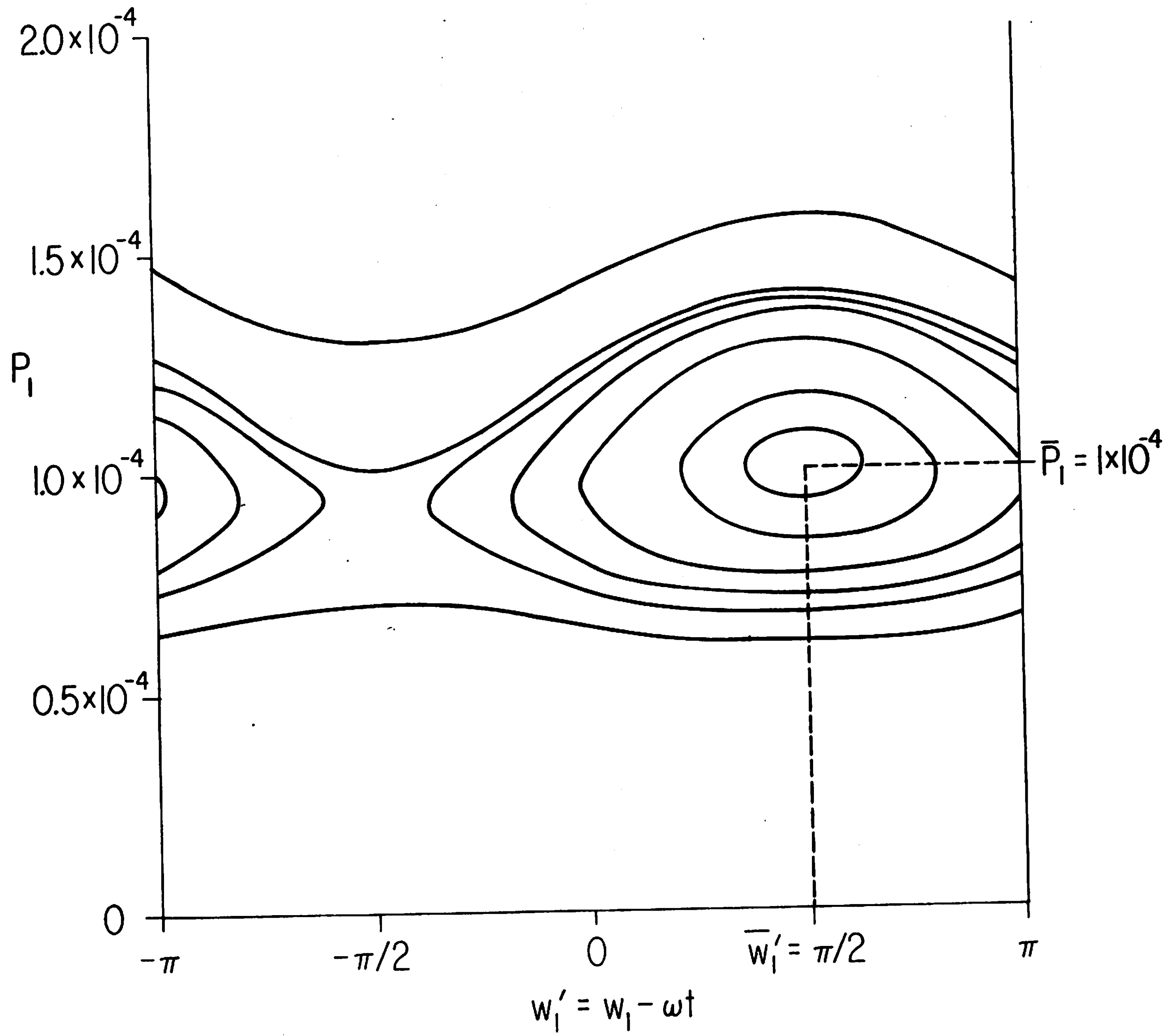

Fig. 7. (a) Hamiltonian curves for electron cyclotron heating after averaging over the bounce period.

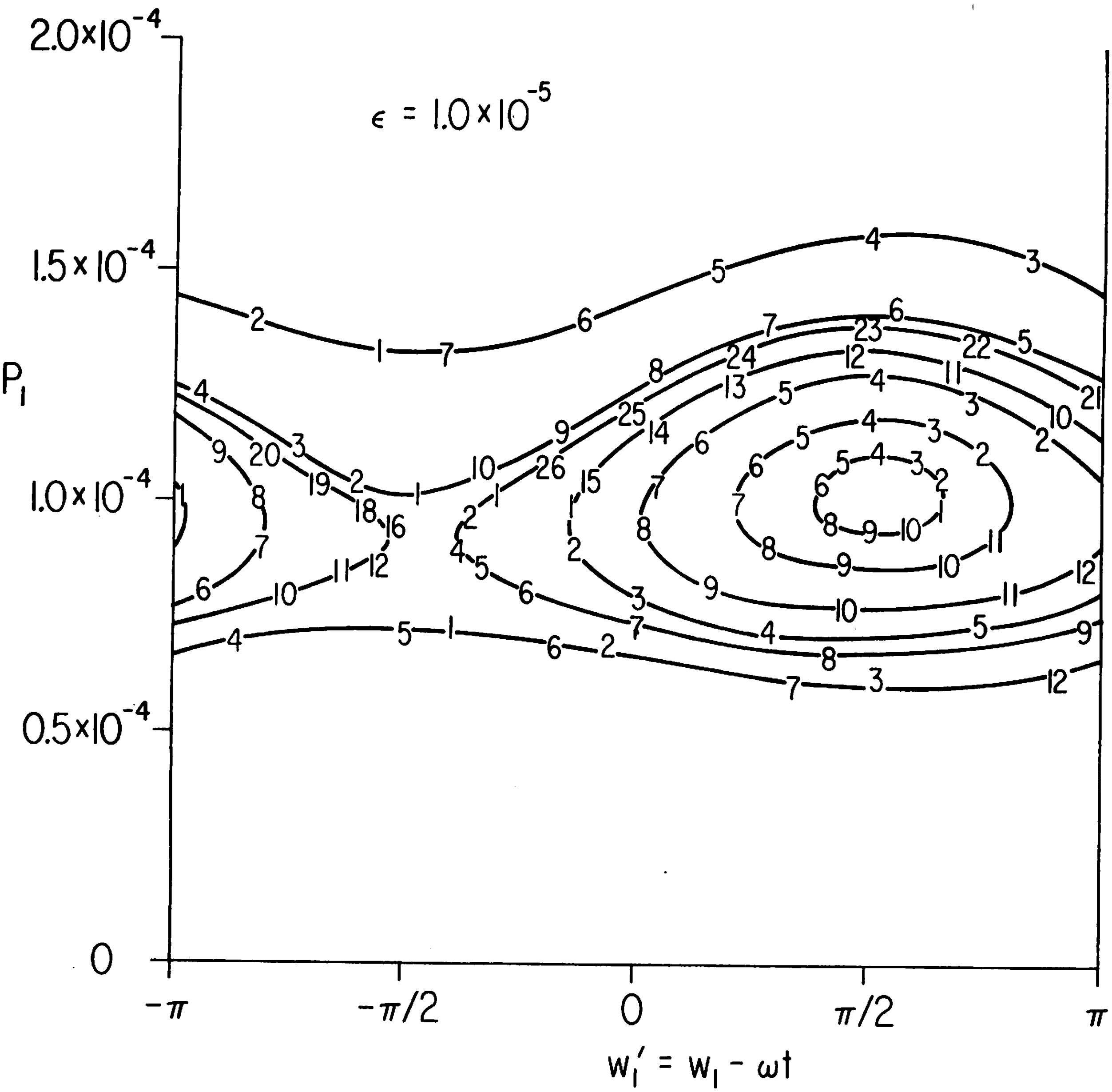

Fig. 7. (b) Hamiltonian curves before averaging over the bounce period.

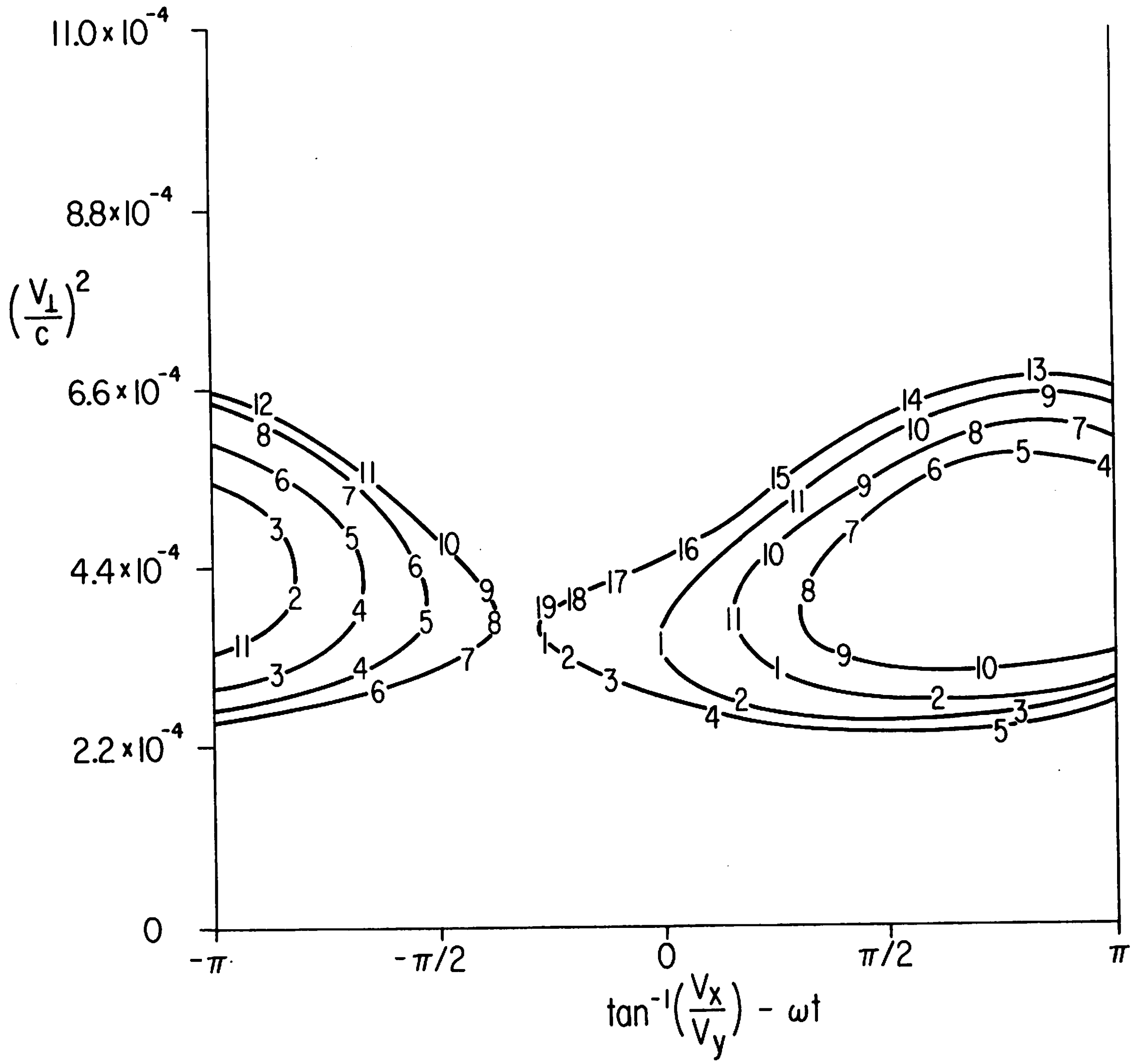

Fig. 7. (c) Phase plot of the exact equations of motion.

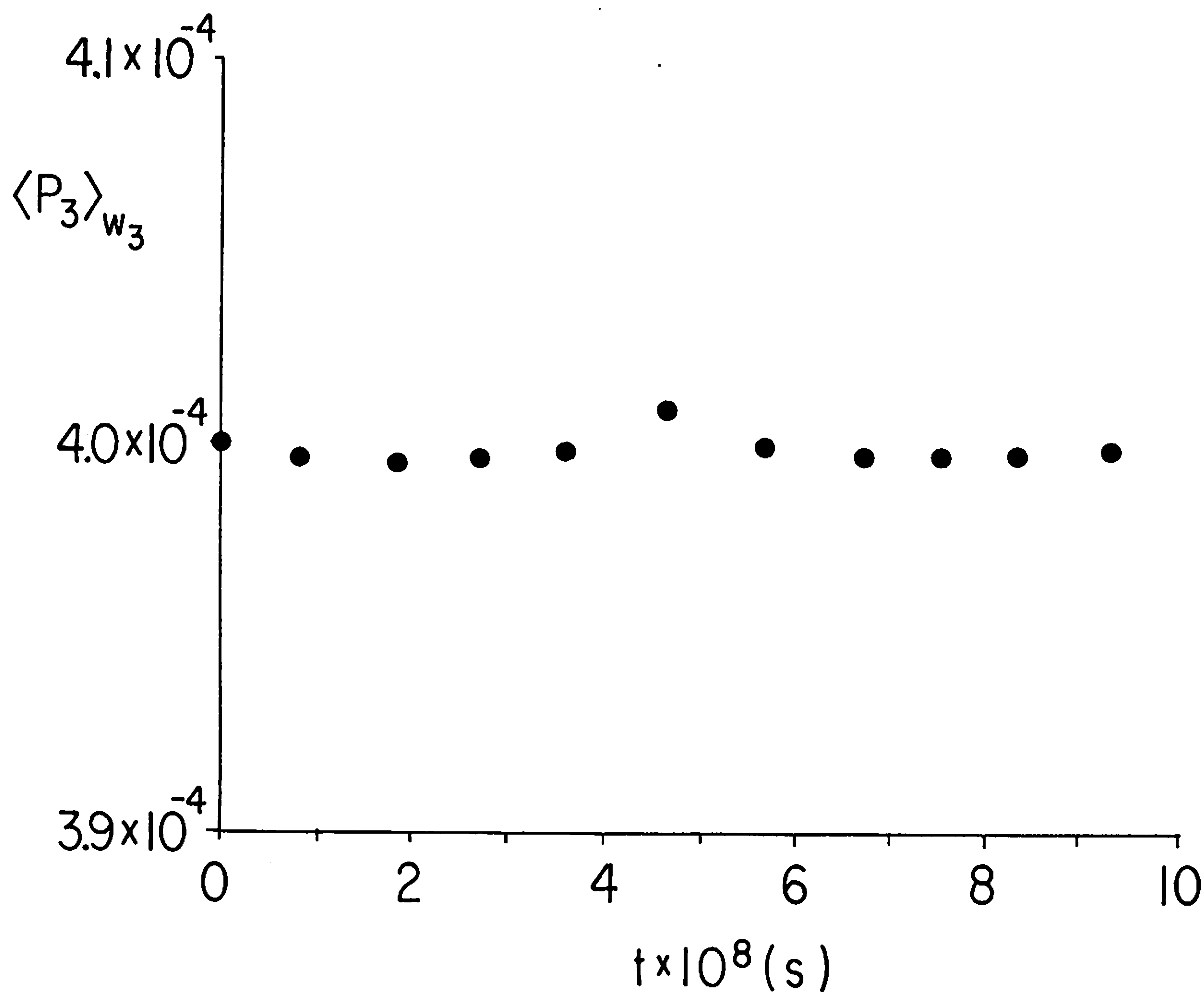

Fig. 8. Plot of the adiabatic invariant of the bounce motion.

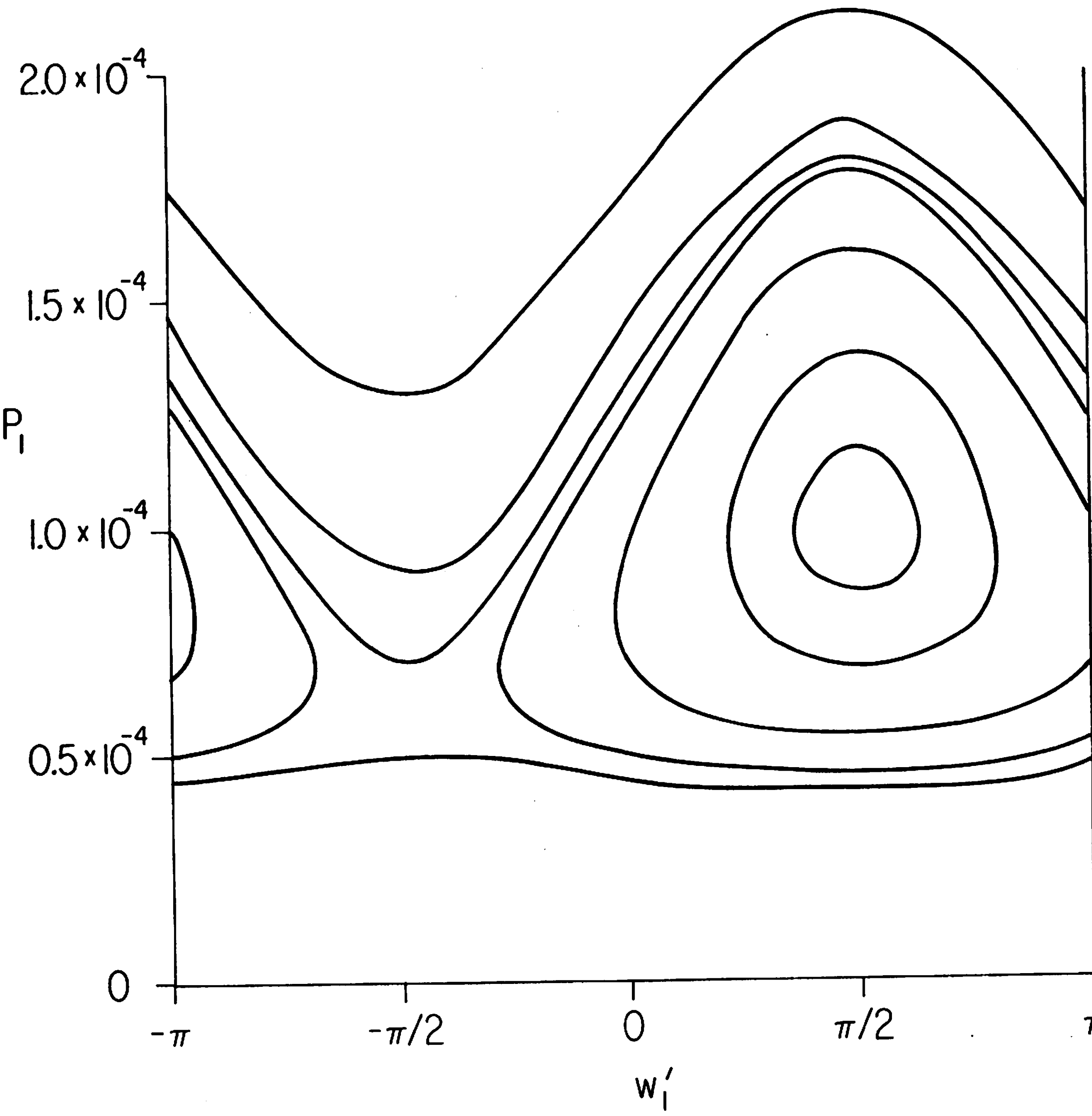

Fig. 9. (a) Hamiltonian curves after averaging for a large perturbing electric field.

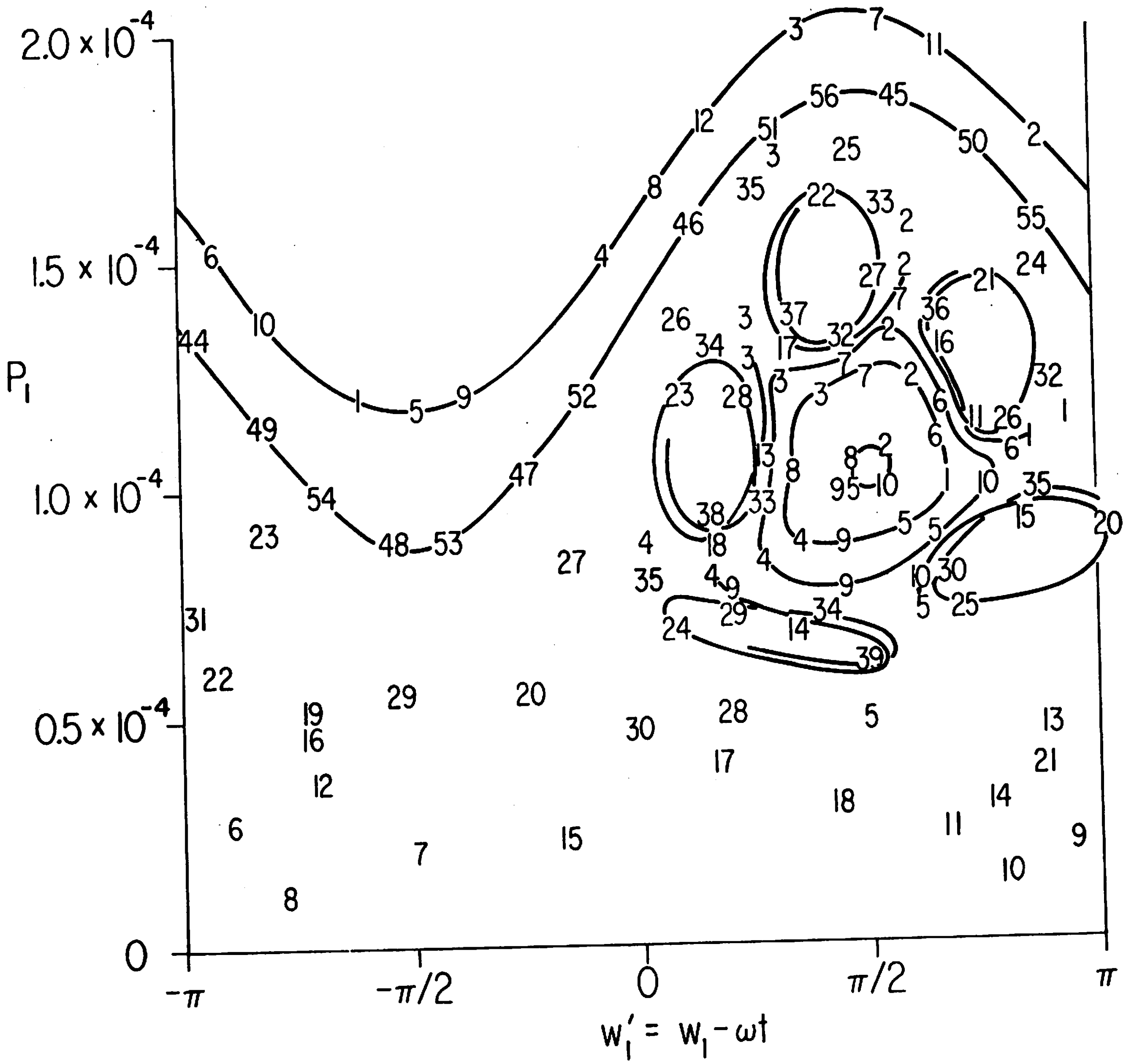

Fig. 9. (b) Hamiltonian curves before averaging for a large perturbing electric field.

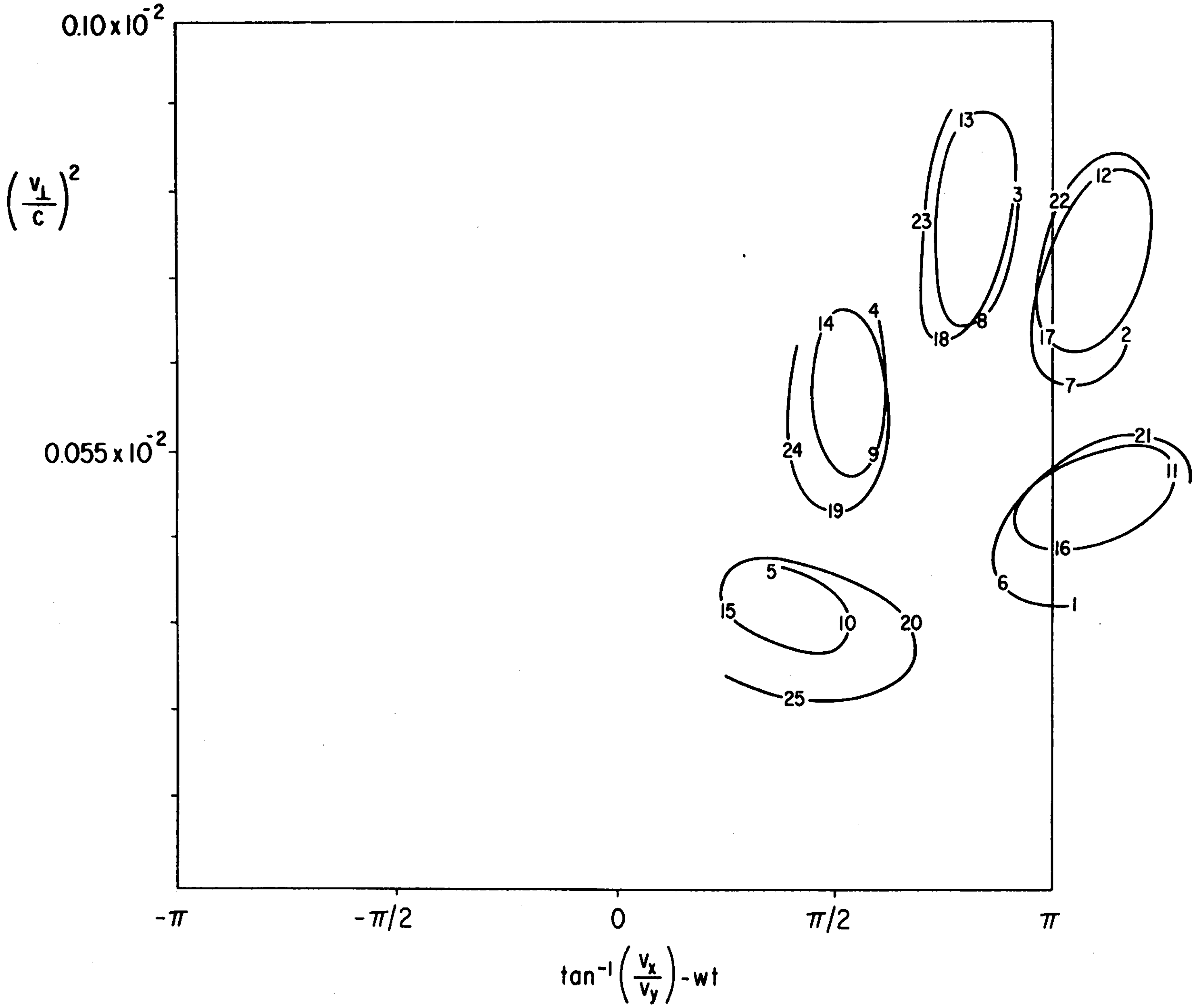

Fig. 9. (c) Phase plot of the exact equations of motion showing only the five-island trajectory for a large perturbing electric field.

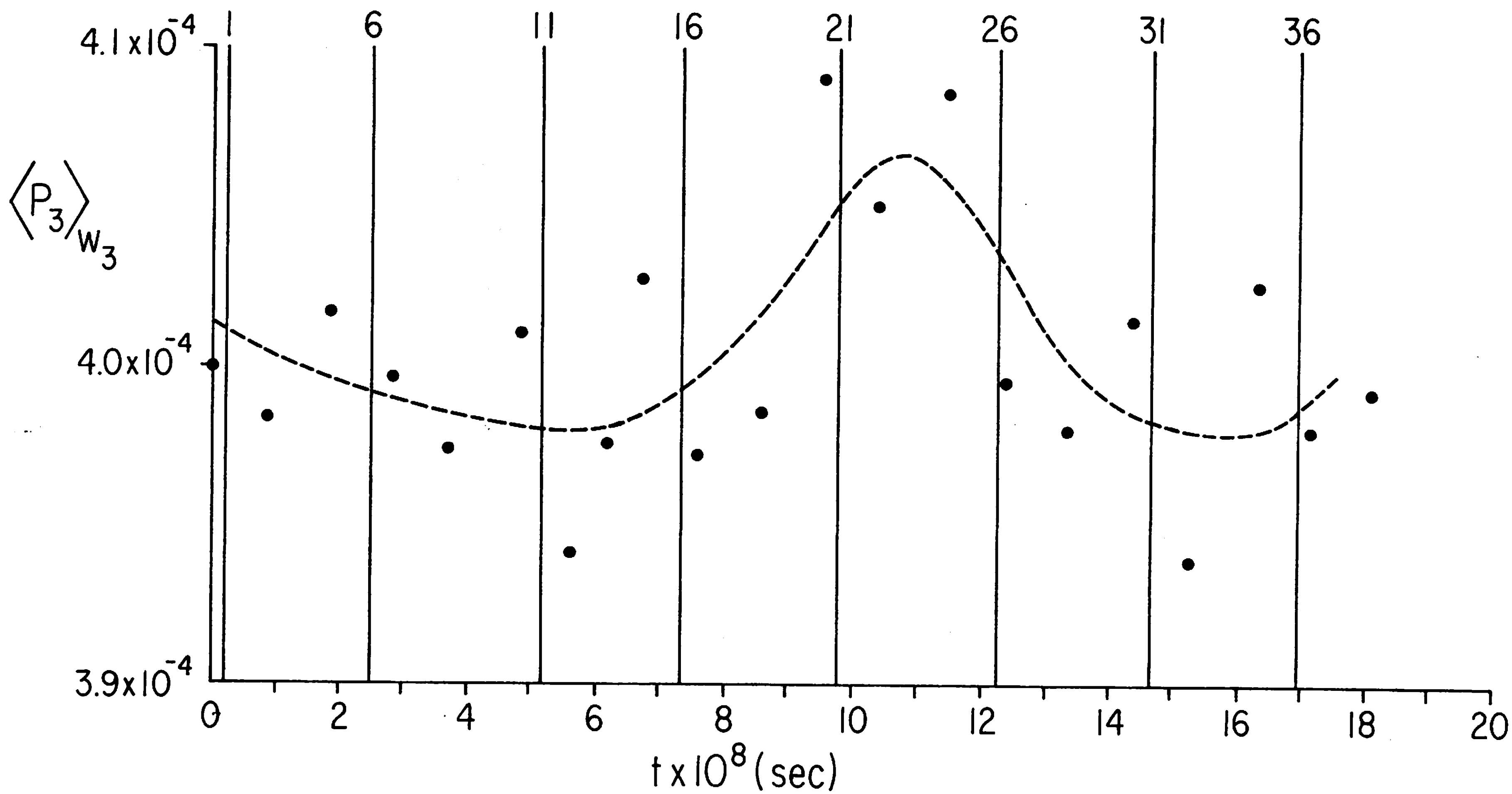

Fig. 10. Adiabatic bounce invariant for a large perturbing electric field.

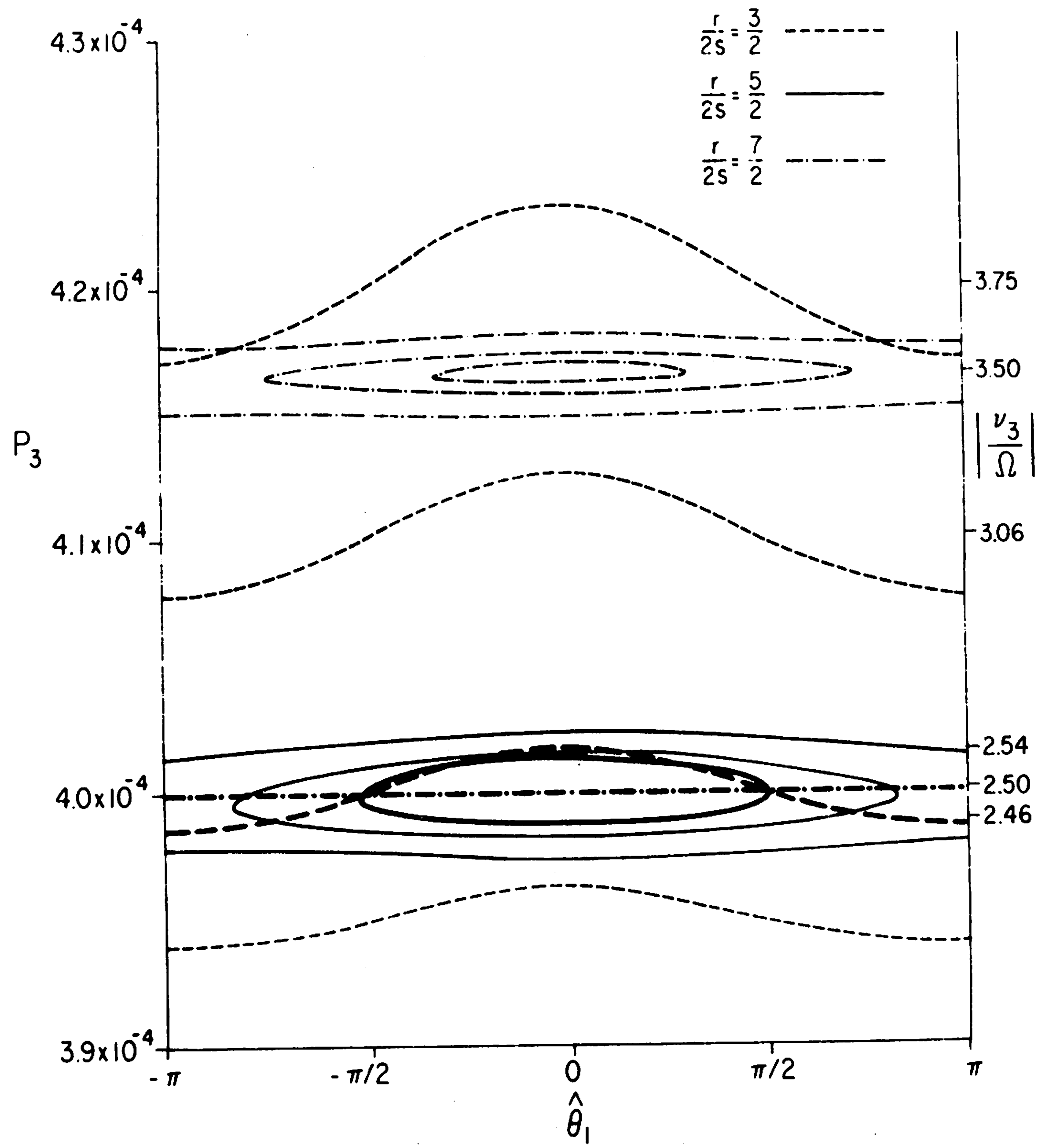

Fig. 11. Phase space of the oscillation of the longitudinal invariant due to resonance corresponding to the islands of Fig. 9b.

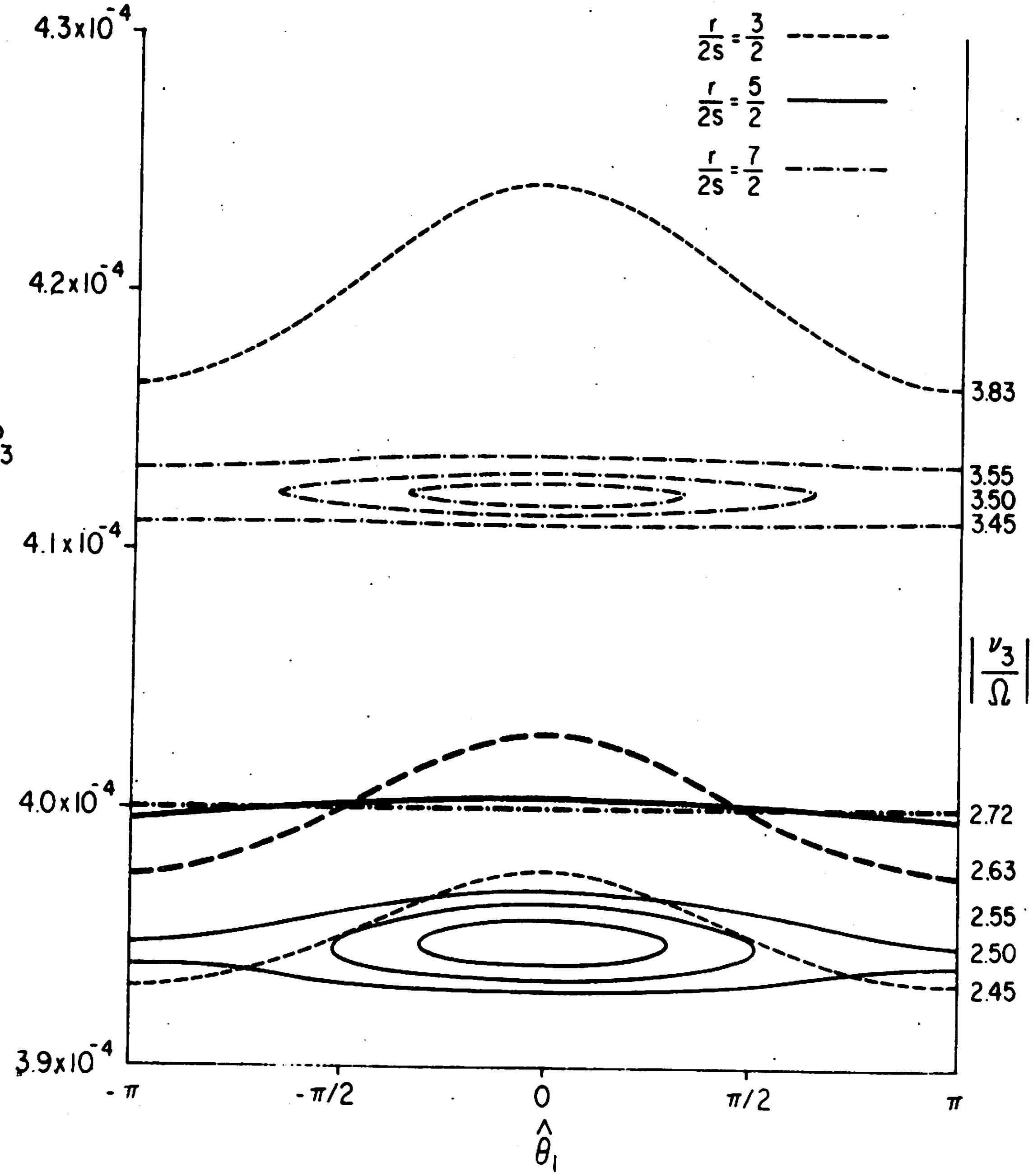

Fig. 12. Phase space of the longitudinal invariant corresponding to random motion in Fig. 9b.

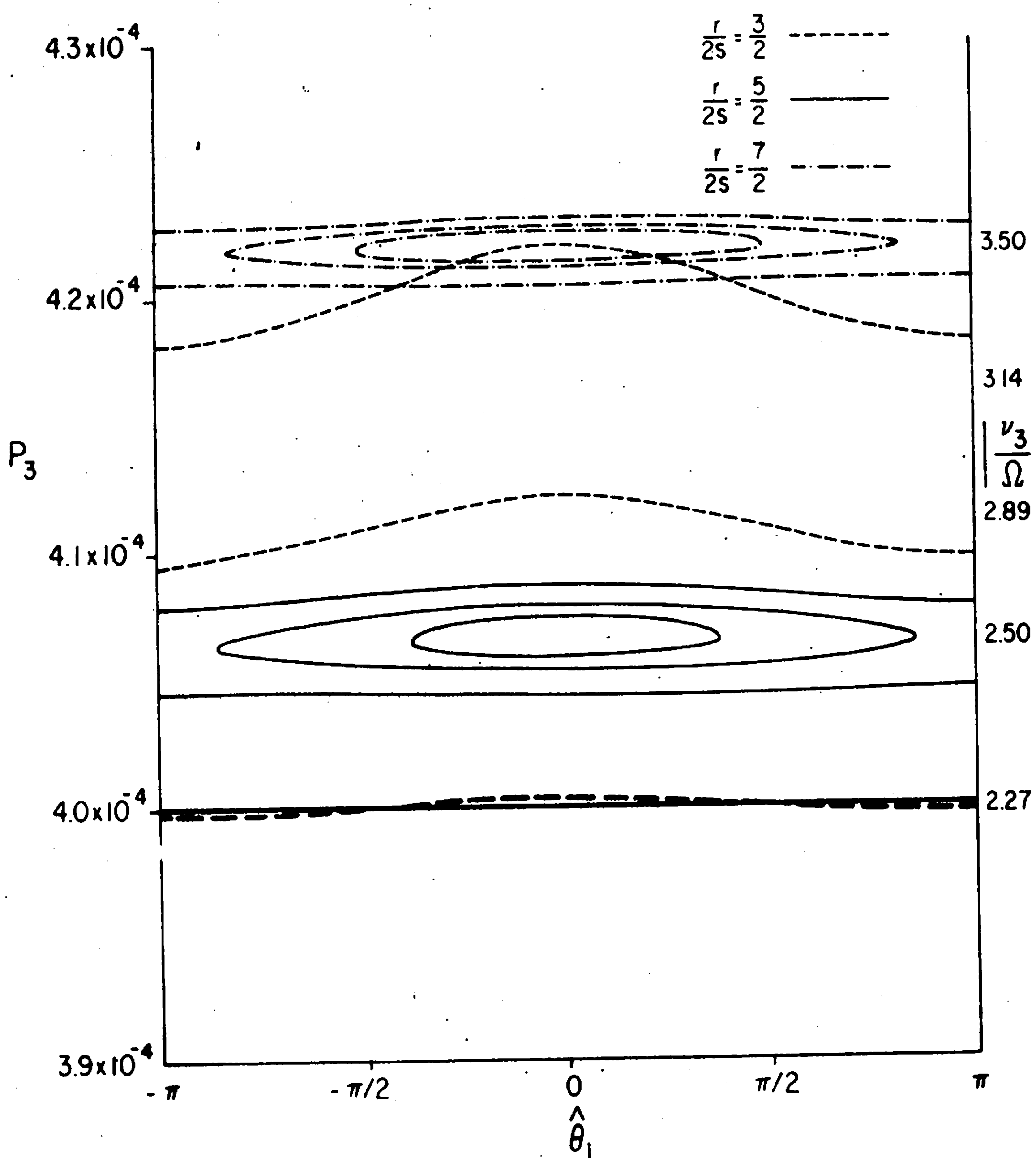

Fig. 13. Phase space of the longitudinal invariant corresponding to the well behaved curves near the elliptic singularity of Fig. 9b.