THE k-FACTOR CONJECTURE IS TRUE

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Abstract. We show that if sequences $\langle d_i \rangle$, $\langle d_i - k \rangle$ are graphical then there exists a graph G with degrees d_i which has a factor with k lines at each vertex.

To every graph G whose vertices are labelled v_i, $1 \leq i \leq n$, one can associate the degree sequence $\langle d_i \rangle$ where d_i = degree of the vertex v_i. G is called a representing graph for $\langle d_i \rangle$ and the sequence itself is called graphical. The degree sequence is an invariant of a graph. It is a rather 'weak' invariant and there are almost always more than one graph with same degree sequence [2]. This 'incompleteness' of degree sequences allows one to raise many existence problems about representing graphs. In that regard, the following conjecture was made by A. R. Rao and S. B. Rao [5] and also by B. Grünbaum.

If $\langle d_i \rangle$, $\langle d_i - k \rangle$ are graphical sequences then there exists a graph G with degrees $\langle d_i \rangle$ which has a factor with k lines at each vertex.

Also a similar conjecture for digraphs was made by A. R. Rao and S. B. Rao [5]. We shall prove a generalized version for each of them,

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first for graphs and then for digraphs. The ideas involved in the two cases are very similar.

1. UNDIRECTED GRAPHS.

We shall assume that graphs have no multiple lines and loops. All graphs are drawn on a fixed set of vertices $V = \{v_1, v_2, \cdots, v_n\}$. Therefore it is convenient to identify a graph G with the subset of unordered pairs $\{(v_i, v_j)\}$, where (v_i, v_j) are lines of G. A sequence (of n integers) $\langle d_i: 1 \leq i \leq n \rangle$ is called <u>graphical</u> if there exists a graph G with degree of v_i being equal to d_i for all i. We say that G is a representing graph of $\langle d_i \rangle$. For a given sequence $\langle k_i: 0 \leq k_i \leq d_i \rangle$ a subgraph $F \subseteq G$ is called a <u>subfactor</u> if F has at most k_i lines at v_i. Call v_i a saturated vertex (with respect to F) if F has exactly k_i lines at v_i. We shall denote by $S = S(F)$ the set of saturated vertices. F is called a <u>factor</u> if $S = V$. If $k_i = k$, $1 \leq i \leq n$, F is called a k-factor. We often consider two graphs G, H simultaneously. To distinguish their lines we shall put colors on (v_i, v_j) as follows: lines of G (resp. H) not in H (resp. G) are colored red (resp. blue), the lines common to G and H are colored green and all other lines are colored white. We shall write r = red, b = blue, g = green, w = white and $c(v_i, v_j)$ for the color of line (v_i, v_j). A few other notations like $r = g-b$, $g = r+b$, $w = b-b$ etc. will be useful. We shall let $E_c(v_i)$ denote the set of lines at vertex v_i with color c, c = r,b,g, and $E_c = \cup E_c(v_i)$, union over all v_i. Admittedly $|E_r(v_i)| + |E_g(v_i)|$, $|E_b(v_i)| + |E_g(v_i)|$ are respectively the degrees of v_i in G and H. Finally, an <u>alternating path</u> $P = (x_0, x_1)$, (x_1, x_2), (x_2, x_3),... is a path whose lines are distinct and $c(x_i, x_{i+1}) =$

r or b according as i even or odd .

<u>THEOREM 1.1</u>. Let $\langle d_i \rangle$, $\langle d_i - k_i \rangle$ be two graphical sequences such that for some $k \geq 0$, $k \leq k_i \leq k+1$ for $1 \leq i \leq n$. Then there is a graph G with degree sequence $\langle d_i \rangle$ and having a $\langle k_i \rangle$-factor.

<u>Proof of Theorem 1.1</u>.

Consider two graph G', H' with degree sequences $\langle d_i \rangle$, $\langle d_i - k_i \rangle$ respectively and the associated coloring of the lines (v_i, v_j) in white, red, blue and green. Clearly, $|E_r(v_i)| = |E_b(v_i)| + k_i$ for all vertex v_i. Let $F' \subseteq E_r$* be a $\langle k_i \rangle$-subfactor; F' is possibly empty. Suppose that the graphs G, H and a subfactor F are so chosen that $|F| + |E_g|$ has maximum value among all possible choices of G', H', F'. If all vertices v_i are saturated in F we are done. We shall show that it is indeed so. This is accomplished in several steps. Let $S = S(F)$ and assume that $S \neq V$.

1^0]. If x_0, x_1, x_2, x_3 are four distinct vertices such that $c(x_0, x_1) = b = c(x_2, x_3)$, $c(x_1, x_2) = r$ and $(x_1, x_2) \notin F$ then $c(x_0, x_3) = b$.

<u>Proof</u>. If $c(x_0, x_3) = r$ or g then changing the colors $c(x_0, x_1)$, $c(x_2, x_3)$ from b to $g = b+r$, $c(x_1, x_2)$ to w and $c(x_0, x_3)$ to $c(x_0, x_3) - r$ we increase $|E_g|$ by two or one according as $c(x_0, x_3) = r$ or g. In the worst case, when $(x_0, x_3) \in F$ we form the new subfactor $F - (x_0, x_3)$. In any case, $|F| + |E_g|$ has been increased, a contradiction.

If $c(x_0, x_3) = w$ then change each of (x_0, x_1), (x_2, x_3) to a white line

*All subfactors will be a subset of red lines.

while adding blue to $c(x_1,x_2)$ and $c(x_0,x_3)$. The result is an increase in $|E_g|$ without changing F. Thus $c(x_0,x_3) = b$. Note that the changes in colors did not disturb the equations $|E_r(v_i)| + |E_g(v_i)| = d_i$, $|E_b(v_i)| + |E_g(v_i)| = d_i - k_i$, $1 \leq i \leq n$. This will always be the case in all recolorings.

2^0]. Let $(x_0,x_1), (x_1,x_2),\cdots, (x_{2t},x_{2t+1})$, $t \geq 1$, $x_0 \neq x_{2t+1}$ be an alternating path P which is line disjoint with F and $(x_0,x_{2t+1}) \notin P$. Then $c(x_0,x_{2t+1}) = r$.

Proof. Suppose $c(x_0,x_{2t+1}) \neq r$. We show as in 1^0 that by suitable recoloring of the lines of P and the line (x_0,x_{2t+1}) we can increase $|E_g|$, F remaining unchanged. For example, if $c(x_0,x_{2t+1}) = b$ or g then change the color of all red lines of P to green by adding blue to them, change the color of all blue lines of P to white and $c(x_0,x_{2t+1})$ to $c(x_0,x_{2t+1}) - b$. If $c(x_0,x_{2t+1}) = w$ then change it to red, $c(x_{2i},x_{2i+1})$ to white for $0 \leq i \leq t$ and $c(x_{2i+1},x_{2i+2})$ to green for $0 \leq i \leq t - 1$.

Next, observe that for each virtex $v_i \in V-S$, there are at least $1 + |E_b(v_i)|$ red lines not in F which are incident with v_i whereas for $v_i \in S$, $|E_r(v_i)-F| = |E_b(v_i)|$. This is straight forward from the definition of S. Also note that a red line with both end points in V-S is necessarily in F. (Otherwise we can add it to F!) Choose a vertex x_0 $\in$ in V-S and a red line $(x_0,x_1) \notin F$; $x_1 \in S$. There is a blue line, say (x_1,x_2) and thus a red line $(x_2,x_3) \notin F$ (x_3 is possibly same as x_0) and a blue line (x_3,x_4) if $x_3 \notin V-S$. One can proceed in this way and get an alternating path (line) disjoint with F and terminating at a vertex in

V-S. Let $P = (x_0,x_1), (x_1,x_2),\dots, (x_{2t},x_{2t+1})$, be an alternating path with smallest number of lines among all the alternating paths from x_0 terminating in V-S and being disjoint with F. It is shown in appendix 1 that $t = 1$ or 2; moreover, if $t = 2$ we have $x_1 = x_4$ (see fig. 1). For $t = 2$, there are two possibilities: $x_0 \neq x_3$ (fig. 2) and $x_0 = x_3$. The case $x_0 = x_3$ will be taken up in 4^0 and 5^0 while 3^0 deals with the other cases.

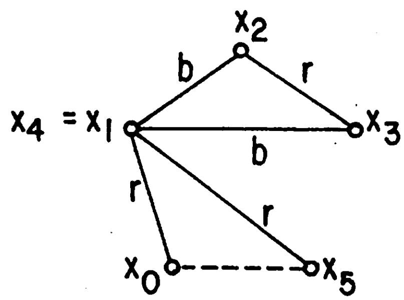


Fig. 1. $t = 2$. The broken line is in F.

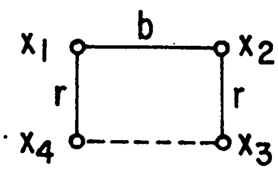


Fig. 2. $t = 1$ and $x_0 \neq x_3$.

3^0]. Each of the following gives a contradiction. The alternating path P has 1) 5 lines, 2) three lines and $x_0 \neq x_3$.

<u>Proof</u>. Let us write $y = x_5$ or x_3 according as we are in 1) or 2). By 2^0, $c(x_0,y) = r$ and x_0, y being in V-S, $(x_0,y) \in F$. Thus $k(y) \geq 2$ (where $k(y) = k_i$ if $y = v_i$) and therefore by the hypothesis of the theorem $k(x_1) \geq 1$. Let $(x_1,u) \in F$. Note that in the case 2) we can assume that

$u \neq y = x_3$ because otherwise $k(y)$ is in fact ≥ 3 and thus $k(x_1)$ being at least 2 we can find a vertex $v \neq x_3$, such that $(v,x_1) \in F$. In case 1) obviously $u \neq y$. Let us define $F' = F - (u_1x_1) - (x_1,x_0)$; $|F'| = |F|$. Consider the path Q from u to y obtained by replacing (x_0,x_1) in P with (u,x_1). Since u and y are unsaturated w.r.t. F' and Q is disjoint from F', by 2^0, $(u,y) \in F'$ and hence $(u,y) \in F$. We can now say that $k(y) \geq 3$ and obtain another vertex $u' \neq u$, such that u' is not incident with any line of P and $(u',x_1) \in F$. Repeating the same argument again and again we obtain $k(y)$ is arbitrarily large which is certainly impossible.

4^0]. $t = 1$ and $x_0 = x_3$. Then there is a blue line at x_0.

Proof. Suppose not. Then $k(x_0) = |E_r(x_0)| \geq 2$, and therefore $k(x_1) \geq 1$. Obtain a vertex u such that $(u,x_1) \in F$. Define, $F' = F - (u,x_1) + (x_0,x_1)$ as before and consider the path (u,x_1), (x_1,x_2), (x_2,x_0). But then we are back in 3^0 which is just shown not possible.

5^0]. $t = 1$ and $x_0 = x_3$. There cannot be a blue line at x_0.

Proof. Suppose there is a blue line at $x_3 = x_0$, say (x_3,x_4). Consider all possible alternating paths Q from x_0 to some point of V-S which contains P properly and disjoint with F. This is possible because for all points $v_i \in S$, at v_i there are as many red lines not belonging to $F \cup P$ as there are blue lines not in $F \cup P$. No such path Q 'returns' to the vertex x_0 'after' x_3.* Let $P' = P \cup P_0$ have smallest number of

*Because it will imply the existence of an even cycle C disjoint with F and whose lines are alternately blue and red. But then one can increase $|E_g|$ ($|F|$ remaining fixed) by making the red lines of C white and blue lines of C green.

lines among all Q. We show that P_0 has two lines only.

By 1^0 and proper choices of three lines one can show that $c(x_1,x_4) = c(x_2,x_4) = b$. If P_0 has three or more lines let (x_3,x_4), (x_4,x_5), (x_5,x_6) be the first three lines. $x_1,x_2,\ldots,x_5$ are all distinct. $c(x_4,x_5) = r$ implies, by 2^0, $c(x_5,x_i) = r$, $i \leq i \leq 3$. Thus x_6 is different from x_i, $1 \leq i \leq 5$. But $c(x_0,x_6) = b$ (by 1^0) and hence $(x_0,x_6) \notin P \cup P_0$ implies we can replace the subsequence (x_3,x_4), (x_4,x_5), (x_5,x_6) in P_0 by (x_3,x_6) and we get a shorter alternating path contradicting minimality of P_0. Thus $P_0 = (x_3,x_4)$, (x_4,x_5). (see fig. 3.) Now consider the path $P'' = (x_0,x_1)$, (x_1,x_4), (x_4,x_5). P'' is line disjoint with F. However, existence of such an alternating path is shown to be impossible in 3^0.

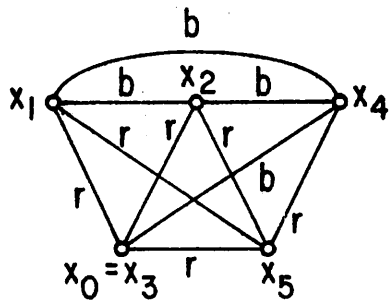


Fig. 3. x_0, x_5 are in V-S.

The contradictions in 3^0 together with 4^0, 5^0 show that $V-S \neq \phi$ is impossible. Therefore $V = S$ and we have proved the theorem.

<u>Remark</u>. If there exists a graph with degree sequence $\langle d_i \rangle$ and containing a $\langle k_i \rangle$-factor then, trivially, $\langle d_i - k_i \rangle$ is graphical.

The following examples show that the theorem is not true if two k_i's differ by two or more.

Example 1. Let $\langle d_i \rangle = \langle 5,5,4,3,3,2 \rangle$ and $\langle k_i \rangle = \langle 3,3,1,3,3,1 \rangle$. Each of $\langle d_i \rangle$, $\langle k_i \rangle$, $\langle d_i - k_i \rangle = \langle 2,2,3,0,0,1 \rangle$ is a graphical sequence. If Theorem 1.1 was true for these $\langle d_i \rangle$, $\langle k_i \rangle$ then it would be possible to find a graph G with degree sequence $\langle d_i \rangle$ that contains the unique graph H (fig. 4) with degree sequence $\langle d_i - k_i \rangle$. Unfortunately, there is no such G, as can be seen easily.

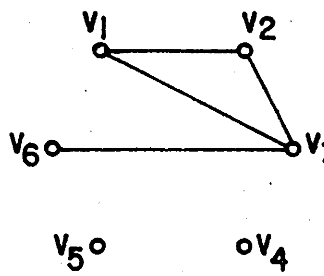


Fig. 4. The graph H with degree sequence $\langle d_i - k_i \rangle$.

Example 2. The sequences $\langle d_i \rangle = \langle 4,3,2,2,1 \rangle$, $\langle k_i \rangle = \langle 3,1,2,2,0 \rangle$, $\langle d_i - k_i \rangle = \langle 1,2,0,0,1 \rangle$ are graphical. But there is no graph G whose degree sequence is $\langle d_i \rangle$ and which contains the graph F with degrees $\langle k_i \rangle$.

COROLLARY 1.2. If $\langle d_i \rangle$ is graphical then there is a graph G with degree sequence $\langle d_i \rangle$ and having a k-factor if and only if $\langle d_i - k \rangle$ is graphical.

A result of A. R. Rao and S. B. Rao [4,5] on connected factors implies rather immediately

COROLLARY 1.3. There exists a graph with degree sequence $\langle d_i \rangle$ and containing a hamiltonian cycle if and only if $\langle d_i \rangle$, $\langle d_i - 2 \rangle$ are graphical and for all $p < \frac{n}{2}$, $\sum_{i=1}^{p} d_i^* < p(n-p-1) + \sum_{n-p+1}^{n} d_i^*$ where $\langle d_i^* \rangle$ is a rearrangement of $\langle d_i \rangle$ into a non increasing sequence.

<u>COROLLARY 1.4</u>. If $\langle d_i \rangle$, $\langle k_i \rangle$ are arbitrary graphical sequences such that $k \leq d_i - k_i \leq k+1$ for some k then there exists a graph G with degree sequence $\langle d_i \rangle$ and containing a $\langle k_i \rangle$ factor.

Proof. Interchange the role of $\langle k_i \rangle$ and $\langle d_i - k_i \rangle$ in Theorem 1.1. We have a graph G with degree sequence $\langle d_i \rangle$ which contains a factor F having $d_i - k_i$ lines at vertex v_i. The lines G-F form the required factor (compare examples 1 and 2).

<u>COROLLARY 1.5</u>. If there exists a graph with degree sequence $\langle d_i \rangle$ and containing a k-factor then for $0 < \ell < k$, $\ell \equiv k \pmod 2$ if n is odd, there is graph with degree sequence $\langle d_i \rangle$ and containing an ℓ-factor.

Proof. One simply notes that under the hypothesis of the corollary $\langle d_i - \ell \rangle$ is graphical. This follows easily from a theorem of Fulkerson [1] on the existence of (0,1) matrices with given row sums and column sums and zero diagonal elements.

The case $\ell = k \pmod 2$ for arbitrary n was obtained earlier in [4 , 5].

The next theorem gives a n.s.c. for existence of graphs G with given degree sequences and containing a given graph F. It happens that we have to assume a lot more than before in order that $G \supseteq F$ and at the same time such extra assumptions allow more flexibility, though not as much as one would wish, on the choice of F than those given by the degree sequences $\langle k_i \rangle$, $k \leq k_i \leq k+1$. This time the proof is by induction.

<u>THEOREM 1.6</u>. Let F be a graph not containing an isomorphic copy of the

one in fig. 5. There exists a graph G with degree sequence $\langle d_i \rangle$ and G containing the graph F if and only if for every graph $F' \subseteq F$ we have $\langle d_i - k_i \rangle$ is graphical where $\langle k_i \rangle$ is the degree sequence of F'.

Proof. The necessity is trivial. We prove 'if' part by induction on $|F|$. If $|F| = 1$ then (1.6) is same as (1.1). Let $|F| = m \geq 2$ and let the theorem be true for all graphs F having m-1 or less lines. Let (x,y) be a line in F and $F_0 = F - (x,y)$. F_0 does not contain a copy of the subgraph in fig. 5. Thus there exists a graph $G \supseteq F_0$ with degree sequence $\langle d_i \rangle$, and let (x,y) not be in any such G. Also there exists a graph $H \supseteq F_0$ which has d_i lines at all vertices $v_i \neq x,y$ where it has only d_i-1 lines. Consider the pair of graphs G, $H-F_0$ and associated coloring of the lines (v_i,v_j) in r,b,g,w. Suppose that G, H are so chosen that $|E_g|$ is maximum ($E_g \cap F_0 = \phi$). It is easy to see that there is an alternating path from x to y which is line disjoint with F_0 and has one more red lines than blue lines; let P be one such path with minimum number of lines. If $(x,y) \notin P$ then as in 2^0 of (1.1) one can perform recoloring on $P \cup (x,y)$ so that (x,y) becomes green or red and we have proved the theorem. Let us therefore assume that $(x,y) = (x_i,x_{i+1})$* and $P = (x = x_0,s_1), (x_1,x_2), \ldots, (x_i = x, x_{i+1} = y), (x_{i+1},x_{i+2}), \ldots, (x_{2t},x_{2t+1} = y)$; $c(x_i,x_{i+1}) = b$. Observe that P 'arrives' at y only at x_{i+1} and x_{2t+1}. Consider the subpath P' of P: from (x_1,x_2) to (x_i,x_{i+1}). If $c(x_1,x_{i+1})$ is blue then we can form alternating path $P' = (x_0,x_1)\ (x_1,x_{i+1}), (x_{i+1},x_{i+2}), \ldots (x_{2t},x_{2t+1})$ not containing (x,y). If $c(x_1,x_{i+1}) = w,g$, or $c(x_1,x_{i+1}) = r$ and $(x_1,x_{i+1}) \notin F_0$ then one can increase $|E_g|$ by a recoloring of $P' \cup (x_1,x_{i+1})$. Thus: $(x_i,x_{i+1}) \in F_0$.

*(x,y) cannot be (x_{i+1},x_i).

Considering the path (x_i,x_{i-1}) $(x_{i-1},x_{i-2}),\ldots$ (x_i,x_0), (x_0,x_{i+1}), $(x_{i+1},x_{i+2})\ldots$ (x_{2t},x_{2t+1}) one can show that $(x_{i-1},x_{i+1}) \in F_0$. Similarly (x_i,x_{i+2}), $(x_i,x_{2t}) \in F_0$. But $x_1 \neq x_{i=1}$, $x_{i+2} \neq x_{2t}$ implies we have F contains the subgraph shown in fig. 5, a contradiction.

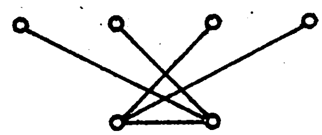


Fig. 5. The excluded subgraph in F.

This proves the theorem.

Following example shows that Theorem 1.6 may not be true if F does contain the graph in fig. 5.

Example 3. Let $\langle d_i \rangle = \langle 4,4,4,4,4,4 \rangle$ and F be $K_{3,3}$. F has 9 edges and thus 2^9 subgraphs. Since d_i's are same it is enough to check that $\langle d_i - k_i \rangle$ is graphical for different isomorphic subgraphs $F' \subseteq F$. For $|F'| \leq 5$ they are shown in table 1. There is no graph $G \supseteq F$ with degree sequence $\langle d_i \rangle$. You may find it strange that there is a graph G which contains all but one edge of $K_{3,3}$ (fig. 6.).

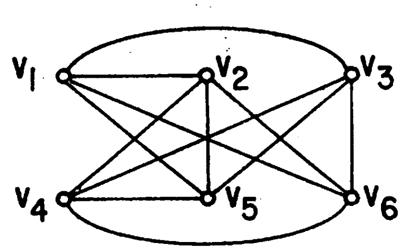
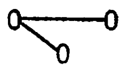
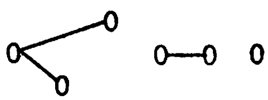
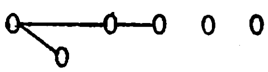
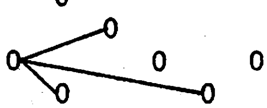
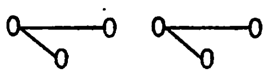
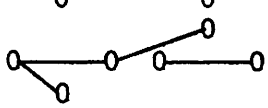
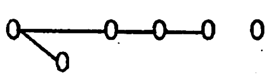
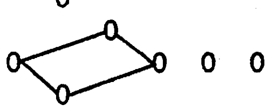
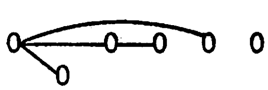
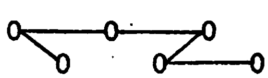


Fig. 6. Graph G containing all but one edge of F.

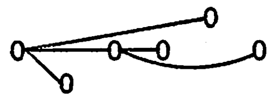
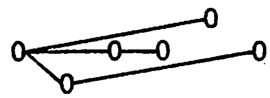
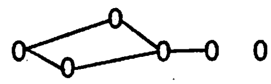
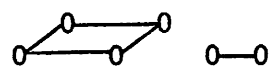
Table 1.

$\lvert F' \rvert$	isomorphic type* of $F' \subseteq F$	$\langle d_i - k_i \rangle$	representing graph
0	0 0 0 0 0 0	$\langle 4,4,4,4,4,4 \rangle$	G in fig. 6.
1	0—0 0 0 0 0	$\langle 3,3,4,4,4,4 \rangle$	$G_1 = G - (v_1,v_2)$
2	0—0 0—0 0 0	$\langle 3,3,3,3,4,4 \rangle$	$G_2 = G - \{(v_1,v_2),(v_3,v_4)\}$
		$\langle 2,3,3,4,4,4 \rangle$	$G_3 = G - \{(v_1,v_2),(v_1,v_3)\}$
3	0—0 0—0 0—0	$\langle 3,3,3,3,3,3 \rangle$	$G_4 = G - \{(v_1,v_3),(v_4,v_6),(v_2,v_5)\}$
		$\langle 2,3,3,3,3,4 \rangle$	$G_5 = G_3 - (v_4,v_5)$
		$\langle 2,3,2,3,4,4 \rangle$	$G_6 = G_3 - (v_3,v_4)$
		$\langle 1,3,3,4,3,4 \rangle$	$G_7 = G_3 - (v_1,v_5)$
4		$\langle 2,3,3,2,3,3 \rangle$	$G_8 = G_5 - (v_4,v_6)$
		$\langle 2,3,2,3,3,3 \rangle$	$G_9 \approx G_8$†
		$\langle 3,2,2,2,3,4 \rangle$	$G_{10} = G_6 - (v_4,v_5)$
		$\langle 2,2,2,2,4,4 \rangle$	$G_{11} = G_6 - (v_2,v_4)$
		$\langle 1,3,2,3,3,4 \rangle$	$G_{12} = G_7 - (v_4,v_5)$
5		$\langle 2,3,2,2,2,3 \rangle$	$G_{13} = G_9 - (v_4,v_5)$.

*The vertices can be thought of as $v_1,v_2,..,v_6$ in that order.

† $\approx$ stands for isomorphism.

table 1 cont.

	$\langle 1,3,1,3,3,3 \rangle$	$G_{14} = G_{12} - (v_3,v_6)$
	$\langle 1,2,2,3,3,3 \rangle$	$G_{15} = G_{12} - (v_2,v_6)$
	$\langle 2,2,2,1,3,4 \rangle$	$G_{16} = G_{11} - (v_4,v_5)$
	$\langle 2,2,2,3,2,3 \rangle$	$G_{17} \approx G_{13}$

For $|F'| \geq 6$ the subgraphs are obtained by removing a subgraph of $9 - |F'|$ lines from F. There are 8 of them. The sequences $\langle d_i - k_i \rangle$ are listed below. They are all graphical as can be checked easily.

$\|F'\|$	$\langle d_i - k_i \rangle$
9	$\langle 1,1,1,1,1,1 \rangle$
8	$\langle 2,2,1,1,1,1 \rangle$
7	$\langle 2,2,2,2,1,1 \rangle$
	$\langle 3,2,2,1,1,1 \rangle$
6	$\langle 2,2,2,2,2,2 \rangle$
	$\langle 2,3,2,2,2,1 \rangle$
	$\langle 2,3,3,2,1,1 \rangle$
	$\langle 2,4,2,2,1,1 \rangle$

COROLLARY 1.7. There exists a graph G with degree sequence $\langle d_i \rangle$ and disjoint from F if and only if $\langle d_i+k_i \rangle$ are graphical where $\langle k_i \rangle$ is degree sequence of an arbitrary graph $F' \subseteq F$.

Proof. Note that $\langle d_i+k_i \rangle$ is graphical if and only if $\langle (n-1-d_i)-k_i \rangle$ is graphical. Rest is easy.

If $F = \{(v_1,v_2),(v_3,v_4),(v_5,v_6)\}$ and $\langle d_i \rangle = \langle 2,2,1,1,3,3 \rangle$ then the sequence $\langle d_i-k_i \rangle$ is graphical for all $F' \subseteq F$ except when $F' = \{(v_1,v_2),(v_3,v_4)\}$. And there is no graph G containing F with degree secquence $\langle d_i \rangle$.

COROLLARY 1.8. Let $\langle d_i \rangle$, $\langle k_i - d_i \rangle$ be two graphical sequences where $k \le k_i \le k+1$, $1 \le i \le n$ and $d_i \le k_i \le n-1$. Then there are disjoint graphs with degree sequences $\langle d_i \rangle$, $\langle k_i - d_i \rangle$.

Proof. There exists a graph G with degree sequences $\langle (n-1) - d_i \rangle$ and G containing a $\langle (n-1) - k_i \rangle$ - factor F. The graphs k_n -G and G-F satisfy the corollary (k_n is the complete graph).

Appendix. 1.

We said in the proof of Theorem 1.1 that the shortest alternating path $P = (x_0,x_1), (x_1,x_2)\ldots, (x_{2t},x_{2t+1})$, starting at x_0 and terminating at a vertex in V-S, has at most 5 lines. P is line disjoint with F. A proof is given below.

Proof. It is useful to regard the lines of P being oriented in the direction from x_i to x_{i+1}, $0 \leq i \leq 2t$; we write them as arc $(x_i,x_{i+1})^-$ Then at a vertex v_i, there is at most one line, of each color, directed from and one line directed into v_i that belong to P. For example, if there are two blue lines directed from v_i one of them preceeds the other as one traverses P. But this implies that P 'enters' v_i with a red line after it had left v_i by the first blue line. In other words there is an even cycle whose lines are alternately blue and red. As we have seen earlier this would imply that $|F| + |E_g|$ is not maximum, contrary to the assumption.

Suppose P has five or more lines.

If possible, let there be three consecutive lines of P as follows: $c(x_i,x_{i+1}) = c(x_{i+2},x_{i+3}) = b$, $c(x_{i+1},x_{i+2}) = r$ and $x_i \neq x_{i+3}$. By 1^0, $c(x_i,x_{i+3}) = b$; (x_i,x_{i+3}) must be a line of P (otherwise we can replace the sequence $(x_i,x_{i+1}),\ldots, (x_{i+2},x_{i+3})$ by (x_i,x_{i+3})). Further, the line (x_{i+3},x_i) is oriented into x_i (fig. 7).

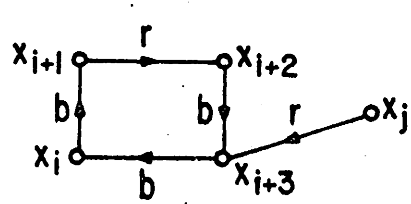


Fig. 7.

Let $(x_{j-1},x_j)^-$, $(x_j,x_{j+1} = x_{i+3})^-$ be the two lines of P immediately preceeding $(x_{i+3},x_i)^-$; they are respectively blue and red and the blue line exists if $x_j \neq x_0$. Observe that $x_j \neq x_p$: $i \leq p \leq i + 3$, and $x_{j-1} \neq x_i$. By 1^0, $c(x_i,x_{j-1}) = b$, and it cannot be in P. However, this contradicts the minimality of P (as the seq. $(x_{j-1},x_j)^-$, $(x_j,x_{i+3})^-$, $(x_{i+3},x_i)^-$ can be replaced by $(x_{j-1},x_i)^-$). Thus $x_j = x_0$. Similarly, the red line following $(x_{i+3},x_i)^-$ in P must be (x_i,x_{2t+1}). But then (x_0,x_{i+3}), (x_{i+3},x_i) (x_i,x_{2t+1}) is an alternating path disjoint with F, a contradiction. Thus $x_i = x_{i+3}$. Then the path P can be written as

$$P = (x_0,x_1),\ (x_1,x_2)\ (x_2,x_3)\ (x_3,x_1),\ (x_1,x_5),\ \dots\ (x_{2t},x_{2t+1})$$

If $t \geq 3$, then the 6th line (x_5,x_3) is blue and we can replace the first six arcs by $(x_0,x_1)^-$, $(x_1,x_3)^-$ to obtain an alternating path with less lines than P. Thus $t \leq 2$ and in case P has five lines it conforms to the description that $x_1 = x_4$.

2. DIRECTED GRAPHS

We shall assume that digraphs have no multiple arcs and loops and all diagraphs are drawn on vertices $v_1,\ldots,v_n$. A pair of arcs (v_i,v_j), (v_j,v_i) is possible. Note that an arc from v_i to v_j is written as (v_i,v_j). Given a sequence of ordered pairs of non negative integers, $\langle (d_i^+,d_i^-)\rangle$, we say it is graphical is there exists a digraph $\vec{G}$ with out-degree and indegree of vertex v_i being equal to respectively d_i^+ and d_i^-. We say $\vec{G}$ has degree sequence $\langle (d_i^+,d_i^-)\rangle$. We shall identify $\vec{G}$ with the set of arcs in $\vec{G}$. A n.s.c. for a sequence $\langle (d_i^+,d_i^-)\rangle$ to be graphical is obtained by Fulkerson, D. R. [1]. Most of the terminology introduced for graphs in §1 has a natural extension to digraphs. A subdigraph $\vec{F} \subseteq \vec{G}$ is called a subfactor with respect to $\langle (k_i^+,k_i^-)\rangle$ if $\vec{F}$ has at most k_i^+ arcs from v_i and k_i^- arcs into v_i. We shall let $S^+ = S^+(\vec{F})$ $(S^- = S^-(\vec{F}))$ denote the vertices v_i having k_i^+ (resp k_i^-) arcs from (into) v_i in $\vec{F}$. A vertex in $S^+(S^-)$ is called outer (inner) saturated. A vertex that is both outer and inner saturated is simply called saturated and $S = S^+ \cap S^-$ is the set of saturated vertices. Notations $E_c^+(v_i)$, $E_c^-(v_i)$ will be used with their obvious meanings. $E_c = \cup E_c^+(v_i) = \cup E_c^-(v_i)$.

We shall prove the following Theorem.

<u>THEOREM 2.1</u>. Let the sequence $d = \langle (d_i^+,d_i^-)\rangle$ be graphical and let $\langle (k_i^+,k_i^-)\rangle$ be a sequence such that for some $k \geq 0$, $k = k_i^-$ (or for that matter $k = k_i^+$), $1 \leq i \leq n$. Then there exists a digraph $\vec{G}$ with degree sequence $\langle (d_i^+,d_i^-)\rangle$ containing a $\langle (k_i^+,k_i^-)\rangle$-factor if and only if the sequence $d-k = \langle (d_i^+-k_i^+,\ d_i^--k_i^-)\rangle$ is graphical.

That the sequence $\langle (d_i^+-k_i^+,d_i^--k_i^-)\rangle$ be graphical is clearly necessary.

The theorem says that it is also sufficient. The special case $k_i^+ = k = k_i^-$, $1 \leq i \leq n$ was conjectured by A. R. Rao and S. B. Rao along with their conjecture on undirected graphs (see §1). The proof of (2.1) is a modification of that of (1.1) to accomodate arcs instead of lines.

Proof of Theorem 2.1. Let $\vec{G}$ and $\vec{H}$ be representing digraphs for the sequences d and d-k respectively. Consider the coloring of arcs (v_i, v_j) in r, b, g and w as before, namely, $c(v_i, v_j) = r$ if $(v_i, v_j) \in \vec{G} \setminus \vec{H}$, b if $(v_i, v_j) \in \vec{H} \setminus \vec{G}$ etc. One has $|E_r^+(v_i)| + |E_g^+(v_i)| = d_i^+$, $|E_b^+(v_i)| + |E_g^+(v_i)| = d_i^+ - k_i^+$ and similar equations for indegrees. We choose a subfactor $\vec{F} \subseteq E$. Let us assume that $\vec{G}$, $\vec{H}$, $\vec{F}$ have been so chosen that $|E_g| + |\vec{F}|$ has maximum value. We show that $\vec{F}$ is a factor. For brevity let $k = k_i^-$, $1 \leq i \leq n$ and let $S \neq V$. We observe that i) $S^+ \neq V \neq S^-$; ii) $|E_r^+(v_i) - \vec{F}| \geq |E_b^+(v_i)|$, $|E_r^-(v_i) - \vec{F}| \geq |E_b^-(v_i)|$ for all v_i and the equality holds precisely for $v_i \in S^+$ and $v_i \in S^-$ respectively; iii) There does not exist distinct arcs (y_0, y_1), (y_2, y_3), $(y_2, y_3), \ldots, (y_{2t}, y_{2t+1})$, (y_0, y_{2t+1}) whose colors are red and blue alternately in that order such that all the red arcs are in $E_r - \vec{F}$. Property iii) is almost trivial. One can change $c(y_{2m}, y_{2m+1})$ to white and $c(y_{2m+2}, y_{2m+1})$ to green for $0 \leq m \leq t$ and increase $|E_g|$, keeping $|\vec{F}|$ unchanged, contradicting that $|E_g| + |\vec{F}|$ was maximum. The sequence (y_0, y_1), $(y_2, y_1), \ldots, (y_{2t}, y_{2t+1})$ is said to constitute an <u>alternating chain</u>.

Take a vertex $x_0 \in V - S^+$ and let (x_0, x_1) be a red arc not in $\vec{F}$. x_1 is necessarily in S^- and let (x_2, x_1) be a blue arc; there exists a red arc (x_2, x_3) not in $\vec{F}$. We can continue in this way to build an alternating chain P until it 'terminates' at a vertex in $V - S^-$. Let P =

(x_0,x_1), (x_2,x_1), (x_2,x_3),..., (x_{2t},x_{2t+1}), $x_{2t+1} \in V\text{-}S^-$.

We show that $c(x_0,x_{2t+1}) = r$ if $x_0 \neq x_{2t+1}$ and thus $(x_0,x_{2t+1}) \in \vec{F}$. This is easy once we show that $(x_0,x_{2t+1}) \notin P$. The proof is similar to the one in 2^0, Theorem 1.1. Suppose (x_0,x_{2t+1}) is in P. If $c(x_0,x_{2t+1}) = b$ then P would contain a closed alternating chain as in iii) because in P blue arcs are traversed in opposite direction. If $c(x_0,x_{2t+1}) = r$ and $(x_0,x_{2t+1}) = (x_i,x_{i+1})$, then (x_{i+2},x_{i+1}), (x_{i+2}, x_{i+3}), ..., (x_{2t},x_{2t+1}) is a closed alternating chain as in iii) except all arcs have reverse orientation. Thus we conclude that $(x_0,x_{2t+1}) \notin P$.

Now we prove by contradiction that an alternating chain P does not exist.

Case I.* $|P| = 3$, $x_0 = x_3$. Since $k^-(x_1) = k^-(x_0) \geq 1$ there exists $x_4 \neq x_0$, x_1, x_2 such that $(x_4,x_1) \in \vec{F}$. Consider $\vec{F}' = \vec{F} - (x_4,x_1) + (x_0,x_1)$; $|\vec{F}'| = |\vec{F}|$. The chain (x_4,x_1), (x_2,x_1), (x_2,x_3) implies $(x_4,x_3) \in \vec{F}'$ because $x_4 \in V\text{-}S^+(\vec{F}')$, $x_3 \in V\text{-}S^-(\vec{F}')$. Hence $(x_4,x_3) \in \vec{F}$ and $k^-(x_0) \geq 2$ which implies there exists $x_5 \neq x_i$, $0 \leq i \leq 4$, such that $(x_5,x_1) \in \vec{F}$. We can prove, as before, that $(x_5,x_3) \in \vec{F}$ and so on, finally obtaining $k^-(x_0)$ is arbitrarily large. This is impossible.

Case II. $|P| = 3$, $x_0 \neq x_3$. Same as above as long as x_4,x_5, .. remain different from x_3. If $x_i = x_3$, then with respect to $\vec{F}' = \vec{F} - (x_i,x_1) + (x_0,x_1)$ and $P = (x_i,x_1)$, (x_2,x_1), (x_2,x_3) we are in Case I.

*Compare the corresponding situation in Theorem 1.1.

Case III. $|P| = 2t+1$, $t \geq 2$, $x_0 = x_{2t+1}$. As in Case I there exists $(x_{2t+2},x_1) \in \vec{F}$ and let $\vec{F}'$, P' be defined as $\vec{F}' = \vec{F} - (x_{2t+2},x_1) + (x_0,x_1)$, $P' = P + (x_{2t+2},x_1) - (x_0,x_1)$. P' is an alternating chain and $(x_{2t+2},x_{2t+1}) \notin P'$. It follows that $(x_{2t+2},x_{2t+1}) \in \vec{F}'$ and thus is in $\vec{F}$ and $k^-(x_{2t+1}) \geq 2$. A contradiction is ahead as in Case I.

Case IV. $|P| = 2t+1$, $t \geq 2$, $x_0 \neq x_{2t+1}$. It can be reduced to Case III or a contradiction otherwise.

COROLLARY 2.2.

Let $\langle (d_i^+,d_i^-) \rangle$ be graphical. Suppose $\langle (k_i^+,k_i^-) \rangle$ is graphical and $d_i^+ \geq k_i^+$, $d_i^- \geq k_i^-$, $1 \leq i \leq n$ and either $\langle d_i^+-k_i^+ \rangle$ or $\langle d_i^--k_i^- \rangle$ is a sequence of constant terms. Then a graph G with degree sequence $\langle (d_i^+,d_i^-) \rangle$ and a $\langle (k_i^+,k_i^-) \rangle$-factor exists.

Example 4. In the following we have the sequences d, k, d-k all graphical and yet there is no graph with degree sequence d and having $\langle k_i^+,k_i^- \rangle$-factor. The sequence $\langle k_i^+ \rangle$ vary only by 1.

$d = \langle (4,4), (3,3), (2,2), (2,2), (1,1) \rangle$

$k = \langle (1,1), (1,2), (1,0), (0,0)\ (1,1) \rangle$

$d-k = \langle (3,3), (2,1), (1,2), (2,2), (0,0) \rangle$

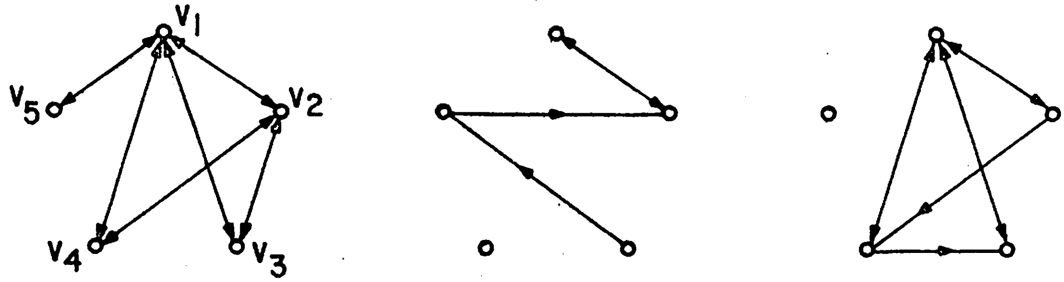


Fig. 8. Digraphs with degree sequences d, k, d-k respectively. There is a unique digraph with degree sequence d-k.

Note. In contrast with the undirected case the shortest alternating path in directed case can be of arbitrary length and thus we don't use them in the proof.

Corresponding to Theorem 1.6 we have,

THEOREM 2.3. Let $\vec{F}$ be a given digraph. There exists a digraph $\vec{G}$ with degree sequence $\langle (d_i^+, d_i^-) \rangle$ and $\vec{G}$ containing $\vec{F}$, if and only if for every subdigraph $\vec{F}' \subseteq \vec{F}$ the sequence $\langle d_i^+ - k_i^+, d_i^- - k_i^- \rangle$ is graphical where $\langle (k_i^+, k_i^-) \rangle$ is degree sequence of $\vec{F}'$.

Proof. The necessity is trivial. We shall prove sufficiency by induction on the number of arcs in $\vec{F}$.

1^0. Let $\vec{F} = (x,y)$ and $\langle (k_i^+, k_i^-) \rangle$ be the degree sequence of $\vec{F}$. Suppose there is no digraph $\vec{G} \supseteq \vec{F}$. We shall obtain a contradiction. Consider digraphs $\vec{G}$, $\vec{H}$ with degree sequences $\langle (d_i^+, d_i^-) \rangle$, $\langle (d_i^+ - k_i^+, d_i^- - k_i^-) \rangle$, respectively, such that $|E_g|$ is maximum in the corresponding coloring; $c(x,y) = b$ or w. It is easy to see that there is an alternating chain P from x to y such that $P \subseteq E_r \cup E_b$, $(x,y) \notin P$ (because that would imply P traverses back to x by an even cycle whose arcs are alternately red and blue and this in turn implies that $|E_g|$ is not maximum). But then $C = P \cup (x,y)$ is an even cycle. Perform a suitable recoloring of c such that $c(x,y) = g$ or r. We obtain a digraph $\vec{G} \geq \vec{F}$.

2^0. Suppose the theorem is true for all digraphs with m-1 or less arcs and $\vec{F}$ has m arcs. Let (x,y) be an arc of $\vec{F}$; write $\vec{F}_0 = \vec{F} - (x,y)$. Thus there are digraphs $\vec{G}$, $\vec{H}$ containing $\vec{F}_0$ with degree sequence $\langle (d_i^+, d_i^-) \rangle$ except that in $\vec{H}$ x(y) has outdegree (indegree) one less than that in $\vec{G}$. Consider a pair of $\vec{G}$, $\vec{H}$ such that in the associated coloring of $\vec{G}$, $\vec{H} - \vec{F}_0$, $|E_g|$ is maximum. Without loss of generality we can assume that $c(x,y) \neq r, g$. There exists an alternating chain $P \subseteq (E_r - \vec{F}_0) \cup E_b$ from x to y since for every vertex v_i, $|E_r^+(v_i) - \vec{F}_0| \geq |E_b^+(v_i)|$, $|E_r^-(v_i) - \vec{F}_0| \geq |E_b^-(v_i)|$ with strict inequality respectively for x and y. As before $(x,y) \notin P$. There exists a recoloring of the even cycle $C = P \cup (x,y)$ such that $c(x,y) = r$ or g. Then we have a digraph $\vec{G} \supseteq \vec{F}$.

For digraphs one can state and prove theorems as in (1.7), (1.8). For example the following is true.

COROLLARY 2.4 Suppose $\langle (d_i^+, d_i^-) \rangle$, $\langle k_i^+ - d_i^+, k_i^- - d_i^- \rangle$ are graphical sequences where k_i^+ (or k_i^-) are same for all i and k_i^+, $k_i^- \leq n-1$. Then there are disjoint representing digraphs for them.

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