NONLINEAR N-PORTSL I--CHARACTERIZATION, CLASSIFICATION, AND REPRESENTATION

by

L. O. Chua and Y-F Lam

Memorandum No. ERL-M307

10 September 1971

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NONLINEAR N-PORTS: I--CHARACTERIZATION,

CLASSIFICATION, AND REPRESENTATION

by

L. O. Chua[†] and Y-F Lam[††]

ABSTRACT

This paper is concerned with the foundational aspects of nonlinear n-ports. Differential geometry provides the natural setting for this study. The p-atlas proposed herein for characterizing algebraic n-ports is general enough to provide the "closure property" desirable in the formulation of a unified theory of nonlinear n-ports. It is shown that most n-ports of practical interest may be characterized as an immersion in R^{2n}. The rank of the immersion provides a logical definition for the dimension of an n-port. An "n"-port is then classified as singular, regular or dense, depending on whether its rank "m" is less than, equal to, or greater than "n".

Most results in this paper are concerned with regular n-ports and their representations. The existence of a hybrid or a transmission representation is interpreted geometrically as equivalent to the commutativity of a certain projection diagram. From the global point of view, a regular n-port is classified into various subclasses such as increasing n-ports, homeomorphic n-ports, proper n-ports, etc. Some of these classifications have the desirable property of being an invariant of the n-port. From the local point of view, the important subclass of reciprocal n-ports is characterized. Reciprocity and antireciprocity criteria for various modes of representations are derived.

Research sponsored by the U. S. Navy Electronics Command, Contract N00039-71-C-0255 to the University of California, and by the National Research Council of Canada, Grant A 7113.

[†]Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California 94720.

[††]Department of Electrical Engineering, University of Waterloo, Waterloo, Ontario, Canada.

I. INTRODUCTION

An electrical n-port is a black box with n pairs of external terminals called "ports" such that the current entering a terminal of each port is equal to the current leaving the second terminal. The theory of n-ports is probably the most fundamental aspect of network theory since most network theoretic concepts such as reciprocity, passivity, losslessness, etc. are defined only for n-ports. Indeed, with the help of the "connection n-port" recently introduced by Brayton [1], any network may be viewed as an interconnection of appropriate n-ports[1]. Although network theorists have succeded in developing a unified theory of linear n-ports during the last two decades [3-6], very little has yet been done for nonlinear n-ports. The relatively slow progress in the theory of nonlinear n-ports is due not only to the difficulty in the mathematics involved, but also to the lack of a precise and logical characterization and classification of n-ports. The class of nonlinear n-ports is much too large -- indeed, it includes all n-ports! In order to obtain useful results, it would be necessary to characterize and classify n-ports into appropriate hierarchies. Our main objective in this paper is to present an in-depth study of the characterization, classification, and representation of n-ports. The results presented will serve as the foundation for a subsequent paper dealing with the circuit theoretic properties of nonlinear n-ports.

II. MATHEMATICAL CHARACTERIZATION OF N-PORTS

Let $v = [v_1, v_2, \ldots v_n]$, $i = [i_1, i_2, \ldots, i_n]$, $\varphi = [\varphi_1, \varphi_2, \ldots, \varphi_n]$, and $q = [q_1, q_2, \ldots, q_n]$ denote respectively the port <u>voltage</u>, <u>current</u>, <u>flux-linkage</u>, and <u>charge</u> vectors[2] associated with each n-port N, where $v_j = \dot{\varphi}_j$ and $i_j = \dot{q}_j$. If we define the "mixed port vectors" $x = [x_1, x_2, \ldots, x_n]$ and $y = [y_1, y_2, \ldots, y_n]$, where $\{x_j, y_j\} \subset \{v_j, i_j, \varphi_j, q_j\}$, $x_j \neq y_j$, $x_j \neq \dot{y}_j$ and $\dot{x}_j \neq y_j$ $\forall\, j = 1, 2, \ldots, n$, then N is said to be an <u>algebraic</u> <u>n-port</u> if it is characterized by m functional relationships

$$f_j(x, y, t) = 0, \quad j = 1, 2, \ldots, m \tag{1}$$

where m need not equal n. An algebraic n-port is said to be <u>time-invariant</u> if the time variable t is not explicitly present in (1). Otherwise, it is said to be <u>time-varying</u>. An algebraic n-port is said to be an <u>n-port resistor</u> if $\{x, y\} = \{v, i\}$, an <u>n-port inductor</u> if $\{x, y\} = \{\varphi, i\}$, an <u>n-port capacitor</u> if $\{x, y\} = \{v, q\}$, and an <u>n-port memristor</u>[3] if $\{x, y\} = \{\varphi, q\}$. Since (1) does not involve either time derivatives or integrals of x and y, there is no loss of generality in restricting our study to time-invariant algebraic n-ports. Moreover, to avoid redundancy, we will address this paper only to n-port resistors. Analogous results would apply <u>mutatis-mutandis</u> to any other algebraic n-port.

In order to motivate and justify the large hierachy of n-ports to be introduced in this paper, consider the following examples illustrating the types of <u>composite</u> v-i relationships that could arise as a result of inter-connecting n-ports together.

<u>Example 1</u>. The simplest case consists of connecting two 1-ports R_1 and R_2 in series as shown in Fig. 1, where six interesting possibilities are shown. Observe that with only 2 segments per v_j - i_j curve, j = 1, 2, it is possible to obtain a composite curve with self-intersections, as in (a) and (b), or with a finite perimeter, as in (c). More complicated v-i curves can be obtained as in (d), (e) and (f) with only 3 segments per v_j - i_j curve. Here, it is possible to obtain two disjoint branches.

<u>Example 2</u>. The circuit in Fig. 2(a) consists of a 2-port N (in fact, a current-controlled current source) and two 1-ports R_A and R_B characterized by the v_j - i_j curves shown in Figs. 2(b) and (c). The composite v-i curve shown in Fig. 2(d) consists of the union of a closed line segment and two isolated points. It is

easy to see that if we replace R_B by a short circuit, the composite v-i curve would reduce to 3 isolated points; namely, (0, -1), (0, 0), and (0, 1). If we also replace R_A by an open circuit, the composite v-i curve degenerates into one point at the origin and becomes a nullator [6].

Example 3. The circuit in Fig. 3(a) consists of a 2-port N and three 1-ports. With the v_j - i_j curves shown in Figs. 3(b) and (c) for R_A and R_B, the composite v-i relationship covers an entire area, as shown in Fig. 3(d). In fact, if we replace R_A and R_C by short circuits, and R_B by an open circuit, the composite v-i relationship would cover the entire v-i plane and become a norator [6].

Example 4. To show that an unicursal v-i curve [8] $v = v(\rho)$, $i = i(\rho)$, $\rho \in P \subset R^1$, could be synthesized, including those with self-intersections and cusps, we introduce a new linear 3-port N_ρ in Fig. 4(a), called the "unicursal 3-port" and characterized by the hybrid matrix:

$$\begin{bmatrix} i_1 \\ v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \\ v_3 \end{bmatrix}$$

If we connect two voltage-controlled resistors R_A and R_B (characterized by $i_A = g_A(v_A)$ and $i_B = g_B(v_B)$ respectively) across ports 2 and 3 of N_ρ, as shown in Fig. 4(b), we obtain a 1-port with the composite v_1 - i_1 curve $i_1 = g_A(\rho)$, $v_1 = g_B(\rho)$. To show that the unicursal 3-port is nothing exotic, we offer a simple realization using only a voltage-controlled voltage source N_a and a current-controlled voltage source N_b, as shown in Fig. 4(c).

Example 5. To show that actually any subset of points in the v-i plane could be realized, we introduce yet another linear 3-port in Fig. 5(a), called a "union 3-port" and characterized by $v_1 = v_2 = v_3$ and $i_1 = -i_2 = -i_3$. If we connect two 1-ports R_A and R_B across ports 2 and 3 of a union 3-port as in Fig. 5(b), then the composite v_1 - i_1 relationship of the resulting 1-port is simply the point set union of the v-i curves for R_A and R_B. Indeed, a v-i curve with any number of branches could be synthesized with the help of a "union n-port". For example, the union 4-port N_U^4 synthesized in Fig. 5(c) can be used to generate a v-i curve with 3 distinct branches. To show that even a "union 3-port" is not too exotic, we offer a simple realization in Fig. 5(d) using only three common 2-ports; namely, two voltage-controlled voltage sources N_a and N_c, and a current-controlled current source N_b.

The preceding examples show that very exotic v-i curves could result from interconnecting n-ports with "simple" characteristics. It is clear, therefore, that in order to endow our theory with some "closure property"[4], it is necessary to allow our n-ports to be characterized by a union of points, curves, hyper-surfaces, and even continuum of points. In order to obtain concrete results, however, it would also be necessary for us to classify n-ports into various more manageable subclasses. We will now proceed to formulate such a theory with the help of some basic concepts from differential geometry [9].

<u>Def. 1.</u> <u>Parametric chart and atlas</u>

Let $S_j \subset R^{2n}$ be a subset of R^{2n}. A function

$$\mu_j: \quad P_j \subset R^{m_j} \to R^{2n}, \qquad 0 \le m_j \le 2n \tag{2}$$

is said to be a <u>parametric</u> <u>chart</u> (p-chart) of S_j if $\mu_j(P_j) = S_j$. In this case, P_j is said to be the <u>parametric</u> <u>space</u>. A collection $\mathcal{M} = \{\mu_1, \mu_2, \ldots\}$ of p-charts is said to be a <u>parametric</u> <u>atlas</u> (p-atlas) of $S = \bigcup_j S_j$ in R^{2n} if for each $x \in S$, $\exists$ a $\mu_j \in \mathcal{M}$ such that $x = \mu_j(\rho_j)$, $\rho_j \in P_j$.[5]

A point [v, i] on the characteristic surface of an n-port will henceforth be considered as a point in R^{2n}. Hence an algebraic n-port is completely characterized by a p-atlas since every point on the characteristic surface in R^{2n} is identified by at least one <u>local</u> <u>coordinate</u> <u>system</u> $\mu_j(\cdot)$. For example, a 1-port characterized by $v = i^3$ is parameterized by μ: $R^1 \to R^2$, where $\mu(\rho) \equiv [v(\rho) = \rho^3,\ i(\rho) = \rho]$, $\rho \in R^1$. A nullator is parametrized by μ: $R^0 \to R^2$, where $\mu(0) \equiv [v(0) = 0,\ i(0) = 0]$, $0 \in R^0$. A norator is parametrized by $\mu: R^2 \to R^2$, where $\mu(\rho) \equiv [v(\rho) = \rho_1,\ i(\rho) = \rho_2]$, $\rho = [\rho_1, \rho_2] \in R^2$. Observe that all of these n-ports are characterized by one p-chart. If an n-port is characterized by a single p-chart $\mu: P \subset R^m \to R^{2n}$ of class C^k [9], it is said to be a <u>C^k-parametrizable</u> <u>n-port</u>.

<u>Def. 2.</u> <u>Immersed n-port and its dimension</u>

A C^k-parametrizable n-port ($k \geq 1$) characterized by the single p-chart

$$\mu: R^m \to R^{2n}, \quad 0 \leq m \leq 2n \tag{3}$$

is said to be a <u>C^k-immersed</u> <u>n-port</u> with <u>dimension</u> "m" if $\mu(.)$ is a C^k-immersion[6] [9].

The class of immersed n-ports would cover almost all parametrizable n-ports of interest. For example, in the case of 1-ports, it includes any unicursal curve with self-intersections so long as there are no "cusps" on the curve. In fact, the class of <u>immersed</u> <u>n-ports</u> is probably the most general hierachy that will submit to a rigorous mathematical study. It is also the largest class that makes practical sense, as shown by the following important theorem due to Whitney (see p.58, Theorem 4.2 of [9]) which we now restate in circuit theoretic terms:

<u>N-port Approximation Theorem</u>

Any C^k-parametrizable ($k \geq 2$) n-port N characterized by $\mu: R^m \to R^{2n}$ can be approximated arbitrarily closely by a C^k-immersed n-port N' with dimension m provided $m \leq n$. Moreover, N' will remain an immersed n-port with dimension m under arbitrary infinitesimal parameter variation.

Roughly speaking, this theorem says that if $m \leq n$, then the n-port is not sensitive to parameter variations. Therefore, an immersed n-port is <u>generic</u> in the sense of Thom [10]. This observation motivates our next definition.

Def. 3. Regular generic n-ports

A C^k-immersed n-port is said to be generic if it has dimension $m \le n$, and dense if it has dimension $m > n$. A generic n-port is regular if $m = n$ and singular if $m < n$. A regular n-port is strongly regular if $\mu(R^n)$ is a closed n-dimensional submanifold of R^n.

In view of Def. 2 and 3, we can now classify quite logically a nullator ($v = 0$, $i = 0$) as a 0-dimensional singular 1-port, a resistor characterized by $v = v(\rho)$ and $i = i(\rho)$ as a 1-dimensional regular 1-port, and a norator ($v = \rho_1$, $i = \rho_2$) as a 2-dimensional dense 1-port. In the same context, a time-dependent voltage source ($v = v_s(t)$) or current source ($i = i_s(t)$) can be classified as a time-varying 1-dimensional regular 1-port. Similarly, a current-controlled voltage source ($v_1 = k\rho_1$, $v_2 = 0$, $i_1 = \rho_2$, $i_2 = \rho_1$) as well as the remaining 3 types of controlled sources can be classified as 2-dimensional regular 2-ports. Observe also that whereas the "unicursal 3-port" is a 3-dimensional regular 3-port, the "union 3-port" is a 2-dimensional singular 3-port since it is characterized by the immersion μ: $R^2 \to R^6$.

A generic n-port N need not be a submanifold in R^{2n} since the hyper-surface may intersect itself. However, if N is singular, then we can assert:

Theorem 1.

The characteristic surface of any singular n-port N can be approximated arbitrarily closely by an immersed submanifold [9] in R^{2n}.

Proof. N is singular means that N is characterized by an immersion μ: $R^m \to R^{2n}$, where $m < n$. Now Theorem 4.3 on p.61 of [9] implies that $\mu(.)$ can be approximated arbitrarily closely by an injective immersion $\hat{\mu}(.)$ in R^{2n} where $\hat{\mu}$: $R^m \to R^{2n}$. Since $\hat{\mu}(R^m)$ is an immersed submanifold in R^{2n}, our conclusion follows. Q.E.D.

III. EQUIVALENT REPRESENTATIONS OF REGULAR N-PORTS

We will henceforth restrict our discussion to the most common and important type of nonlinear n-ports; namely, the class of regular n-ports characterized by a C^k-immersion μ: $R^n \to R^{2n}$. Hence, a regular n-port can always be characterized by:

$$v = v(\rho), \qquad i = i(\rho) \tag{4}$$

where $\rho \epsilon R^n$, $v \epsilon R^n$, and $i \epsilon R^n$. It is usually possible to recast (4) into various equivalent modes of representations with the help of the two commutative diagrams [9] shown in Fig. 6. In both diagrams, $\pi_x(.)$ represents a projection map from R^{2n} to $X = R^n$, while $\pi_y(.)$ represents a projection map from R^{2n} to $Y = R^n$. Observe that diagram (a) commutes if, and only if, the composition $\hat{x} \equiv \pi_x o \mu$ is bijective. Under this condition, (4) can be recast into the form $f: X = R^n \to Y = R^n$, or $y = f(x)$, $x \epsilon R^n$, $y \epsilon R^n$. A dual statement applies to diagram (b). In either case, the vector $[x, y]$ can be considered as obtained from a permutation of the vector $[v, i]$; namely;

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = \Sigma \begin{bmatrix} v \\ i \end{bmatrix} \tag{5}$$

where Σ is a 2n x 2n permutation matrix [11]. We will sometimes abbreviate (5) by writing $[x, y] = \sigma\,[v, i]$, where $\sigma(.)$ is the associated bijective permutation function.

Def. 4. Regular representation

A regular n-port is said to admit a regular representation if there exists a permutation matrix Σ which makes either diagram (a) or diagram (b) in Fig. 6 commute, i.e., either $\hat{x} \equiv \pi_x o \mu$ or $\hat{y} \equiv \pi_y o \mu$ is bijective. In particular, a regular representation is said to be a hybrid representation if $A = D$ and $B = C$.

It is easy to show that the permutation matrix Σ defining a hybrid representation is orthogonal, unimodular, and elementary. Moreover, the n x n submatrices $A = D$ and $B = C$ are diagonal and $A + B = 1_n$, $AB = BA = 0_n$,

$AA = A$, $BB = B$ where 1_n and 0_n denote the identity and the zero matrix, respectively. Although there are 2^n distinct hybrid representations, only four are commonly encountered in practice; namely, the <u>conductance representation</u>, the <u>resistance representation</u>, the <u>hybrid I representation</u>, and the <u>hybrid II representation</u>. For future reference, these four representations are defined in Table 1 (rows 1 to 4). Closely related to the hybrid representations I and II are the four <u>conjugate hybrid representations</u> defined in Table 1 (rows 5 to 8). These representations are extremely convenient for studying the potential functions of reciprocal n-ports [12].

Besides the four hybrid and the four conjugate hybrid representations, there are two slightly modified forms of regular representations which are useful in studying the transformation properties of nonlinear n-ports (n is a positive even integer); namely, the <u>transmission representation I</u> and <u>II</u> defined in Table 1 (rows 9 and 10).

Generally speaking, a regular n-port which admits of one hybrid representation may fail to admit another representation. It is desirable therefore to derive conditions which prevent this from happening as done in [13, 14]. More recently, Desoer and Oster [15] have obtained conditions which guarantee a reciprocal n-port to admit all 2^n hybrid representations. We now generalize their results to arbitrary n-ports.

<u>Lemma 1</u>.

Let a regular n-port be characterized by a C^1-hybrid representation:

$$y = \begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} h_a(x_a, x_b) \\ h_b(x_a, x_b) \end{bmatrix} \equiv h(x) = [h_1(x), h_2(x), \ldots, h_n(x)] \qquad (6)$$

where $[x, y] = \sigma\,[v, i]$, $x_a \epsilon X_a = R^m$, $x_b \epsilon X_b = R^{n-m}$, $y_a \epsilon Y_a = R^m$ and $y_b \epsilon Y_b = R^{n-m}$. Suppose the following conditions are satisfied:

(i) If $n = 2$, $\det [\partial h_a(x)/\partial x_a] \neq 0 \ \forall \ x \in R^n$

If $n \neq 2$, $\det [\partial h_a(x)/\partial x_a] > 0$ (or < 0) $\forall \ x \in R^n$, except possibly for at most a set S_a of isolated points in R^n.

(ii) $\lim_{||x_a|| \to \infty} ||h_a(x_a, x_b)|| = \infty \ \ \forall \ x_b \in X_b$.

Then N admits of the following equivalent hybrid representation:

$$\begin{bmatrix} x_a \\ y_b \end{bmatrix} = \begin{bmatrix} g_a(y_a, x_b) \\ g_b(y_a, x_b) \end{bmatrix} = g(y_a, x_b) \tag{7}$$

where $g(.,.)$ is a continuous function on $Y_a \times X_b = R^n$.

Proof. It follows from the generalized global implicit function theorem derived in [16] that there exists a continuous function g_a: $Y_a \times X_b \to X_a$ such that $x_a = g_a(y_a, x_b)$. Hence, $y_b = h_b(x_a, x_b) = h_b(g_a(y_a, x_b), x_b) \equiv g_b(y_a, x_b)$. Clearly, $g_b(.,.)$ is continuous since composition of continuous maps is continuous. Q.E.D.

Theorem 2. Hybrid Representation Theorem.

If a regular n-port N admits of one hybrid representation $y = h(x)$, where $[x, y] = \sigma [v, i]$ and h: $R^n \to R^n$, then the following two conditions are sufficient to guarantee that N admits all 2^n distinct hybrid representations:

(i) $\partial h(x)/\partial x$ is a P-matrix[7], $\forall \ x \in R^n$ (8)

(ii) $\lim_{||x|| \to \infty} |h_j(x)| = \infty$, $j = 1, 2, \ldots, n$. (9)

Proof. Let $[\tilde{x}, \tilde{y}] = \tilde{\sigma}[v, i]$ where $\tilde{\sigma}$ is any one of the 2^n distinct permutations of $[v, i]$. Since $[x, y] = \sigma [v, i]$, $[\tilde{x}, \tilde{y}] = \tilde{\sigma} \circ \sigma^{-1}[x, y] \equiv \hat{\sigma} [x, y]$. Let A and B be the permutation submatrices associated with $\hat{\sigma}$. Then $\tilde{x} = Ax + By \equiv [\tilde{x}_a, \tilde{x}_b]$ and $\tilde{y} = Bx + Ay \equiv [\tilde{y}_a, \tilde{y}_b]$, where $\tilde{x}$ and $\tilde{y}$ are partitioned arbitrarily into m and n-m components. If $B = 0_n$, then the theorem is trivially true since $A = 1_n$ and $[\tilde{x}, \tilde{y}] = [x, y]$. If $A = 0_n$, then $B = 1_n$ and we have $[\tilde{x}, \tilde{y}] = [y, x]$. Hence, the theorem is again true since (8) and (9) imply h is homeomorphic onto [18].

It suffices therefore to consider the case $A \neq 0_n$ and $B \neq 0_n$. Rearrange the n hybrid equations $y_j = h_j(x_1, x_2, \ldots, x_n)$, $j = 1, 2, \ldots, n$ into the form:

$$y' \equiv \begin{bmatrix} y_A \\ y_B \end{bmatrix} = \begin{bmatrix} h_A(x_A, x_B) \\ h_B(x_A, x_B) \end{bmatrix} = \begin{bmatrix} h_A(x') \\ h_B(x') \end{bmatrix} \equiv h'(x')$$

where the variables y_j and x_j associated with the non-zero columns of B are lumped together in y_B and x_B, respectively. Clearly, $\partial h_B(x')/\partial x_B$ is a <u>principal submatrix</u> of $\partial h(x)/\partial x$ and has therefore a positive determinant in view of (8). Moreover, (9) implies that

$$\lim_{||x_B|| \to \infty} || h_B(x_A, x_B) || = \infty \quad \forall \quad x_A \in R^m.$$

It follows from Lemma 1 that N admits of the equivalent representation

$$x_B = g_B(x_A, y_B)$$

$$y_A = g_A(x_A, y_B) = g_A(x_A, g_B(x_A, y_B)) \equiv \tilde{g}_A(x_A, y_B).$$

$$\therefore \quad \tilde{y} \equiv \begin{bmatrix} y_A \\ x_B \end{bmatrix} = \begin{bmatrix} \tilde{g}_A(x_A, y_B) \\ g_B(x_A, y_B) \end{bmatrix} \equiv \tilde{h}(\tilde{x}). \qquad \text{Q.E.D.}$$

IV. GLOBAL CHARACTERIZATION OF REGULAR N-PORTS

It is well known that the qualitative properties of nonlinear networks depend to a great extent on the "global" characteristics of the elements' nonlinearity. For networks made up of interconnection of 1-ports, various sufficient conditions have been obtained which ensure either the existence and uniqueness of solutions for resistive networks [19, 20], or the global stability of dynamic networks [21-23]. Some of these conditions require the resistors to be characterized by strictly monotonically-increasing functions. Others require the v-i curve to be either voltage-controlled or current-controlled. Still others require the v-i curve to be surjective or bijective. A precise classification of n-ports in terms of their global characteristics is fundamental not only to the analysis of nonlinear networks but to synthesis as well [24].

In attempting to generalize the various global characterization of 1-ports to n-ports, many subtleties and complications may arise. For example, whereas any "injective" 1-port is strictly monotonic, there exist injective n-ports which are not monotonic. Moreover, an n-port which is bijective with respect to one hybrid representation may fail to be bijective with respect to another representation. On the other hand, there are other characterizations which are independent of coordinate systems, and are therefore <u>invariants</u> of the n-ports. These possibilities make it necessary for us to define many seemingly redundant but distinct global characterizations.

<u>Def. 5</u>. <u>Non-decreasing n-ports</u>

A regular n-port N characterized by (4) is said to be non-decreasing if $\alpha(\rho_a, \rho_b)\,\beta(\rho_a, \rho_b) \geq 0 \;\; \forall\, \rho_a, \rho_b \in R^n$, where $\alpha(\rho_a, \rho_b) \equiv \langle v(\rho_a) - v(\rho_b), \rho_a - \rho_b \rangle$ and $\beta(\rho_a, \rho_b) \equiv \langle i(\rho_a) - i(\rho_b), \rho_a - \rho_b \rangle$. If N admits of a hybrid representation $y = h(x)$, then N is <u>non-decreasing</u> if $\langle x_a - x_b, h(x_a) - h(x_b) \rangle \geq 0$ $\forall\, x_a, x_b \in R^n$.[8]

<u>Def. 6</u>. <u>Increasing and uniformly increasing n-ports</u>

Let N be a regular n-port characterized by a hybrid representation $y = h(x)$ and let $\alpha(x_a, x_b) \equiv \langle x_a - x_b, h(x_a) - h(x_b) \rangle$. Then N is said to be <u>increasing</u> if $\alpha(x_a, x_b) > 0 \;\forall\, x_a \neq x_b \in R^n$. N is said to be x-uniformly increasing if there exists a constant $c > 0$ such that $\alpha(x_a, x_b) \geq c\, \| x_a - x_b \|^2$. If, in addition, $h(.) \in C^1$ and the Jacobian matrix of $h(.)$ is bounded on R^n, then N is said to be <u>strongly</u> <u>uniformly-increasing</u>.

<u>Def. 7</u>. <u>Proper n-port</u>

A regular n-port N which can be characterized by a hybrid representation $y = h(x)$ is said to be x-proper. If, in addition, $h(.)$ is surjective on R^n, then N is said to be <u>proper</u>.

Theorem 3. Invariant Characterization.

The definitions for an increasing, non-decreasing, strongly uniformly-increasing, and proper n-port are independent of the mode of the hybrid representation and are therefore invariants of the n-port.

Proof. Let $y = h(x)$, $[x, y] = \sigma[v, i]$ and $\tilde{y} = \tilde{h}(\tilde{x})$, $[\tilde{x}, \tilde{y}] = \tilde{\sigma}[v, i]$ be any two hybrid representations for N. Since the permutation functions $\sigma(.)$ and $\tilde{\sigma}(.)$ are bijective, we can write $[x, y] = \sigma\circ\tilde{\sigma}^{-1}[\tilde{x}, \tilde{y}]$ and $[\tilde{x}, \tilde{y}] = \tilde{\sigma}\circ\sigma^{-1}[x, y]$. Hence, there exists a permutation matrix $\Sigma = \Sigma^{-1}$ such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \quad , \quad \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Now let $[v_a, i_a]$ and $[v_b, i_b]$ be any two points in R^{2n} and let $[x_a, y_a] = \sigma[v_a, i_a]$, $[\tilde{x}_a, \tilde{y}_a] = \tilde{\sigma}[v_a, i_a]$, $[x_b, y_b] = \sigma[v_b, i_b]$, and $[\tilde{x}_b, \tilde{y}_b] = \tilde{\sigma}[v_b, i_b]$. Now using the properties $AA = A$, $BB = B$, $AB = BA = 0_n$ and $A + B = 1_n$, we obtain:

$$\begin{aligned} <(\tilde{y}_a - \tilde{y}_b), (\tilde{x}_a - \tilde{x}_b)> &= <B(x_a - x_b), A(x_a - x_b)> + <B(x_a - x_b), B(y_a - y_b)> \\ &\quad + <A(y_a - y_b), A(x_a - x_b)> + <A(y_a - y_b), B(y_a - y_b)> \\ &= <B(x_a - x_b), (y_a - y_b)> + <A(x_a - x_b), (y_a - y_b)> \\ &= <(x_a - x_b), (y_a - y_b)> . \end{aligned}$$

Hence, the definitions for increasing and non-decreasing n-ports are invariants of the hybrid representations. It has been shown in [13, 14] that if N admits a hybrid representation $y = h(x)$ where the describing function $h(.)$ is uniformly increasing with bounded Jacobian matrix on R^n, then N can be represented by any one of the 2^n possible hybrid representations and the describing function of each hybrid representation is uniformly increasing with bounded Jacobian matrix of R^n. Hence, strongly uniformly-increasing n-ports are independent of the mode of the hybrid representation. It remains to prove that the definition of proper n-ports is also invariant.

In view of Def. 4 and the commutative diagrams on Fig. 6, N admits the hybrid representations $h(.)$ and $\tilde{h}(.)$ implies that $y = h(x) = \hat{y}o\hat{x}^{-1}(x)$ and $\tilde{y} = \tilde{h}(\tilde{x}) = \hat{\tilde{y}}o\hat{\tilde{x}}^{-1}(\tilde{x})$, where $\hat{x}(.)$ and $\hat{\tilde{x}}(.)$ are bijective maps. Now suppose $h(.)$ is surjective, then $\hat{y}(.)$ is also surjective since $\hat{x}^{-1}(.)$ is. It follows from the bijection $[\tilde{x}, \tilde{y}] = \tilde{\sigma}o\sigma^{-1}[x, y]$ that $\hat{\tilde{y}}(.)$ is also surjective. Since composition of surjective maps is surjective, we have $\tilde{h}(.) = \hat{\tilde{y}}o\hat{\tilde{x}}^{-1}(.)$ is surjective. Hence N is proper with respect to $h(.)$ if, and only if, N is proper with respect to $\tilde{h}(.)$.

Q.E.D.

Def. 8. Homeomorphic and bijective n-ports

An n-port characterized by a hybrid representation $y = h(x)$ is said to be x-homeomorphic [x-bijective] if $h(.)$ is an injection [bijection].

Remarks:

1. The basis for defining an "x-homeomorphic" n-port in terms of an "injection" is given by Brouwer's theorem on the invariance of domain [25, 26]: "any injective C^0-function $h: R^n \to R^n$ is homeomorphic".

2. It can be shown that any increasing n-port characterized by $y = h(x)$ is x-homeomorphic [12,16]. However, the converse is false for $n \geq 2$, as shown by the following counter-example: Let N be characterized by $i = g(v)$, where $i_1 = v_1 + v_2$, $i_2 = v_1 - v_2$. N is clearly v-homeomorphic. However, N is not increasing since $\alpha(a, b) \equiv \langle g(a) - g(b), (a - b)\rangle = -1$ when $a = (1, 1)$, $b = (1, 0)$ and $\alpha(a, b) = 1$ when $a = (1, 1)$, $b = (0, 1)$.

3. The reason for attaching the prefix "x" to Def. 8 for homeomorphic and bijective n-ports is because this definition is not invariant of the mode of hybrid representation. The following example is a case in point: Let N be characterized by the two equivalent representations $y = h(x)$, $y = [i_1, i_2, i_3]$, $x = [v_1, v_2, v_3]$ and $\tilde{y} = \tilde{h}(\tilde{x})$, $\tilde{y} = [v_1, v_2, i_3]$, $\tilde{x} = [i_1, v_2, v_3]$, where $h(.)$ and $\tilde{h}(.)$ are defined respectively by:

$$h(x) = Hx = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ and } \tilde{h}(\tilde{x}) = \tilde{H}\tilde{x} = \begin{bmatrix} -1 & 3 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Since det $H \neq 0$, but det $\tilde{H} = 0$, it follows that N is x-homeomorphic and x-bijective, but it is neither $\tilde{x}$-homeomorphic nor $\tilde{x}$-bijective.

4. The following collection of properties can be readily proved [12, 16]:

(a) N is v-homeomorphic [v-increasing] and i-proper if, and only if, N is i-homeomorphic [i-increasing] and v-proper.

(b) Every x-uniformly increasing n-port is increasing, x-homeomorphic, x-bijective, y-homeomorphic and y-bijective, where $[x, y] = \sigma[v, i]$.

(c) A regular n-port characterized by $v = v(\rho)$ and $i = i(\rho)$ is non-decreasing if the Jacobian matrices $\partial v(\rho)/\partial\rho$ and $\partial i(\rho)/\partial\rho$ are both positive semi-definite or both negative semi-definite.

(d) An n-port characterized by $i = g(v)$ is non-decreasing if, and only if, $\partial g(v)/\partial v$ is positive semi-definite; v-uniformly increasing if, and only if, $\partial g(v)/\partial v$ is uniformly positive definite [13], and increasing if $\partial g(v)/\partial v$ is almost positive definite [16]. The dual case is also true.

V. LOCAL CHARACTERIZATION OF REGULAR N-PORTS

A regular n-port can be characterized locally according to whether it is reciprocal or not. The concept of reciprocity is extremely useful not only in network theory, but also in other physical theories such as thermodynamics. Two important works on reciprocal n-ports have appeared recently [1, 15]. In this paper, we define reciprocity in a slightly different context from that used by Brayton [1]. Our motivations in offering an equivalent though distinct definition of reciprocity are: (1) our definition does not involve exterior derivatives and is therefore more familiar to engineers; (2) our definition is identical in form to the well-known "Lorentz reciprocity" definition for linear n-ports [6];

(3) Ours is an "operational" definition and can be interpreted as taking incremental port measurements when the n-port is biased at some operating point. In the following, we let N be a regular n-port characterized by a C^k-immersion $(k \geq 1)\mu: R^n \to R^{2n}$, where $\mu(\rho) \equiv [v(\rho), i(\rho)]$. Let $T_\rho(R_n)$ denote the n-dimensional tangent space [9] at the point $\rho \in R^n$.

Def. 9. Reciprocal and anti-reciprocal n-ports

A regular n-port is said to be reciprocal at $\rho \in R^n$ if

$$\langle di(\rho),\ d'v(\rho)\rangle = \langle d'i(\rho),\ dv(\rho)\rangle \tag{10}$$

where $[di(\rho), dv(\rho)]$ and $[d'i(\rho), d'v(\rho)]$ are any two distinct elements of the tangent space $T_\rho(R^n)$. It is said to be anti-reciprocal at ρ if,

$$\langle di(\rho),\ d'v(\rho)\rangle = -\langle d'i(\rho),\ dv(\rho)\rangle. \tag{11}$$

The n-port is said to be reciprocal [anti-reciprocal] if it is reciprocal [anti-reciprocal] at all points $\rho \in R^n$. The pair $[di(\rho), dv(\rho)]$ and $[d'i(\rho), d'v(\rho)]$ can be interpreted as the Frechet differential [26] of $v(.)$ and $i(.)$ at the point ρ corresponding to the differentials $d\rho$ and $d'\rho$, respectively. Physically, they can be approximated by two sets of incremental measurements $[\Delta v(\rho), \Delta i(\rho)]$ and $[\Delta' v(\rho), \Delta' i(\rho)]$ when the n-port is biased at the point $[v(\rho), i(\rho)]$. Observe that we can write $dv(\rho) = J_v(\rho)\, d\rho$, $di(\rho) = J_i(\rho) d\rho$, $d'v(\rho) = J_v(\rho) d'\rho$ and $d'i(\rho) = J_i(\rho) d'\rho$, where $J_v(\rho)$ and $J_i(\rho)$ are the Jacobian matrices of $v(.)$ and $i(.)$, respectively.

Theorem 4. Reciprocity Criterion.[9]

A regular n-port is reciprocal [anti-reciprocal] if, and only if, its associated reciprocity matrix

$$\mathcal{J}(\rho) \equiv [J_i(\rho)]^t\, [J_v(\rho)] \tag{12}$$

is symmetric [skew-symmetric].

Proof. $\langle d'i(\rho),\ dv(\rho)\rangle = \langle J_i(\rho) d'\rho,\ J_v(\rho) d\rho\rangle = \langle d'\rho, \mathcal{J}(\rho) d\rho\rangle$ (13)

$$\langle di(\rho),\ d'v(\rho)\rangle = \langle d\rho,\ \mathcal{J}(\rho) d'\rho\rangle = \langle d'\rho,\ [\mathcal{J}(\rho)]^t\, d\rho\rangle \tag{14}$$

Substituting (13) and (14) into (10) and (11), we obtain the desired result.

Q.E.D.

Corollary.

An n-port characterized by a C^1-hybrid representation $y = h(x)$, where $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}$, is reciprocal [anti-reciprocal] if, and only if, its associated hybrid reciprocity matrix

$$\mathcal{J}(x) \equiv [B + AJ_h(x)]^t\,[A + BJ_h(x)] \tag{15}$$

is symmetric [skew-symmetric], where $J_h(x)$ is the Jacobian matrix of $h(.)$.

Proof. Since the permutation matrix Σ is orthogonal and symmetric, we can write $v(x) = Ax + Bh(x)$ and $i(x) = Bx + Ah(x)$. Substituting $J_i(x) = B + AJ_h(x)$ and $J_v(x) = A + BJ_h(x)$ into (12), we obtain (15). Q.E.D.

The hybrid reciprocity matrix $\mathcal{J}(x)$ defined in (15) is extremely convenient to work with since it can be used to derive the reciprocity or anti-reciprocity criteria associated with each of the hybrid or conjugate hybrid representation tabulated in Table 1. Likewise, the matrix $\mathcal{J}(\rho)$ can be used to derive the analogous criteria for the two transmission representations in Table 1.

VI. CONCLUDING REMARKS

The p-atlas introduced in Section II should cover all conceivable algebraic n-ports, thereby providing the "closure property" to our theory. Of course, such a representation is too general to be useful. It appears that the class of immersed n-ports should provide the proper setting for future research. Such n-ports are not only more tractable mathematically, but also possess some generic properties of physical significance. For many n-ports of practical interest, the associated p-atlas may be endowed with an additional differential structure so that these n-ports can be interpreted as closed submanifolds imbedded in R^{2n} [27, 28]. Such a submanifold may be interpreted geometrically as the union of disjoint smooth non-self-intersecting hypersurfaces in R^{2n}.

It is hoped that the global and local characterizations in Sections IV and V would provide the foundation for the synthesis of algebraic n-ports. The basic philosophy would be to decompose a prescribed n-port into an interconnection of component n-ports chosen from among the various subclasses defined in this paper. Such an approach will be exploited in another paper.

ACKNOWLEDGEMENT

The first author would like to thank Professor S. Smale for pointing out the existence of the Whitney Approximation Theorem, thereby justifying the significance of immersed n-ports. He would also like to thank Professor G. Oster for many hours of stimulating discussions.

REFERENCES

1. R.K. Brayton, "Nonlinear reciprocal networks", IBM report RC 2606 (#12427), September 3, 1969.

2. E.A. Guillemin, Synthesis of Passive Networks, John Wiley, New York, 1957.

3. D.C. Youla, L.J. Castriota, and H.J. Carlin, "Bounded real scattering matrices and the foundations of linear passive network theory", IEEE Trans. Circuit Theory, vol. CT-14, March 1959, pp. 102-124.

4. R.W. Newcomb, Linear Multiport Synthesis, McGraw-Hill, New York, 1966.

5. E.S. Kuh and R.A. Rohrer, Theory of Linear Active Networks, Holden-Day, San Francisco, 1967.

6. H.J. Carlin and A.B. Giordano, Network Theory, Prenctice Hall, Englewood Cliffs, N.J., 1964.

7. L.O. Chua, "Memristor -- the missing circuit element", IEEE Trans. on Circuit Theory, vol. CT-18, no. 5, September 1971, pp. 507-519.

8. L.O. Chua and R.A. Rohrer, "On the dynamic equations of a class of nonlinear RLC networks", IEEE Trans. on Circuit Theory, vol. CT-12, Dec. 1965, pp.475-489.

9. S. Sternberg, Lectures on Differential Geometry, Prentice-Hall, Englewood Cliffs, N.J., 1964.

10. R. Abraham and J. Robbin, Transversal Mappings and Flows, Benjamin, New York, 1967.

11. L. Mirsky, An Introduction to Linear Algebra, The Clarendon Press, Oxford, 1955.

12. L.O. Chua and Y-F Lam, "Foundations of nonlinear n-port network theory", Purdue University report, TR-EE 71-20, June 1971.

13. T. Ohtsuki and H. Watanabe, "State variable analysis of RLC networks containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-16, February 1969, pp. 26-38.

14. M. Vehovec, "Comments on state variable analysis of RLC networks containing nonlinear coupling elements", IEEE Trans. on Circuit Theory, vol. CT-17, August 1970, pp. 451-452.

15. C.A. Desoer and G.F. Oster, "Globally reciprocal stationary systems", to appear.

16. L.O. Chua and Y-F Lam, "Global homeomorphism of vector-valued functions", to appear in J. of Math. Analysis and Applications.

17. M. Fiedler and V. Ptak, "On matrices with non-positive off-diagonal elements and positive principal minor", Czech. Math. J., vol. 12, no. 3, 1962, pp.382-400.

18. R.S. Palais, "Natural operations on differential forms", Trans. Amer. Math Soc., vol. 92, 1959, pp. 125-141.

19. I.W. Sandberg and A.N. Willson, Jr., "Some theorems on properties of dc equations of nonlinear networks", Bell Syst. Tech. J., vol. 48, January 1969, pp. 1-34.

20. T. Fujisawa and E.S. Kuh, "Some results on existence and uniqueness of solutions of nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-18, no. 5, September 1971, pp. 501-506.

21. T.E. Stern, Theory of Nonlinear Networks and Systems, Addison-Wesley, Reading, Mass., 1965.

22. R.K. Brayton and J.K. Moser, "A theory of nonlinear networks -- I and II", Quart. Appl. Math., vol. 22, April 1964, pp. 1-33, and July, 1964, pp. 81-104.

23. P.P. Varaiya and R.W. Liu, "Normal form and stability of a class of coupled nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-13, Dec. 1966, pp. 413-418.

24. L.O. Chua, "The linear transformation converter and its application to the synthesis of nonlinear networks", IEEE Trans. on Circuit Theory, vol. CT-17, November 1970, pp. 584-594.

25. I. Bers, Topology, Courant Institute of Mathematical Science, New York University, New York, 1956.

26. J.T. Schwartz, Nonlinear Functional Analysis, Gordon and Breach, New York, 1969.

27. S. Smale, "On the mathematical foundations of electrical circuit theory", to appear.

28. C.A. Desoer and F.F. Wu, "Networks viewed as flows on manifolds", to appear.

29. L. O. Chua and B. J. Leon, "A unified theory of linear n-port networks," Proc. 3rd Annual Allerton Conference on Circuit and System Theory, October 20-22, 1965, Monticello, Illinois.

Manuscript revised ______________________________

1. To economize on symbols, we will use the same index "n" for different "n"-ports. We will also assume that whenever necessary, our n-ports are provided with internal isolation transformers so that arbitrary interconnections will not introduce circulation currents [2].
2. Throughout this paper, we let R^k denote the Euclidean k-space and $||.||$ the usual Euclidean norm. Vectors are denoted by lower case letters and matrices by upper case letters. A column vector will usually be denoted by $x = [x_1, x_2, \ldots, x_k]$. Since we will be dealing mostly with vector quantities, we will distingush the scalar component of vectors by arabic subscripts. A literal subscript will normally denote sub-vectors. For example, we usually partition a vector $x = [x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n] \epsilon R^n$ into $x = [x_a, x_b]$, where $x_a = [x_1, x_2, \ldots, x_k]$ and $x_b = [x_{k+1}, x_{k+2}, \ldots, x_n]$.
3. A memristor is a new circuit element characterized by a relationship between φ and q [7]. More general algebraic n-ports involving couplings among the 4 basic port variables v_j, i_j, φ_j, and q_j may obviously be defined. However, their significance remains to be established.
4. The term "closure property" is used loosely in this paper to mean that if two objects, systems, theories, etc. possess certain common property P, or are members of a certain class C, then their combination according to some prescribed rules must retain the same property P, or belong to the same class C.
5. Notice that we do not require our p-chart to be injective or differentiable. Neither do we provide any differential structure on our p-atlas. However, if S is a differentiable manifold [9], then the associated local charts and atlas automatically qualify as p-charts and p-atlas on S.

6. A C^k-function μ: $R^m \to R^{2n}$ is said to be a C^k-immersion if $\mu(.)$ is of rank m; i.e., if the Jacobian of $\mu(.)$ has rank m for every point in R^m.

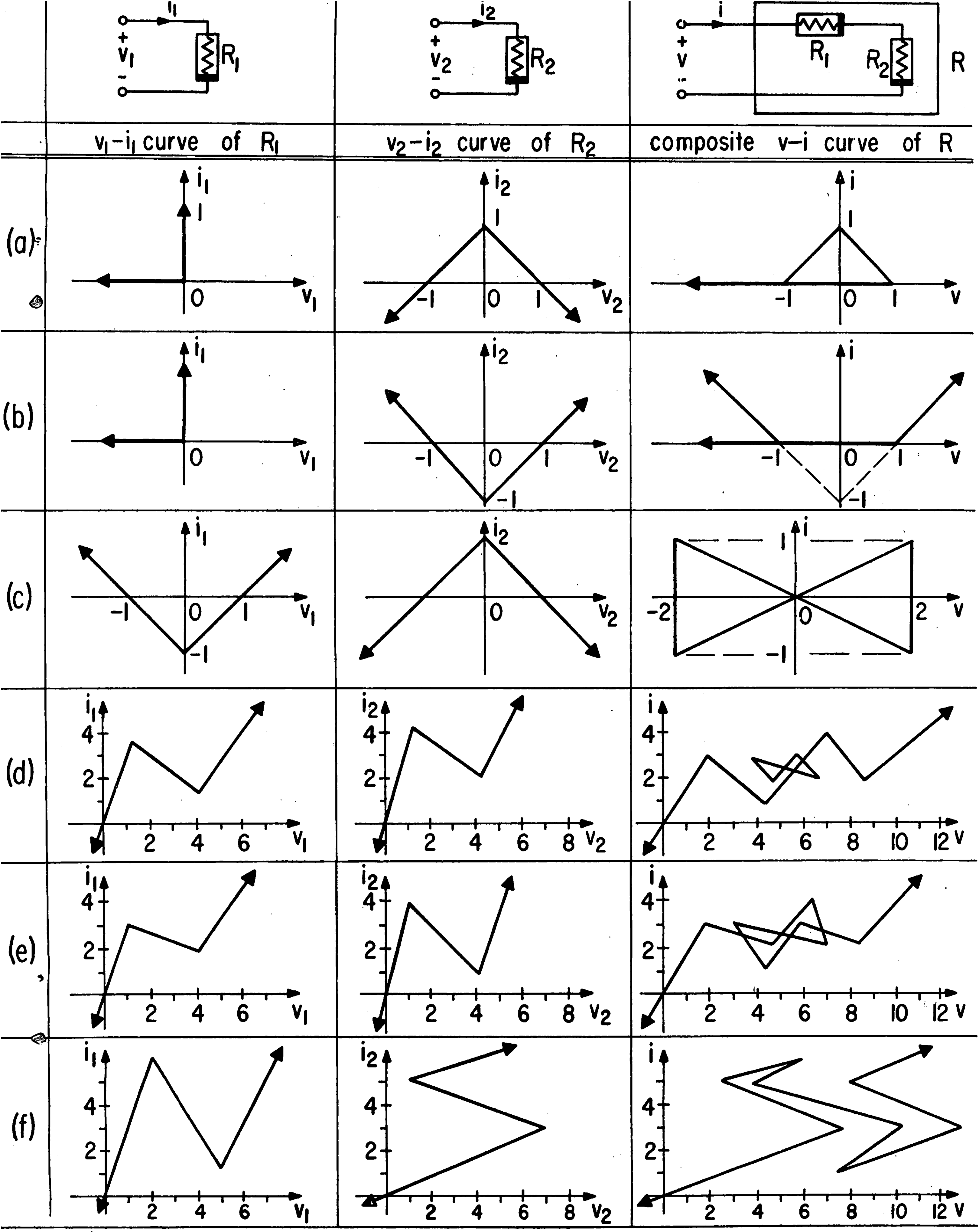
7. An n x n matrix A is said to be a P-matrix if all its principal submatrices have positive determinants [17].

8. A non-decreasing n-port may not admit a hybrid representation. A v-i curve containing both vertical and horizontal segments is a case in point.

9. This reciprocity criterion was first derived in this coordinate-free form for linear n-ports in [29]. This criterion had also been derived for non-linear n-ports by Brayton [1].

Table 1: Representation of Nonlinear N-Ports and Their Criteria for Reciprocity and Anti-Reciprocity.

Mode of Representation	Defining Equations $i=\begin{bmatrix} i_a \\ i_b \end{bmatrix}$, $v=\begin{bmatrix} v_a \\ v_b \end{bmatrix}$	Jacobian Matrix	Necessary and Sufficient Conditions for Reciprocity	Necessary and Sufficient Conditions for Anti-Reciprocity
1. Conductance Representation	$i = g(v)$	$J_g = \frac{\partial g}{\partial v}$	The Jacobian Matrix J_g is symmetric	The Jacobian Matrix J_g is skew-symmetric
2. Resistance Representation	$v = f(i)$	$J_f = \frac{\partial f}{\partial i}$	The Jacobian Matrix J_f is symmetric	The Jacobian Matrix J_f is skew-symmetric
3. Hybrid Representation I	$i_a = h_a^1(v_a, i_b)$ $v_b = h_b^1(v_a, i_b)$	$J_{h^1} = \begin{bmatrix} \frac{\partial h_a^1}{\partial v_a} & \frac{\partial h_a^1}{\partial i_b} \\ \frac{\partial h_b^1}{\partial v_a} & \frac{\partial h_b^1}{\partial i_b} \end{bmatrix}$	(a) $\partial h_a^1/\partial v_a$ is symmetric (b) $\partial h_b^1/\partial i_b$ is symmetric (c) $\partial h_a^1/\partial i_b = -[\partial h_b^1/\partial v_a]^t$	(a) $\partial h_a^1/\partial v_a$ is skew-symmetric (b) $\partial h_b^1/\partial i_b$ is skew-symmetric (c) $\partial h_a^1/\partial i_b = -[\partial h_b^1/\partial v_a]^t$
4. Hybrid Representation II	$v_a = h_a^2(i_a, v_b)$ $i_b = h_b^2(i_a, v_b)$	$J_{h^2} = \begin{bmatrix} \frac{\partial h_a^2}{\partial i_a} & \frac{\partial h_a^2}{\partial v_b} \\ \frac{\partial h_b^2}{\partial i_a} & \frac{\partial h_b^2}{\partial v_b} \end{bmatrix}$	(a) $\partial h_a^2/\partial i_a$ is symmetric (b) $\partial h_b^2/\partial v_b$ is symmetric (c) $\partial h_a^2/\partial v_b = -[\partial h_b^2/\partial i_a]^t$	(a) $\partial h_a^2/\partial i_a$ is skew-symmetric (b) $\partial h_b^2/\partial v_b$ is skew-symmetric (c) $\partial h_a^2/\partial v_b = -[\partial h_b^2/\partial i_a]^t$
5. Conjugate Hybrid Representation I	$i_a = \hat{h}_a^1(v_a, i_b^*)$ $v_b = \hat{h}_b^1(v_a, i_b^*)$ $(i_b^* \equiv -i_b)$	$J_{\hat{h}^1} = \begin{bmatrix} \frac{\partial \hat{h}_a^1}{\partial v_a} & \frac{\partial \hat{h}_a^1}{\partial i_b^*} \\ \frac{\partial \hat{h}_b}{\partial v_a} & \frac{\partial \hat{h}_b^1}{\partial i_b^*} \end{bmatrix}$	The Jacobian Matrix $J_{\hat{h}^1}$ is symmetric	(a) $\partial \hat{h}_a^1/\partial v_a$ is skew-symmetric (b) $\partial \hat{h}_b^1/\partial i_b^*$ is skew-symmetric (c) $\partial \hat{h}_a^1/\partial i_b^* = [\partial \hat{h}_b^1/\partial v_a]^t$
6. Conjugate Hybrid Representation II	$v_a = \hat{h}_a^2(i_a, v_b)$ $i_b^* = \hat{h}_b^2(i_a, v_b)$ $(i_b^* \equiv -i_b)$	$J_{\hat{h}^2} = \begin{bmatrix} \frac{\partial \hat{h}_a^2}{\partial i_a} & \frac{\partial \hat{h}_a^2}{\partial v_b} \\ \frac{\partial \hat{h}_b^2}{\partial i_a} & \frac{\partial \hat{h}_b^2}{\partial v_b} \end{bmatrix}$	The Jacobian Matrix $J_{\hat{h}^2}$ is symmetric	(a) $\partial \hat{h}_a^2/\partial i_a$ is skew-symmetric (b) $\partial \hat{h}_b^2/\partial v_b$ is skew-symmetric (c) $\partial \hat{h}_a^2/\partial v_b = [\partial \hat{h}_b^2/\partial i_a]^t$
7. Conjugate Hybrid Representation III	$i_a = \hat{h}_a^3(v_a^*, i_b)$ $v_b = \hat{h}_b^3(v_a^*, i_b)$ $v_a^* \equiv -v_a$	$J_{\hat{h}^3} = \begin{bmatrix} \frac{\partial \hat{h}_a^3}{\partial v_a^*} & \frac{\partial \hat{h}_a^3}{\partial i_b} \\ \frac{\partial \hat{h}_b^3}{\partial v_a^*} & \frac{\partial \hat{h}_b^3}{\partial i_b} \end{bmatrix}$	The Jacobian Matrix $J_{\hat{h}^3}$ is symmetric	(a) $\partial \hat{h}_a^3/\partial v_a^*$ is skew-symmetric (b) $\partial \hat{h}_b^3/\partial i_b$ is skew-symmetric (c) $\partial \hat{h}_a^3/\partial i_b = [\partial \hat{h}_b^3/\partial v_a^*]^t$
8. Conjugate Hybrid Representation IV	$v_a^* = \hat{h}_a^4(i_a, v_b)$ $i_b = \hat{h}_b^4(i_a, v_b)$ $v_a^* \equiv -v_a$	$J_{\hat{h}^4} = \begin{bmatrix} \frac{\partial \hat{h}_a^4}{\partial i_a} & \frac{\partial \hat{h}_a^4}{\partial v_b} \\ \frac{\partial \hat{h}_b^4}{\partial i_a} & \frac{\partial \hat{h}_b^4}{\partial v_b} \end{bmatrix}$	The Jacobian Matrix $J_{\hat{h}^4}$ is symmetric	(a) $\partial \hat{h}_a^4/\partial i_a$ is skew-symmetric (b) $\partial \hat{h}_b^4/\partial v_b$ is skew-symmetric (c) $\partial \hat{h}_a^4/\partial v_b = [\partial \hat{h}_b^4/\partial i_a]^t$
9. Transmission Representation I	$v_a = t_v^1(v_b, i_b^*)$ $i_a = t_i^1(v_b, i_b^*)$ $i_b^* \equiv -i_b$	$J_{t^1} = \begin{bmatrix} \frac{\partial t_v^1}{\partial v_b} & \frac{\partial t_v^1}{\partial i_b^*} \\ \frac{\partial t_i^1}{\partial v_b} & \frac{\partial t_i^1}{\partial i_b^*} \end{bmatrix}$	(a) $\left[\frac{\partial t_v^1}{\partial v_b}\right]^t \left[\frac{\partial t_i^1}{\partial v_b}\right]$ is symmetric (b) $\left[\frac{\partial t_v^1}{\partial i_b^*}\right]^t \left[\frac{\partial t_i^1}{\partial i_b^*}\right]$ is symmetric (c) $\left[\frac{\partial t_v^1}{\partial v_b}\right]^t \left[\frac{\partial t_i^1}{\partial i_b^*}\right] - \left[\frac{\partial t_i^1}{\partial v_b}\right]^t \left[\frac{\partial t_v^1}{\partial i_b^*}\right] = 1_n$	(a) $\left[\frac{\partial t_v^1}{\partial v_b}\right]^t \left[\frac{\partial t_i^1}{\partial v_b}\right]$ is skew-symmetric (b) $\left[\frac{\partial t_v^1}{\partial i_b^*}\right]^t \left[\frac{\partial t_i^1}{\partial i_b^*}\right]$ is skew-symmetric (c) $\left[\frac{\partial t_v^1}{\partial v_b}\right]^t \left[\frac{\partial t_i^1}{\partial i_b^*}\right] + \left[\frac{\partial t_i^1}{\partial v_b}\right]^t \left[\frac{\partial t_v^1}{\partial i_b^*}\right] = 1_n$
10. Transmission Representation II	$v_b = t_v^2(v_a, i_a^*)$ $i_b = t_i^2(v_a, i_a^*)$ $i_a^* = -i_a$	$J_{t^2} = \begin{bmatrix} \frac{\partial t_v^2}{\partial v_a} & \frac{\partial t_v^2}{\partial i_a^*} \\ \frac{\partial t_i^2}{\partial v_a} & \frac{\partial t_i^2}{\partial i_a^*} \end{bmatrix}$	(a) $\left[\frac{\partial t_v^2}{\partial v_a}\right]^t \left[\frac{\partial t_i^2}{\partial v_a}\right]$ is symmetric (b) $\left[\frac{\partial t_v^2}{\partial i_a^*}\right]^t \left[\frac{\partial t_i^2}{\partial i_a^*}\right]$ is symmetric (c) $\left[\frac{\partial t_v^2}{\partial v_a}\right]^t \left[\frac{\partial t_i^2}{\partial i_a^*}\right] - \left[\frac{\partial t_i^2}{\partial v_a}\right]^t \left[\frac{\partial t_v^2}{\partial i_a^*}\right] = 1_n$	(a) $\left[\frac{\partial t_v^2}{\partial v_a}\right]^t \left[\frac{\partial t_i^2}{\partial v_a}\right]$ is skew-symmetric (b) $\left[\frac{\partial t_v^2}{\partial i_a^*}\right]^t \left[\frac{\partial t_i^2}{\partial i_a^*}\right]$ is skew-symmetric (c) $\left[\frac{\partial t_v^2}{\partial v_a}\right]^t \left[\frac{\partial t_i^2}{\partial i_a^*}\right] + \left[\frac{\partial t_i^2}{\partial v_a}\right]^t \left[\frac{\partial t_v^2}{\partial i_a^*}\right] = 1_n$



R1
R2
R
v1-i1 curve of R1
v2-i2 curve of R2
composite v-i curve of R
(a)
(b)
(c)
(d)
(e)
(f)

FIGURE 1

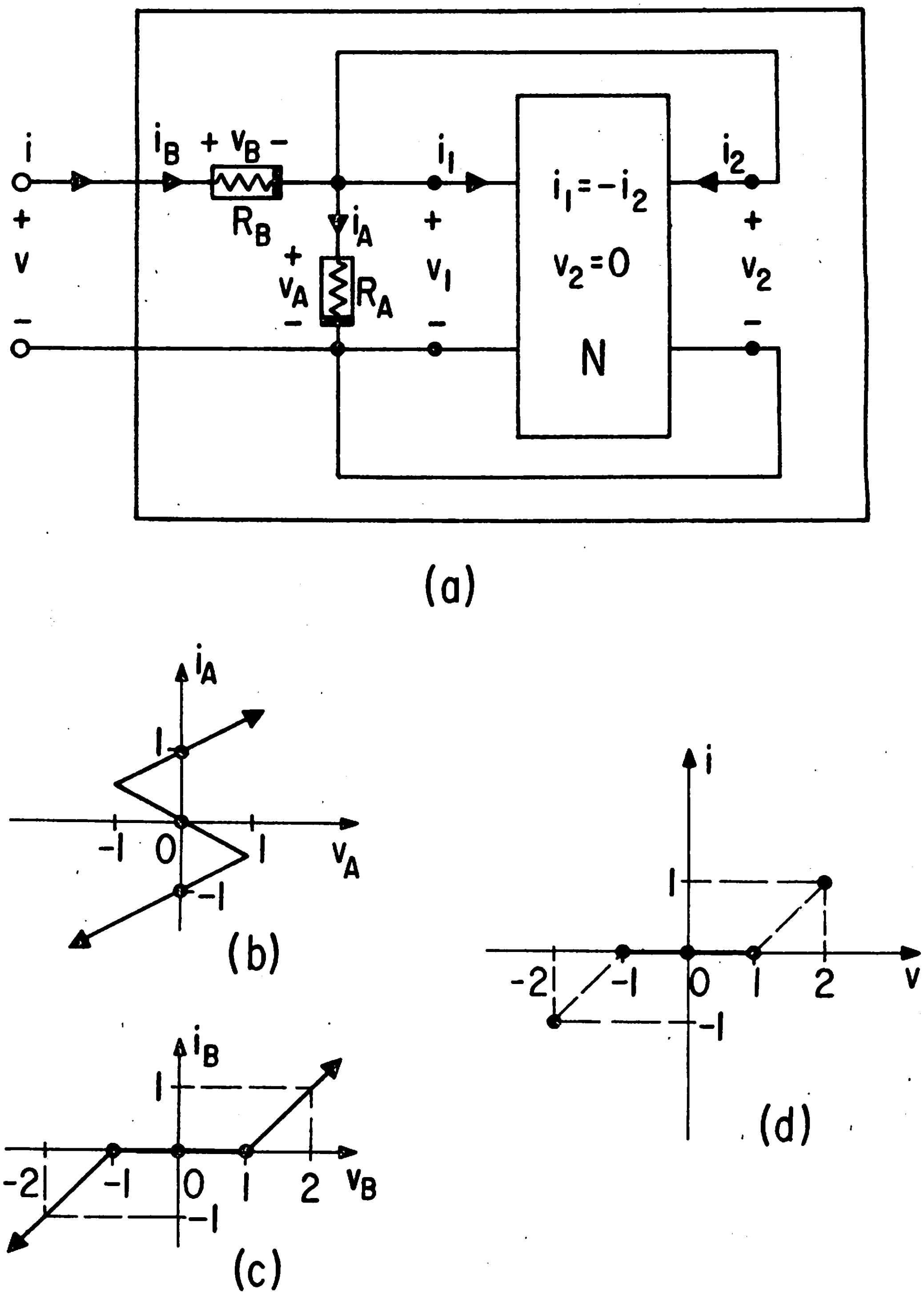
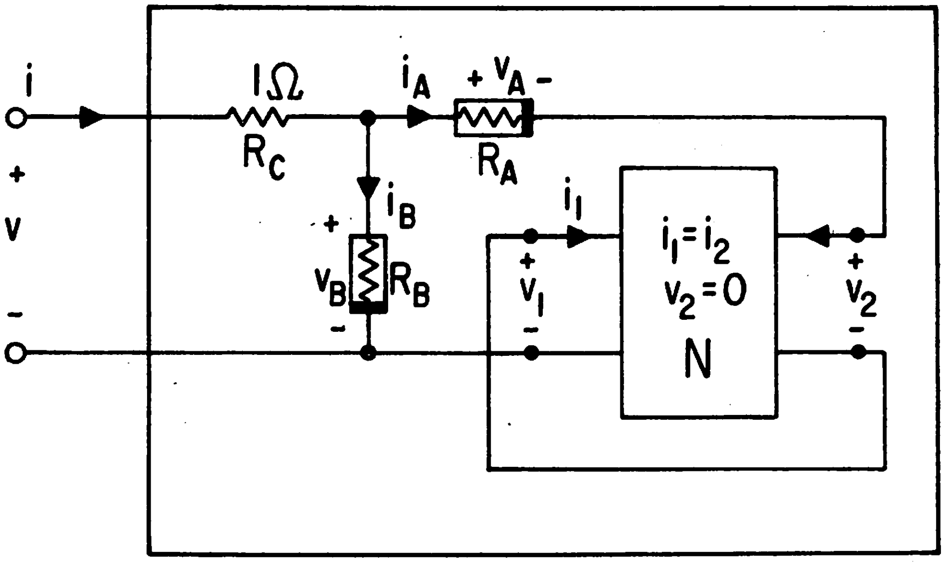
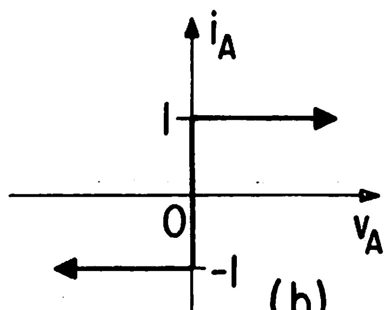


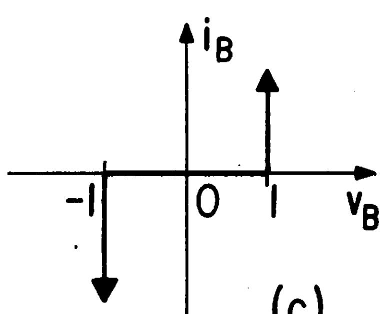
FIGURE 2



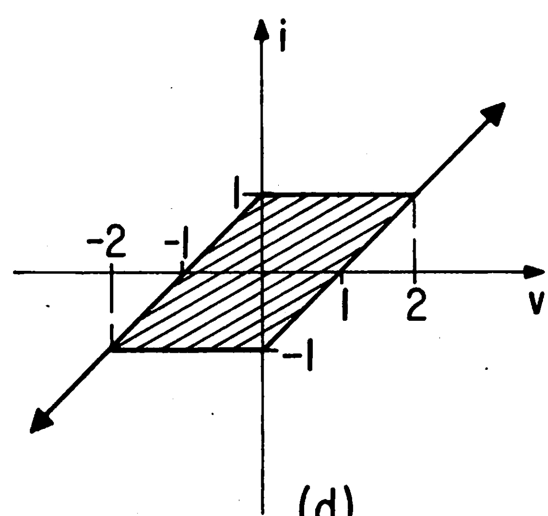
(a)



(b)

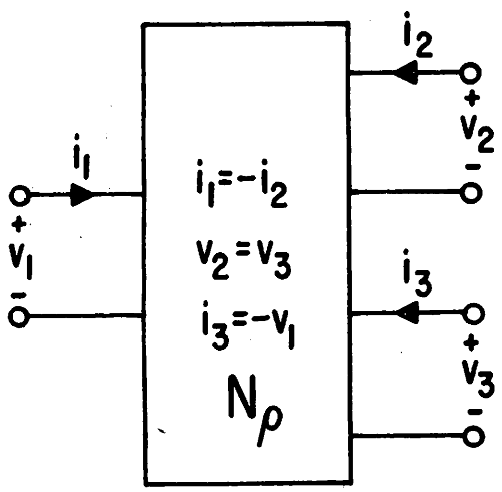


(c)

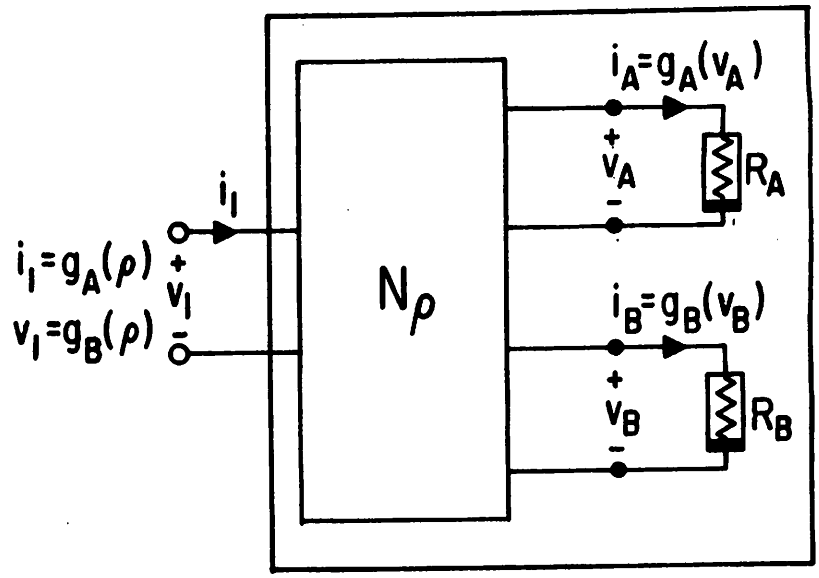


(d)

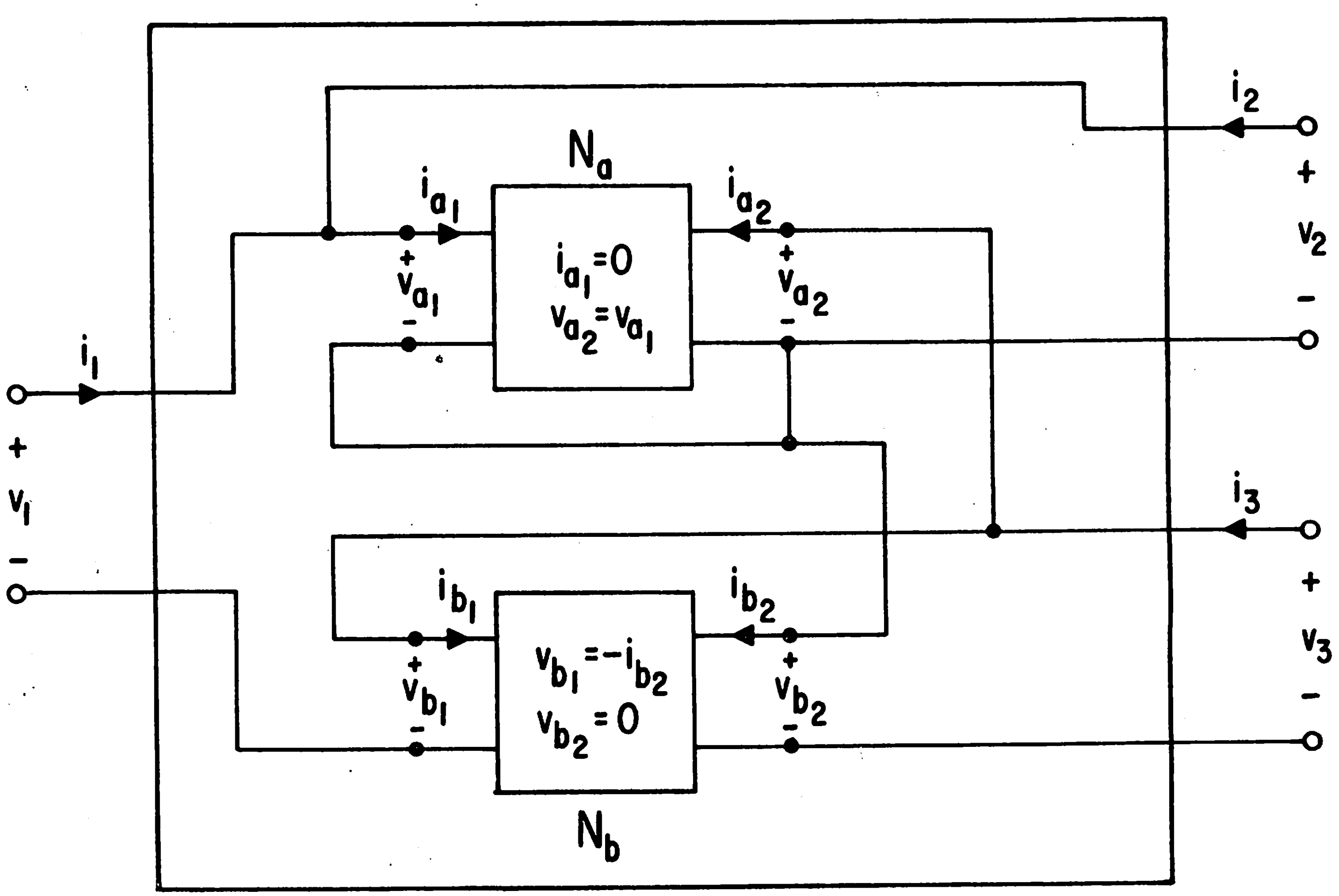
FIGURE 3



(a) A unicursal 3-port

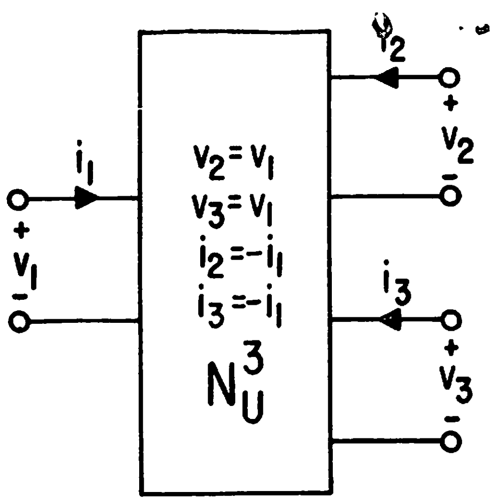


(b)

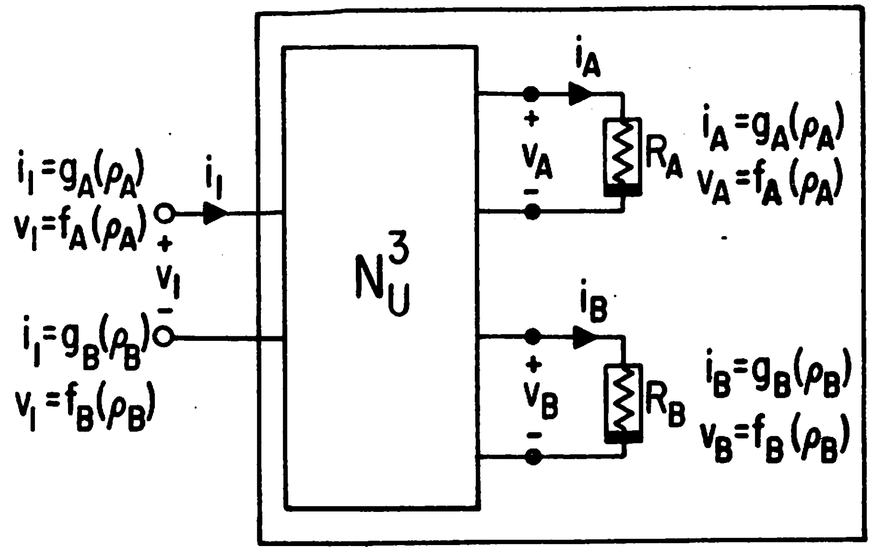


(c) Circuit realization of a unicursal 3-port N_ρ

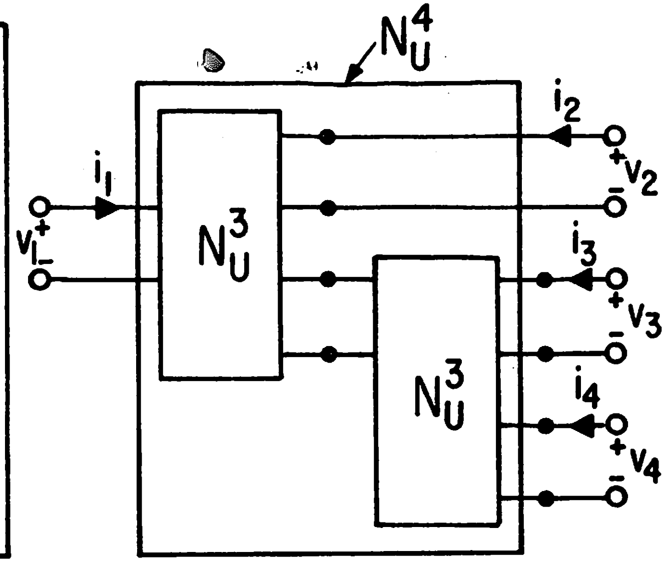
FIGURE 4



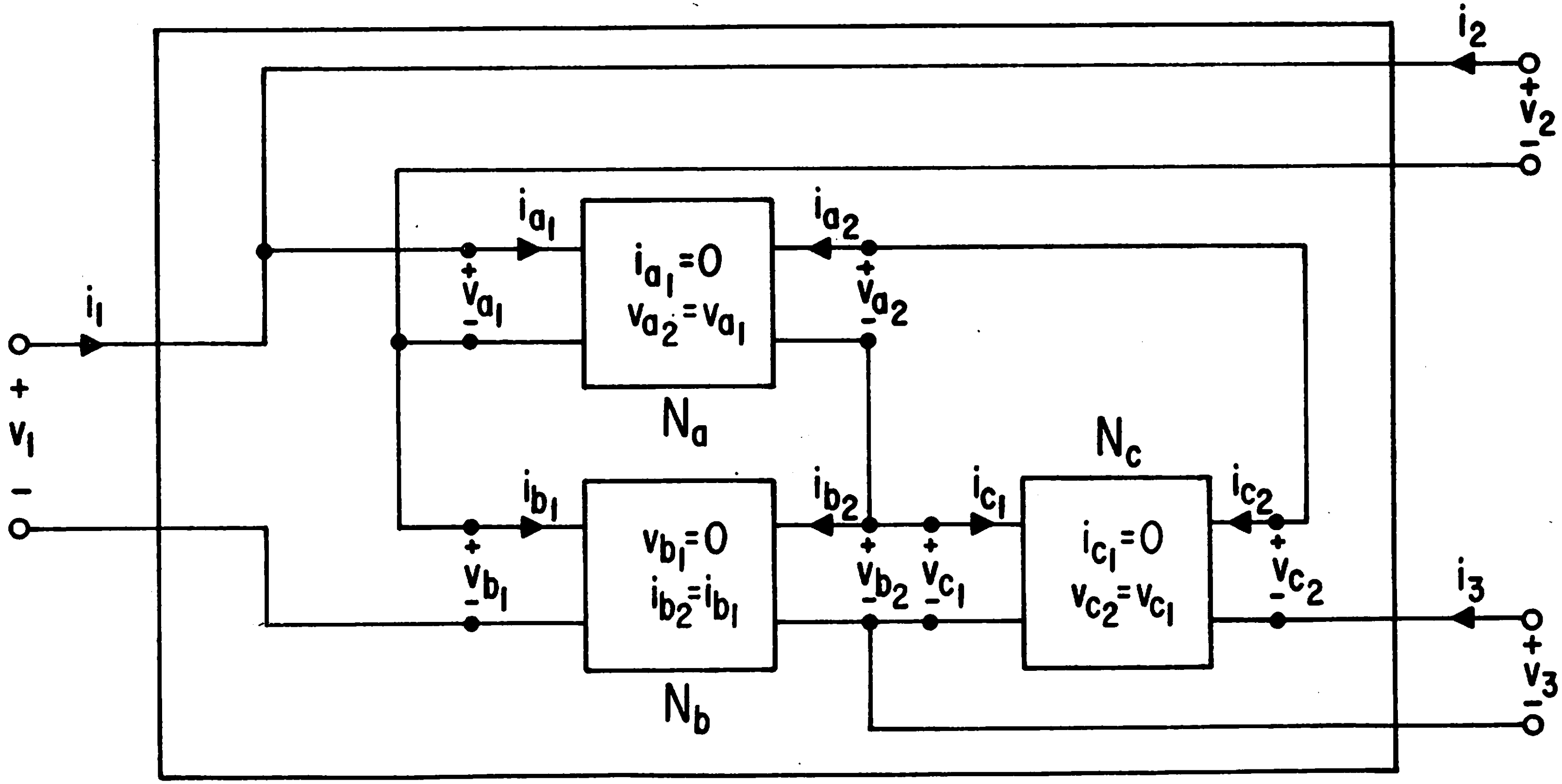
(a) A "union 3-port"



(b)



(c) A "union 4-port"



(d) Circuit realization of a "union 3-port" N_U^3

FIGURE 5

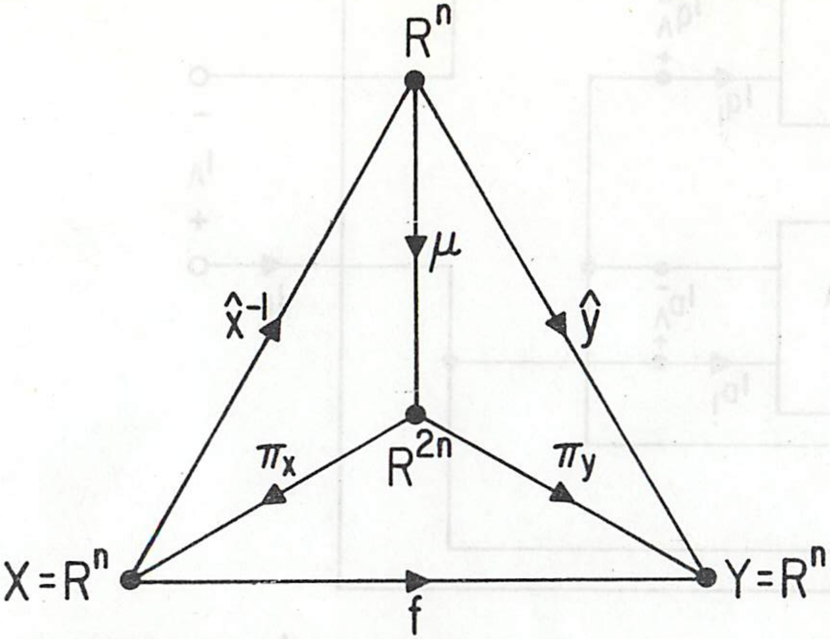


DIAGRAM (a)

$f : X \rightarrow Y$

$f = \hat{y} \circ \hat{x}^{-1}$

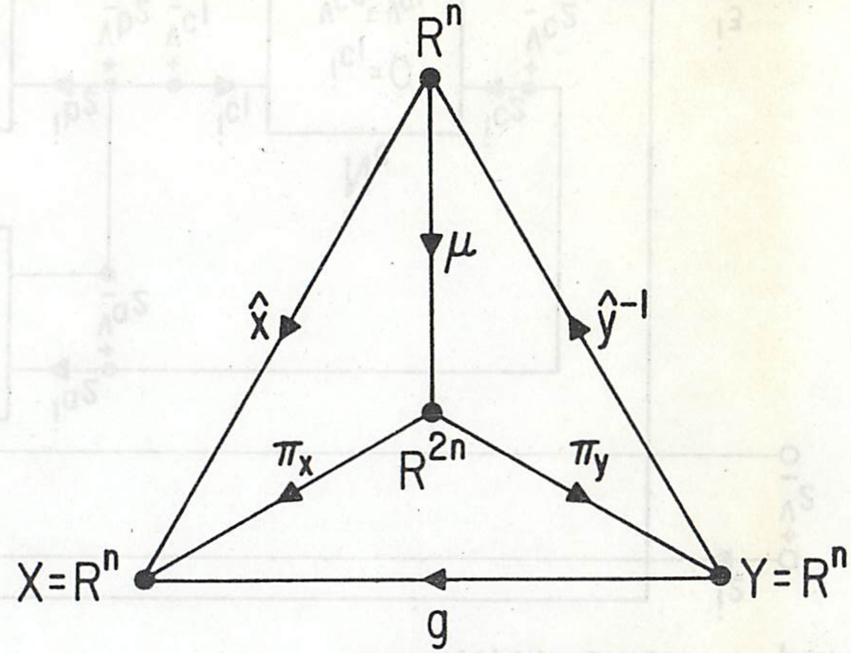


DIAGRAM (b)

$g : Y \rightarrow X$

$g = \hat{x} \circ \hat{y}^{-1}$

FIGURE 6