A MATRIX DECOMPOSITION-REDUCTION PROCEDURE FOR THE POLE-ZERO CALCULATION OF TRANSFER FUNCTIONS

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Tatsuo Ohtsuki and Lap-Kit Cheung

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Abstract

This paper is concerned with eigenvalue approaches for the pole-zero calculations of the transfer function of a linear time-invariant network. Combinatorial and sparse matrix algorithms are fully used to increase numerical accuracy and computational speed in the two-sets-of-eigenvalues approach. Some matrix decomposition-reduction algorithms are presented to simplify and stabilize the numerical eigenvalue-finding procedures.

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I. Introduction

Sandberg and So have presented a two-sets-of eigenvalues approach to the determination of poles and zeros of a transfer function [1]. This approach begins with the state-space representation

$$s\underset{\sim}{x} = A\underset{\sim}{x} + Bu \tag{1}$$

$$y = C\underset{\sim}{x} + du \tag{2}$$

where s is the differential operator. As is well known, the poles of the transfer function can be computed as the eigenvalues of A. They have presented an algorithm for obtaining a matrix A^* whose eigenvalues are zeros of the transfer function. As an independent work, Pottle also has considered the same problem [2] based on the concept of "inverse system" [3]. The eigenvalues of A and A^* are calculated by means of Francis QR method [4] ch. 8, [5], which is considered to be the best available by numerical analysts. The main advantage of the eigenvalue approach over those based on calculating polynomial coefficients is that the former is much more numerically stable in the sense of sensitivity than the latter [4] pp. 413-423.

It is the purpose of this paper to present some combinatorial and numerical procedures for increasing numerical accuracy and computational speed in the two-sets-of-eigenvalues approach. One of the disadvantages of the existing methods is that the eigenvalue-finding stage often encounters singular matrices due to topological structure of the network. If it is the case, there is no way to estimate accuracy in terms of relative errors. Another disadvantage arises when the given single-input

single-output system is a cascade of several 2-port blocks. Such a situation is very likely in practice, and it is well-known that the numerator polynomial can be decomposed into a product of several factors, each of which corresponds to a block. The existing methods are not suitable for the factorization of the numerator polynomials.

These disadvantages are caused by the formulation of the transfer function in terms of the A, B, C, d matrices. More specifically, the combinatorial structure of these matrices would not keep the information on number of zeros at zero and/or infinity frequencies and on decomposability of the numerator polynomial. To overcome these problems, the presented method formulates the transfer function as a ratio of the determinants of two sparse matrices, called denominator and numerator tableaus[1]. These tableaus correspond to the most primitive set of network equations: node-edge incidence relation, Ohm's law and input-output configuration.

In this paper, several combinatorial and numerical techniques are presented to decompose or reduce the tableaus taking advantage of their sparseness structure. This approach enables us to obtain a nonsingular matrix whose eigenvalues are the poles, and a set of nonsingular matrices whose eigenvalues are the nonzero, finite transfer zeros. With this, the eigenvalue calculations are much simplified.

In Section II, the outline of the presented approach is briefly sketched. In subsequent sections, details of the combinatorial and numerical algorithms are described.

II. Sparse Tableau Formulation

First of all, let us define some notations. A real matrix is denoted by a capital letter. An identity matrix is denoted by I, and a subscript is attached, if necessary, to specify its dimension. The determinant, inverse and transpose of a matrix, say M, is denoted by det M, M^{-1}, and M^T, respectively. In addition to real values, the differential operator, denoted by s, is involved in matrix operations treated in this paper. Any matrix involving s is denoted by a capital letter with a ^ above it. Throughout this paper, we assume that the given network consists of resistors, inductors and capacitors without mutual coupling[2] and that the input-output pair is specified as shown in Fig. 1[3]. The following variables and matrices are involved in the system to be analyzed.

$\underset{\sim}{v}_R$ ($\underset{\sim}{v}_L$, $\underset{\sim}{v}_C$) : voltages across resistors (inductors, capacitors)

$\underset{\sim}{v}$: $\underset{\sim}{v}^T \triangleq [(\underset{\sim}{v}_R)^T, (\underset{\sim}{v}_L)^T, (\underset{\sim}{v}_C)^T]$

$\underset{\sim}{p}$: node voltages with 2' as the reference[4]

$\underset{\sim}{i}_R$($\underset{\sim}{i}_L$, $\underset{\sim}{i}_C$) : currents through resistors (inductors, capacitors)

j : value of the current source

$\underset{\sim}{i}$: $\underset{\sim}{i}^T = [(\underset{\sim}{i}_R)^T, (\underset{\sim}{i}_L)^T, (\underset{\sim}{i}_C)^T]$

$\underset{\sim}{q}(\underset{\sim}{\phi})$: charges (fluxes) of capacitors (inductors)

$\underset{\sim}{z}$: $\underset{\sim}{z}^T \triangleq [\underset{\sim}{v}^T, \underset{\sim}{p}^T, \underset{\sim}{i}^T, \underset{\sim}{q}^T, \underset{\sim}{\phi}^T]$

$G(L,C)$: branch conductance (inductance, capacitance) matrix

$[F_I : F_I^{(j)}]$: incidence matrix relating the edges (including the current source) to the nodes (excluding 1')[4]

F_V : incidence matrix relating the edges (excluding the current source) to the nodes (excluding 2')[4]

The equations which associate the input with the output are the following.

Input Equation: $$u = j \tag{3}$$

Kirchhoff's Current Law: $$F_I \underset{\sim}{i} + F_I^{(j)} \cdot j = \underset{\sim}{0} \tag{4}$$

Kirchhoff's Voltage Law: $$-\underset{\sim}{v} + [F_T]^T \underset{\sim}{p} = \underset{\sim}{0} \tag{5}$$

$$-\underset{\sim}{i}_R + G\underset{\sim}{v}_R = \underset{\sim}{0} \tag{6}$$

Ohm's Law: $$-\underset{\sim}{\phi} + L\underset{\sim}{i}_L = \underset{\sim}{0} \tag{7}$$

$$-\underset{\sim}{q} + C\underset{\sim}{v}_C = \underset{\sim}{0} \tag{8}$$

$$-\underset{\sim}{v}_L + s\underset{\sim}{\phi} = \underset{\sim}{0} \tag{9}$$

Differential Equation:

$$-\underset{\sim}{i}_C + s\underset{\sim}{q} = \underset{\sim}{0} \tag{10}$$

Output Equation: $$y - p_2 = 0 \tag{11}$$

Combining (3) ~ (11), the input-output relation is expressed in the matrix form,

$$\begin{bmatrix} u \\ 0 \\ \sim \end{bmatrix} = \hat{M} \begin{bmatrix} y \\ z \\ \sim \end{bmatrix}. \tag{12}$$

Then the transfer function is given by

$$\frac{y}{u} = \frac{\det \hat{N}}{\det \hat{M}}, \tag{13}$$

where $\hat{N}$ is a submatrix of $\hat{M}$ obtained by deleting the first row and column from $\hat{M}$. In subsequent discussions, $\hat{M}$ and $\hat{N}$ will be referred to as denominator and numerator tableaus, respectively.

The problem of pole-zero calculations is essentially that of finding the set of complex values of s which satisfies

$$\det \hat{U} = 0, \tag{14}$$

where $\hat{U}$ has the following structure[5],

$$\hat{U} = \left[\begin{array}{c|c} P & Q \\ \hline R & sI_{\ell}-T \end{array}\right]. \tag{15}$$

The algorithms to be described in this paper enable us to obtain a real matrix U which satisfies

$$\begin{aligned} &\det \hat{U} := s^{\ell_0} \cdot \det[sI_{\ell-\ell_0-\ell_\infty} - U]; \\ &\det U \neq 0 \end{aligned} \tag{16}$$

where := means that, for all s, the two sides are equal to within a constant multiplier. Once det $\hat{U}$ is transformed into the form of (16), eigenvalues of U together with ℓ_0 superfluous zeros constitute the solutions of (14). This paper is not concerned with how the eigenvalues of a matrix are calculated.[6]

Basically, the size of $\hat{U}$ is reduced by the following three operations.

i) Pivotal condensation

As long as there exists a non-singular submatrix, say A, in the P-part of $\hat{U}$, we can reduce the dimension of $\hat{U}$ by means of the following identity,

$$\det\begin{bmatrix} A & B \\ C & \hat{D} \end{bmatrix} = \det A \cdot \det[\hat{D}-CA^{-1}B]. \qquad (17)$$

ii) Elimination of s

When the submatrix P of $\hat{U}$ is singular, i.e. the degree of the polynomial det $\hat{U}$ is less than ℓ, then we can eliminate some differential operators.

iii) Extraction of s

When $\hat{U}$ is singular at s = 0, then we can extract some differential operators as factors of the polynomial det $\hat{U}$.

ii) and iii) will be discussed in detail in sections III and IV.

From the viewpoint of efficiency in reducing a matrix by the above-mentioned operations, we should take the matrix sparseness structure into consideration. In the special case when all the matrix entries involved in the reduction procedures depend only on the topology of a network, no

numerical calculation is involved because of the total unimodularity of incidence matrices[7]. This type of operation is combinatorial in nature. Zero entries resulted from such an operation are independent of the values of network elements. On the other hand, if some entries depending on the values of network elements are involved in the reduction process, floating-point zeros due to numerical cancellation may be formed in the resulting matrix. We may call this type of operations numerical operations. As far as sparseness structure is concerned, these floating-point zeros are regarded as non-zero entries. In order to preserve sparseness of matrices and to avoid unnecessary round-off errors, combinatorial reduction should be performed prior to numerical reduction.

The procedure for calculating the poles and the zeros of (13) consists of the following five parts.

Part 1

The denominator tableau is reduced as follows.

$$\det \hat{M} := s^{m_0} \det[sI_m - M];$$

$$m = \ell - m_0 - m_\infty, \tag{18}$$

where m_0 and m_∞ are nonnegative integers such that $m_0 + m_\infty \leq \ell$. This part will be discussed in Section V.

Part 2

If the network is a cascade of two or more 2-port blocks, the numerator tableau is factored as follows.

$$\det \hat{N} = \prod_{k} \det \hat{N}_k . \tag{19}$$

This part will be discussed in Section VI.

Part 3

For each block, $\det \hat{N}_k$ is reduced to the form,

$$\det \hat{N}_k := s^{m_0^{(k)}} \det \left[\begin{array}{c|c} s\, I_{m^{(k)}} - A_k & -B_k \\ \hline C_k & d_k \end{array}\right] ;$$

$$m^{(k)} = \ell^{(k)} - m_0^{(k)} - m_\infty^{(k)} , \tag{20}$$

where $\ell^{(k)}$ is the number of inductors and capacitors in the k^{th} block, and $m_0^{(k)}$ and $m_\infty^{(k)}$ are nonnegative integers such that $m_0^{(k)} + m_\infty^{(k)} \leq \ell^{(k)}$.

As an auxiliary calculation for extracting transfer zeros at $s = 0$, $\det \hat{N}_k$ is reduced to the following form using another reduction procedure[8].

$$\det \hat{N}_k := s^{\ell^{(k)} - m_\infty^{(k)}} \det \left[\begin{array}{c|c} s^{-1} I_{m^{(k)}} - A_k' & -B_k' \\ \hline C_k' & d_k' \end{array}\right] , \tag{21}$$

where

$$\begin{cases} A_k' \triangleq A_k^{-1} \\ B_k' \triangleq -A_k^{-1}B_k \\ C_k' \triangleq C_kA_k^{-1} \\ d_k' \triangleq d_k - C_kA_k^{-1}B_k \quad . \end{cases} \tag{22}$$

This part will be discussed in Section VII.

Part 4

Whenever $d_k = 0$ and/or $d_k' = 0$, (20) is further reduced as follows.

$$\begin{aligned} \det \hat{N}_k &:= s^{m_0^{(k)} + n_0^{(k)}} \det[sI_{n^{(k)}} - N_k]; \\ n^{(k)} &= m^{(k)} - n_0^{(k)} - n_\infty^{(k)}, \end{aligned} \tag{23}$$

where $n_0^{(k)}$ and $n_\infty^{(k)}$ are nonnegative integers such that $n_0^{(k)} + n_\infty^{(k)} \leq m^{(k)}$.

This part will be discussed in Section VIII.

Part 5

Combining (18), (19) and (23), we have

$$\frac{y}{u} := s^{n_0} \frac{\prod_k \det[sI_{n^{(k)}} - N_k]}{\det[sI_m - M]}, \tag{24}$$

where

$$n_0 = \sum_k m_0^{(k)} + \sum_k n_0^{(k)} - m_0 \tag{25}$$

and

$$n_\infty = \sum_k m_\infty^{(k)} + \sum_k n_\infty^{(k)} - m_\infty \tag{26}$$

give the number of transfer zeros at $s = 0$ and $s = \infty$, respectively. The number of transfer zeros at finite nonzero values of s is given by

$$\sum_k n^{(k)} = m - n_0 - n_\infty . \tag{27}$$

Now the remaining procedure is to obtain the eigenvalues of M and each N_k, which is not treated in this paper.

III. First Elimination and Extraction of s

This section presents a technique to eliminate and extract some differential operators from a matrix characterized by (15). The following two lemmas are used as formulas to perform these operations.

[Lemma 1]

Let $\hat{U}_1$ and $\hat{U}_3$ be two square matrices characterized by Fig. 2 (a) and (b), respectively. Then

$$\det \hat{U}_1 = (-1)^{\mu} \det \hat{U}_3. \tag{28}$$

Proof Let $\hat{U}_2$ be the result of pivotal condensations on $\hat{U}_1$ taking $\underset{\sim}{r}^4$ - $\underset{\sim}{c}^2$ position as pivots. Then we have from (17) that

$$\det \hat{U}_1 = (-1)^{\mu} \det \hat{U}_2. \tag{29}$$

$\hat{U}_2$ is simply obtained from $\hat{U}_1$ by 1) deleting the rows $\underset{\sim}{r}^4$ and columns $\underset{\sim}{c}^2$, and 2) replacing 0 at $\underset{\sim}{r}^5$ -$\underset{\sim}{c}^3$ position with sR. Now $\hat{U}_3$ can be regarded as the matrix obtained from $\hat{U}_2$ by subtracting the columns $\underset{\sim}{c}^4$, post-multiplied by R, from the columns $\underset{\sim}{c}^3$. Then as is well-known in linear algebra,

$$\det \hat{U}_2 = \det \hat{U}_3. \tag{30}$$

Combining (29) and (30), we have (28). Q.E.D.

This formula shows that μ differential operators are eliminated from $\hat{U}_1$ without involving numerical operations.

[Lemma 2]

Let $\hat{U}_4$ and $\hat{U}_6$ be two square matrices characterized by Fig. 2(c) and (d), respectively, where $Q_{\nu\nu}$ is a nonsingular square matrix of dimension ν such that

$$Q_{\mu\mu} - Q_{\mu\nu}Q_{\nu\nu}^{-1}Q_{\nu\mu} = 0. \tag{31}$$

Then,

$$\det \hat{U}_4 = s^{\mu} \det \hat{U}_6. \tag{32}$$

Proof Let $\hat{U}_5$ be the matrix obtained from $\hat{U}_4$ by replacing $Q_{\mu\mu}$, $Q_{\mu\nu}$ and 0 in the rows $\underset{\sim}{r}^4$ with 0, 0 and $-sQ_{\mu\nu}Q_{\nu\nu}^{-1}$, respectively. Then $\hat{U}_5$ is regarded as the matrix obtained from $\hat{U}_4$ by subtracting the rows $\underset{\sim}{r}^5$, premultiplied by $Q_{\mu\nu}Q_{\nu\nu}^{-1}$, from the rows $\underset{\sim}{r}^4$. Thus we have

$$\det \hat{U}_4 = \det \hat{U}_5. \tag{33}$$

Now the pivotal condensations on $\hat{U}_5$ taking $\underset{\sim}{r}^4$-$\underset{\sim}{c}^3$ position as pivots result in $\hat{U}_6$. From (17), we have

$$\det \hat{U}_5 = s^{\mu} \hat{U}_6. \tag{34}$$

Combining (33) and (34), we have (32). Q.E.D.

This formula enables us to extract μ differential operators from $\hat{U}_4$. And if the submatrix determined by rows $\underset{\sim}{r}^4$, $\underset{\sim}{r}^5$ and columns $\underset{\sim}{c}^1$, $\underset{\sim}{c}^2$ is totally unimodular, no numerical calculation is involved.

IV. Further Elimination and Extraction of s

This section is concerned with techniques to reduce the polynomial

$$\psi(s) = \det\left[\begin{array}{c|c} sI_m - A & -B \\ \hline C & d \end{array}\right] \tag{35}$$

to the form

$$\psi(s) := s^{n_0}\det[sI_n - N];$$

$$n = m - n_0 - n_\infty, \tag{36}$$

where n_∞ and n_0, respectively, are the number of eliminated and extracted differential operators. We assume that A is non-singular[9]. And, if all the superfluous roots of $\psi(s) = 0$ have been extracted, N should also be non-singular.

First, we recall the formulas, which are available in [1] and [8], for determining n_∞ and n_0. For convenience, we give them in an alternative form using the following sequence of real numbers.

$$\begin{cases} \zeta(0) = 0 \\ \zeta(1) = d \\ \zeta(i) = CA^{i-2}B \qquad ; i \geq 2 \\ \zeta(-1) = CA^{-1}B - d \\ \zeta(-i) = CA^{-i}B \qquad ; i \geq 2 \end{cases} \tag{37}$$

[Lemma 3]

n_∞ is equal to the integer such that $\zeta(n_\infty+1) \neq 0$ and $\zeta(i) = 0$ for all $i = 0, 1, \cdots, n_\infty$.

[Lemma 4]

n_0 is equal to the integer such that

$$\zeta(-n_0-1) \neq 0 \text{ and } \zeta(-i) = 0 \text{ for all } i = 0, 1, \cdots, n_0.$$

In order to extract all the zero roots of $\psi(s) = 0$, we use the following formula presented by Ishizaki [8].

[Lemma 5]

Let $\theta(s)$ be the polynomial defined by

$$\theta(s) = \det \left[\begin{array}{c|c} SI_m - A^{(0)} & -B^{(0)} \\ \hline C^{(0)} & d^{(0)} \end{array}\right], \tag{38}$$

then

$$\psi(s) = s^{n_0} \cdot \theta(s), \tag{39}$$

where $A^{(0)} = A$, $B^{(0)} = B$ and

$$C^{(0)} = \begin{cases} C & ; n_0 = 0 \\ CA^{-n_0} & ; n_0 \geq 1, \end{cases} \tag{40}$$

$$d^{(0)} = \begin{cases} d & ; n_0 = 0 \\ 0 & ; n_0 \geq 1. \end{cases} \tag{41}$$

By means of this lemma, the problem of putting $\psi(s)$ into the form of (36) is reduced to that of putting $\theta(s)$ into the form of

$$\theta(s) := \det[sI_n - N] \tag{42}$$

It is clear from (39) that $\theta(s)$ is of degree $m-n_0-n_\infty$. Now we shall give a numerical procedure for eliminating $(n_\infty+n_0)$ differential operators from (38) when $(n_\infty+n_0) \geq 1$. As long as the system to be analyzed is non-trivial, $B^{(0)}$ has at least one nonzero entry. Without loss of generality, we assume that the 1st entry $b_1^{(0)}$ of $B^{(0)}$ is nonzero. Let $C_1^{(0)}$ be the 1st entry of $C^{(0)}$ and $a_{11}^{(0)}$ be the 1-1 entry of $A^{(0)}$. For any matrix, say A, we use ${}_1A$ and A_1 to denote the 1st row and column, respectively, of A. If the subscript $\bar{1}$ is attached to the left or the right of a matrix, it indicates the deletion of the 1st row or column, respectively. Now we shall state the following theorem.

[Theorem 1]

$$\theta(s) = \det\left[\begin{array}{c|c} sI_m - A^{(0)} & -B^{(0)} \\ \hline C^{(0)} & 0 \end{array}\right] = b_1^{(0)} \det\left[\begin{array}{c|c} sI_{m-1} - A^{(1)} & -B^{(1)} \\ \hline C^{(1)} & d^{(1)} \end{array}\right] \tag{43}$$

where

$$A^{(1)} = {}_{\bar{1}}A^{(0)}_{\bar{1}} - \frac{1}{b_1^{(0)}} {}_{\bar{1}}B^{(0)} \; A^{(0)}_{1\bar{1}} \tag{44}$$

$$B^{(1)} = E^{(1)} + \frac{1}{b_1^{(0)}} A^{(1)} \; {}_{\bar{1}}B^{(0)} \; ; \tag{45}$$

$$\begin{cases} E^{(1)} = \underline{A}_{1\,1}^{(0)} - \dfrac{a_{11}^{(0)}}{b_1^{(0)}}\,\underline{B}_{1}^{(0)} & (46) \\ C^{(1)} = \underline{C}_{1}^{(0)} & (47) \\ d^{(1)} = c_1^{(0)} + \dfrac{1}{b_1^{(0)}}\,\underline{C}_{1}^{(0)}\,\underline{B}_{1}^{(0)} & (48) \end{cases}$$

Proof By permuting rows and columns of the original matrix, we have

$$\theta(s) = -\det \left[\begin{array}{c|c|c} sI_{m-1} - \underline{A}_{1\,1}^{(0)} & -\underline{A}_{1\,1}^{(0)} & -\underline{B}_{1}^{(0)} \\ \hline \underline{C}_{1}^{(0)} & C_1^{(0)} & 0 \\ \hline -\underline{A}_{1\,1}^{(0)} & s-a_{11}^{(0)} & -b_1^{(0)} \end{array}\right].$$

Performing the pivotal condensation taking $-b_1^{(0)}$ as the pivot, we have

$$\theta(s) = b_1^{(0)} \det \left[\begin{array}{c|c} sI_{m-1} - \underline{A}_{1\,1}^{(1)} & -E^{(1)} - \dfrac{1}{b_1^{(0)}}\, s\cdot \underline{B}_{1}^{(0)} \\ \hline \underline{C}_{1}^{(0)} & C_1^{(0)} \end{array}\right]$$

By adding the left-hand side columns, post-multiplied by $\dfrac{1}{b_1^{(0)}}\,\underline{B}_{1}^{(0)}$,

to the rightmost column, we have (43). Q.E.D.

Remark

An alternative formula to this theorem has been presented by Sand-

berg-So [1]. As will be compared in Appendix A, their formula involves additional calculations.

It is clear that $d^{(1)}$ vanishes again if $n_\infty + n_0 \geq 2$. In general, Theorem 1 can be repeated $(n_\infty+n_0)$ times. While differential operators are eliminated one by one, $A^{(k)}$, $B^{(k)}$, $E^{(k)}$, $C^{(k)}$ and $d^{(k)}$; $k = 1, 2, \cdots, n_\infty + n_0$ are successively defined as in Theorem 1. Finally, it terminates when (38) is reduced into

$$\theta(s) := \det \left[\begin{array}{c|c} sI_{n_\infty+n_0} - A^{(n_\infty+n_0)} & -B^{(n_\infty+n_0)} \\ \hline C^{(n_\infty+n_0)} & d^{(n_\infty+n_0)} \end{array} \right] ; \; d^{(n_\infty+n_0)} \neq 0 \qquad (49)$$

As an immediate consequence of Theorem 1, we have the following formula.

[Corollary]

$$d^{(n_\infty+n_0)} = \frac{1}{b} C^{(0)} [A^{(0)}]^{n_\infty+n_0-1} B^{(0)}, \qquad (50)$$

where b is the product of the nonzero entries of $B^{(0)}$, $B^{(1)}$, $\cdots$, $B^{(n_\infty+n_0-1)}$, which are chosen for processing the calculation of (43). $C^{(n_\infty+n_0)}$ in (49) is simply obtained from $C^{(0)}$ by deleting its $(n_\infty+n_0)$ entries corresponding to the chosen nonzero entries of $B^{(k)}$; $k = 0,1, \cdots, n_\infty+n_0-1$.

Now the desired nonsingular matrix N is obtained by

$$N = A^{(n_\infty+n_0)} - \frac{1}{d^{(n_\infty+n_0)}} B^{(n_\infty+n_0)} C^{(n_\infty+n_0)}. \qquad (51)$$

Unlike the procedure described in Section III, numerical operations must be involved in the extraction or elimination of differential operators using Theorem 1. However, we can determine the integers n_∞ and n_0 in a combinatorial way as will be shown in Appendix C. Thus, we are able to terminate the iterative reduction procedure correctly.

V. Reduction of Denominator Tableau

The matrices, whose eigenvalues give the poles of a transfer function, are not unique. As is well-known in state-space technique, the A-matrix is one of such matrices. Particularly, if we choose a set of state-variables in the sense of degrees of freedom [9], we can obtain a non-singular A-matrix. Then m_∞ and m_0 in (18) are given by

m_∞ = number of independent C-only loops and L-only cutsets. (52)

m_0 = number of independent C-only cutsets and L-only loops. (53)

In this section, we shall give an efficient reduction procedure to obtain M from $\hat{M}$, which corresponds to the state-space technique presented in [10].

Step 1: There are two singleton rows in $\hat{M}$ determined by j and y; the rows corresponding to (3) and (11), respectively. Obviously, the deletion of these two rows and columns does not change the determinant except for the sign.

Step 2: We enlarge the matrix by introducing the additional variables ϕ^* and q^* such that

$$\begin{cases} \underset{\sim}{\phi} - \underset{\sim}{\phi}^* = \underset{\sim}{0} & (54) \\ \underset{\sim}{q} - \underset{\sim}{q}^* = \underset{\sim}{0} & (55) \end{cases}$$

and by replacing (9) and (10) by

$$\begin{cases} -\underset{\sim}{v}_L + s\underset{\sim}{\phi}^* = \underset{\sim}{0} & (56) \\ -\underset{\sim}{i}_c - s\underset{\sim}{q}^* = \underset{\sim}{0} \, . & (57) \end{cases}$$

This operation does not change the determinant, either.

Step 3: We eliminate all the columns corresponding to p by means of pivotal condensations. Then (5) is reduced to the form of

$$-\underset{\sim}{v}_\ell + F^T \underset{\sim}{v}_t = \underset{\sim}{0}, \tag{58}$$

where $\underset{\sim}{v}_t$ and $\underset{\sim}{v}_\ell$ denote the voltages of tree-branches and links, respectively, of a tree. The pivots are chosen from $(F_V)^T$ in such a way that as many rows corresponding to capacitors and as few rows corresponding to inductors as possible are deleted. This is a matrix interpretation of the graph-theoretic procedure of finding a C-normal tree [10].

Using the principal part F^T of the fundamental loop matrix, we can replace (4) by

$$\underset{\sim}{i}_t + F\underset{\sim}{i}_\ell = \underset{\sim}{0} \tag{59}$$

without changing the determinant except for the sign.

Step 4: We eliminate all the capacitor currents in $\underset{\sim}{i}_t$ and all the inductor voltages in $\underset{\sim}{v}_\ell$ by means of pivotal condensations, where the pivots are taken within the identity submatrix determined by either (59) and $\underset{\sim}{i}_t$ or (58) and $\underset{\sim}{v}_\ell$.

Step 5: The structure of the matrix at this stage is characterized by Fig. 2(a), where columns $\underset{\sim}{c}^2$ correspond to the independent set of C-only loops and L-only cutsets. Now Lemma 1 enables us to reduce the matrix into the form shown by Fig. 2(b), while m_∞ differential operators are eliminated. Note that each column of $\underset{\sim}{c}^4$ in $\hat{U}_3$ in Fig. 2(b) corresponds to a linear combination of capacitor charges or inductor fluxes. Next, the variables in $\underset{\sim}{\phi}^*$ and $\underset{\sim}{q}^*$ corresponding to C-only loops or L-only cutsets are eliminated by taking pivots at $\underset{\sim}{r}^2 - \underset{\sim}{c}^3$ position of $\hat{U}_3$ in Fig. 2(b).

Step 6: Now we can arrange the matrix into the form of Fig. 2(c), where $Q_{\nu\nu}$ can be found by means of an L-normal tree [10]. And using Lemma 2, we can reduce the matrix into the form of Fig. 2(d), while m_0 differential operators are extracted. Note that the rows $\underset{\sim}{r}^4$ correspond to the linear dependencies among the currents of C-only cutsets or among the voltages of L-only loops. It should also be noted that the columns $\underset{\sim}{c}^4$ in Fig. 2(d) correspond to the set of state-variables defined in eqs. (36) and (37) of [10]. At this stage $\hat{M}$ has been reduced in the form,

$$\det \hat{M} := s^{m_0} \left[\begin{array}{c|c} P & Q \\ \hline R & sI_m \end{array}\right] \tag{60}$$

Step 7: The desired matrix M is obtained by

$$M = RP^{-1}Q. \tag{61}$$

Unlike the procedures from Step 1 to 6, some numerical calculations should be involved in (61). The efficiency and accuracy in the calculation of

(61) are improved by taking advantage of the following decomposition.

i) If the rows and columns are arranged in such a way that those corresponding to resistors are placed first and those corresponding to inductors are placed next, then P has a block upper triangular form as follows:

$$P = \begin{bmatrix} P_R & X \\ 0 & \begin{matrix} P_L & 0 \\ 0 & P_C \end{matrix} \end{bmatrix}, \tag{62}$$

Note that (62) corresponds to the representation of a state equation in terms of three one-element-kind networks as in Fig. 1 of [10].

ii) In many cases, P_R, P_L or P_C can be further decomposed into a block diagonal form, where each diagonal block corresponds to an unhinged subnetwork of the one-element-kind networks.

iii) If an unhinged subnetwork consists only of one element, the corresponding diagonal block can be arranged into an upper triangular form.

Remarks

The three-stage decomposition described above gives the Dulmage-Mendelsohn canonical representation of the P-part, and it is shown that this decomposition is essentially unique [11]. This decomposition indicates to us, when we calculate (61), to take the pivots in the diagonal blocks of P. This pivoting strategy is very important for both efficiency and accuracy [12].

Example 1

A 2-port with an inductor and four capacitors are shown in Fig. 3(a). By taking a pair of C-normal tree T_c and L-normal tree T_L, we reduce the denominator tableau up to Step 6. The matrix of Fig. 3(b) with deletion of the last row and column corresponds to the matrix in the right-hand side of (60). As we can see, one of the differential operator is eliminated and another is extracted, i.e. $m_0 = m_\infty = 1$. The three-stage decomposition of P is also illustrated.

VI. Factorization of Numerator Tableau

Before reducing $\hat{N}$ to N, it is advantageous to decompose $\hat{N}$ into a block-triangular form. With this, we can work on matrices of smaller dimension. This section is concerned with how to block-triangularize $\hat{N}$. Let us consider the following theorem.

[Theorem 2]

If a 2-port is a cascade of two blocks as shown in Fig. 4, then $\hat{N}$ can be arranged in the upper block-triangular form as follows,

$$\hat{N} = \left[\begin{array}{c|c} \hat{N}_1 & \hat{N}_{12} \\ \hline 0 & \hat{N}_2 \end{array}\right] \tag{63}$$

where $\hat{N}_1$ and $\hat{N}_2$ are two nontrivial square matrices.

Proof We arrange the matrix so that the following rows and columns determine $\hat{N}_2$. The rows, i.e. equations, of $\hat{N}_2$ consist of i) β voltage equations for all branches in Block (2), ii) $(\nu+\nu')$ current equations at all nodes, in Block (2) except 3 and 3', iii) β Ohm's laws for all branches in Block (2), iv) λ differential equations for all reactive elements in Block (2) and v) the output equation. The columns, i.e. variables, of $\hat{N}_2$ consist of i) 2β branch voltages and currents in Block (2), ii) $(\nu+\nu'+1)$ node voltages of all the nodes in Block (2). and iii) λ charges and fluxes of all the reactive elements in Block (2). With this arrangement, $\hat{N}_2$ is a square matrix of dimension $(2\beta+\lambda+\nu+\nu'+1)$. This arrangement also implies that the submatrix at the southwest corner consists only of zero entries. Q.E.D.

Remarks

Theorem 2 shows that the determinant calculation of $\hat{N}$ is reduced to those of $\hat{N}_1$ and $\hat{N}_2$, which are of relatively small size. As we can see from the proof, the specific choice of the reference nodes given in Fig. 4 is essential to the block-triangular decomposition of $\hat{N}$.

For the efficiency of subsequent numerical analyses, it is desirable to decompose the network into a cascade of minimal 2-ports[10]. In Appendix B, the outline of our algorithm for this purpose is described.

If a 2-port block consists only of one element, the corresponding diagonal block of $\hat{N}$ is further arranged into an upper triangular form. Then the result gives the Dulmage-Mendelsohn canonical representation [11] of $\hat{N}$.

Example 2

Fig. 5 shows a network and the corresponding upper block-triangularized $\hat{N}$. It can be seen that each diagonal block of $\hat{N}$ corresponds to a 2-port in the cascaded network.

VII. First Reduction of Numerator Tableau

In the following sections, we drop the superscript k and the superscript (k) from the notations as in (19), since each diagonal block of the numerator tableau can be discussed independently. We simply denote a diagonal block of the numerator tableau to be analyzed by $\hat{N}$. For convenience, we use the enlarged matrix $\hat{N}_1$ as defined below.

$$\hat{N}_1 = \left[\begin{array}{c|c|c} 0 & Y & -1 \\ \hline Z & \hat{N} & 0 \\ \hline 1 & 0 & 0 \end{array}\right]. \tag{64}$$

Then, as we can see, $\hat{N}_1$ has the same determinant as $\hat{N}$ for any Y and Z. If, in particular, Y and Z are chosen so that they represent the incidence relations at the input node 1 and at the output node 2, respectively, then $\hat{N}_1$ has exactly the same structure as the numerator tableau of the network which consists only of this block with the same input-output configuration as in Fig. 1. Furthermore, noting that the first column of $\hat{N}_1$ corresponds to the node voltage of 2, we can see that the deletions of the last row and column results, with pertinent permutations, in the reduced denominator tableau at the end of Step 1 of Section V.

Now we can apply the procedure of Steps 2-6 in Section V to reduce $\hat{N}$ to the form of

$$\det \hat{N} := s^{m_0} \det \left[\begin{array}{cc|c} P & Q & \\ & & U \\ \hline R & sI_{m'} & \\ \hline V & & w \end{array}\right] \tag{65}$$

in the same way as for the reduction of $\hat{M}$ into (60), except that the additional calculations for the last row and column of $\hat{N}_1$ are involved. The matrix above can be viewed as the state-space representation of a single-input single-output system, where its last column and row correspond to the input and the output, respectively. By performing pivotal condensations with respect to P based on the decomposition in Step 7, the desired matrices A, B, C, d in (20) are obtained.

In the subsequent discussions, (65) with the three-stage decomposition of P will be referred as <u>normal form canonical representation</u> of a state equation.

<u>Example 3</u>

We consider the same 2-port as for Example 1. Although this is not a minimal 2-port, we regard it as a block. Processing Steps 2-6, we obtain the normal form canonical representation as shown in Fig. 3(b).

If we take the pivots in the diagonal blocks, the sparceness structure of A, B, C, d will be

$$\left[\begin{array}{c|c} A & B \\ \hline C & d \end{array}\right] = \left[\begin{array}{ccc|c} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & 0 \\ \hline 0 & x & x & 0 \end{array}\right]. \qquad (66)$$

Once each diagonal block of the numerator tableau has been reduced to the form of (35), we can perform the further reduction based on the procedure described in Section IV. Before using the algorithm of Theorem 1, we should detect n_0 by means of the sequence defined by (37),

which means we should obtain A^{-1}. If A^{-1} is calculated by performing Gaussian eliminations on A, then each member of the sequence $\zeta(-1)$, $\zeta(-2)$, $\cdots$ will be some small number when it should be zero. Here we have to decide whether the small, but nonzero floating point number should be treated as zero or not. For example the calculated result of $d-CA^{-1}B$ would, when $n_0 \neq 0$, be a very small number rather than pure zero due to the floating-point arithmetic. To avoid this, we shall take advantage of the A-inverse form suggested by Ishizaki [8].

The A-inverse form of the state equation

$$\begin{bmatrix} s\underset{\sim}{x} \\ \hline u \end{bmatrix} = \begin{bmatrix} A & B \\ \hline C & d \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{x} \\ \hline y \end{bmatrix} \tag{67}$$

is defined by

$$\begin{bmatrix} \underset{\sim}{x} \\ \hline u \end{bmatrix} = \begin{bmatrix} A' & B' \\ \hline C' & d' \end{bmatrix} \begin{bmatrix} s\underset{\sim}{x} \\ \hline y \end{bmatrix}, \tag{68}$$

where A', B', C' and d' are given by (22) with removal of the subscript k.

In the remainder of this section, we describe the procedure to obtain A', B', C', d' matrices from the original numerator tableau, rather than from the A, B, C, d matrices.

As in the normal form representation, we start from the matrix $\hat{N}_1$ defined by (64). Then we apply the procedure of Steps 2-7 in Section V with the following modifications.

Step 2': (54) and (35) are replaced by

$$\begin{cases} \underset{\sim}{v}_L - \underset{\sim}{v}_L^* = \underset{\sim}{0} & (54)' \\ \underset{\sim}{i}_c - \underset{\sim}{i}_c^* = \underset{\sim}{0} & (55)' \end{cases}$$

(56) and (57) are replaced by

$$\begin{cases} -\underset{\sim}{\phi} + s^{-1} \underset{\sim}{v}_L^* = \underset{\sim}{0} & (56)' \\ -\underset{\sim}{q} + s^{-1} \underset{\sim}{i}_c^* = \underset{\sim}{0} & (57)' \end{cases}$$

The determinant of the enlarged matrix is equal to $s^{-\ell}$ det $\hat{N}$.

Step 3': In the pivotal condensations for obtaining F, the priority order of choosing capacitors and inductors is reversed. That is, the corresponding tree is an L-normal tree [10] in place of a C-normal tree.

Step 4': $\underset{\sim}{v}_L$ and $\underset{\sim}{i}_c$ are eliminated by taking pivots at the identity submatrices determined by (54)' and (55)'.

Step 5': The structure of the matrix at this stage is characterized by the transpose of $\hat{U}_1$ in Fig. 2(a) except that s is replaced by s^{-1}, where the rows $\underset{\sim}{c}^2$ correspond to the linear dependencies among the currents of C-only cutsets or among the voltages of L-only loops. By Lemma 1, m_0 inverse differential operators are eliminated.

Step 6': The structure of the matrix is characterized by the transpose of $\hat{U}_4$ in Fig. 2(c) except that s is replaced by s^{-1}, where $Q_{\nu\nu}$ can be found by means of a C-normal tree. By Lemma 2, m_∞ inverse differential

operators are extracted. Note that columns $\underset{\sim}{r}_4$ correspond to the independent set of C-only loops and L-only cutsets. The columns $\underset{\sim}{r}^5$ correspond to the time derivative of state-variables defined in eqs. (36) and (37) of [10]. At this stage, $\hat{N}$ has been reduced in the form

$$\det \hat{N} := s^{\ell - m_\infty} \begin{bmatrix} P' & Q' & U' \\ R' & s^{-1} I_m & \\ V' & & w' \end{bmatrix} \qquad (65)'$$

Step 7': Suppose the same pair of T_c and T_L is used for deriving both (65) and (65)', then the desired matrices A', B', C' and d' are obtained by taking P' as pivots.

i)' If the rows and columns are arranged in such a way that those corresponding to inductors are placed first and those corresponding capacitors are placed next, P' has another block upper triangular form as follows:

$$P' = \begin{bmatrix} P'_L & 0 & X' \\ 0 & P'_c & \\ 0 & & P'_R \end{bmatrix} . \qquad (62)'$$

Note that (62)' corresponds to the A-inverse form of state equation in Fig. 1(b) of [8].

The block-diagonal decomposition of P'_R, P'_L and P'_c as in ii) may be possible depending on the topological structure of the network. This decomposition of P' also gives the Dulmage-Mendelsohn canonical representation.

In the subsequent discussions, (65)' with the three-stage decomposition of P' will be referred as A-inverse form canonical representation of a state equation.

Example 4

We consider the same network and choose the same pair of T_C and T_L as in Example 3. The A-inverse form of state equation is shown in Fig. 3(c). If we take the pivots in the diagonal blocks, the sparseness structure of A', B', C', d' will be

$$\left[\begin{array}{c|c} A' & B' \\ \hline C' & d' \end{array}\right] = \left[\begin{array}{ccc|c} 0 & x & x & 0 \\ x & x & x & x \\ x & x & x & x \\ \hline 0 & 0 & x & 0 \end{array}\right] . \qquad (66)'$$

Note that if (67)' was calculated by (22) using A, B, C, d, it would be a full matrix due to floating point arithmetic.

VIII. Further Reduction of Numerator Tableau

In this section, we assume that each diagonal block of the numerator tableau has been reduced to the forms of (20) and (21). In order to further reduce (20) to the form of (23), the procedure described in Section IV is used.

When we are to determine n_0, the A-inverse form reduction (21) is used. In terms of A', B', C' and d', the sequence $\zeta(-i)$; $i \geq 1$ defined by (37) is represented by

$$\begin{cases} \zeta(-1) = - d' \\ \zeta(-i) = - C'(A')^{i-2}B'; \ i \geq 2. \end{cases} \tag{67}$$

The row matrix $C^{(0)}$ in (40) can also be represented in terms of the A-inverse form as follows.

$$C^{(0)} = - C'(A')^{n_0-1}; \ n_0 \geq 1. \tag{68}$$

In order to obtain $\zeta(-i)$ for $i = 3 \sim (n_0+1)$ efficiently when $n_0 \geq 2$, A' is successively post-multiplied to C' and then the row matrix $C'(A')^{i-2}$ and the column matrix B' are multiplied together for each i. This process terminates at $i = (n_0+1)$, when

$$\zeta(-n_0-1) = C'(A')^{n_0-1}B' \neq 0. \tag{69}$$

The intermediate result of the calculation of (69), i.e. before B' is post-multiplied, gives the matrix $C^{(0)}$.

The sequence $\zeta(i)$; $i \geq 1$ are calculated using A, B, C, d matrices. Thus, (20) is used for determining n_∞.

Once n_0 and $C^{(0)}$ have been obtained, the numerator tableau can be represented by

$$\det \hat{N} := s^{m_0+n_0}\, \theta(s), \tag{70}$$

where $\theta(s)$ is defined by (38). Now $(n_\infty+n_0)$ differential operators are eliminated from (38) which is further reduced to the form of (49), using Theorem 1 $(n_\infty+n_0)$ times. In each repetition of the algorithm of Theorem 1, it is not necessary to calculate $d^{(k)}$; $k = 1, 2, \cdots, (n_\infty+n_0-1)$ because they are known to be zero. From the corollary of Theorem 1, $d^{(n_\infty+n_0)}$ is given by (50). As is clear from the definition of $\theta(s)$, the matrices for A-inverse form are not required when $n_0 = 0$. If $n_0 \geq 1$, they are required only to evaluate $C^{(0)} = CA^{-n_0}$. In this case, (50) can be replaced by

$$d^{(n_\infty+n_0)} = \begin{cases} \frac{1}{b}\, d & ; \quad n_\infty = 0 \\ \\ \frac{1}{b}\, CA^{n_\infty-1} B; & n_\infty \geq 1 \end{cases} \tag{71}$$

using (40). Hence $d^{(n_\infty+n_0)}$ can be calculated using the matrices of the normal form only. Finally, the desired matrix N is obtained by (51). At the same time, n_∞ differential operators are eliminated and n_0 differential operators are extracted. It should be noted here that n_∞ and n_0 are the number of transfer zeros at $s = \infty$ and $s = 0$, respectively,

caused by the 2-port block under consideration.

The transfer zeros at $s = \infty$ or $s = 0$ are caused by topological structure of the network unless some special dependencies exist among the element values. This type of transfer zeros involves no numerical calculation in determining n_∞ and n_0 using (37) and (67), respectively. The topological structures which cause zeros at $s = \infty$ and $s = 0$ are illustrated in Fig. 6(a) and (b), respectively. In Appendix C, the relation between this kind of topological structure and the sparseness structure of A, B, C, d matrices and A', B', C', d' matrices are discussed. And it is shown that n_∞ and n_0 can be determined in a combinatorial way.

IX. Conclusions

An efficient procedure for calculating transfer functions based on the two-set-of-eigenvalues approach has been presented. Poles are identified as eigenvalues of a nonsingular matrix, while zeros at finite, nonzero frequencies are identified as eigenvalues of a set of nonsingular matrices.

The presented approach is essentially based on state-space technique. In addition, sparse matrix techniques are fully used so that factorization of the numerator polynomial and transfer zeros at 0 or ∞ frequencies can be determined in a combinatorial way. This increases numerical accuracy and reduces problem complexity. The matrix decomposition and reduction algorithms make the existing state-space methods good for practical uses, as far as pole-zero calculation is concerned.

The results obtained in this paper can readily be extended to networks having mutual couplings or controlled sources.

Appendix A: Comparison of Theorem 1 with Sanberg-So Algorithm

The following theorem, which is equivalent to Theorem 1 of [1], is an alternative of Theorem 1 of this paper.

[Theorem 1']

$$\theta(s) = b_1^{(0)} \det \left[\begin{array}{c|c} sI_{m-1} - A^{(1)*} & -B^{(1)*} \\ \hline C^{(1)} & d^{(1)} \end{array}\right], \tag{A1}$$

where

$$A^{(1)*} = A^{(1)} - \frac{1}{b_1^{(0)}} \underline{B}^{(0)} C^{(1)} \tag{A2}$$

$$B^{(1)*} = B^{(1)} - \frac{d^{(1)}}{b_1^{(0)}} \underline{B}^{(0)} \quad . \tag{A3}$$

Proof is straightforward from the identity

$$\left[\begin{array}{c|c} I_{m-1} & \underline{B}^{(0)}/b_1^{(0)} \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} sI_{m-1} - A^{(1)} & -B^{(1)} \\ \hline C^{(1)} & d^{(1)} \end{array}\right] = \left[\begin{array}{c|c} sI_{m-1} - A^{(1)*} & -B^{(1)*} \\ \hline C^{(1)} & d^{(1)} \end{array}\right]. \tag{A4}$$

Combining (A2) and (A3) with (44) and (45), we have

$$A^{(1)*} = \underline{A}^{(0)}_{\underline{1}\,\underline{1}} - \frac{1}{b_1^{(0)}} \underline{B}^{(0)} \left(\underline{A}^{(0)}_{1\,\underline{1}} + C^{(1)} \right) \tag{A5}$$

$$B^{(1)*} = E^{(1)} + \frac{1}{b_1^{(0)}} \left(A^{(1)} - d^{(1)} I_{m-1}\right) \underline{B}^{(0)}. \tag{A6}$$

Thus the calculations of (A5) and (A6) involve 2(m-1) more additions or subtractions than those of (44) and (45). On the other hand, Theorems 1 and 1' theoretically give the same result as far as the matrix N in (42) is concerned. For example, when $d^{(1)} \neq 0$,

$$N = A^{(1)*} - \frac{1}{d^{(1)}} B^{(1)*} C^{(1)} = A^{(1)} - \frac{1}{d^{(1)}} B^{(1)} C^{(1)}. \tag{A7}$$

(A7) shows that the second terms of (A2) and (A3) are cancelled out finally.

The second terms of (A2) and (A3) do not make much difference in computation speed. As we can see from (A5) and (A6), however, Theorem 1 is more desirable than Theorem 1' in the sense of keeping sparseness of the matrix and avoiding unnecessary round-off errors.

Appendix B: Decomposition of a 2-Port into a Cascade of Minimal 2-Ports

Dulmage and Mendelsohn presented an efficient method for block-triangularization of a matrix [13]. However, their algorithm is too general to apply to the numerator tableau $\hat{N}$, since $\hat{N}$ is related to Ohm's laws which have nothing to do with decomposability of the 2-port network. The only information required here is the incidence relation of the network and specification of input-output pair.

Now we recall that Dulmage-Mendelsohn algorithm consists of two steps, namely, (1) finding a complete matching and (2) determining a partial ordering, which is unique, of the diagonal blocks of $\hat{N}$.

The complete matching involved in the first step corresponds to a pair of node-disjoint paths going from input to output. Such a pair always exists unless the given 2-port is hinged. In the simple case where the 2-port is common grounded, the grounded node is regarded as one of the paths. To find a path, which does not pass through the grounded node, is a very simple task. In the general case, a pair of node-disjoint paths can be found using, for example, the algorithm presented by Frisch [14].

Such a pair of node-disjoint paths has an important property that these two paths go through any node pairs which separate output from input. By means of the incidence matrix, we can easily find the set of such node pairs, so that the given 2-port is decomposed into a cascade of several 2-ports. The ordering of these 2-ports is determined by that of the node-pairs with respect to the pair of node-disjoint paths. When a 2-port block determined by the above-mentioned node

pairs, has only two terminals we decompose it into a parallel connection of two-terminal networks. With this, the set of minimal 2-port blocks is determined. Now each minimal 2-port block corresponds to a diagonal block of the Dulmage-Mendelsohn canonical representation of $\hat{N}$.

Appendix C; Combinatorial Determination of n_∞ and n_0

First of all, we shall recall the normal form canonical representation given in (65) and the two-stage decomposition of P. If there exists a subnetwork characterized by Fig. 6(a), it is clear that the sparseness structure has the property

$$w = VU = 0. \tag{C1}$$

Thus we can combinatorially check whether or not $d = 0$, i.e. $n_\infty \geq 1$. In order to consider the case of $n_\infty \geq 2$, we shall partition the set of diagonal blocks of P into subsets $\{\Lambda_i;\ i = 1, 2, \cdots, \nu\}$ and $\{\Gamma_i;\ i = 0, 1, 2, \cdots, \nu\}$ and the set of state variables $\underset{\sim}{x}$ into subsets $\{\underset{\sim}{x}^{(i)};\ i = 1, 2, \cdots, \nu\}$. These subsets are successively defined in the following way. For the time being we assume that there is no C-only cutset nor L-only loop.

Γ_0: set of diagonal blocks of P_R such that the submatrices of U determined by rows of these diagonal blocks are nonzero.

$\underset{\sim}{x}^{(1)}$: set of state-variables such that the submatrices of R or U determined by the rows of $\dot{\underset{\sim}{x}}^{(1)}$ and the columns of Γ_0 or u are nonzero.

Λ_1: set of diagonal blocks of P_L or P_c such that the submatrices of Q determined by their rows and the columns of $\underset{\sim}{x}^{(1)}$ are nonzero.

Γ_1: set of diagonal blocks of P_R such that the submatrices of P determined by their rows and the columns of Λ_1 are nonzero.

$\underset{\sim}{x}^{(2)}$; set of state-variables such that the submatrices of R determined by the rows of $\dot{\underset{\sim}{x}}^{(2)}$ and the columns of Λ_1 or Γ_1 are nonzero.

. . .

Finally, this sequence terminates at $i = \nu$, when the submatrix of V determined by the columns of Λ_ν or Γ_ν is nonzero. Let $\underset{\sim}{x}^{(\nu+1)}$, $\Lambda_{\nu+1}$ and $\Gamma_{\nu+1}$ be the remainders of $\underset{\sim}{x}$, of the diagonal blocks of P_L or P_C and of the diagonal blocks of P_R, respectively. Then we replace $\underset{\sim}{x}^{(\nu)}$, Λ_ν and Γ_ν by $\underset{\sim}{x}^{(\nu)} \cup \underset{\sim}{x}^{(\nu+1)}$, $\Lambda_\nu \cup \Lambda_{\nu+1}$ and $\Gamma_\nu \cup \Gamma_{\nu+1}$, respectively. With this, $\{\Lambda_i\}$and $\{\Gamma_i\}$ are regarded as a partitioning of the set of reactive elements and resistive elements into ν and $(\nu+1)$ subsets, respectively. This partitioning has the following properties.

P1: Each Λ_i contains a set of inductors and/or capacitors which constitutes a topological structure characterized by Fig. 6(a).

P2: Λ_i and $\underset{\sim}{x}^{(i)}$; $i = 1, 2, \cdots, \nu$ are nonempty, while Γ_i may be an empty set.

P3: All the capacitors (inductors) in a C-only loop (L-only cutset) belongs to only one of Λ_i's.

P4: This partitioning is unique independent of the choice of set of state-variables.

Based on this partioning, the 2-port block can be represented as a cascade connection of several subnetworks as shown in Fig. 7(a), where, for simplicity, the case of $\nu = 4$ is illustrated. From Fig. 7(a) we see that i) $\dot{\underset{\sim}{x}}^{(1)}$ is a function of only u, $\underset{\sim}{x}^{(1)}$ and $\underset{\sim}{x}^{(2)}$, ii) $\dot{\underset{\sim}{x}}^{(i)}$; $i = 2, 3, \cdots, \nu-1$ is a function of only $\underset{\sim}{x}^{(i-1)}$, $\underset{\sim}{x}^{(i)}$ and $\underset{\sim}{x}^{(i+1)}$, iii) $\dot{\underset{\sim}{x}}^{(\nu)}$ is a

function of only $\underset{\sim}{x}^{(\nu-1)}$ and $\underset{\sim}{x}^{(\nu)}$ and iv) y is a function of only $\underset{\sim}{x}^{(\nu)}$. Therefore the sparseness structure of A, B, C, d matrices in (20) is characterized by Fig. 7(b). The sparseness structure of A can be referred as a block-tridiagonal form, and ν is equal to the number of blocks. Now comparing this sparceness structure with the definition of the sequence $\zeta(i)$; $i \geq 2$, we see that ν is equal to the number of $\zeta(i)$'s which vanish independently of the numerical values of A, B, C, d matrices. It should be noted here that, the pivoting strategy suggested by the two-stage decomposition of P is essential to keeping this sparseness structure.

In the general cases where C-only cutsets and/or L-only loops exist, some restrictions are imposed on choosing the set of state-variables[12]. Suppose, there exists a C-only cutset spreading through two or more Λ_i's, then one of the state-variables should be eliminated. If the state-variable correspond to the leftmost Λ_i is eliminated, the resultant sparseness structure is characterized by Fig. 7(c). The sparseness structure of A can be referred to as a block upper Hessenberg form. Now we see that the number of blocks ν is still equal to the number of $\zeta(i)$'s which vanish. More specifically, nonzero entries above the diagonal blocks do not affect the $\zeta(i)$'s from vanishing, as far as combinatorial properties are concerned. The same consideration is required for L-only loops. Now, from the sparseness structure of A, B and C, we have

$$\nu = n_{\infty}, \tag{C2}$$

where it is assumed that there is no special dependency among the element values. Note that there is no need to actually arrange the matrix in the

form of Fig. 7(b) or (c), since the calculations of $\zeta(i)$ are independent of the ordering of rows and columns of the A, B, C, d matrices.

By the same way as in the case of the normal form, the A-inverse form canonical representation (65)' uniquely defines a partitioning of diagonal blocks of P'. Suppose P'_L and P'_c are decomposed into ν' subsets $\{\Lambda'_j;\ j = 1, 2, \cdots, \nu'\}$, then we have

$$\nu' = n_0. \tag{C3}$$

And the sparseness structure of A' is also a block tridiagonal or block upper Hessenberg form with ν' blocks. The properties of $\{\Lambda'_j\}$ are the same as $\{\Lambda_i\}$ except that the roles of inductors and capacitors are interchanged. For example, each Λ'_i contains a set of inductors and/or capacitors which constitutes a topological structure characterized by Fig. 6(b).

Finally, we shall list up the restrictions on choosing a pair of C-normal tree T_c and L-normal tree T_L, so that the corresponding set of state-variables is consistent with (C2) and (C3), simultaneously.

R1: All inductor (capacitors) contained in T_c(T_L) are also contained in T_L(T_c). This is required for obtaining A, B, C, d matrices and A', B', C', d' matrices. (see [10]).

R2: Each inductor (capacitor) not contained (contained) in $T_c \cup T_L$ ($T_c \cap T_L$) belongs to Λ_i such that no inductor (capacitor) in the corresponding L-only loop (C-only cutset) belongs to any Λ_α; $\alpha < i$. This is required for (C2) to be valid.

R3: Each inductor (capacitor) contained (not contained) in $T_c \cap T_L$ ($T_c \cup T_L$) belongs to Λ_j' such that no inductor (capacitor) in the corresponding L-only cutset (C-only loop) belongs to any Λ_β'; $\beta < j$. This is required for (C3) to be valid.

A pair of T_c and T_L which satisfies the above requirements is simply obtained i) by taking an arbitrary pair of T_c and T_L satisfying R1, ii) by modifying T_L based on $\{\Lambda_i\}$ so that R2 is satisfied and iii) by modifying T_c based on $\{\Lambda_j'\}$ so that R3 is satisfied.

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Footnotes

Manuscript received_____

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Tatsuo Ohtsuki is with the Central Research Laboratories, Nippon Electric Company, Kawasaki, Japan.

Lap-Kit Cheung is with the Electronics Research Laboratory, University of California, Berkeley, Ca. 94720.

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1. The presented method can be viewed as an application of the sparse tableau approach [6] to the special problem of pole-zero calculations.
2. This assumption is imposed in order to simplify the discussion. Actually our results are applicable to more general networks with controlled sources, although some additional considerations are required.
3. Modification of the formulation for another input-output configulation such as the voltage source drive is trivial.
4. This specific choice of the reference nodes is essential to the factorization of the numerator polynomial, which will be discussed in Section VI. If the input and output are common grounded, say $1' = 2'$, this grounded node can be taken as the reference for both voltage and current equations.
5. In the original tableaus $\hat{M}$ and $\hat{N}$, T is a zero matrix and ℓ is the number of capacitors and inductors in the network.

6. We assume that the eigenvalues are calculated by Francis QR algorithm [4] ch. 8, [5].

7. The incidence matrix of a graph is known to be totally unimodular. It is also known that, if a matrix of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is totally unimodular, then so are $A^{-1}B$, CA^{-1} and $D-CA^{-1}B$ [7].

8. This auxiliary procedure is for obtaining $A^{(k)'}$, $B^{(k)'}$, $C^{(k)'}$ and $d^{(k)'}$ without calculating the inverse of $A^{(k)}$. This increases the accuracy of the calculations involved in the next part.

9. The formula given in Lemma 2 guarantees the nonsingularity of A, unless particular dependencies exist among the element values of the network.

10. Minimal 2-port is a 2-port which cannot be decomposed into a cascade of two or more 2-port blocks.

11. The former (latter) is a subnetwork which separates the output from the input when capacitors (inductors) are short-circuited and inductors (capacitors) are open-circuited.

12. There is no restriction if each C-only cutset or L-only loop exists within one of Λ_i's.

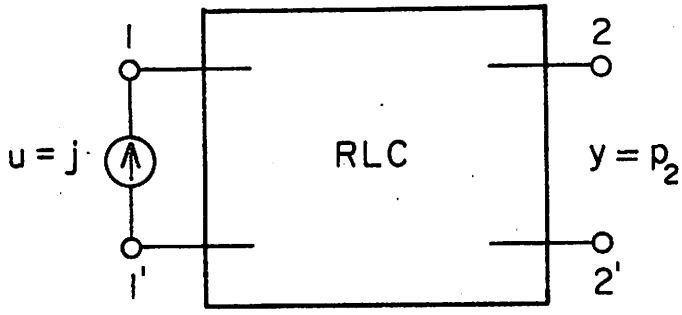


Fig. 1 A 2-port

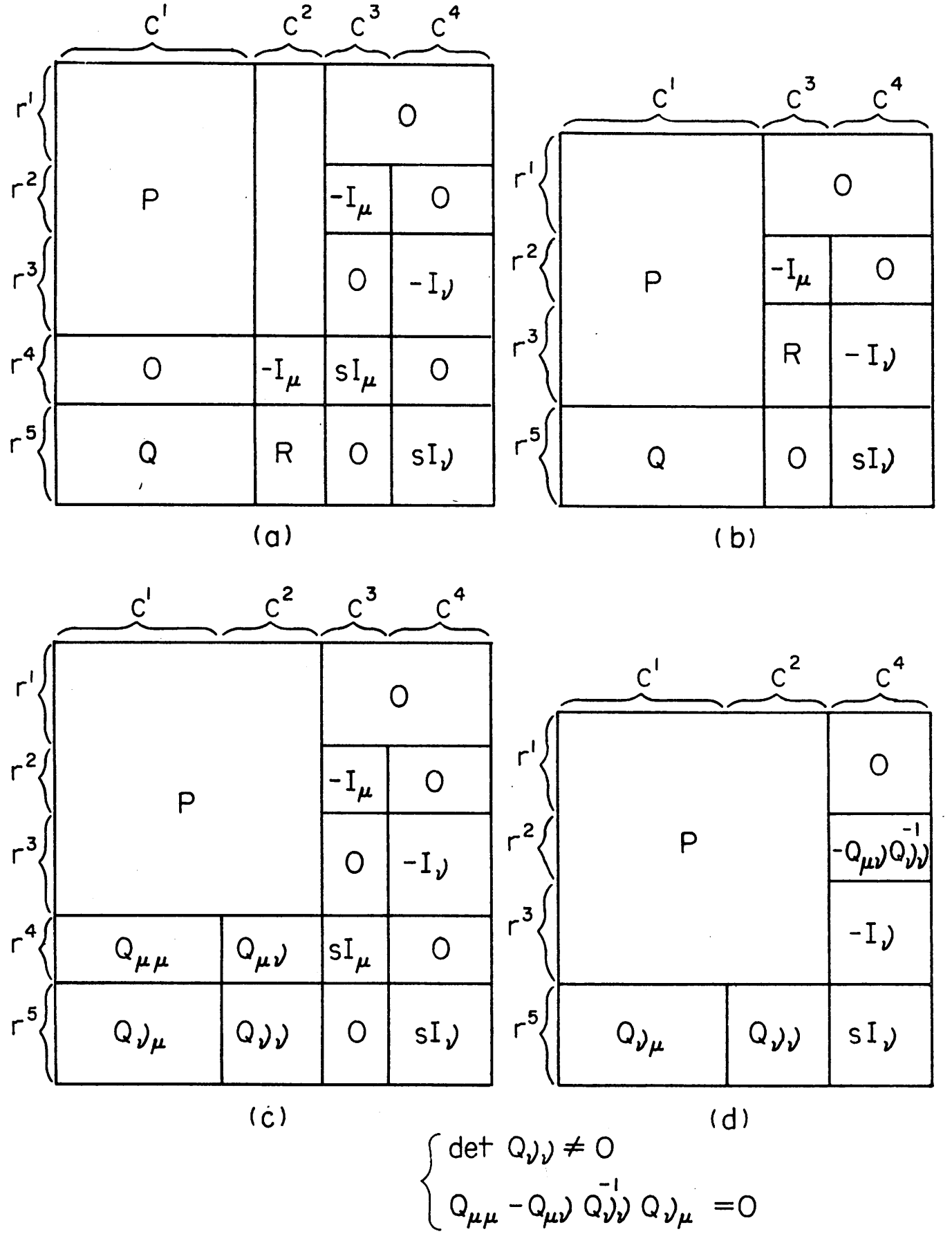
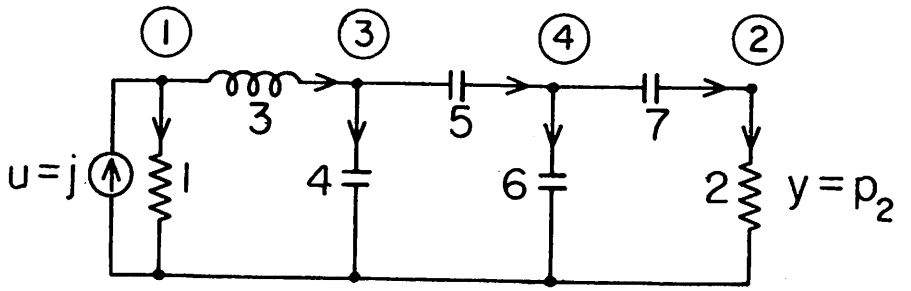


Fig. 2 Combinatorial elimination and extraction of s

(a) $\hat{U}_1$ (b) $\hat{U}_3$ (c) $\hat{U}_4$ (d) $\hat{U}_6$



(a)

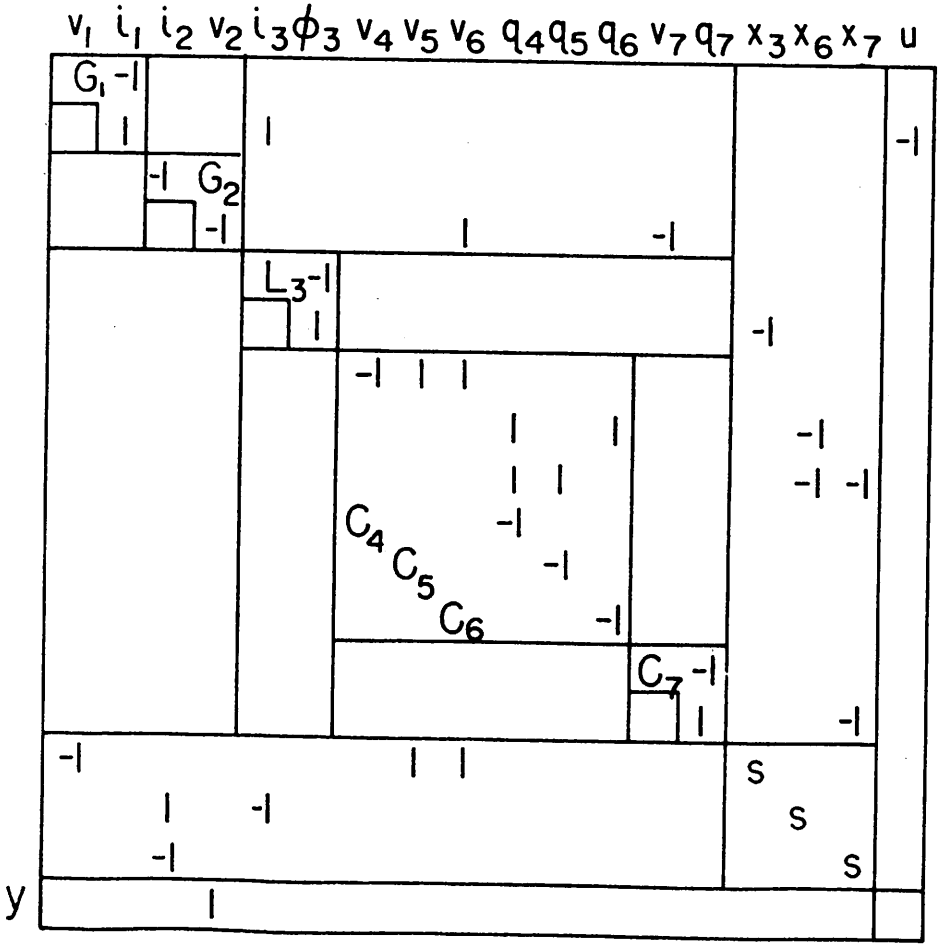
$T_C = \{1, 5, 6, 7\}$

$T_L = \{1, 2, 3, 5\}$

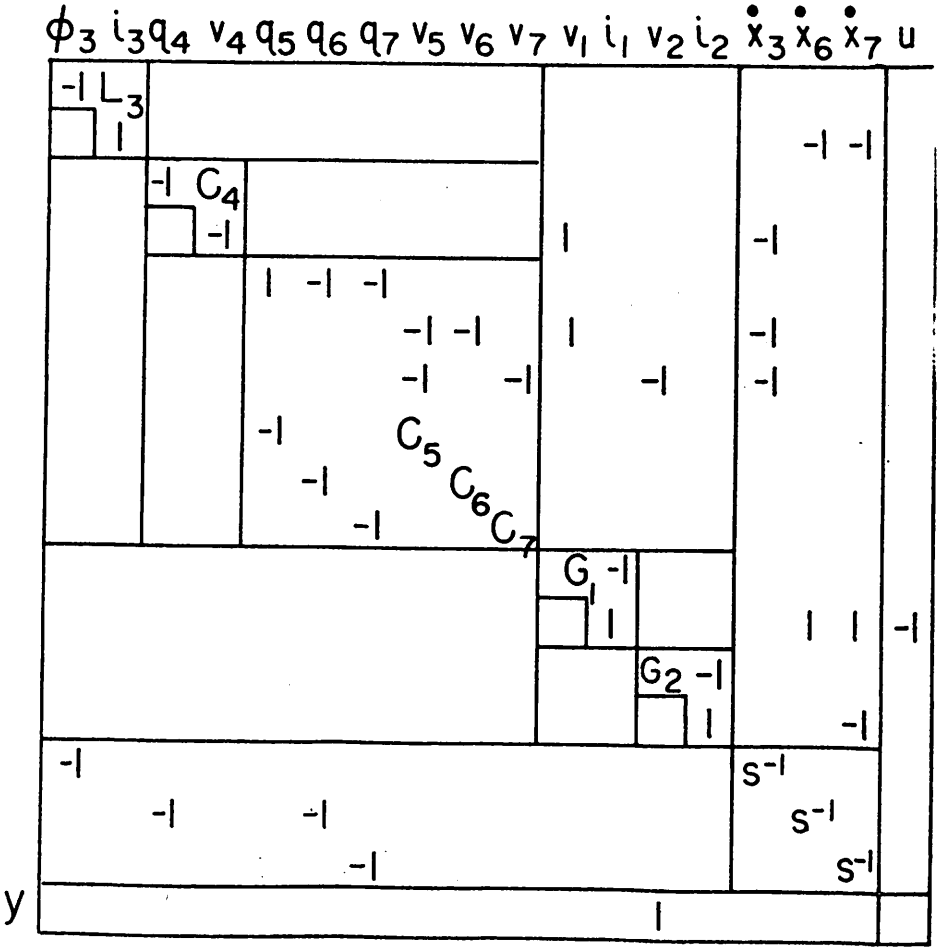
$x_3 = \phi_3$, $\dot{x}_3 = v_3$

$x_6 = q_6 + q_4$, $\dot{x}_6 = i_6 + i_4$

$x_7 = q_7$, $\dot{x}_7 = i_7$



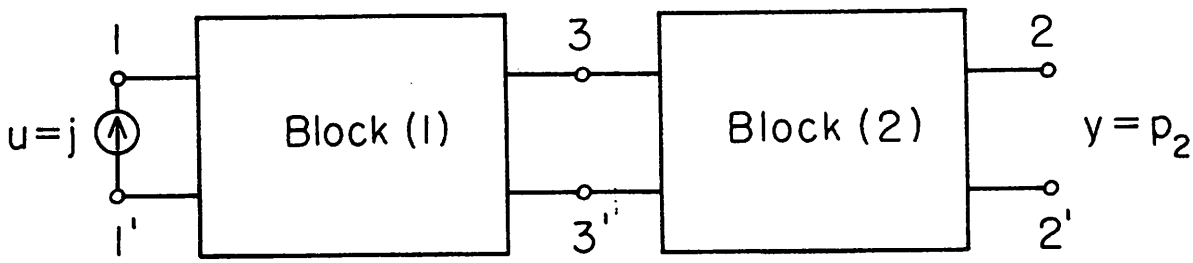
(b)



(c)

Fig. 3 Canonical representations of the state equation of a 2-port

(a) a 2-port (b) normal form (c) A-inverse form



a) $1'$: current reference $2'$: voltage reference

b) $1 \neq 1'$, $2 \neq 2'$, $3 \neq 3'$

c) β = number of branches in Block (2)

λ = number of L's and C's in Block (2)

ν = number of nodes in Block (2) except 2, 2', 3 and 3'

d) $$\nu' = \begin{cases} 0 \text{ if } 2 = 3,\ 2' = 3' \\ 1 \text{ if } 2 \neq 3,\ 2' = 3' \\ 2 \text{ if } 2 \neq 3,\ 2' \neq 3' \end{cases}$$

Fig. 4 Cascade connection of two 2-ports

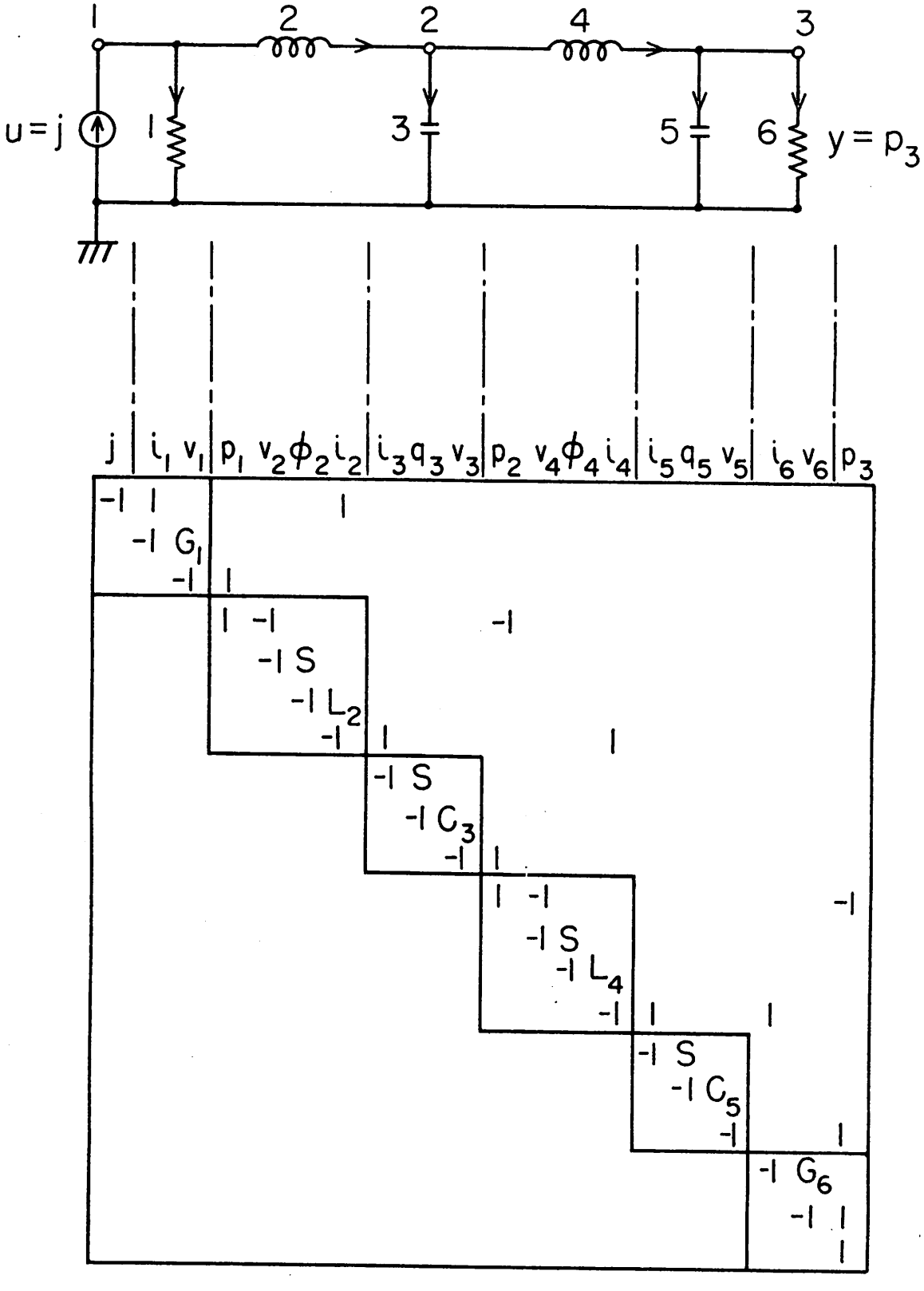
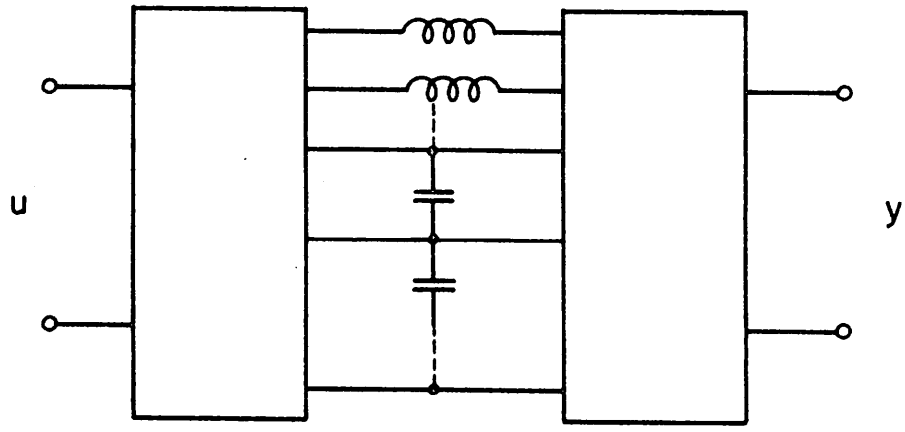
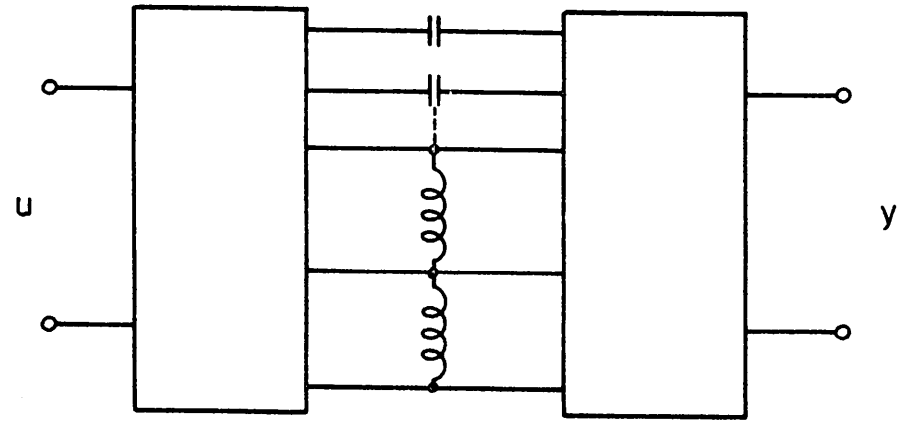


Fig. 5 Block-triangular decomposition of $\hat{N}$



(a)



(b)

Fig. 6 Topological structures causing transfer zeros at $s = \infty$ and $s = 0$

(a) $s = \infty$ (b) $s = 0$

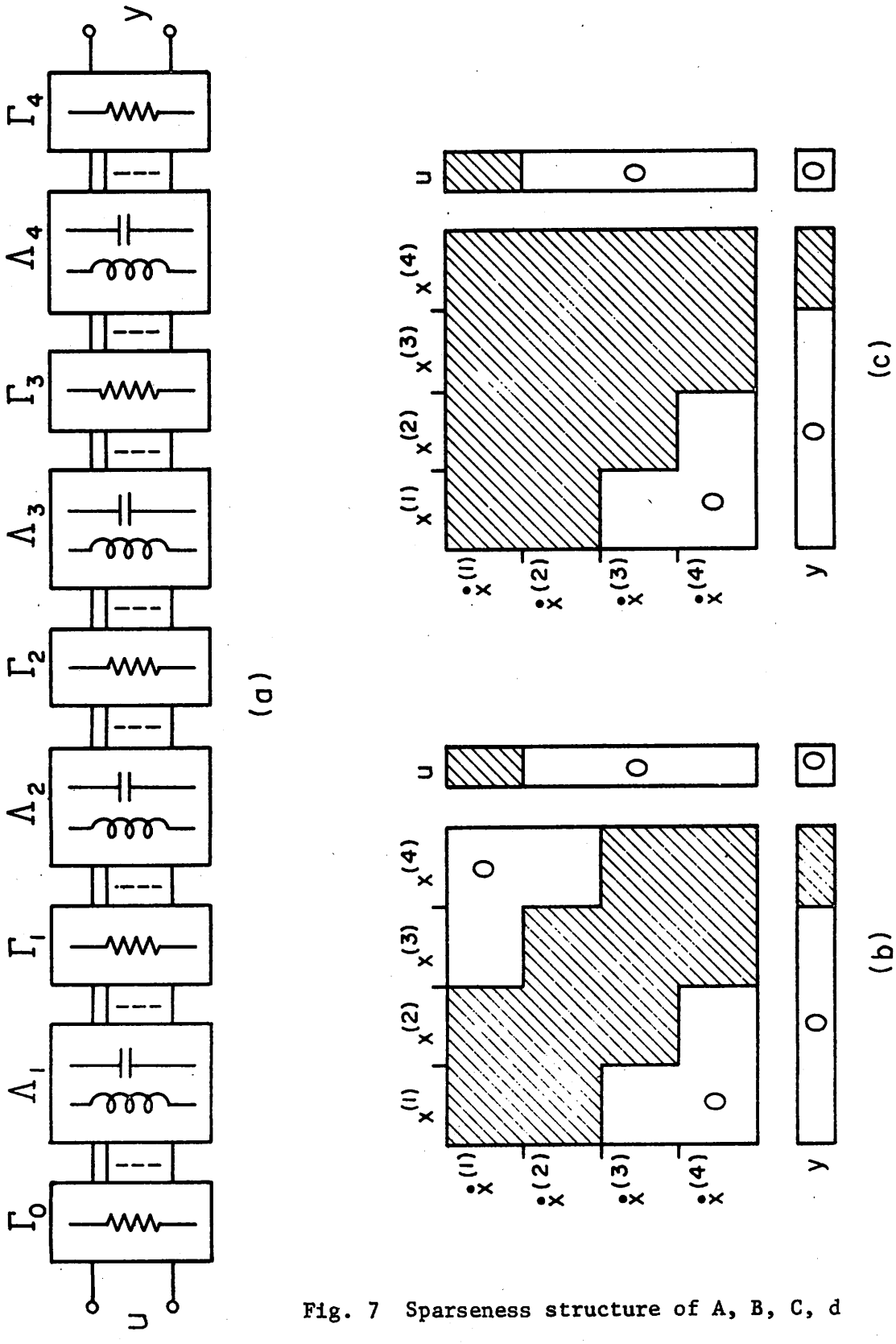


Fig. 7 Sparseness structure of A, B, C, d

(a) Partitioning of set of network elements

(b) block-tridiagonal form

(c) upper block-Hessenberg form