AN ADAPTIVE ALGORITHM FOR UNCONSTRAINED OPTIMIZATION WITH AN APPLICATION TO OPTIMAL CONTROL

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An Adaptive Algorithm for Unconstrained Optimization with an Application to Optimal Control

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Abstract: There are many unconstrained minimization problems in which the calculation, with high precision, of values and derivatives of the objective function can be extremely expensive. This paper presents an efficient gradient algorithm which uses function and gradient approximations which are improved adaptively. It is then shown how, in conjunction with certain numerical integration techniques, this algorithm can be used to solve unconstrained optimal control problems.

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1. Introduction.

The algorithm to be presented in Section 3 of this paper was developed with unconstrained min max and unconstrained optimal control problems in mind. It may therefore be helpful to review one of these problems so as to highlight the peculiarities which make an adaptive algorithm highly desirable. Thus, consider the problem

1.1 $$\min g(x(T,u))$$

subject to

1.2 $$\frac{d}{dt} x(t,u) = f(x(t,u),u(t),t),\ t \in [0,T],\ x(0,u) = x_0,\quad u \in L_\infty^m [0,T]$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Under suitable assumptions, to be stated later, the function $\phi(u(\cdot)) \triangleq g(x(T,u(\cdot)))$, from $L_\infty^m [0,T]$ into $\mathbb{R}^1$, is continuously differentiable, and its gradient at $u(\cdot)$ is given by

1.3 $$\text{grad } \phi(u)(t) = \left[\frac{\partial f(x(t,u),u(t),t)}{\partial u}\right]^T p(t,u),\ 0 \le t \le T,$$

where $p(\cdot,u)$ is computed from

1.4 $$\frac{d}{dt} p(t,u) = -\left[\frac{\partial f(x(t,u),u(t),t)}{\partial x}\right]^T p(t,u),\ 0 \le t \le T,\quad p(T) = \nabla g(x(T,u)).$$

Because of this, it is natural to suggest that problem (1.1)-(1.2) be solved by the following gradient method.

1.5 <u>Algorithm</u>:

<u>Step 0</u>: Select a $u_0(\cdot) \in L_\infty^m [0,T]$ and set $i = 0$.

<u>Step 1</u>: Compute $x(t,u_i)$ for $0 \leq t \leq T$ by solving (1.2) for $u = u_i$, (forwards).

<u>Step 2</u>: Compute $p(t,u_i)$, for $0 \leq t \leq T$, by solving (1.4)(backwards) with $p(T,u_i) = \nabla g(x(T,u_i))$.

<u>Step 3</u>: Find the smallest integer $j \geq 0$ such that

1.6 $$\phi(u_i-(\tfrac{1}{2})^j \text{grad } \phi(u_i)) - \phi(u_i) + (\tfrac{1}{2})^{j+1} \int_0^T \|\text{grad } \phi(u_i)(t)\|^2 dt \leq 0,$$

where $\|\cdot\|$ denotes the euclidean norm.

<u>Step 4</u>: Set $u_{i+1}(\cdot) = u_i(\cdot) - (\tfrac{1}{2})^j$ grad $\phi(u_i)(\cdot)$.

<u>Step 5</u>: set $i = i+1$ and go to Step 1. $\square$

It is quite clear that in this algorithm, as well as in any other algorithm which computes $\phi(u_i)$ and grad $\phi(u_i)(\cdot)$ at each iteration, the major computational time is taken up with integration of differential equations. Consequently, in such an algorithm, the target of an adaptive feature should be the numerical integration procedure. Qualitatively speaking, one should aim at a stable approximation scheme which is quite coarse (and hence fast) when $u_i(\cdot)$ is far from being optimal and which becomes progressively refined as an optimal solution is approached. As we shall see, the proposed algorithm does just that, and experimental results indicate that it can be from 3-10 times faster than comparable schemes using a fixed integration procedure.

The adaptive approximation scheme which we shall describe has evolved from the work on algorithm models and methods of implementing conceptual algorithms described in [6] and [7]. The model approach to algorithms is very powerful and hence we shall also adopt it in this paper. Because of this, we shall present our work in three stages, each more specific and more complex than the preceding: a general model and convergence theorem in Section 2, a general adaptive gradient method in Section 3, and an adaptive gradient method for unconstrained optimal control in Section 4.

2. An Abstract Prototype.

Let $\mathcal{Z}$ be a normed linear space (with norm $\|\cdot\|_{\mathcal{Z}}$) containing a set of desirable points, which we denote by Δ. The algorithm model we are about to present is designed to compute points in Δ. The specific structure of this model is dictated by the manner in which we propose to carry out numerical integration in Section 4.

Thus, let $\{D_j\}_{j=0}^{\infty}$ be a sequence of subsets of $\mathcal{Z}$ such that

2.1 $$D_j \subset D_{j+1} \quad j = 0,1,2,\cdots.$$

Let

2.2 $$D = \bigcup_{j=0}^{\infty} D_j \ .$$

Next, let $\{A_j\}_{j=0}^{\infty}$, $A_j : D_j \to 2^{D_j}$ be a sequence of search functions, and let $\{c_j\}_{j=0}^{\infty}$, $c_j : D_j \to \mathbb{R}^1$, be a sequence of test functions. We shall assume that these functions have the following properties.

2.2 Assumptions: (i) $\bar{D} = \mathcal{Z}$

(ii) There exists a set $M \subset \mathcal{Z}$, satisfying $M \cap \Delta \neq \phi$, such that for every $z \in M$, $z \notin \Delta$, there exists an $\varepsilon(z) > 0$ a $\delta(z) > 0$ amd a interger $N(z) \geq 0$, such that

2.3 $$c_j(z'') - c_j(z') \leq -\delta(z) \quad \forall z' \in B_j(z,\varepsilon(z)) \cap M$$
$$\forall z'' \in A_j(z'), \ \forall j \geq N(z),$$

where

2.4 $$B_j(z,\varepsilon(z)) = \{z' \in D_j \mid \|z'-z\| \leq \varepsilon(z)\} \cdot$$

(iii) There exists a continuous function $c: \mathcal{Z} \to \mathbb{R}^1$ such that for the set M introduced in (ii) there exists a sequence $\{\beta_s\}_{s=0}^{\infty}$, possibly depending on M, with $\beta_s > 0$ for $s = 0,1,2,\cdots$, such that

2.5 $$\sum_{s=0}^{\infty} \beta_s < \infty$$

and

2.6 $$|c_j(z) - c(z)| \leq \beta_s \quad \forall z \in M \cap D_j, \ \forall j \geq s$$

2.7 Remark: Assumption (2.2)(iii) can be interpreted as follows. First, for (2.5) to hold, we must have $\beta_s \to 0$ faster than the arithmetical progression $\{1/s\}_{s=1}^{\infty}$. Hence, (2.6) implies that

2.8 $$|c_j(z) - c(z)| \to 0 \text{ as } j \to 0, \forall z \in D,$$

faster than an arithmetical progression. Next, $c_j(z)$ is usually the result of j iterations of an infinite subprocedure for computing c(z). Thus, (2.6) states that we must use a subprocedure for computing c(z) which converges at least as fast as an arithmetical progression. Since most algorithms converge at least linearly, we see that assumption (2.2) (iii) should not be too difficult to satisfy. □

2.9 <u>Algorithm Model</u>: Assume $z_0 \in D_0$, $\varepsilon_0 > 0$ and $\alpha \in (0,1)$ are given.

<u>Step 0</u>: Set $i = 0$, $j = 0$, $q(0) = 0$ and $\varepsilon = \varepsilon_0$.

<u>Step 1</u>: Compute a $y \in A_j(z_i)$.

<u>Step 2</u>: If $c_j(y) - c_j(z_i) \leq -\varepsilon$, go to Step 3; else, set $j = j+1$, set $\varepsilon = \alpha\varepsilon$ and go to Step 1.

<u>Step 3</u>: Set $z_{i+1} = y$, set $\varepsilon_{i+1} = \varepsilon$, set $q(i+1) = j$.

<u>Step 4</u>: Set $i = i+1$ and go to Step 1. □

<u>Comment</u>: The sequences $\{\varepsilon_i\}$ and $\{q(i)\}$ are introduced only for the purpose of the proofs to follow. They need not be computed. Note that by construction

2.10 $$\varepsilon_i = \alpha^{q(i)}\varepsilon_0 \quad i = 0,1,2,\cdots$$

2.11 $$z_i \in D_{q(i)} \quad i = 0,1,2,\cdots$$

2.12 $$c_{q(i+1)}(z_{i+1}) - c_{q(i+1)}(z_i) \leq -\varepsilon_{i+1} = -\alpha^{q(i+1)}\varepsilon_0, i = 0,1,2,\ldots$$

Also, since $D_{i+1} \supset D_i$, for $i = 0,1,2,\cdots$, we see that $A_j(z_i)$ and $c_j(z_i)$ are well defined for all $j \geq q(i)$ and hence Step 1 in (2.9) can always be executed. $\square$

2.13 Remark: Suppose that there exists a map $A: \mathcal{Z} \to 2^{\mathcal{Z}}$ such that $A(z) = \lim_{j\to\infty} A_j(z)$ for all $z \in D$. Then the algorithm model (2.9) can be thought of as an implementation of the following "conceptual" algorithm (see (1.3.9) in [1]).

2.14 Algorithm Model: Assume $z_0 \in D$ is given.

Step 0: set $i = 0$.

Step 1: Compute a $y \in A(z_i)$.

Step 2: If $c(y) - c(z_i) < 0$, set $z_{i+1} = y$ and go to Step 3; else stop.

Step 3: Set $i = i+1$ and go to Step 1. $\square$

We shall now establish the convergence properties of algorithm (2.9).

2.15 Lemma: Suppose that assumptions (2.2) are satisfied, and that algorithm (2.9) has generated an infinite sequence $\{z_i\}_{i=0}^{\infty} \subset M$, (see (2.2)) which has an accumulation point z^*. Then the corresponding sequence $\{\varepsilon_i\}_{i=0}^{\infty}$ satisfies $\varepsilon_i \to 0$ as $i \to \infty$.

Proof: Since $\{\varepsilon_i\}_{i=0}^{\infty}$ is a monotonically decreasing sequence which is bounded from below by zero, it must converge. Suppose, therefore, that

2.16 $$\varepsilon_i \to \varepsilon^* > 0 \text{ as } i \to \infty.$$

We shall show that this leads to a contradiction. Since $\varepsilon_i = \alpha^{q(i)}\varepsilon_0$, (2.16) implies that there exists an integer $N \geq 0$ such that

2.17 $$\varepsilon_i = \varepsilon^* \text{ and } q(i) = q^* \ \forall i \geq N.$$

But this implies (see Step 2 of (2.9)) that

2.18 $$c_{q(i+1)}(z_{i+1}) - c_{q(i+1)}(z_i) = c_{q^*}(z_{i+1}) - c_{q^*}(z_i) \leq -\varepsilon^* \quad \forall\, i \geq N.$$

Thus, (2.18) implies that $c_{q^*}(z_i) \to -\infty$ as $i \to \infty$.

Now, suppose that $z_i \to z^*$ as $i \to \infty$ for $i \in K \subset \{0,1,2,\ldots\}$. Since $\{z_i\}_{i \in K} \subset M$, there exists by (2.2)(iii) $\beta_{q^*} > 0$ such that

2.19 $$|c_{q^*}(z_i) - c(z_i)| \leq \beta_{q^*} \ \forall i \in K, \ \ \forall i \geq N.$$

Since $z_i \to z^*$ as $i \to \infty$ for $i \in K$, and since $c(\cdot)$ is continuous by (2.2)(iii), there exists a $k \in K$, $k \geq N$ such that

2.20 $$|c(z_i) - c(z^*)| \leq \beta_{q^*} \ \ \forall i \in K, \ i \geq k.$$

Hence, from (2.19) and (2.20), we obtain that

2.21 $$c_{q^*}(z_i) \geq c(z^*) - 2\beta_{q^*} \ \forall i \in K, \ i \geq k,$$

and therefore we have obtained a contradiction, of our previous conclusion that $c_{q^*}(z_i) \to -\infty$. Thus, the lemma must be true. □

2.22 <u>Lemma</u>. Suppose that (2.2)(ii) is satisfied and that algorithm (2.9) jams up, for $i = s$, at $z_s \in M \cap D_{q(s)}$ cycling between Steps 1 and 2, so that a point z_{s+1} cannot be constructed. Then $z_s \in \Delta$.

<u>Proof</u>: Suppose that $z_s \notin \Delta$. We shall show that this leads to a contradiction. Since we assume that $z_s \in M \cap D_{q(s)}$ and that $z_s \notin \Delta$, it follows from (2.2)(ii) that there exist a $\delta_s > 0$ and an integer $N_s \geq 0$ such that

2.23 $$c_j(z'') - c_j(z_s) \leq -\delta_s \quad \forall z'' \in A_j(z_s),\ \forall j \geq N_s$$

(where $N_s \geq q(s)$ so that $c_j(z_s)$ and $A_j(z_s)$ are well defined for all $j \geq N_s$). Since the algorithm is cycling between Steps 1 and 2, it must be constructing an infinite sequence $\{y_r\}_{r=0}^{\infty}$ such that

2.24 $$y_r \in D_{q(s)+r},\ r = 0,1,2,\cdots,$$

2.25 $$y_r \in A_{q(s)+r}(z_s),\ r = 0,1,2,\cdots,$$

2.26 $$c_{q(s)+r}(y_r) - c_{q(s)+r}(z_s) > -\alpha^{q(s)+r}\varepsilon_0,\ r = 0,1,2,\cdots .$$

Let t be an integer such that $\alpha^{q(s)+t}\varepsilon_0 \leq \delta_s$ and $q(s)+t \geq N_s$. Then for $r \geq t$ (2.25) and (2.26) contradict (2.23). Hence we must have $z_s \in \Delta$. □

2.27 <u>Definition</u>: Given an infinite subsequence $K \subset \{0,1,2,\cdots\}$ we shall associate with it an <u>index</u> <u>function</u> k: $K \to K$ defined by

2.28 $$k(i) = \min\{j \in K \mid j \geq i+1\}$$ □

2.29 <u>Lemma</u>: Let $\{z_i\}_{i=0}^{\infty}$ be any sequence constructed by algorithm (2.9), with an associated sequence $\{q(i)\}_{i=0}^{\infty}$, and let K be any infinite subsequence of $\{0,1,2,\cdots,\}$, with index function k. If $\{z_i\}_{i=0}^{\infty}$ is contained in the set M (see (2.2)) and (2.2)(iii) is satisfied, then

2.30 $$c_{q(k(i))}(z_{k(i)}) \leq 2 \sum_{j=q(i+1)}^{q(k(i))} \beta_j + c_{q(i+1)}(z_{i+1}), \ \forall i \in K.$$

<u>Proof</u>: Let b: $\{1,2,3,\cdots,\} \to \{0,1\}$ be defined by

2.31 $$b(i) = \begin{cases} 0 & \text{if } q(i) = q(i-1) \\ 1 & \text{otherwise.} \end{cases}$$

Now, from (2.12), we must have

2.32 $$c_{q(i+1)}(z_{i+1}) \leq c_{q(i+1)}(z_i) - \varepsilon_{i+1}, \ i = 0,1,2,\ldots$$

Suppose that for some i, $q(i) = q(i+1)$; then, for this i, (2.32) becomes, because of 2.31,

2.33 $$c_{q(i+1)}(z_{i+1}) \leq c_{q(i)}(z_i) - \varepsilon_{i+1} \leq c_{q(i)}(z_i) = c_{q(i)}(z_i) + 2b(i+1)\beta_{q(i)},$$

where $\beta_{q(i)} \in \{\beta_s\}_{s=0}^{\infty}$, with $\{\beta_s\}_{s=0}^{\infty}$ as in (2.2)(iii).

Next, suppose that for some i, $q(i) \neq q(i+1)$, then, again making use of (2.31), and of (2.6) twice, we obtain from (2.32),

2.34 $$\begin{aligned} c_{q(i+1)}(z_{i+1}) &\leq c_{q(i+1)}(z_i) - \varepsilon_{i+1} \\ &\leq c(z_i) + \beta_{q(i)} - \varepsilon_{i+1} \\ &\leq c_{q(i)}(z_i) + 2\beta_{q(i)} - \varepsilon_{i+1} \\ &\leq c_{q(i)}(z_i) + 2\beta_{q(i)} = c_{q(i)}(z_i) + 2b(i+1)\beta_{q(i)}. \end{aligned}$$

Combining (2.33) with (2.34), we obtain,

2.35 $$c_{q(i+1)}(z_{i+1}) \leq c_{q(i)}(z_i) + 2b(i+1)\beta_{q(i)}, \ i = 0,1,2,\cdots.$$

Hence, for any $i \in K$, we obtain from (2.35),

2.36
$$c_{q(k(i))}(z_{k(i)}) \le 2 \sum_{j=i+2}^{k(i)} b(j)\beta_{q(j-1)} + c_{q(i+1)}(z_{i+1})$$
$$\le 2 \sum_{j=q(i+1)}^{q(k(i))} \beta_j + c_{q(i+1)}(z_{i+1})$$

the last inequality holding because $b(j) \in \{0,1\}$, $\beta_s > 0$ for $s = 0,1,2,\cdots$, and because $q(j) = q(j-1)$ whenever $b(j) = 0$. □

2.37 Theorem: Suppose that assumptions (2.2) are satisfied, and consider a sequence $\{z_i\}$, with the associated sequences $\{\varepsilon_i\}$ and $\{q(i)\}$, constructed by algorithm (2.9). If $\{z_i\} \subset M$, where $M \subset \mathcal{Z}$ is as defined in (2.2)(ii), then, either $\{z_i\}$ is finite, because the algorithm has jammed up at its last element z_s, in which case $z_s \in \Delta$, or $\{z_i\}$ is infinite, in which case every accumulation point of $\{z_i\}$ is in Δ. (When $\{z_i\}_{i=0}^{\infty}$ has no accumulation points, the theorem is vacuous).

Proof: The case of $\{z_i\}$ finite was established in lemma (2.22). Hence, let us suppose that $\{z_i\}$ is infinite and that it has at least one accumulation point, $\hat{z}$. Suppose that K is an infinite subset of $\{0,1,2,\cdots,\}$, with index function $k(\cdot)$(see (2.27), such that the subsequence $\{z_i\}_{i \in K}$ converges to $\hat{z}$.

By lemma (2.15), we must have $\varepsilon_i \to 0$, and hence $q(i) \to \infty$ as $i \to \infty$. By (2.2)(iii) there exists a continuous function $c\colon \mathcal{Z} \to \mathbb{R}^1$ and a sequence $\{\beta_s\}_{s=0}^{\infty}$ such that $\beta_s > 0$, $s = 0,1,2,\cdots$, $\sum_{s=0}^{\infty} \beta_s < \infty$ and

2.38 $$c_{q(i)}(z_i) \geq c(z_i) - \beta_{q(i)}, \quad i = 0,1,2,\cdots.$$

Since the function $c(\cdot)$ stipulated in (2.2)(iii) is continuous, we must have

2.39 $$c(z_i) \to c(\hat{z}) \text{ as } i \to \infty, \text{ for } i \in K.$$

Consequently, since $\beta_s \to 0$ as $s \to \infty$, we obtain from (2.38) and (2.39) that

2.40 $$\underline{\lim_{i \in K}}\, c_{q(i)}(z_i) \geq \underline{\lim_{i \in K}}\, \{c(z_i) - \beta_{q(i)}\} = c(\hat{z}).$$

Now suppose that $\hat{z} \notin \Delta$. We shall show that this leads to a contradiction of (2.40).

By (2.2)(ii), there exist $\hat{\varepsilon} > 0$, $\hat{\delta} > 0$ and an integer $\hat{N}$ such that

2.41 $$c_j(z'') - c_j(z') \leq -\hat{\delta} \ \forall z' \in B_j(\hat{z},\hat{\varepsilon}) \cap M, \ \forall z'' \in A_j(z'), \ \forall j \geq \hat{N},$$

where $B_j(\hat{z},\hat{\varepsilon})$ is defined as in (2.4). Since $z_i \to \hat{z}$ as $i \to \infty$, $i \in K$, and $q(i) \to \infty$ as $i \to \infty$, there exists an integer N_1 such that for all $i \in K$, with $i \geq N_1$ (see (2.4) and (2.11)),

2.42 $$z_i \in B_{q(i)}(\hat{z},\hat{\varepsilon}) \cap M$$

and

2.43 $$q(i) \geq \hat{N}$$

Since $z_{i+1} \in A_{q(i+1)}(z_i)$ by construction, (2.41)-(2.43) yield, 2.44 $$c_{q(i+1)}(z_{i+1}) - c_{q(i+1)}(z_i) \leq -\hat{\delta} \;\; \forall i \in K,\ i \geq N_1.$$

Now, let

2.45 $$\gamma_n = \sum_{j=0}^{n} \beta_j$$

Then $\{\gamma_n\}_{n=0}^{\infty}$ is a monotonically increasing sequence which is bounded from above because of (2.5), and hence $\{\gamma_n\}_{n=0}^{\infty}$ must converge, i.e., it must be a Cauchy sequence. Hence, since $q(i) \to \infty$ as $i \to \infty$, there must exist an integer $N_2 \geq N_1$ such that

2.46 $$|\gamma_{q(k(i))} - \gamma_{q(i+1)}| \leq \hat{\delta}/8 \;\; \forall i \geq N_2,\ i \in K,$$

where $k(\cdot)$ is the index function of K (see (2.27)). Next, since $\beta_s \to 0$ as $s \to \infty$, there must exist an integer $N_3 \geq N_2$ such that $\beta_{q(i)} \leq \hat{\delta}/8$ for all $i \geq N_3$, and hence

2.47 $$|c_{q(i)}(z_i) - c(z_i)| \leq \hat{\delta}/8 \;\; \forall i \geq N_3,\ i \in K$$

and also

2.48 $$|c_{q(i+1)}(z_i) - c(z_i)| \leq \hat{\delta}/8 \;\; \forall i \geq N_3,\ i \in K.$$

Combining (2.47) and (2.48) we now obtain

2.49 $$|c_{q(i+1)}(z_i) - c_{q(i)}(z_i)| \leq \hat{\delta}/4 \;\; \forall i \geq N_3,\ i \in K.$$

Now making use of (2.30), (2.45) and (2.46), we obtain

2.50 $$c_{q(k(i))}(z_{k(i)}) - c_{q(i)}(z_i) \le 2(\gamma_{q(k(i))} - \gamma_{q(i+1)}) + c_{q(i+1)}(z_{i+1}) - c_{q(i)}(z_i)$$
$$\le \hat{\delta}/4 + c_{q(i+1)}(z_{i+1}) - c_{q(i)}(z_i), \quad \forall i \in K,\ i \ge N_3.$$

Finally, adding and substrating $c_{q(i+1)}(z_i)$ to the right hand side of (2.50) and making use of (2.44) and of (2.49), we obtain,

2.51 $$c_{q(k(i))}(z_{k(i)}) - c_{q(i)}(z_i) \le \hat{\delta}/4 + c_{q(i+1)}(z_{i+1}) - c_{q(i+1)}(z_i) + c_{q(i+1)}(z_i) - c_{q(i)}(z_i) \le \hat{\delta}/4 - \hat{\delta} + \hat{\delta}/4 = -\hat{\delta}/2 \quad \forall i \in K,\ i \ge N_3.$$

Since (2.51) implies that $c_{q(i)}(z_i) \to -\infty$ as $i \to \infty$, $i \in K$, we see that (2.51) contradicts (2.40), and hence we must have $\hat{z} \in \Delta$, which completes our proof. □

2.52 <u>Remark</u>: It is clear from the above theorem that whenever $M \cap \Delta = \phi$, contrary to assumption (2.2)(ii), and the sequence $\{z_i\}$ constructed by algorithm (2.9) is in M, then $\{z_i\}$ cannot have any accumulation points. □

2.53 <u>Remark</u>: Since by lemma (2.15), $\{\varepsilon_i\}$ must converge to zero for $\{z_i\}$ to have an accumulation point, it appears reasonable to insert into (2.9) the following "Step 3': If $i/k = 0$ modulo k, set $\varepsilon = \alpha\varepsilon$ and go to Step 4; else, go to Step 4," where $k > 0$ is an integer. Theorem (2.37) remains valid with this modification. □

3. The Algorithm.

As in Section 2, let $\mathcal{Z}$ be a normed linear space, with norm $\|\cdot\|_{\mathcal{Z}}$, and let $\phi: \mathcal{Z} \to \mathbb{R}^1$ be a continuous function. Consider the problem

3.1 $$\min\{\phi(z) \mid z \in \mathcal{Z}\}.$$

As we shall see in the next section, we may not be justified in assuming that the function $\phi(\cdot)$ is Fréchet differentiable or even continuous on $\mathcal{Z}$. However, we may assume that there exists a subset M on which $\phi(\cdot)$ is continuous and a function $h(\cdot)$ which, for all purposes, acts as the gradient of $\phi(\cdot)$ on M. Hence we shall only be able to solve problem (3.1) if $\hat{z}$ is in this set M.

3.2 Assumptions: We shall assume that

(i) there exists a set $M' \subset \mathcal{Z}$ such that $\phi(\cdot)$ is continuous on M';

(ii) for any $z \in M'$ and any $y \in \mathcal{Z}$

$$\phi'(z,y) = \lim_{\lambda \downarrow 0} \frac{\phi(z+\lambda y) - \phi(z)}{\lambda} \tag{3.3}$$

exists;

(iii) there exists a function $h: \mathcal{Z} \to \mathcal{Z}$ which is continuous on M' and a set $M \subset M'$ such that (a) for every $z \in M$ there exists a $\delta(z) > 0$ such that $(z'+\lambda h(z')) \in M'$ for all $\lambda \in [0,\delta(z)]$, for all $z' \in M$ satisfying $\|z' - z\| \leq \delta(z)$, and (b) for every $z \in M'$ and every $y \in \mathcal{Z}$,

$$\phi'(z,y) = \langle h(z),y \rangle_{\mathcal{Z}} \tag{3.4}$$

where $\langle \cdot,\cdot \rangle_{\mathcal{Z}}$ is a continuous scalar product on $\mathcal{Z}$. □

The following result is obvious.

3.5 Proposition: Suppose that $\hat{z} \in M$ minimizes $\phi(z)$ for $z \in \mathcal{Z}$. Then $h(\hat{z}) = 0$. □

We shall assume that we have an infinite sequence of linear subspaces $\{D_j\}_{j=0}^{\infty}$ of $\mathcal{Z}$ such that $D_{j+1} \supset D_j$ for $j = 0,1,2,\cdots$ and such that $\overline{\bigcup_{j=0}^{\infty} D_j} = \mathcal{Z}$, together with approximation functions $\phi_j: D_j \to \mathbb{R}^1$, $j = 0,1,2$, and $h_j: D_j \to D_j$, $j = 0,1,2,\cdots$, which satisfy the following assumption.

3.6 Assumption: Given any $z \in M'$ and any $\gamma > 0$, there exists an $\varepsilon(z,\gamma) > 0$ and an integer $N(z,\gamma) \geq 0$ such that

3.7 $$|\phi_j(z') - \phi(z')| \leq \gamma$$

3.7' $$\|h_j(z') - h(z')\|_{\mathcal{Z}} \leq \gamma$$

for all $z' \in B(z,\varepsilon(z,\gamma)) \cap D_j \cap M'$, for all $j \geq N(z,\gamma)$, where

3.8 $$B(z,\varepsilon) = \{z' \in \mathcal{Z} \,|\, \|z'-z\|_{\mathcal{Z}} \leq \varepsilon\} \;. \quad \square \;.$$

We now present a steepest descent type algorithm which uses adaptively the approximating functions $\phi_j(\cdot)$ and $h_j(\cdot)$. This algorithm approximates a gradient method due to Armijo [1].

3.9 Algorithm:

Initialization: Select an integer $j_0 \geq 0$. Then select a $z_0 \in D_{j_0} \cap M$, $\varepsilon_0 > 0$, $\alpha \in (0,1)$, $\beta \in (0,1)$ and $\lambda_{min} \in (0,1]$.

Step 0: Set $i = 0$, $j = j_0$, $q(0) = j_0$, and $\varepsilon = \varepsilon_0$.

Step 1: Compute $h_j(z_i)$.

Step 2: Set $\lambda = 1$.

Step 3: Compute

3.10 $$\theta_j(z_i,\lambda) = \phi_j(z_i-\lambda h_j(z_i)) - \phi_j(z_i) + \frac{\lambda}{2}\langle h_j(z),h_j(z)\rangle_{\mathcal{Z}}$$

Step 4: If $\theta_j(z_i,\lambda) > 0$, set $\lambda = \beta\lambda$ and go to Step 5; else, set $y = z_i - \lambda h_j(z_i)$ and go to Step 6.

Step 5: If $\lambda \geq \varepsilon\lambda_{min}$, go to Step 3; else, set $y = z_i$ and go to Step 6.

Step 6: If $\phi_j(y) - \phi_j(z_i) \leq -\varepsilon$, go to Step 8; else, go to Step 7.

Step 7: Set $j = j+1$, set $\varepsilon = \alpha\varepsilon$, and go to Step 1.

Step 8: Set $z_{i+1} = y$, set $q(i+1) = j$, set $\varepsilon_{i+1} = \varepsilon$, set $i = i+1$, and go to Step 1. $\square$

3.11 <u>Comment</u>: The sequences $\{q(i)\}$ and $\{\varepsilon_i\}$ are in correspondence with those appearing in the algorithm model (2.9). They are only introduced for the convenience of the proofs to follow and need not be computed in practice. □

3.12 <u>Remark</u>: As we shall see in the next section, the set M can be chosen to be a very large set containing the origin, and that $z_i \notin M$ for some i can be interpreted as an indication that the sequence $\{z_i\}$ generated by algorithm (3.9) cannot converge to a solution of (3.1). □

Since we make no convexity assumptions for the function $\phi(\cdot)$, the best we can hope to establish for algorithm (3.9) is that if $\hat{z} \in M$ is an accumulation point of $\{z_i\}_{i=0}^{\infty}$ or if $z_s \in M$ is a point at which the algorithm has jammed up, then $h(\hat{z}) = 0$, or $h(z_s) = 0$. We now proceed to show that algorithm (3.9) has exactly this property. To be specific, we shall show that algorithm (3.9) is of the form of the algorithm model (2.9) and that the assumptions (2.2) are satisfied by it.

For this purpose, we make the following identifications:

3.13 <u>Definition</u>: We set $\Delta = \{z \in M | h(z) = 0\}$, $c(\cdot) = \phi(\cdot)$, $c_j(\cdot) = \phi_j(\cdot)$, and we define $A_j(\cdot)$ by the calculations in Steps 1 to 5 of algorithm (3.9). □

We shall need the following results

3.14 <u>Proposition</u>: Given any $z \in M'$ and any $\gamma > 0$, there exists an $\overline{\varepsilon}(z,\gamma) > 0$ and an integer $\overline{N}(z,\gamma) \geq 0$, such that

3.15 $$|\phi_j(z') - \phi(z)| \leq \gamma$$

3.16 $$\|h_j(z') - h(z)\|_{\mathcal{Z}} \leq \gamma$$

for all $z' \in B(z,\bar{\varepsilon}(z,\gamma)) \cap D_j \cap M'$, for all $j \geq \bar{N}(z,\gamma)$. □

The above proposition follows directly from assumption (3.6) and from the continuity of $\phi(\cdot)$ and $h(\cdot)$ on M'.

3.17 Lemma: Suppose that $z \in M$ is such that $h(z) \neq 0$. Then there exists an $\varepsilon(z) > 0$ and an integer $k(z) > 0$ such that

3.18 $$\phi(z'-\beta^{k(z)}h(z')) - \phi(z') + \frac{\beta^{k(z)}}{2} \langle h(z'),h(z') \rangle_{\mathcal{Z}} \leq 0$$

for all $z' \in B(z,\varepsilon(z)) \cap M$, where $\beta \in (0,1)$ is as defined in algorithm (3.9).

Proof: Since $z \in M$, it follows from the assumptions (3.2) that there exist a $\delta(z) > 0$ and an integer $k(z) \geq 0$ such that $[z', z'-\beta^{k(z)}h(z')] \subset M'$ for all $z' \in B(z,\delta(z)) \cap M$, and such that

3.19 $$\phi(z-\beta^{k(z)}h(z)) - \phi(z) + \frac{\beta^{k(z)}}{2} \langle h(z),h(z) \rangle_{\mathcal{Z}} \leq -\frac{\beta^{k(z)}}{4} \langle h(z),h(z) \rangle_{\mathcal{Z}} .$$

Now, since $\phi(\cdot)$, $h(\cdot)$, and $\langle \cdot,\cdot \rangle_{\mathcal{Z}}$ are continuous on M', there exists an $\varepsilon(z) \in (0,\delta(z)]$ such that for all $z' \in B(z,\varepsilon(z)) \cap M$

3.20 $$\Big|\phi(z'-\beta^{k(z)}h(z')) - \phi(z') + \frac{\beta^{k(z)}}{2} \langle h(z'),h(z') \rangle_{\mathcal{Z}} - \phi(z-\beta^{k(z)}h(z)) + \phi(z) - \frac{\beta^{k(z)}}{2} \langle h(z),h(z) \rangle_{\mathcal{Z}}\Big| \leq \frac{\beta^{k(z)}}{4} \langle h(z),h(z) \rangle_{\mathcal{Z}} .$$

It now follows from (3.19) and (3.20) that (3.18) must hold for $k(z) \geq 0$ and $\varepsilon(z) > 0$ defined so as to satisfy (3.19) and (3.20). □

3.20 Corollary: Suppose that $z \in M$ is such that $h(z) \neq 0$. Then there exists an $\varepsilon(z) > 0$ and integers $k(z) \geq 0$, $N(z) \geq 0$, such that

3.21 $$\phi_j(z' - \beta^{k(z)} h_j(z')) - \phi_j(z') \leq - \frac{\beta^{k(z)}}{4} \langle h(z'), h(z') \rangle_{\mathcal{Z}}$$

for all $z' \in D_j \cap M \cap B(z, \varepsilon(z))$, for all $j \geq N(z)$. □

Corollary (3.20) follows directly from proposition (3.14), lemma (3.17) and the continuity of $\phi(\cdot)$ and $h(\cdot)$.

3.22 Theorem: Suppose that there exists a sequence $\{\beta_s\}_{s=0}^{\infty}$, $\beta_s > 0$ for $s = 0,1,2,\cdots$, such that $\sum_{s=0}^{\infty} \beta_s < \infty$ and such that

3.23 $$|\phi_j(z) - \phi(z)| \leq \beta_s \quad \forall z \in M' \cap D_j, \ \forall j \geq s.$$

Consider any sequence $\{z_i\}$ constructed by algorithm (3.9) and suppose that $\{z_i\} \subset M$. If $\{z_i\}$ is finite, because the algorithm has jammed up at z_s, its last element, then $h(z_s) = 0$; if $\{z_i\}$ is infinite, then every accumulation point of $\{z_i\}$, which satisfies $\hat{z} \in M$ also satisfies $h(\hat{z}) = 0$.

Proof: Theorem (3.22) follows directly from theorem (2.37) because of the identification in (3.13) and because assumptions (2.2) are then satisfied as a result of the assumptions (3.2), corollary (3.20), and of the assumption stated in theorem (3.22). □

4. An Application to Optimal Control

Consider the problem

4.1 $$\text{minimize } g(x(T,u))$$

subject to

4.2 $$\frac{dx(t,u)}{dt} = f(x(t,u),u(t),t), \; t \in [0,T]; \; x(0,u) = x_0,$$

where g: $\mathbb{R}^n \to \mathbb{R}^1$, f: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^n$, u: $[0,T] \to \mathbb{R}^m$, and $T > 0$ is given.

4.3 <u>Assumption</u>: We shall assume that

(i) $g(\cdot)$ is continuously differentiable;

(ii) $f(\cdot,\cdot,\cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1$,

(iii) $u \in \tilde{M}' = \{u \in \mathcal{R}^m\,[0,T] \mid \|u\|_\infty = \operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\| \le 2w\}$,

where $w \in (0,\infty)$ is some very large number, $\|\cdot\|$ denotes the euclidean norm, and $\mathcal{R}^m[0,T]$ is the space of m-vector valued, regulated functions [2];

(iv) for every $u \in \tilde{M}'$, (4.2) has a unique, absolutely continuous solution $x(\cdot,u)$. □.

The reason for defining the set $\tilde{M}'$, in (iii) above, in the specific manner chosen, stems from the need to satisfy the assumptions (3.2) together with the assumptions which we shall shortly need in conjunction with numerical integration formulas.

4.4 <u>Definition</u>: For any $u \in \tilde{M}'$, let $\tilde{h}(u)(\cdot)$ be defined as follows:

4.4 $$\tilde{h}(u)(t) = \frac{\partial f(x(t,u),u(t),t)}{\partial u}^T p(t,u), \; t \in [0,T]$$

where $x(t,u)$ is the solution of (4.2) corresponding to the given u, and $p(t,u)$ is given by

4.5 $$\frac{d}{dt} p(t,u) = \frac{-\partial f(x(t,u),u(t),t)}{\partial x}^T p(t,u), \; t \in [0,T],$$

with the boundary condition

4.6 $$p(T,u) = \frac{\partial g(x(T,u))^T}{\partial x} . \square$$

4.7 Definition: for any $u \in \tilde{M}'$ let $\tilde{\phi}(u)$ be defined by

4.8 $$\tilde{\phi}(u) = g(x(T,u)),$$

where $x(T,u)$ is the solution at T of (4.2), corresponding to the given u. $\square$

In view of theorems (A.1) and (A.13) in the appendix, the following result is obvious.

4.9 Theorem: Let $u^* \in \tilde{M}'$ and $\varepsilon > 0$ be arbitrary. Then there exist $k > 0$ and $\gamma > 0$ such that

4.10 $$\sup_{t \in [0,T]} \|x(t,u) - x(t,u^*)\| \leq k\|u-u^*\|_2$$

and

4.11 $$\operatorname{ess\,sup}_{t \in [0,T]} \|\tilde{h}(u)(t) - \tilde{h}(u^*)(t)\| \leq \varepsilon$$

for all $u \in \tilde{B}(u^*,\gamma)$ where

4.12 $$\tilde{B}(u^*,\gamma) = \{u \in \tilde{M}' \mid \|u-u^*\|_2 \leq \gamma\},$$

and $\|u\|_2 = (\int_0^t \|u(t)\|^2 dt)^{1/2}$. $\square$

The following result follows directly from [3].

4.13 Proposition: For any $u^* \in \tilde{M}'$ and any $u \in \mathcal{R}^m[0,T]$

4.14 $$\tilde{\phi}'(u^*,u) = \lim_{\lambda \downarrow 0} \frac{\tilde{\phi}(u^*+\lambda u) - \tilde{\phi}(u^*)}{\lambda}$$

exists and is given by

$$\tilde{\phi}'(u^*,u) = \langle \tilde{h}(u^*),u \rangle_2 \triangleq \int_0^T \langle \tilde{h}(u^*)(t),u(t) \rangle dt, \tag{4.15}$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. □

For the purpose of establishing the convergence properties of the algorithm to be described for solving problem (4.1)-(4.3), which can be written in the form $\min\{\tilde{\phi}(u) | u \in \tilde{M}'\}$, it is convenient to embed it into $L_2^m[0,T]$,[†] as follows.

First, we define $\tilde{M} \subset \tilde{M}'$ by

$$\tilde{M} = \{u \in \tilde{M}' | \|u\|_\infty \leq w\}. \tag{4.16}$$

4.17 <u>Definition</u>: Let $\mathcal{Z} = L_2^m[0,T]$, let M', $M \subset L_2^m[0,T]$ be the sets of classes of functions incuced in $L_2^m[0,T]$ by the sets of regulated functions $\tilde{M}'$, $\tilde{M}$, respectively, and let ϕ: $M' \to \mathbb{R}^1$, h: $M' \to L_2^m[0,T]$, and ϕ': $M \times L_2^m[0,T] \to \mathbb{R}^1$ be defined as natural extensions of $\tilde{\phi}(\cdot)$, $\tilde{h}(\cdot)$ and $\tilde{\phi}'(\cdot,\cdot)$, i.e., if $u^* \in \tilde{M}'$, $u \in \mathcal{R}^m[0,T]$, $v^* \in M'$ and $v \in L_2^m[0,T]$ are such, that $u^* \in v^*$ and $u \in v$, then $\phi(v^*) = \tilde{\phi}(u^*)$, $h(v^*) = \tilde{h}(u^*)$ and $\phi'(v^*,v) = \tilde{\phi}'(u^*,u)$.

In view of definition (4.17), we see that problem (4.1)-(4.3) can

[†]By $L_2^m[0,T]$ we denote the space of equivalence classes of square integrable functions from [0,T] into $\mathbb{R}^m$, with norm $\|\cdot\|_2$ defined by $\|u\|_2 = (\int_0^T \|u(t)\|^2 dt)^{\frac{1}{2}}$, where $\|\cdot\|$ denotes the euclidean norm.

be embedded into the problem $\min\{\phi(u) | u \in M'\}$.

4.18 Lemma: Assumption (3.2) is satisfied by the set M and the functions $\phi(\cdot)$, $\phi'(\cdot,\cdot)$, $h(\cdot)$ and $\langle \cdot,\cdot \rangle_2$, defined in (4.17).

Proof: This lemma follows directly from theorem (4.9), proposition (4.13), and the fact $\langle \cdot,\cdot \rangle_2$ is continuous on $L_2^m[0,T]$. □

We are now ready to construct the various approximations needed to define an algorithm for solving the problem (4.1)-(4.3).

4.19 Definition: For any integer $j \geq 0$, let

4.20 $$\eta_j = T/2^j, \; j = 0,1,2,\cdots$$

and let

4.21 $$t_{js} = s\eta_j, \; s = 0,1,2,\cdots,2^j.$$

Furthermore, for $j = 0,1,2,\cdots$, let $\tilde{D}_j$ denote the space of functions $u\colon [0,T] \to \mathbb{R}^m$, such that the components $u^i(\cdot)$ of $u(\cdot)$, $i = 1,2,\cdots,m$, are real valued step functions with mesh points at t_{js}, $s = 0,1,\cdots,2^j$, which are continuous from the right and which satisfy $u^i(T) = u^i(T-\eta_j)$, $i = 1,2,\cdots,m$. □ .

4.22 Definition: For $j = 0,1,2,\cdots$, let $F_1\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^n$ and $F_2\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^{2n}$ be one step integration functions for the differential equations

4.23 $$\frac{d}{dt}x(t,u) = f(x(t,u),u(t),t), \; t \in [0,T],$$

and

$$2.24 \qquad \frac{d}{dt}\begin{pmatrix} x(t,u) \\ \text{------} \\ p(t,u) \end{pmatrix} = \begin{pmatrix} f(x(t,u),u(t),t) \\ \text{------------------------} \\ -\frac{\partial f(x(t,u),u(t),t)^T}{\partial x} p(t,u) \end{pmatrix},$$

respectively, where f is as in (4.2). (For example, F_1, F_2 are part of a Range-Kutta formula of the form (4.25) or (4.26) below). Then, for any $u \in D_j$, we define the sequences $\{x_j(s,u)\}_{s=0}^{2^j}$, $\{x_j'(s,u)\}_{j=0}^{2^j}$ and $\{p_j(s,u)\}_{j=0}^{2^j}$ as follows:

$$4.25 \qquad x_j(s+1,u) = x_j(s,u) + \eta_j F_1(x_j(s,u),u(t_{js}),t_{js},\eta_j) \qquad s = 0,1,2,\cdots, 2^j-1,$$

with $x_j(0,u) = x_0$, and

$$4.26 \qquad \begin{pmatrix} x_j'(s,u) \\ \text{-----} \\ p_j(s,u) \end{pmatrix} = \begin{pmatrix} x_j'(s+1,u) \\ \text{-------} \\ p_j(s+1,u) \end{pmatrix} + \eta_j F_2(x_j'(s+1,u),p_j(s+1,u),u(t_{js}), t_{js},\eta_j), \quad s = 0,1,2,\cdots,2^j-1,$$

with $x_j'(2^j,u) = x_j(2^j,u)$ and $p_j(2^j,u) = \dfrac{(\partial g(x_j(2^j,u))^T}{\partial x}$.

Eventually one must associate piecewise continuous functions of t with the sequences $\{x_j(s,u)\}$, $\{x_j'(s,u)\}$ and $\{p_j(s,u)\}$. For our purposes, it is most convenient to do this as follows. For $j = 0,1,2,\cdots$, we define the functions $\bar{x}_j$: $[0,T] \times \tilde{D}_j \to \mathbb{R}^n$, $\bar{x}_j'$: $[0,T] \times D_j \to \mathbb{R}^n$ and $\bar{p}_j$: $[0,T] \times \tilde{D}_j \to \mathbb{R}^n$, as follows:

4.27 $$\bar{x}_j(t,u) = x_j(s,u) \text{ for } t \in [s\eta_j,(s+1)\eta_j),\ s = 0,1,\cdots,2^j-1,$$
$$= x_j(2^j-1,u) \text{ for } t = T,$$

4.28 $$\bar{x}_j'(t,u) = x_j'(s+1,u) \text{ for } t \in [s\eta_j,(s+1)\eta_j),\ s = 0,1,2,\cdots,2^{j-1},$$
$$= x_j'(2^j,u) \text{ for } t = T,$$

4.29 $$\bar{p}_j(t,u) = p_j(s+1,u) \text{ for } t \in [s\eta_j,(s+1)\eta_j),\ s = 0,1,2,\cdots,2^{j-1},$$
$$= p_j(2^j,u) \text{ for } t = T,$$

where $x_j(s,u)$, $x_j'(s,u)$ and $p_j(s,u)$, $s = 0,1,\cdots,2^j$, are defined by (4.25) and (4.26). □

4.30 <u>Assumption</u>: We shall assume that the integration formulas (4.25) and (4.26) are such that given any $\varepsilon > 0$ and any $w > 0$, there exist an integer $J \geq 0$ and $K \in (0,\infty)$, such that:

4.31 $$\sup_{t \in [0,T]} \|x(t,u) - \bar{x}_j(t,u)\| \leq \frac{K}{2^j}.$$

and

4.32 $$\sup_{t \in [0,T]} (\|x(t,u) - \bar{x}_j'(t,u)\|^2 + \|p(t,u) - \bar{p}_j(t,u)\|^2)^{1/2} \leq \varepsilon$$

for all $u \in \tilde{M} \cap \tilde{D}_j$, for all $j \geq J$. □

We shall show in the appendix that, for suitable conditions on f, assumption (4.30) is satisfied by the first order Runge-Kutta formula. It appears that it is also satisfied for higher order Runge-Kutta formulas as well.

We are now ready to define the approximation functions $\phi_j(\cdot)$, and

$h(\cdot)$.

4.33 Definition: For $j = 0,1,2,\ldots$, we define the functions $\tilde{\phi}_j: \tilde{D}_j \to \mathbb{R}^1$ and $\tilde{h}_j: \tilde{D}_j \to \tilde{D}_j$ as follows:

4.34 $$\tilde{\phi}_j(u) = g(x_j(2^j,u))$$

4.35 $$\tilde{h}_j(u)(t) = \frac{\partial f(x_j'(s+1),u(s\eta_j),s\eta_j)^T}{\partial u} p_j(s+1,u),$$

$$\text{for } t \in [s\eta_j,(s+1)\eta_j),\ s = 0,1,2,\ldots,2^j-1$$

4.36 $$\tilde{h}_j(u)(T) = \tilde{h}_j(u)(T-\eta_j)\ .\quad \square$$

4.37 Definition: For $j = 0,1,2,\cdots$, we define $D_j \in L_2^m[0,T]$ to be the subspace induced by the set of functions $\tilde{D}_j$. For $j = 0,1,2,3,\cdots$, we define the functions $\phi_j: D_j \to \mathbb{R}^1$, $h_j: D_j \to D_j$ by obvious extension of $\tilde{\phi}_j(\cdot)$, $\tilde{h}_j(\cdot)$, i.e. for any $u \in \tilde{D}_j$ and a $v \in D_j$ such that $u \in v$, $\phi_j(v) = \tilde{\phi}_j(u)$ and $h_j(v)$ is the class of functions in D_j containing $\tilde{h}_j(u)$. $\square$.

Remark: Obviously, the space $L_2^m[0,T]$, sets D_j, and the functions $\phi_j(\cdot)$ and $h_j(\cdot)$, above, are only introduced for convenience of defining convergence properties of the algorithm below, because the space of regulated functions to which numerical calculations are confined is not a normed space under the $\|\cdot\|_2$ norm. $\square$

4.38 Algorithm (Solves problem (4.1)-(4.3).)

Initialization: Select an integer $j_0 \geq 0$. Then select $u_0 \in \tilde{D}_{j_0}$, $\varepsilon_0 > 0$, $\alpha \in (0,1)$, $\beta \in (0,1)$ and $\lambda_{min} \in (0,1]$.

Step 0: Set $i = 0$, $j = j_0$, $\varepsilon = \varepsilon_0$.

Step 1: Compute $\tilde{h}_j(u)(\cdot)$ according to (4.35), (4.36) by means of (4.25), (4.26).

Step 2: Set $\lambda = 1$.

Step 3: Compute $\theta_j(u_i,\lambda) = \tilde{\phi}_j(u_i-\lambda\tilde{h}_j(u_i)) - \tilde{\phi}_j(u_i) + \frac{\lambda}{2}\|\tilde{h}_j(u_i)\|_2^2$, according to (4.34), by means of (4.25), (4.35) and (4.36).[†]

Step 4: If $\theta_j(u_i,\lambda) > 0$, set $\lambda = \beta\lambda$ and go to Step 5; else, set $y = u_i - \lambda\tilde{h}_j(u_i)$ and go to Step 6.

Step 5: If $\lambda \geq \varepsilon\lambda_{min}$, go to Step 3; else set $y = u_i$ and go to Step 6.

Step 6: Compute $\tilde{\phi}_j(y)$ according to (4.34), by means of (4.25).

Step 7: If $\tilde{\phi}_j(y) - \phi_j(u_i) \leq -\varepsilon$, go to Step 9; else, go to Step 8.

Step 8: Set $j = j+1$, set $\varepsilon = \alpha\varepsilon$, and go to Step 1.

Step 9: Set $u_{i+1} = y$, set $i = i+1$ and go to Step 1. □.

4.39 **Theorem:** Suppose that assumption (4.30) is satisfied by the integration formulas used in algorithm (4.38). Consider any sequence $\{u_i\}$ constructed by algorithm (4.38) in the process of solving problem (4.1)-(4.3) and suppose that there exists $w > 0$ such that $\{u_i\} \subset \tilde{M}$. If $\{u_i\}$ is finite because the algorithm has jammed up at u_s, its last element, then $\tilde{h}(u_s) = 0$; if $\{u_i\}$ is infinite then any $\hat{u} \in \tilde{M}$ satisfying $\lim_{j\to\infty} \|u_{i_j} - \hat{u}\|_2 = 0$, where $\{u_{i_j}\}$ is any subsequence of $\{u_i\}$, must satisfy $\tilde{h}(\hat{u}) = 0$

Proof: This theorem follows directly from theorem (3.22), once we recall that problem (4.1)-(4.3) can be embedded in the problem $\min\{\phi(u)|u \in M'\}$, where ϕ and M are as in (4.17), by making use of lemma (4.18) and assumption (4.30). □

[†]Note that $\|\tilde{h}_j(u_i)\|_2^2$ is simple to compute since $\tilde{h}_j(u_i)(\cdot)$ is a step function.

Appendix A: Continuity of $x(t,\cdot)$ and $p(t,\cdot)$

A.1 **Theorem:** Under the assumptions and definitions of Section 4, given any $u^* \in \tilde{M}'$, there exist $k > 0$ and $\gamma > 0$ such that

A.2
$$\sup_{t \in [0,T]} \|x(t,u) - x(t,u^*)\| \leq k\|u-u^*\|_2$$

for all $u \in \tilde{B}(u^*,\gamma) = \{u \in \tilde{M}' \,|\, \|u-u^*\|_2 \leq \gamma\}$.

Proof: Let $w' \in (0,\infty)$ be such that

A.3
$$\sup_{t \in [0,T]} \|x(t,u^*)\| < w'.$$

Then, because of assumption (4.3)(ii) and the generalized mean value theorem, there exist $K_1 \in (0,\infty)$ and $K_2 \in (0,\infty)$ such that

A.4
$$\|f(x_1,v_1,t) - f(x_2,v_2,t)\| \leq K_1\|x_1 - x_2\| + K_2\|v_1 - v_2\|$$

for all $x_1, x_2 \in \{x' \in \mathbb{R}^n \,|\, \|x'\| \leq 2w'\}$, for all $v_1, v_2 \in \{v' \in \mathbb{R}^n | \|v'\| \leq 2w\}$, and for all $t \in [0,T]$. (w is defined in (4.3)(iv).) Now set

A.5
$$\gamma = \min\{ \frac{w'}{2} , (K_2\sqrt{T} \; e^{K_1 T})^{-1}\}.$$

Choose any $u \in \tilde{B}(u^*,\gamma)$. Since $x(\cdot,u)$ is continuous and $\|x(0,u)\| = \|x_0\| < w'$, there must exist a $\bar{t} > 0$ such that

A.6
$$\|x(t,u)\| \leq 2w' \quad \forall t \in [0,\bar{t}].$$

Then from (A.4), we have

A.7
$$\|f(x(t,u), u(t),t) - f(x(t,u^*),u^*(t),t)\|$$
$$\leq K_1\|x(t,u) - x(t,u^*)\| + K_2\|u(t) - u^*(t)\|$$

for all $t \in [0,\bar{t}]$. Consequently, for any $t \in [0,\bar{t}]$,

A.8 $$\|x(t,u) - x(t,u^*)\| = \left\| \int_0^t [f(x(t',u),u(t'),t') - f(x(t',u^*), u^*(t'),t')]dt' \right\|$$
$$\leq \int_0^t K_1\|x(t',u) - x(t',u^*)\|dt' + \int_0^t K_2\|u(t')-u^*(t')\|dt'$$

However, the Hölder Inequality gives us

A.9 $$\int_0^t K_2\|u(t')-u^*(t')\|dt' \leq \int_0^T K_2\|u(t')-u^*(t')\|dt' \leq K_2\sqrt{T}\|u-u^*\|_2$$

Thus using (A.9) and the Bellman-Gronwall inequality in (A.8), we get

A.10 $$\|x(t,u)-x(t,u^*)\| \leq K_2\sqrt{T}\,\|u-u^*\|_2 \exp\{\int_0^t K_1 dt'\}$$
$$\leq K_2\sqrt{T}\,\|u-u^*\|_2\, e^{K_1 T}$$

for all $t \in [0,\bar{t}]$. Now we show by contradiction that $\bar{t} \geq T$. If $\bar{t} < T$, we can assume without loss of generality that

A.11 $$\|x(\bar{t},u)\| = 2w'.$$

Then from (A.5), (A.10), and (A.11) we find that

A.12 $$\|x(\bar{t},u^*)\| \geq \|x(\bar{t},u)\| - \frac{w'}{2} = \frac{3w'}{2} .$$

But (A.12) contradicts (A.3) and we must conclude that $\bar{t} \geq T$. Consequently (A.10) and the fact that u was arbitrary in $\tilde{B}(u^*,\gamma)$ imply the desired result. □

A.13 <u>Theorem</u>: Under the assumptions and definitions of Section 4, given any $u^* \in M'$ and any $\varepsilon > 0$, there exists a $\gamma > 0$ such that

A.14
$$\sup_{t \in [0,T]} \| p(t,u) - p(t,u^*) \| \leq \varepsilon$$

for all $u \in \tilde{B}(u^*,\gamma) = \{u \in \tilde{M}' \mid \|u-u^*\|_2 \leq \gamma\}$.

<u>Proof</u>: Let $K_3 = \sup_{t \in [0,T]} \|p(t,u^*)\|$. Then, by assumption (4.3)(ii) and (A.1), there exist $K_4 \in (0,\infty)$ and $\gamma_1 > 0$ such that

A.15
$$\left\| \frac{\partial}{\partial x} f(x(t,u),\, u(t),t)^T \right\| \leq K_4$$

A.16
$$\left\| \frac{\partial}{\partial x} f(x(t,u),\, u(t),t)^T - \frac{\partial}{\partial x} f(x(t,u^*),\, u^*(t),t)^T \right\| \leq \frac{\varepsilon e^{-K_4 T}}{2K_3 T}$$

for all $u \in \tilde{B}(u^*,\gamma_1)$ and for all $t \in [0,T]$. By assumption (4.3)(i), (4.6), and (A.1), there exists $\gamma \in (0,\gamma_1]$ such that

A.17
$$\|p(T,u) - p(T,u^*)\| \leq \frac{\varepsilon}{2} e^{-K_4 T}$$

for all $u \in \tilde{B}(u^*,\gamma)$. Now using (4.5) and adding and subtracting appropriate terms, we obtain from (A.15), (A.16), and (A.17)

A.18
$$\begin{aligned}\|p(T-s,u) - p(T-s,u^*)\| = \|p(T,u) - p(T,u^*) \\ + \int_0^s [-\frac{\partial}{\partial x} f(x(T-s',u),u(T-s'),T-s')^T\, p(T-s',u) \\ + \frac{\partial}{\partial x} f(x(T-s',u^*),u^*(T-s'),T-s')^T\, p(T-s',u^*)]ds'\| \\ \leq \frac{\varepsilon}{2} e^{-K_4 T} + \int_0^s K_4 \|p(T-s',u) - p(T-s',u^*)\| ds'\end{aligned}$$

$$+ \frac{\varepsilon}{2K_3T} e^{-K_4T} \int_0^s \| p(T-s',u^*) \| ds'$$

$$\leq \varepsilon e^{-K_4T} + \int_0^s K_4 \| p(T-s',u) - p(T-s',u^*) \| ds'$$

for all $u \in \tilde{B}(u^*,\gamma)$ and for all $s \in [0,T]$. If we apply the Bellman-Gronwall inequality to (A.18) we obtain

A.19 $$\| p(T-s,u) - p(T-s,u^*) \| \leq \varepsilon e^{-K_4T} \exp[\int_0^s K_4 dt'] \leq \varepsilon$$

for all $u \in \tilde{B}(u^*,\gamma)$ and for all $s \in [0,T]$. Hence the proof is complete. □

Appendix B: Error Analysis of the First Order Runge-Kutta Integration Formula

In addition to the assumptions of Section 4, we make the following assumption.

B.1 Assumption: For $w \geq 0$ defined in (4.3)(iv), there exists a $K \in (0,\infty)$ such that

B.2 $$\|f(\xi_1,v,t) - f(\xi_2,v,t)\| \leq K\|\xi_1-\xi_2\|$$

and

B.3 $$\left\| \begin{bmatrix} -f(\xi_1,v,t) \\ \frac{\partial}{\partial x} f(\xi_1,v,t)^T \zeta_1 \end{bmatrix} - \begin{bmatrix} -f(\xi_2,v,t) \\ \frac{\partial}{\partial x} f(\xi_2,v,t)^T \zeta_2 \end{bmatrix} \right\| \leq K(\|\xi_1-\xi_2\|^2 + \|\zeta_1-\zeta_2\|^2)^{1/2}$$

for all ξ_1, ξ_2, ζ_1, $\zeta_2 \in \mathbb{R}^n$, for all $v \in \{v' \in \mathbb{R}^m | \|v'\| \leq 2w\}$, and for all $t \in [0,T]$. □

B.4 Definition: Let the one step integration functions F_1 and F_2 used in (4.25) and (4.26) be defined by

B.5 $$F_1(\xi,v,t,\sigma) = f(\xi,v,t)$$

B.6 $$F_2(\xi,\zeta,v,t,\sigma) = \begin{bmatrix} -f(\xi,v,t) \\ \frac{\partial}{\partial x} f(\xi,v,t)^T \zeta \end{bmatrix}.$$

(Then (4.25) and (4.26 become first order Runge-Kutta integration formulas.) □

B.7 Definition: For each $j \geq 0$, we define the truncation errors $\tau_j(s,u)$ and $\mu_j(s,u)$ by

B.8 $$\tau_j(s+1,u) = \frac{1}{\eta_j}(x(t_{js+1},u) - x(t_{js},u))$$

$$- F_1(x(t_{js},u),\ u(t_{js}),\ t_{js},\eta_j) \quad s = 0,1,\ldots,2^j-1$$

B.9 $$\tau_j(0,u) = 0$$

B.10 $$\mu_j(s-1,u) = \frac{1}{\eta_j}\left(\begin{bmatrix} x(t_{js-1},u) \\ p(t_{js-1},u) \end{bmatrix} - \begin{bmatrix} x(t_{js},u) \\ p(t_{js},u) \end{bmatrix}\right)$$

$$- F_2(x(t_{js},u),\ p(t_{js},u),\ u(t_{js-1}),\ t_{js-1},\eta_j)$$

$$s = 1,2,\ldots,2^j$$

B.11 $$\mu_j(2^j,u) = 0$$

for all $u \in \tilde{D}_j$.

The following result is an obvious consequence of Theorem (8.1.1) in [4].

B.12 Lemma: Under the assumptions of Section 4 and (B.1), there exists $Q \in (0,\infty)$ such that

B.13 $$\|x_j(s,u) - x(t_{js},u)\| \leq Q\bar{\tau}_j(u) \quad s = 0,1,\ldots,2^j$$

where

B.14 $$\bar{\tau}_j(u) \triangleq \max\{\|\tau_j(s,u)\| \mid s = 0,1,\ldots,2^j\}$$

and

B.15 $$\left\| \begin{bmatrix} x_j'(s,u) \\ p_j(s,u) \end{bmatrix} - \begin{bmatrix} x(t_{js},u) \\ p(t_{js},u) \end{bmatrix} \right\| \leq Q\left(\left\| \begin{bmatrix} x_j'(2^j,u) \\ p_j(2^j,u) \end{bmatrix} - \begin{bmatrix} x(T,u) \\ p(T,u) \end{bmatrix} \right\| + \bar{\mu}_j(u) \right) \quad s = 0,1,\ldots,2^j$$

where

B.16 $$\bar{\mu}_j(u) \triangleq \max\{\|\mu_j(s,u)\| \mid s = 0,1,\ldots,2^j\}$$

for all $u \in \tilde{D}_j$ and for all $j = 0,1,\ldots$. □

B.17 Theorem: Under the assumptions of Section 4 and (B.1), there exists $Q_1 \in (0,\infty)$ such that

B.18 $$\sup_{t\in[0,T]} \|\bar{x}_j(t,u) - x(t,u)\| \leq Q_1/2^j$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$ and for all $j = 0,1,\ldots$, where $\bar{x}_j$ is defined by (4.26) with F_1 as in (B.4).

Proof: First we note that (B.1) and the Bellman-Gromwell inequality imply that there exists $Q_2 \in (0,\infty)$ such that

B.19 $$\sup_{t\in[0,T]} \|x(t,u)\| \leq Q_2$$

for all $u \in \tilde{M}'$. Since $u \in \tilde{D}_j \cap \tilde{M}'$ implies that $\dot{x}(t,u)$ exists for all $t \in (t_{js},t_{js+1})$ $s = 0,1,\ldots, 2^j-1$, we can apply the mean value theorem to (B.8) to find $\bar{t}_{js} \in (t_{js},t_{js+1})$ such that

B.20 $$\tau_j(s+1,u) = f(x(\bar{t}_{js},u), u(\bar{t}_{js}), \bar{t}_{js}) - f(x(t_{js},u), u(t_{js}), t_{js}) \quad s = 0,1,\ldots, 2^j-1$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$. Now, since $u \in \tilde{D}_j \cap \tilde{M}'$ is constant on $[t_{js}, t_{js+1})$, we see that $\frac{d}{dt} f(x(t,u), u(t),t)$ exists for all $t \in (t_{js}, \bar{t}_{js})$ and that $f(x(t,u),u(t),t)$ is continuous for all $t \in [t_{js}, \bar{t}_{js}]$. Thus we may apply the mean value theorem to (B.20) to find $\tilde{t}_{js} \in (t_{js}, \bar{t}_{js})$ such that

B.21 $$\tau_j(s+1,u) = \frac{d}{dt} f(x(\tilde{t}_{js},u),u(\tilde{t}_{js}),\tilde{t}_{js})\ (\bar{t}_{js}-t_{js}) \quad s = 0,1,\ldots,2^j-1$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$. Also, for $u \in \tilde{D}_j \cap \tilde{M}'$,

B.22 $$\frac{d}{dt} f(x(t,u),u(t),t) = \frac{\partial}{\partial x} f(x(t,u),u(t),t) f(x(t,u),u(t),t) + \frac{\partial}{\partial t} f(x(t,u),u(t),t)$$

for all $t \in (t_{js}, t_{js+1})$. Thus by assumption (4.3)(ii), (B.19), (B.20), and (B.22), there exists $Q_3 \in (0,\infty)$ such that

B.23 $$\|\tau_j(s+1,u)\| \leq Q_3(\bar{t}_{js}-t_{js}) \leq Q_3\eta_j \quad s = 0,1,\ldots, 2^j-1$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$. But now from (4.19), (B.9), (B.13), (B.14), and (B.23),

B.24 $$\|x_j(s,u) - x(t_{js},u)\| \leq QQ_3T/2^j. \quad s = 0,1,\ldots, 2^j.$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$. Then, as before, we can apply the mean value theorem and (B.19) to show that there exists $Q_4 \in (0,\infty)$ such that

B.25 $$\|x(t,u) - x(t_{js},u)\| \leq Q_4\eta_j \quad s = 0,1,\ldots, 2^j-1$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$ and for all $t \in [t_{js}, t_{js+1})$. The theorem now follows directly from (4.19), (4.26), (B.24), and (B.25). □

B.26 <u>Lemma</u>: Under the assumptions of Section 4, there exists $Q_6 \in (0,\infty)$

such that

B.27 $$\sup_{t\in[0,T]} \|p(t,u)\| \leq Q_6$$

for all $u \in \tilde{M}'$.

Proof: Because (B.19) holds, (4.3) implies that there exists $Q_5 \in (0,\infty)$ such that

B.28 $$\left\| \frac{\partial}{\partial x} f(x(t,u),u(t),t)^T \right\| \leq Q_5$$

and

B.29 $$\|p(T,u)\| \leq Q_5$$

for all $u \in \tilde{M}'$ and for all $t \in [0,T]$. Then from (4.5),

B.30 $$\|p(T-s,u)\| = \|p(T,u) - \int_0^s \frac{\partial}{\partial x} f(x(T-s',u),u(T-s'),T-s')^T p(T-s',u) \cdot ds'\|$$
$$\leq Q_5 + \int_0^s Q_5 \|p(T-s',u)\| ds'$$

for all $u \in \tilde{M}'$ and for all $s = [0,T]$. Thus, by the Bellman-Gronwall inequality,

B.31 $$\|p(T-s,u)\| \leq Q_5 e^{Q_5 s} \leq Q_5 e^{Q_5 T}$$

for all $u \in \tilde{M}'$ and for all $s \in [0,T]$ which is the desired result. □

B.32 Theorem: Suppose that the assumptions of Section 4 and (B.1) hold. Then, given any $\varepsilon > 0$, there exists an integer $J(\varepsilon) \geq 0$ such that

B.33 $$\sup_{t\in[0,T]} (\|x(t,u) - \bar{x}_j'(t,u)\|^2 + \|p(t,u) - \bar{p}_j(t,u)\|^2)^{1/2} \leq \varepsilon$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$ and for all $j \geq J(\varepsilon)$ where $\bar{x}_j'$ and $\bar{p}_j$ are defined by (4.27) and (4.28) with F_2 as in (B.4).

Proof: First we note that (B.19) holds i.e.

B.34 $$\sup_{t\in[0,T]} \|x(t,u)\| \leq Q_2$$

for all $u \in \tilde{M}'$. Now suppose that we let $\mu_j^1(s,u)$ denote the first n components of $\mu_j(s,u)$ and $\mu_j^2(s,u)$ denote the second n components of $\mu_j(s,u)$. Then proceeding as in the proof of (B.17), we can show that there exists $Q_3 \in (0,\infty)$ such that

B.35 $$\|\mu_j^1(s-1,u)\| \leq Q_3\eta_j \quad s = 1,2,\ldots, 2^j$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$. Consequently, from (B.11) we find

B.36 $$\mu_j^1(s,u) \leq Q_3\eta_j \quad s = 0,1,\ldots, 2^j$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$. From (B.6) and (B.10),

B.37 $$\mu_j^2(s-1,u) = \frac{1}{\eta_j}[p(t_{js-1},u) - p(t_{js},u) - \eta_j \frac{\partial}{\partial x} f(x(t_{js},u),u(t_{js-1}),t_{js})^T p(t_{js},u)] \quad s = 1,\ldots, 2^j.$$

Since for $u \in \tilde{D}_j \cap \tilde{M}'$, $p(\cdot,u)$ is continuous on $[0,T]$ and $\dot{p}(t,u)$ exists for $t \in (t_{js-1},t_{js})$, we can apply the mean value theorem to (B.37) to find a $\bar{t}_{js} \in (t_{js-1},t_{js})$ such that (notice that $t_{js-1} - t_{js} = -\eta_j$)

B.38 $$\mu_j^2(s-1,u) = \frac{\partial}{\partial x} f(x(\bar{t}_{js},u),u(\bar{t}_{js}),\bar{t}_{js})^T p(\bar{t}_{js},u) - \frac{\partial}{\partial x} f(x(t_{js},u),u(t_{js-1}),t_{js})^T p(t_{js},u) \quad s = 1,\ldots, 2^j.$$

Also, $u(t_{js-1}) = u(\bar{t}_{js})$ since $u \in \tilde{D}_j \cap \tilde{M}'$. Thus (4.3)(ii), (B.26), and (B.34) imply that the right hand side of (B.38) converges to zero uniformly for all $u \in \tilde{D}_j \cap \tilde{M}'$ as $j \to \infty$. Hence, from (B.11) and (B.36), we conclude that there exists an integer $J_1 \geq 0$ such that

B.39 $$\mu_j(u) \leq Q^{-1} \frac{\varepsilon}{4}$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$ and for all $j \geq J_1$. Furthermore, from (4.3)(i), (4.22) and (B.24), there must exist an integer $J_2 \geq J_1$ such that

B.40 $$[\|x_j'(2^j,u) - x(T,u)\|^2 + \|p_j(2^j,u) - p(T,u)\|^2]^{1/2} \leq Q^{-1} \frac{\varepsilon}{4}$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$ and for all $j \geq J_2$. Now, applying (B.39) and (B.40) to (B.15) yields

B.41 $$[\|x_j'(s,u) - x(t_{js},u)\|^2 + \|p_j(s,u) - p(t_{js},u)\|^2]^{1/2} \leq \frac{\varepsilon}{2} \quad s = 0,1,\ldots, 2^j$$

for all $u \in \tilde{D}_j \cap \tilde{M}'$ and for all $j \geq J_2$. To complete the proof, we note that $x(\cdot,u)$ and $p(\cdot,u)$ are continuous on $[0,T]$ and for $u \in \tilde{D}_j \cap \tilde{M}'$, $\dot{x}(t,u)$ and $\dot{p}(t,u)$ exist for $t \in (t_{js-1},t_{js})$. Also, in view of (B.34) and Lemma (B.26), $\dot{x}(t,u)$ and $\dot{p}(t,u)$ are bounded on $(t_{js-1},t_{js}) \times \tilde{D}_j \cap \tilde{M}'$. Thus by applying the mean value theorem, we can find an integer $J(\varepsilon) \geq J_2$ such that

B.42 $$\left[\|x(t_{js},u) - x(t,u)\|^2 + \|p(t_{js},u) - p(t,u)\|^2\right]^{1/2} \le \frac{\varepsilon}{2}$$

$$s = 1,2,\ldots, 2^j$$

for all $t \in [t_{js-1}, t_{js})$, for all $u \in \tilde{D}_j \cap \tilde{M}'$, and for all $j \geq J(\varepsilon)$. The theorem now follows directly from (4.25), (4.26), (B.41), and (B.42). □

Conclusion: To evaluate the performance of the optimal control algorithm (4.38) we have performed a number of numerical experiments. These experiments were done as follows: For a given problem, we would choose an initial point, an integration formula (Runge-Kutta order 1 or 4) and an integer j_{max} with j_{max} = 10, or 8 respectively. Then we would solve this problem numerically twice. The first time always setting $j = j_{max}$ in (4.38), i.e. not using the adaptive procedure. The second time we would set $j_0 = 2$ and use the algorithm as stated, stopping to increase j when $j = j_{max}$ was reached. In all of our experiments we have found that the adaptive method was at least twice as fast as the fixed step size method. In several cases it was five to ten times faster.

Thus, the method presented in this paper is of theoretical and of practical interest. It shows that it is possible to account theoretically for the effects of numerical integration. Also, it shows that the numerical integration process can be handled adaptively, and our experiments indicate that this leads to reduced computational times.

The algorithm presented in this paper is of the humble, first order variety, however, it is not difficult to see that the adaptive integration techniques proposed could also be used in conjunction with Newton-Raphson, or implementable conjugate gradient methods of the type described in [5].

Finally, it should be pointed out that there are also problems outside of optimal control where function approximations must be used and where the algorithm (3.9) presented in this paper could be quite useful.

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