CONVOLUTION FEEDBACK SYSTEMS

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Abstract

This paper considers multi-input multi-output feedback systems characterized by $y = G*e$ and $e = u - y$. Theorem I shows that if the closed loop impulse response H is stable in the sense that $H \in \mathcal{A}^{nxn}(\sigma)$, then $\hat{G}(s) = \hat{P}(s)[\hat{Q}(s)]^{-1}$ where $\hat{P}(s)$, $\hat{Q}(s)$ are also in $\hat{\mathcal{A}}^{nxn}(\sigma)$. Theorem II gives necessary and sufficient conditions for $H \in \mathcal{A}^{nxn}(\sigma)$. Finally Theorem III gives necessary and sufficient condition for stability when $\hat{G}(s)$ has a finite number of multiple poles in $\operatorname{Re} s \geq \sigma$: the case where the leading term of the Laurent expansion at each of these poles is singular is treated in detail.

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I. Introduction

This paper considers linear time-invariant feedback systems with n inputs and n outputs. As it will become apparent, there is no loss of generality in taking the feedback to be unity. The input u, output y and error e are functions from $\mathbb{R}_+$, (defined as $[0,\infty)$), to $\mathbb{R}^n$ or corresponding distributions on $\mathbb{R}_+$. The open loop system is of the convolution type so that we have

$$y = G*e \tag{1}$$

$$e = u-y \tag{2}$$

G is an nxn matrix whose elements are distributions on $\mathbb{R}_+$. We use $\underset{\sim}{G}$ to denote the map $\underset{\sim}{G}$: $e \mapsto G*e$.

We shall repeatedly use the convolution algebra $\mathcal{A}(\sigma)$ [1,2]: f is said to be in $\mathcal{A}(\sigma)$ iff $f(t) = 0$ for $t < 0$ and

$$f(t) = f_a(t) + \sum_0^\infty f_i \delta(t-t_i) \tag{3}$$

where $f_a(t)e^{-\sigma t} \in L^1(0,\infty)$, $f_i \in \mathbb{R}$ for all i, $\sum_0^\infty |f_i| e^{-\sigma t_i} < \infty$ and

$0 = t_0 < t_1 < t_2 \ldots$. Thus f is a distribution of order 0 with support on $\mathbb{R}_+$. An n-vector v (nxn matrix A) is said to be in $\mathcal{A}^n(\sigma)$ ($\mathcal{A}^{nxn}(\sigma)$) iff all its elements are in $\mathcal{A}(\sigma)$. Let $\hat{f}$ denote the Laplace transform of f: f belongs to the convolution algebra $\mathcal{A}(\sigma)$ if and only if $\hat{f}$ belongs to the algebra $\hat{\mathcal{A}}(\sigma)$ (with pointwise product). Similarly, $\hat{v} \in \hat{\mathcal{A}}^n(\sigma)$, $\hat{A} \in \hat{\mathcal{A}}^{nxn}(\sigma)$.

Recently M. Vidyasagar [5] has shown that the class of systems (1), (2) where

$$\hat{G}(s) = \hat{P}(s)[\hat{Q}(s)]^{-1} \tag{4}$$

with $\hat{P},\hat{Q} \in \hat{\mathcal{A}}^{nxn}(\sigma)$ is very useful for distributed networks for example, and he extended some stability results of Desoer, Wu, Baker, Vakharia and Lam [1,2,3,10]. In Theorem I below we prove that, under very mild assumptions on G and on the closed loop system, if the closed loop impulse response $H \in \mathcal{A}^{nxn}(\sigma)$ then $\hat{G}$ is of the form (4). Theorem I is also an extension of a result of Nasburg and Baker [4]: the extension is in two directions, first, the n-input n-output case is considered and, second, the requirements on G are greatly relaxed. Theorem II is a straightforward extension of a result of [4]: it shows the importance of the systems considered by Vidyasagar in the sense that $H \in \mathcal{A}^{nxn}(\sigma)$ if and only if $\hat{G}$ is of the form (4). Finally Theorem III gives the <u>necessary</u> and <u>sufficient</u> conditions for stability of the closed loop system when $\hat{G}$ is of the form (4) with a finite number of poles of finite order in $\mathrm{Re}\, s \geq \sigma$. This theorem culminates a series of investigations starting with [1,2,3,7]. Note that except for [7], all previous work could only prove sufficiency.

II. The relation between $\underset{\sim}{G}$ and $\underset{\sim}{H}$.

<u>Theorem I</u>

Let G be an nxn matrix whose elements are distributions with support on $\mathbb{R}_+$. Suppose that in a neighborhood of the origin, say $V \subset \mathbb{R}$, G includes at most δ-functions (i.e. on V, it is a distribution of at most order 0). For the system defined by (1) and (2), assume that the closed loop response H exists and is uniquely defined by

$$H + G*H = G. \tag{5}$$

Under these conditions, if $H \in \mathcal{A}^{nxn}(\sigma)$, then

(a) G is Laplace transformable and for some $\bar{\sigma} \geq \sigma$, $G \in \mathcal{A}^{nxn}(\bar{\sigma})$.

(b) $\hat{G}$ is of the form

$$\hat{G}(s) = \hat{P}(s)[\hat{Q}(s)]^{-1} \text{ for Re } s > \sigma \tag{6}$$

where $\hat{P}(\cdot)$ and $\hat{Q}(\cdot) \in \hat{\mathcal{A}}^{nxn}(\sigma)$.

(c) $\hat{G}$ can at most have a countable number of poles in Re $s > \sigma$.

Comment. This theorem shows that under mild conditions on G regarding its behavior near $t = 0$; once the closed loop system is well-defined and "stable", then $\hat{G}$ is necessarily of the form (6), can at most have poles in the strip $\sigma <$ Re $s \leq \bar{\sigma}$ and is analytic for Re $s \geq \bar{\sigma}$.

Proof.

(a) By assumption, H is of the form

$$H(t) = H_a(t) + \sum_{i=0}^{\infty} H_i \delta(t-t_i)$$

where $0 = t_0 < t_1 < t_2 < \ldots$. By assumption G can at most have an impulse at the origin. By the Abelian theorem of the Laplace transform [11] and the properties of distributions, if G has an impulse G_0 at $t = 0$, $\hat{G}(s) \to G_0$ as $s \to \infty$ with $\operatorname{Re} s \to \infty$. Clearly from (5), if G_0 is the zero matrix, then $H_0 = 0$. If $G_0 \neq 0$, then by balancing impulses at the origin in (5) we have $(I + G_0)H_0 = G_0$. By assumption H, hence H_0, is uniquely defined by (5), hence $\det(I + G_0) \neq 0$. Furthermore by direct calculation, $(I - H_0)(I + G_0) = I$, so that $\det[I - H_0] \neq 0$.

The function $I - \hat{H}(s)$ is analytic and bounded for $\operatorname{Re} s > \sigma$, and tends to $I - H_0$ as $s \to \infty$ with $\operatorname{Re} s \to \infty$. Consequently, there exists a $\bar{\sigma} \geq \sigma$ such that

$$\inf_{\operatorname{Re} s \geq \bar{\sigma}} |\det[I - \hat{H}(s)]| > 0 \ . \tag{7}$$

From (5), if G had a Laplace transform, we would have $\hat{H} + \hat{G}\hat{H} = \hat{G}$. Now by (7), $\hat{H}(s)[I - \hat{H}(s)]^{-1} \in \hat{\mathcal{A}}^{n\times n}(\bar{\sigma})$, so $\hat{G}(s)$ is equal to that function, by the uniqueness of the convolution algebra of distributions on $\mathbb{R}_+$.

(b) Since $\hat{H}(s)$ is analytic for $\operatorname{Re} s > \sigma$, $[I - \hat{H}(s)]^{-1}$ has at most a countable number of poles in $\operatorname{Re} s > \sigma$ and by analytic continuation

$$\hat{G}(s) = \hat{H}(s)[I - \hat{H}(s)]^{-1} \text{ for } \operatorname{Re} s > \sigma. \tag{8}$$

Choose $\hat{P}(s) = \hat{H}(s)$, $\hat{Q}(s) = I - \hat{H}(s)$. Thus (b) and (c) have been established. ¤

<u>Remark</u>. It is important to reflect on the fact that under the conditions of Theorem I, we have

$$[I + \hat{G}(s)][I - \hat{H}(s)] = I \text{ for } \operatorname{Re} s > \sigma.$$

This expression emphasizes the symmetrical role played by $\underset{\sim}{H}$ and $\underset{\sim}{G}$: $\underset{\sim}{H}$ is obtained from $\underset{\sim}{G}$ by a negative feedback of I; $\underset{\sim}{G}$ is obtained from $\underset{\sim}{H}$ by a negative feedback of $-I$ (to cancel the preceding one!).

<u>Theorem II</u>

Let G be an nxn matrix whose elements are Laplace transformable distributions with support in $\mathbb{R}_+$. For the system defined by (1) and (2), assume that the closed loop transfer function $\hat{H}$ is well-defined for almost all s in the half plane of convergence of $\hat{G}$; i.e.

$$\hat{H}(s) = \hat{G}(s)[I + \hat{G}(s)]^{-1} \tag{9}$$

for almost all s in the half-plane of convergence of $\hat{G}(\cdot)$. Under these conditions,

$$H \in \mathcal{A}^{nxn}(\sigma) \tag{10}$$

if and only if there exists $\hat{P}$, $\hat{Q} \in \hat{\mathcal{A}}^{nxn}(\sigma)$ such that

$$\hat{G}(s) = \hat{P}(s)[\hat{Q}(s)]^{-1} \tag{11}$$

and

$$\inf_{\mathrm{Re}\ s \geq \sigma} |\det[\hat{P}(s) + \hat{Q}(s)]| > 0. \tag{12}$$

<u>Proof</u>:

<u>Necessity</u>. From (9) by algebra

$$\hat{G}(s) = \hat{H}(s)[I - \hat{H}(s)]^{-1} \text{ for } \mathrm{Re}\ s \geq \sigma$$

Choose $\hat{P}(s) = \hat{H}(s) \in \hat{\mathcal{A}}^{nxn}(\sigma)$ and $\hat{Q}(s) = I - \hat{H}(s) \in \hat{\mathcal{A}}^{nxn}(\sigma)$, by (10). Since $\hat{P} + \hat{Q} = I$, (12) holds.

Sufficiency. From (9) and (11)

$$\hat{H}(s) = \hat{P}(s)[\hat{P}(s) + \hat{Q}(s)]^{-1}.$$

In view of (12) $\hat{H} \in \hat{\mathcal{A}}^{nxn}(\sigma)$ as the product of two elements of $\hat{\mathcal{A}}^{nxn}(\sigma)$. ◻

Remark. It is clear from (11) that a given $\hat{G}$ does not define the ordered pair $(\hat{P},\hat{Q})$ uniquely; for example, they might have a matrix as right common factor. In order to be able to express the condition (12) in a form which depends on $\hat{G}$ only, we impose the Vidyasagar no-cancellation condition (N) [5]: the ordered pair (a,b) where $a,b: \mathbb{C} \to \mathbb{C}$ is said to satisfy the no-cancellation condition on a set $A \subset \mathbb{C}$ iff, for all sequences $\{s_k\}$ in A, $a(s_k) \to 0$ implies that $\lim \inf|b(s_k)| > 0$.

It is then easy to show that, [5], if $(\det \hat{Q}(s), \det[\hat{P}(s) + \hat{Q}(s)])$ satisfies (N) on $\operatorname{Re} s \geq \sigma$, then (12) is equivalent to $\inf_{\operatorname{Re} s \geq \sigma} |\det[I + \hat{G}(s)]| > 0$.

III. Necessary and Sufficient Conditions for Stability.

We consider first and in detail the case where $\hat{G}$ has a single pole p of order m in $\operatorname{Re} s \geq \sigma$. The extension to the case of a finite number of poles is straightforward.

We consider the open loop transfer function

$$\hat{G}(s) = \sum_{i=0}^{m-1} R_i(s-p)^{-m+i} + \hat{G}_0(s) \tag{13}$$

where Re $p \geq \sigma$, $\hat{G}_0 \in \hat{\mathcal{A}}^{nxn}(\sigma)$, $r_0 \triangleq \text{rank } R_0 \leq n$ and R_i $(i = 0,1,\ldots, m-1)$ are nxn matrices with complex coefficients. We start by pointing out some facts which will streamline the proof.

Fact 1. Let

(14) $$\hat{R}\left(\frac{1}{s+a}\right) \triangleq \left(\sum_{i=0}^{m-1} R_i (s-p)^{-m+i}\right) \left(\frac{s-p}{s+a}\right)^m ; \; a \triangleq 1-\sigma$$

then $\hat{R}\left(\frac{1}{s+a}\right)$ is nxn complex polynomial matrix in $\left(\frac{1}{s+a}\right)$ of degree m. This is obvious by considering the Laurent expansion of $\hat{R}\left(\frac{1}{s+a}\right)$ about $s = -a$.

Fact 2. (Smith canonical form [12]). For the nxn polynomial matrix $\hat{R}\left(\frac{1}{s+a}\right)$ there exist unimodular (i.e. with nonzero constant determinant) polynomial matrices in $\left(\frac{1}{s+a}\right)$ viz. $\hat{P}\left(\frac{1}{s+a}\right)$ and $\hat{Q}\left(\frac{1}{s+a}\right)$, such that:

(15) $$\hat{Q}\left(\frac{1}{s+a}\right) \hat{R}\left(\frac{1}{s+a}\right) \hat{P}\left(\frac{1}{s+a}\right) =$$

$$\operatorname{diag}\{\underbrace{\hat{a}_1\left(\frac{1}{s+a}\right), \ldots, \hat{a}_j\left(\frac{1}{s+a}\right), \ldots, \hat{a}_r\left(\frac{1}{s+a}\right)}_{r}, \underbrace{0, 0, \ldots, 0}_{n-r}\}$$

where i) r = rank of $\hat{R}\left(\frac{1}{s+a}\right)$ = order of the largest minor of $\hat{R}\left(\frac{1}{s+a}\right)$ whose determinant is not equal to the zero polynomial;

ii) $\hat{a}_j\left(\frac{1}{s+a}\right)$ $j = 1,2,\ldots, r$ are the invariant polynomials of $\hat{R}\left(\frac{1}{s+a}\right)$ and each polynomial $\hat{a}_j(\cdot)$ divides $\hat{a}_{j+1}(\cdot)$, $j = 1,2,\ldots, r-1$;

iii) the diagonal matrix in the R.H.S. of (15) can be obtained by elementary operations.

Fact 3. The polynomial matrices $\hat{P}\left(\frac{1}{s+a}\right)$ and $\hat{Q}\left(\frac{1}{s+a}\right) \in \hat{\mathcal{A}}^{nxn}(\sigma)$ and their inverses are polynomial matrices in $\left(\frac{1}{s+a}\right)$ also in $\hat{\mathcal{A}}^{nxn}(\sigma)$.

Fact 4.

Let $\hat{a}_j(\cdot)$ $j = 1,2,\ldots, r$ be as in (15) and let r_0 be the rank of R_0, then

(a)

$$\begin{cases} \hat{a}_j(1/(p+a)) = 0 \text{ for } r_0 + 1 \leq j \leq r \text{ by definition of } r_0; \\ \hat{a}_j(1/(p+a)) \neq 0 \text{ for } 1 \leq j \leq r_0 \ ; \end{cases} \tag{16}$$

(b)

$$\hat{a}_j\left(\frac{1}{s+a}\right) = \hat{b}_j\left(\frac{1}{s+a}\right)\left(\frac{s-p}{s+a}\right)^{c_j} \quad \text{for } r_0 + 1 \leq j \leq r \tag{17}$$

where c_j is the order of the zero of $\hat{a}_j(\cdot)$ at $s = p$;

$\hat{b}_j(\cdot)$ is a polynomial with

$$\hat{b}_j(1/(p+a)) \neq 0, \text{ (see [13]), and} \tag{18}$$

$$1 \leq c_{r_0+1} \leq c_{r_0+2} \leq \cdots \leq c_r.$$

__Proof__. Set s=p in (15) and note that the L.H.S. becomes $\hat{Q}(1/(p+a)) R_0(p+a)^{-m} \hat{P}(1/(p+a))$. Since $\hat{P}(\cdot)$ and $\hat{Q}(\cdot)$ are unimodular, exactly $(r-r_0)$ polynomials $\hat{a}_j(\cdot)$ are zero at s=p. By ii) of (15) $\hat{a}_j(1/(p+a)) = 0$ for $r_0 + 1 \leq j \leq r$. Hence (16) and (17) follow with the properties of the latter as a consequence of ii) of (15). ⌑

__Remark__. Note that the exponents c_j in (17) may, for some j, be larger than m (in fact $c_r \leq rm$).

Therefore, since the c_j are monotonically increasing and since c_j-m may be of any sign, partition the index set $K = \{r_0+1, r_0+2, \ldots, r\}$ into

(19) $$K_- = \{r_0+1, r_0+2, \ldots, \alpha\} = \{j \mid 1 \leq c_j < m\}$$

(20) $$K_0 = \{\alpha+1, \alpha+2, \ldots, \beta\} = \{j \mid c_j = m\}$$

(21) $$K_+ = \{\beta+1, \beta+2, \ldots, r\} = \{j \mid c_j > m\} \quad .$$

We are now ready for Theorem III.

<u>Theorem III</u>.

Let $\hat{G}(s)$ be given by (13) and let $\hat{P}\left(\frac{1}{s+a}\right)$ and $\hat{Q}\left(\frac{1}{s+a}\right)$ be the polynomial matrices defined in (15). Suppose that the index-sets K_-, K_0, K_+, as defined in (19)-(21), are not empty.

Consider the partitioning

(22) $$\hat{Q}\left(\frac{1}{s+a}\right) [I+\hat{G}_0(s)]\hat{P}\left(\frac{1}{s+a}\right) = \begin{array}{c c} & \begin{array}{cc} \overbrace{\hphantom{\hat{L}_{11}(s)}}^{\alpha} & \overbrace{\hphantom{\hat{L}_{12}(s)}}^{n-\alpha} \end{array} \\ \begin{array}{r} \alpha\{ \\ n-\alpha\{ \end{array} & \left[\begin{array}{c|c} \hat{L}_{11}(s) & \hat{L}_{12}(s) \\ \hline \hat{L}_{21}(s) & \hat{L}_{22}(s) \end{array}\right] \end{array}$$

and let $\hat{b}_j(\cdot)$ be the polynomials defined in (17). Under these conditions,

(10) $$H \in \mathcal{A}^{n\times n}(\sigma)$$

if and only if

(23) $$\inf_{\mathrm{Re}\,s\geq\sigma} \left|\det[I+\hat{G}(s)]\right| > 0$$

and

(C) $$\det\{\hat{L}_{22}(p) + \mathrm{diag}[\hat{b}_{\alpha+1}(1/(p+a)),\ldots,\hat{b}_{\beta}(1/(p+a)),0,0,\ldots,0]\} \neq 0.$$

<u>Proof</u>.

<u>Sufficiency</u>. Since $I - \hat{H}(s) = [I+\hat{G}(s)]^{-1}$, we need only to show that

(24) $$[I+\hat{G}(s)]^{-1} \in \hat{\mathcal{A}}^{n\times n}(\sigma) \quad .$$

By fact 3, (24) is equivalent to

$$\left\{\hat{Q}\left(\frac{1}{s+a}\right) [I+\hat{G}(s)]P\left(\frac{1}{s+a}\right)\right\}^{-1} \in \hat{\mathcal{A}}^{n\times n}(\sigma) \ .$$

Introduce now the following multiplier:

(25) $$\hat{M}(s) \triangleq$$

$$\mathrm{diag}\{\underbrace{\hat{z}(s)^m,\hat{z}(s)^m,\ldots,\hat{z}(s)^m}_{r_0},\underbrace{\hat{z}(s)^{m-c_{r_0+1}},\hat{z}(s)^{m-c_{r_0+2}},\ldots,\hat{z}(s)^{m-c_\alpha}}_{\alpha-r_0},\underbrace{1,\ldots,1}_{n-\alpha}\}$$

with

(26) $$\hat{z}(s) = \frac{s-p}{s+a} \in \hat{\mathcal{A}}(\sigma) \ .$$

By (19) and (26)

(27) $$\hat{M}(s) \in \hat{\mathcal{A}}^{n\times n}(\sigma) \ .$$

Remark that

$$\left\{\hat{Q}\left(\frac{1}{s+a}\right)\ [I+\hat{G}(s)]\hat{P}\left(\frac{1}{s+a}\right)\right\}^{-1} = \hat{M}(s)\hat{N}(s)^{-1} \quad \text{where}$$

(28) $$\hat{N}(s) \triangleq \left\{\hat{Q}\left(\frac{1}{s+a}\right)\ [I+\hat{G}(s)]\hat{P}\left(\frac{1}{s+a}\right)\right\} \hat{M}(s)\ .$$

Clearly by (27) we are done if we can show that

$$\hat{N}(s)^{-1} \in \hat{\mathcal{A}}^{nxn}(\sigma)\ .$$

Therefore by a reasoning of [2], we prove that $\hat{N}(s) \in \hat{\mathcal{A}}^{nxn}(\sigma)$ and $\inf\limits_{Res\geq\sigma} |\det \hat{N}(s)| > 0$.

Rewrite (25), therefore

(29) $$\hat{M}(s) = \hat{z}(s)^m\ \hat{\Delta}(s) \quad \text{where}$$

(30) $$\hat{\Delta}(s) \triangleq$$

$$\text{diag}\{\underbrace{1,1,\ldots,1}_{r_0},\underbrace{\hat{z}(s)^{-c_{r_0+1}},\hat{z}(s)^{-c_{r_0+2}},\ldots,\hat{z}(s)^{-c_\alpha}}_{\alpha-r_0},\underbrace{\hat{z}(s)^{-m},\hat{z}(s)^{-m},\ldots,\hat{z}(s)^{-m}}_{n-\alpha}\}.$$

By (28), (13), (29), (30), (26), (14), (15), (17) and (20), we obtain

(31) $$\hat{N}(s) = \hat{N}_1(s) + \hat{N}_2(s) \quad \text{where}$$

(a)

(32) $$\hat{N}_1(s) = \hat{D}_1(s) \oplus \hat{D}_2(s) \quad \text{with}$$

(33) $\hat{D}_1(s) =$

$$\mathrm{diag}\{\underbrace{\hat{a}_1\left(\frac{1}{s+a}\right),\hat{a}_2\left(\frac{1}{s+a}\right),\ldots,\hat{a}_{r_0}\left(\frac{1}{s+a}\right)}_{r_0},\underbrace{\hat{b}_{r_0+1}\left(\frac{1}{s+a}\right),\hat{b}_{r_0+2}\left(\frac{1}{s+a}\right),\ldots,\hat{b}_{\alpha}\left(\frac{1}{s+a}\right)}_{\alpha-r_0}\}$$

(34) $\hat{D}_2(s) =$

$$\mathrm{Diag}\{\underbrace{\hat{b}_{\alpha+1}\left(\frac{1}{s+a}\right),\hat{b}_{\alpha+2}\left(\frac{1}{s+a}\right),\ldots,\hat{b}_{\beta}\left(\frac{1}{s+a}\right)}_{\beta-\alpha},\underbrace{\hat{b}_{\beta+1}\left(\frac{1}{s+a}\right)\hat{z}(s)^{c_{\beta+1}-m},\hat{b}_{\beta+2}\left(\frac{1}{s+a}\right)\hat{z}(s)^{c_{\beta+2}-m},\ldots,\hat{b}_{r}\left(\frac{1}{s+a}\right)\hat{z}(s)^{c_r-m}}_{r-\beta},\underbrace{0,0,\ldots,0}_{n-r}\}$$

and (b)

(35) $\hat{N}_2(s) = \hat{Q}\left(\frac{1}{s+a}\right)\ [I+\hat{G}_0(s)]\hat{P}\left(\frac{1}{s+a}\right)\hat{M}(s)$.

Immediately

(36) $\hat{N}(s) \in \hat{\mathcal{A}}^{nxn}(\sigma)$.

$\hat{N}_1(s) \in \hat{\mathcal{A}}^{nxn}(\sigma)$ because all its elements $\in \hat{\mathcal{A}}(\sigma)$ (indeed all its nonzero elements are polynomials in $\left(\frac{1}{s+a}\right)$ because there are no negative powers of $\hat{z}(s)$ by (21)) and $\hat{N}_2(s) \in \hat{\mathcal{A}}^{nxn}(\sigma)$ by fact 3, (13) and (27). Finally by (23) and since $\hat{P}\left(\frac{1}{s+a}\right)$ and $\hat{Q}\left(\frac{1}{s+a}\right)$ are unimodular

$$\inf_{\mathrm{Re}s\geq\sigma} \left|\det \hat{Q}\left(\frac{1}{s+a}\right)\ [I+\hat{G}(s)]\hat{P}\left(\frac{1}{s+a}\right)\right| > 0.$$

Hence, since by (25)-(26) det $\hat{M}(s)$ has only one zero for Re $s \geq \sigma$ i.e. at p, we obtain with (28)

$$\inf_{s \in U} |\det \hat{N}(s)| > 0 \tag{37}$$

where U is the half plane Re $s \geq \sigma$ with a small neighborhood of p deleted.

Consider now det $\hat{N}(p)$.

Remark that by (35), (22) and (25)-(26)

$$\hat{N}_2(s) = \begin{array}{c} \\ \alpha\{ \\ \\ \end{array} \begin{array}{c} \overbrace{\hphantom{\hat{K}_{11}(s)}}^{\alpha} \hphantom{\hat{L}_{12}(s)} \\ \left[\begin{array}{c|c} \hat{K}_{11}(s) & \hat{L}_{12}(s) \\ \hline \hat{K}_{21}(s) & \hat{L}_{22}(s) \end{array}\right] \end{array} \tag{38}$$

with

$$\hat{K}_{11}(p) = 0 \tag{39}$$

$$\hat{K}_{21}(p) = 0 \; . \tag{40}$$

Thus by (31), (32), (38)-(40)

det $\hat{N}(p)$ = det $\hat{D}_1(p)$ det$[\hat{L}_{22}(p) + \hat{D}_2(p)]$ with

by (33), (16) and (18)

$$\det \hat{D}_1(p) \neq 0 \tag{41}$$

and by (34), (18), (26) and (21)

(42) $\det[\hat{L}_{22}(p) + \hat{D}_2(p)] =$

$$\det\{\hat{L}_{22}(p) + \mathrm{diag}[\hat{b}_{\alpha+1}(1/(p+a)),\ldots,\hat{b}_{\beta}(1/(p+a)),0,\ldots,0]\}$$

which is nonzero by (C). Hence

(43) $\det \hat{N}(p) \neq 0$.

Since $\hat{N}(s)$ is continuous in Re $s \geq \sigma$, (36), (37) and (43) imply that $[\hat{N}(s)]^{-1} \in \hat{\mathcal{A}}^{n\times n}(\sigma)$. Q.E.D. ¤

<u>Necessity</u>. $\hat{H} \in \hat{\mathcal{A}}^{n\times n}(\sigma)$ by assumption.

(23) follows immediately by [6].

To establish (C) we use contradiction. So by (42) suppose that $\det[\hat{L}_{22}(p) + \hat{D}_2(p)] = 0$. We are going to show that, for some input $u \in L_n^{2\sigma}[0,\infty)$ (i.e. $u(t)\, e^{-\sigma t} \in L_n^2[0,\sigma)$), the system defined by (1)-(2) has an error e and thus also an output $y = u - e$ not in $L_n^{2\sigma}[0,\infty)$. This is a contradiction because $u \in L_n^{2\sigma}[0,\infty)$ and $H \in \mathcal{A}^{n\times n}(\sigma)$ imply $y = H*u \in L_n^{2\sigma}[0,\infty)$, [1] [2].

The Laplace transforms of e and u are related by

(44) $[I + \hat{G}(s)]\hat{e}(s) = \hat{u}(s)$.

Multiply (44) on the left by $\hat{Q}\left(\frac{1}{s+a}\right)$ and define the n-vectors $\overline{e}(s)$ and $\overline{u}(s)$ by

(45) $\hat{P}\left(\frac{1}{s+a}\right) \hat{M}(s)\, \overline{e}(s) = \hat{e}(s)$

(46) $\hat{Q}\left(\frac{1}{s+a}\right) \hat{u}(s) = \overline{u}(s)$.

By (44)-(46) and (28) obtain

(47) $$\hat{N}(s)\,\bar{e}(s) = \bar{u}(s)\ .$$

Because $\det[\hat{L}_{22}(p) + \hat{D}_2(p)] = 0$ we can pick a nonzero vector $\eta \in \mathbb{C}^{n-\alpha}$ in the null space of $[\hat{L}_{22}(p) + \hat{D}_2(p)]$, hence

(48) $$[\hat{L}_{22}(p) + \hat{D}_2(p)]\eta = 0\ .$$

Pick now the vector $\xi \in \mathbb{C}^{\alpha}$ such that

(49) $$\xi \triangleq -\ [\hat{D}_1(p)]^{-1}\ \hat{L}_{12}(p)\eta$$

which is well defined because of (41) and the fact that all elements of $\hat{L}_{12}$ are in $\hat{\mathcal{A}}(\sigma)$.

Hence with

(50) $$\bar{e}(s) = \frac{1}{s-p}\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

and

(51) $$\bar{u}(s) = \begin{pmatrix}\bar{u}_1(s)\\ \bar{u}_2(s)\end{pmatrix}\begin{matrix}\}\,\alpha\\ \}\,n-\alpha\end{matrix}\quad ,$$

and (47), (31), (32), (38), we obtain

(52) $$\bar{u}_1(s) = \{[\hat{D}_1(s) + \hat{K}_{11}(s)]\xi + \hat{L}_{12}(s)\eta\}/(s-p)$$

(53) $$\bar{u}_2(s) = \{\hat{K}_{21}(s)\xi + [\hat{D}_2(s) + \hat{L}_{22}(s)]\eta\}/(s-p)\ .$$

All the components of the numerators of (52) and (53) are in $\hat{\mathcal{A}}(\sigma)$; by virtue of (39)-(40) and (48)-(49) they have at least a first order zero at p. Therefore $\bar{u}_1(s)$ and $\bar{u}_2(s)$ are well behaved and bounded at $s = p$. Thus $\bar{u}(s)$ is analytic for $\mathrm{Re}\, s > \sigma$, bounded on $\mathrm{Re}\, s \geq \sigma$ and as $|\omega| \to \infty$:

$$|\bar{u}(\mathrm{Re}\ s + j\omega)| \text{ is at most } 0\left(\frac{1}{\omega}\right) \text{ for any fixed } \mathrm{Re}\ s \geq 0.$$

It follows therefore that the components of $\bar{u}(s)$ are the Laplace transforms of elements of $L^{2\sigma}[0,\infty)$ [14]. From fact 3 and (46) we conclude that the same is true for the components of $\hat{u}(s)$, hence

$$(54) \qquad u \in L_n^{2\sigma}[0,\infty) \ .$$

Finally by (45), (50), (25)-(26) and since $\eta \neq 0$ and $\hat{P}\left(\frac{1}{s+a}\right)$ is unimodular, there exists at least one component of $\hat{e}(s)$ which has a nonzero residue at p.

Thus

$$(55) \qquad e \notin L_n^{2\sigma}[0,\infty)$$

and by (54) and (55) we have established a contradiction. Q.E.D.

$\square$

Remarks.

1) The theorem above describes in detail what happens when K_-, K_0, K_+ are nonempty. When one or more of these sets are empty the required modifications of (C) and of the multiplier $\hat{M}(s)$ are straightforward.

2) In case there are ℓ poles at $p_1, p_2, \ldots, p_\ell$ of order $m_1, m_2, \ldots, m_\ell$ with real part larger than or equal to σ, one uses a product of

multipliers like $\hat{M}(s)$, one for each pole. Condition (C) is used only to check that det $\hat{N}(s)$ does not vanish at $s = p$. Therefore for the more general case an appropriate condition (C) is required at each pole.

3) We have checked that these techniques can be applied in a straightforward manner for the discrete-time case, thus providing a generalization to the work of Desoer, Wu and Lam [8,9,10].

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