NONLINEAR MONOTONE NETWORKS

by

C. A. Desoer and F. F. Wu

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Charles A. Desoer and Felix F. Wu

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory,
University of California, Berkeley, California 94720

ABSTRACT

Necessary and sufficient conditions are obtained for the existence and uniqueness of solutions for strictly increasing resistive networks and a general class of increasing resistive networks. They are also necessary and sufficient for the existence of solutions for increasing resistive networks. These conditions are sufficient for the existence of solutions for eventually strictly increasing resistive networks, and for a class of eventually increasing resistive networks. The conditions are circuit-theoretic and can readily be used as a criterion in design. The dependence of solutions on the inputs is studied and also a bounded-input bounded-solution result is presented. Existence and uniqueness results for monotone RLC networks are obtained by viewing them as combinations of three one-element-kind subnetworks. Finally two algorithms are given for testing the conditions.

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I. Introduction

In this paper we present some fundamental results on nonlinear networks with uncoupled, monotone-increasing, but not necessarily surjective characteristics. Necessary and sufficient conditions are obtained for the existence and uniqueness of solutions for strictly increasing resistive networks and a general class of increasing resistive networks. In such cases, the solution depends continuously on the inputs. These conditions are also necessary and sufficient for the existence of solutions for increasing resistive networks. They are sufficient for the existence of solutions for eventually strictly increasing resistive networks, and for a class of eventually increasing resistive networks. The dependence of these solutions on the inputs is then studied and a bounded-input bounded-solution result is presented. Existence and uniqueness results for monotone RLC networks are obtained by viewing them as combinations of three one-element-kind subnetworks. Finally two algorithms are given for testing the conditions.

The nature of our necessary and sufficient conditions is circuit-theoretic in the sense that they are checked by considering the network topology and the element characteristics rather than evaluating determinants, or eigenvalues, etc. Moreover, in case one of the conditions fail, the proposed algorithms will pinpoint where the network needs to be modified.

Nonlinear monotone resistive networks were studied by Duffin[1] early in 1946. Later Desoer and Katzenelson[2] considered monotone increasing resistive networks and a class of RLC networks. They have given sufficient conditions for the existence and uniqueness of solutions. Since many semiconductor devices have monotone characteristics, networks with monotone nonlinearities have received considerable attention. Sandberg and Willson[3]-[6] have made significant advances both theoretically and computationally on networks with strictly increasing nonlinearities. The research reported here was stimulated by the work of Sandberg and Willson and is a generalization of the work of Desoer and Katzenelson.

Proofs of the theorems are included in the text because they improve the understanding of the results. For ease of reference, we state three theorems in Appendix I. In Appendix II, we derive two lemmas which are used in the proof of Theorem 1.

II. Formulation

1. Resistors

In this paper, we define a _resistor_ as a two-terminal element that, at any instant time t, is characterized by a _continuous_ f which maps the real line $\mathbb{R}$ into itself and $\sigma = f(\rho)$, where either ρ is the branch-voltage and σ the branch-current, or vice versa. If ρ is the branch-voltage, we say that the resistor is _voltage-controlled_ (v.c.); on the other hand, if ρ is the branch-current, we say that the resistor is _current-controlled_ (c.c.). Such resistors are thus not-necessarily-linear, not-necessarily-time-invariant, uncoupled, and either v.c. or c.c. (and possibly both). In the considerations that follow, conditions are examined for fixed t, so that we formulate, for simplicity, as if the resistors were time-invariant.

A resistor is said to be _increasing_ if its characteristic f satisfies

$f(\rho_2) \geq f(\rho_1)$ whenever $\rho_2 \geq \rho_1$; strictly increasing if $f(\rho_2) > f(\rho_1)$ whenever $\rho_2 > \rho_1$. A resistor is said to be eventually (strictly) increasing if its characteristic is (strictly) increasing on $\{\rho \in \mathbb{R} | |\rho| > M\}$ for some M, which depends on the resistor under consideration.

We say that a resistor is of type U if its characteristic has the property that $f(\rho) \to \infty$ as $\rho \to \infty$ and $f(\rho) \to -\infty$ as $\rho \to -\infty$; of type H if either (i) $f(\rho) \to -\infty$ as $\rho \to -\infty$ and $|f(\rho)| < B$ for some B as $\rho \to \infty$, or (ii) $|f(\rho)| < B$ for some B as $\rho \to -\infty$ and $f(\rho) \to \infty$ as $\rho \to \infty$; of type B if $|f(\rho)| < M$ as $|\rho| \to \infty$.

Clearly the set of all increasing (resp. strictly increasing, eventually increasing, eventually strictly increasing) resistors can be partitioned into type U, type H, and type B resistors.

2. Network

Let $\mathcal{N}$ be an interconnection of a finite number of resistors. Without loss of generality, $\mathcal{N}$ is assumed to have a connected and nonseparable graph. If $\mathcal{N}$ inherently has some independent sources, they may be regarded as increasing resistors, or, by source transformation [7, pp. 409-412], be absorbed in the resistive branches.

3. Network topology

Let the network variables be partitioned into (v_v, v_c) and (i_v, i_c), where subscripts v and c denote those corresponding to the v.c. resistors and c.c. resistors, respectively.[1] Let us pick a tree which contains the

[1] In the case where a resistor is both v.c. and c.c., we can assign it to either class.

maximum number of v.c. resistors. If $\mathcal{N}$ contains any type H resistor, let the reference direction for that branch be so chosen that $|f(\rho)| < M$ for some M as $\rho \to \infty$. The fundamental loop matrix B and the fundamental cutset matrix Q corresponding to such a choice of tree take the form[2]:

$$B = \begin{array}{c} \\ v\ell \\ c\ell \end{array} \begin{array}{c} \begin{array}{cccc} v\ell & c\ell & vt & ct \end{array} \\ \begin{bmatrix} I & 0 & F_{vv} & 0 \\ 0 & I & F_{vc} & F_{cc} \end{bmatrix} \end{array}$$

$$Q = \begin{array}{c} vt \\ ct \end{array} \begin{bmatrix} -F^T_{vv} & -F^T_{vc} & I & 0 \\ 0 & -F^T_{cc} & 0 & I \end{bmatrix}$$

where subscript ℓ (resp. t) denotes links (resp. tree-branches); hence the double-subscript $v\ell$, for example, denotes v.c. link resistors. According to this partition, we have

$$\begin{aligned} i_{v\ell} &= \hat{i}_{v\ell}\,(v_{v\ell}) \\ i_{vt} &= \hat{i}_{vt}\,(v_{vt}) \\ v_{c\ell} &= \hat{v}_{c\ell}\,(i_{c\ell}) \\ v_{ct} &= \hat{v}_{ct}\,(i_{ct}) \end{aligned} \tag{1}$$

4. Independent sources

There are two ways of applying independent sources to a network:

[2]Superscript T denotes transpose of a matrix.

namely, pliers entry and soldering-iron entry. By pliers entry we mean that we enter the network by cutting any branch of the network and connecting the two terminals of a source to the terminals created by the cut. By soldering-iron entry we mean that we enter the network by connecting the two terminals of a source to any two nodes of the network. Throughout the following, we apply a voltage source only by a pliers entry and a current source only by a soldering-iron entry.

5. Network equations

Let e_v (resp. e_c) denote the voltage-source vector around fundamental loops defined by v.c. (resp. c.c.) resistive links; j_v(resp. j_c) denote the current-source vector across fundamental cutset defined by v.c. (resp. c.c.) resistive tree-branches. Kirchhoff laws are thus expressed by

$$\begin{cases} i_{ct} - F_{cc}^T\, i_{c\ell} = j_c \\ v_{v\ell} + F_{vv}\, v_{vt} = e_v \end{cases} \tag{2a}$$

$$\begin{cases} v_{c\ell} + F_{cc}\, v_{ct} + F_{vc}\, v_{vt} = e_c \\ i_{vt} - F_{vv}^T\, i_{v\ell} - F_{vc}^T\, i_{c\ell} = j_v \end{cases} \tag{2b}$$

Substitute (1) into (2) and eliminate i_{ct} and $v_{v\ell}$, we obtain two equations in terms of v_{vt} and $i_{c\ell}$:

$$\begin{bmatrix} -F_{vv}^T & I & 0 & 0 \\ 0 & 0 & I & F_{cc} \end{bmatrix} \begin{bmatrix} \hat{i}_{v\ell}(-F_{vv}\, v_{vt} + e_v) \\ \hat{i}_{vt}\ (v_{vt}) \\ \hat{v}_{c\ell}\ (i_{c\ell}) \\ \hat{v}_{ct}\ (F_{cc}^T i_{c\ell} + j_c) \end{bmatrix} + \begin{bmatrix} 0 & -F_{vc}^T \\ F_{vc} & 0 \end{bmatrix} \begin{bmatrix} v_{vt} \\ i_{c\ell} \end{bmatrix} = \begin{bmatrix} j_v \\ e_c \end{bmatrix} \tag{3}$$

6. Notations

To simplify presentation, we sometimes write (3) in the form:

$$C\,\mathcal{F}(C^T x+u) + Sx = y \tag{3'}$$

where

$$x = \begin{bmatrix} v_{vt} \\ i_{c\ell} \end{bmatrix} \in \mathbb{R}^m \;; \quad y = \begin{bmatrix} j_v \\ e_c \end{bmatrix} \in \mathbb{R}^m$$

$$u = \begin{bmatrix} e_v \\ 0 \\ 0 \\ j_c \end{bmatrix} \in \mathbb{R}^r \;; \quad \mathcal{F} = \begin{bmatrix} \hat{i}_{v\ell} \\ \hat{i}_{vt} \\ \hat{v}_{c\ell} \\ \hat{v}_{ct} \end{bmatrix} : \; \mathbb{R}^r \to \mathbb{R}^r$$

$$C = \begin{bmatrix} -F_{vv}^T & I & 0 & 0 \\ 0 & 0 & I & F_{cc} \end{bmatrix} \in \mathbb{R}^{m\times r} \;; \quad S = \begin{bmatrix} 0 & -F_{vc}^T \\ F_{vc} & 0 \end{bmatrix} \in \mathbb{R}^{m\times m}$$

$\mathbb{R}^m$ denotes Euclidean m-space with scalar product $\langle x|y \rangle = \sum_{k=1}^{m} x_k y_k$ and norm $\|x\| = (\sum_{k=1}^{m} x_k^2)^{1/2}$.

We sometimes consider the map G: $\mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m$ defined by

$$G(x,u) = C\,\mathcal{F}(C^T x+u) + Sx. \tag{4}$$

Note that S is a real skew-symmetric matrix, C is a matrix of full rank, and $\mathcal{F}$ is a "diagonal" map, i.e., $\mathcal{F}(z) = [f_1(z_1) \cdots f_r(z_r)]^T$. The zero element in $\mathbb{R}^m$ is denoted by θ. The null space of a matrix S is denoted by $\mathcal{N}(S)$, i.e., $\mathcal{N}(S) = \{x|Sx = \theta\}$. $D_1G(x,u)$ denotes the derivative map of $G(\cdot,u)$ evaluated at x.

Let us define the cone

$$\mathcal{B}(\mathcal{F}) \triangleq \{z \in \mathbb{R}^r \mid \rho \mapsto \|\mathcal{F}(\rho z)\| \text{ is bounded on } [1,\infty)\} \quad (5)$$

Clearly $z = (z_1,\ldots z_r) \in \mathcal{B}(\mathcal{F})$ if and only if $z_i = 0$ whenever f_i is of type U; $z_i \geq 0$ whenever f_i is of type H (Note that by our convention $|f_i(\rho)| < M$ as $\rho \to \infty$), and z_i = any real number whenever f_i is of type B.

By a <u>solution</u> of a network, we mean a set of branch-voltages and branch-currents $\chi = (v,i)$ (including currents in the voltage sources and voltages across the current-sources) that satisfies both Kirchhoff laws and the branch characteristics. By an <u>input</u>, we mean an independent-source vector $\mu = (e_v,e_c,j_v,j_c)$. We say that a solution <u>depends continuously</u> on the inputs iff considering χ as a function of μ, i.e., $\chi = \hat{\chi}(\mu)$, $\hat{\chi}(\cdot)$ is a continuous function on $\mathbb{R}^r$.

A set of branches belonging to a loop (resp. cutset) in a directed graph is said to be <u>similarly directed</u> if we can assign a reference direction to the loop (resp. cutset) such that the direction of each branch in the set agrees with the reference direction of the loop (resp. cutset).

III. Strictly Increasing Resistive Networks

In this section we prove Theorem 1 for strictly increasing resistive networks. We first prove two lemmas, which, taken together, assert that Theorem 1 is true if all resistor characteristics are C^1 (in such case, the dependence of the unique solution on the inputs is C^1). Then in the proof of Theorem 1 we show that with the aid of two lemmas in Appendix II, the C^1 assumption can be dropped.

Theorem 1. (Existence, uniqueness, and continuous dependence)

Let $\mathcal{N}$ be a finite network made of strictly increasing, continuous, and uncoupled resistors. Then for all independent voltage sources with pliers entries and for all independent current sources with soldering-iron entries, the network $\mathcal{N}$ has one and only one solution and this solution depends continuously on the inputs if and only if the following conditions (i) and (ii) hold:

(i) every loop made of c.c. resistors either contains at least one type U c.c. resistor or if not, then it contains at least two type H c.c. resistors and not all such type H c.c. resistors are similarly directed.

(ii) every cutset made of v.c. resistors either contains at least one type U v.c. resistor or if not, then it contains at least two type H v.c. resistors and not all such type H v.c. resistors are similarly directed.

Comments (a) Uniqueness follows directly (by Tellegen's theorem) from the strictly increasing property of the characteristics.

(b) Condition (i) and (ii) are dual of each other.

(c) If there is a loop (resp. cutset) of type B c.c. (resp. v.c.) resistors, then condition (i) (resp. condition (ii)) does not hold; If there is a loop (resp. cutset) of type B and type H c.c. (resp. v.c.) resistors, in which all type H resistors are similarly directed, then condition (i) (resp. condition (ii)) does not hold.

(d) Condition (i) (resp. (ii)) is equivalent to: (ia) (resp. (iia)) there is no loop (resp. cutset) of only type B c.c. (resp. v.c.) resistors;

(ib) (resp. (iib)) for every type H c.c. (resp. v.c.) resistor b there is a cutset (resp. loop) containing b, and made of v.c., type U c.c., and type H c.c. (resp. c.c., type U v.c., and type H v.c.) resistors, in which all type H c.c. (resp. v.c.) resistors are similarly directed. (By the Colored Arc Lemma (App. I)).

<u>Lemma 1</u>. Consider equation (3')

$$G(x,u) \triangleq C\mathcal{F}(C^T x+u) + Sx = y$$

Suppose that each f_i is C^1 and strictly increasing. If

$$\{x \in \mathbb{R}^m | C^T x \in \mathcal{B}(\mathcal{F}) \text{ and } x \in \mathcal{N}(S)\} = \{\theta\} ,$$

then there is a unique C^1 function Φ: $\mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^m$, satisfying $G(\Phi(u,y),y) = y$, $\forall y \in \mathbb{R}^m$, $\forall u \in \mathbb{R}^r$. (equivalently, $\forall u \in \mathbb{R}^r$, $G(\cdot,u)$ is a diffeomorphism from $\mathbb{R}^m$ onto $\mathbb{R}^m$).

Proof: (1) Claim: $D_1G(x,u)$ is nonsingular, $\forall x \in \mathbb{R}^m$, $\forall u \in \mathbb{R}^r$. Differentiating (4), $D_1G(x,u) = C[D\mathcal{F}(C^T x+u)]C^T + S$. Since each component of $\mathcal{F}$ is strictly increasing, $[D\mathcal{F}(C^T x+u)]$ is diagonal and positive definite for all (x,u). Suppose $D_1G(x,u)$ were singular for some (x,u), then there would be a $\xi \neq \theta$ such that $[D_1G]\xi = \theta$. Note that $\xi \neq \theta$ implies $C^T\xi \neq \theta$ because C is of full rank. Consider $\xi^T[D_1G]\xi = \xi^T C[D\mathcal{F}]C^T\xi = \theta$, which contradicts that $[D\mathcal{F}]$ is positive definite. Therefore, $D_1G(x,u)$ is nonsingular for all (x,u).

(2) Claim: $\|x\| \to \infty \Rightarrow \|G(x,u)\| \to \infty$ $\forall u \in \mathbb{R}^r$. (i.e., for any sequence $\{x^i\}$ such that $\|x^i\| \to \infty$ implies $\|G(x^i,u)\| \to \infty$, $\forall u \in \mathbb{R}^r$).

Let $z = C^T x$, $z \in \mathbb{R}^r$, hence any sequence $\{x^i\}$ defines a sequence $\{z^i\}$ in $\mathbb{R}^r$, where $z^i = C^T x^i$. Let us partition $\mathbb{R}^r$ into its 2^r orthants, $\{z^i\}$ will accordingly be partitioned into at most 2^r subsequences. Note that C^T contains an identity matrix, thus $\{x^i\} \to \infty$ if and only if $\|z^i\| \to \infty$ and hence at least one of the subsequences is unbounded. We will first consider $\{z | z \in \mathcal{B}(\mathcal{F})\}$ which is a closed convex cone consisting of the union of several orthants, and show that $\|z^i\| \to \infty$ with $z^i \in \mathcal{B}(\mathcal{F})$ implies that the corresponding $\|G(x^i,u)\| \to \infty$. Then we consider each orthant for which $\{z | z \notin \mathcal{B}(\mathcal{F})\}$ and show that the same fact holds.

(a) Consider $\{z | z \in \mathcal{B}(\mathcal{F})$ and $z = C^T x\}$, for such $x \neq \theta$ we have $x \notin \mathcal{N}(S)$. We are going to show an equivalent condition of $\|x\| \to \infty \Rightarrow \|G(x,u)\| \to \infty$, namely; given any $M > 0, \exists\ N > 0$ such that $\rho > N \Rightarrow \|G(\rho\xi,u)\| > M \quad \forall\ C^T\xi \in \mathcal{B}(\mathcal{F})$ with $\|\xi\| = 1$. Now

$$\|G(\rho\xi,u)\| \geq \rho\ \|S\xi\| - \|C\| \cdot \|\mathcal{F}(\rho C^T\xi+u)\|$$

Since $C^T\xi \in \mathcal{B}(\mathcal{F})$, $\|C\| \cdot \|\mathcal{F}(C^T\xi+u)\|$ is bounded, say by β. Note that $\{x | C^T x \in \mathcal{B}(\mathcal{F})\}$ is closed and $\{\xi |\ \|\xi\| = 1\}$ is compact, hence the set $\Sigma \triangleq \{\xi |\ \|\xi\| = 1$ and $C^T\xi \in \mathcal{B}(\mathcal{F})\}$ is compact. Moreover $\|S\xi\|$ is a continuous function and $\|S\xi\| > 0$ on the compact set Σ. Therefore,

$$\inf_{\xi \in \Sigma} \|S\xi\| = m > 0$$

So $\|G(\rho\xi,u)\| \geq m\rho - \beta$. Thus given any M, if $\rho > \frac{M+\beta}{m}$, then $\|G(\rho\xi,u)\| > M$ $\forall \xi \in \Sigma$.

(b) Let $\mathcal{O}$ be any orthant of $\mathbb{R}^r$ in which $z = C^T x \notin \mathcal{B}(\mathcal{F})$. Let $\{z^i\}$, $z^i = C^T x^i$, be a sequence in $\mathcal{O}$ such that $\|z^i\| \to \infty$.

Case 1. For all j such that $f_j(z_j^i)$ is of type H in the unbounded half or of type U, the sequence $\{|z_j^i|\}_{i=1}^{\infty}$ is bounded. Again, we have

$$\|G(x^i,u)\| \geq \|Sx^i\| - \|C\| \cdot \|\mathcal{F}(C^T x^i + u)\|$$

The second term is bounded in this case. Hence if we show $\|Sx^i\| \to \infty$ then we are done. Observe that $\mathbb{R}^m = \mathcal{N}(S) \oplus \mathcal{N}(S)^{\perp}$ and the map S restricted to $\mathcal{N}(S)^{\perp}$ is a bijection of $\mathcal{N}(S)^{\perp}$ onto $\mathcal{R}(S)$. [18,p.572] Let $x^i = n^i + p^i$, where $n^i \in \mathcal{N}(S)$ and $p^i \in \mathcal{N}(S)^{\perp}$. We are going to show by contradiction that the assumption $\|x^i\| \to \infty$ implies that $\|p^i\| \to \infty$, but $Sx^i = Sp^i$ so $\|Sx^i\|$ also $\to \infty$. Suppose $\|x^i\| \to \infty$ and $\|p^i\|$ is bounded, hence $\|n^i\| \to \infty$. Note that $\mathcal{B}(\mathcal{F})$ is a closed cone and $\{z | z = C^T x \text{ and } x \in \mathcal{N}(S)\}$ is also a closed cone. By assumption the intersection of these two closed cones contains only $\{\theta\}$, let

$$\delta = \inf \|\xi - \eta\| \qquad \text{where} \quad \begin{aligned} &\|\xi\| = 1 \text{ and } \xi \in \mathcal{B}(\mathcal{F}) \\ &\|\eta\| = 1 \text{ and } \eta = C^T x,\ x \in \mathcal{N}(S). \end{aligned}$$

we have $\delta > 0$. Hence

$$d_i \triangleq \inf_{\xi \in \mathcal{B}(\mathcal{F})} \|C^T n^i - \xi\| \geq \frac{\delta}{\sqrt{2}} \|C^T n^i\|$$

and $d_i \to \infty$ as $\|n^i\| \to \infty$. But this requires that at least one of the components of $C^T n^i$ goes to infinity. Clearly this branch variable belongs to a resistor of type H in the unbounded half or of type U. But $z^i = C^T x^i = C^T n^i + C^T p^i$, and since by assumption such $\{z_j^i\}$ are bounded, to compensate we must have $\|C^T p^i\| \to \infty$, hence $\|p^i\| \to \infty$, we reach the desired contradiction.

Case 2. There is a j such that $|z_j^i| \to \infty$ and $|f_j(z_j^i)| \to \infty$. Recall that z_j^i is either a branch-voltage or a branch-current and $f_j(z_j^i)$ is the corresponding branch-current or branch-voltage. Call the branch for which $|z_j^i| \to \infty$ and $|f_j(z_j^i)| \to \infty$ the branch b_j. In the orthant $\mathcal{O}$, each $z_j^i (i=1,2,...)$ has a definite sign. Let us reassign the reference directions of each type H and of each type U resistor in accordance with the associated sign in the orthant so that if z_k is a branch-voltage (resp. branch-current) and f_k is a type H or a type H v.c. (res. c.c.) resistor, the reference direction is so chosen that the branch-voltage (resp. branch current) z_k (measured with respect to the new reference direction) is positive whenever z is in the orthant $\mathcal{O}$. Now we have three kinds of branches in the graph, namely (i) type U and type H (v.c. and c.c.) resistors, for which we have assigned directions, (ii) type B c.c. resistors, (iii) type B v.c. resistors. By the Colored Arc Lemma one of the following alternatives must occur;

Alternative I: There is a loop $\mathcal{L}$ containing b_j, of type U and type H (v.c. and c.c.) resistors, all of which are similarly-directed, and of type B c.c. resistors.

Alternative II: There is a cutset $\mathcal{C}$ containing b_j, of type U and type H (v.c. and c.c.) resistors, all of which are similarly-directed, and of type B v.c. resistors.

If Alt. I occurs, note that the branch-voltage of a type U or a type H v.c. resistor agrees with the reference direction and the branch-voltage of a type U or a type H c.c. resistor either agrees with the reference direction (the direction of its current flow) or, if it is opposite to the reference direction, is bounded. Moreover, the voltage in a type B c.c. resistor is bounded. Hence, in order that KVL be satisfied for $\mathcal{L}$, we have to have a unbounded voltage source e_c to compensate the unbounded voltage in b_j. Dually if Alt. II occurs, we need a unbounded current source j_v to compensate the unbounded current in b_j. Therefore, in either case $\|G(x^i,u)\| = \|(j_v,e_c)\| \to \infty$.

Thus, it follows from Global Implicit Function Theorem [App. I] that there is a unique C^1 function Φ: $\mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying $G(\Phi(u,y),u) = y$. ∎

Lemma 2 If conditions (i) and (ii) of Theorem 1 hold, then

$$\mathcal{S} \triangleq \{x \in \mathbb{R}^m | C^T x \in \mathcal{B}(\mathcal{F}) \text{ and } x \in \mathcal{N}(S)\} = \{\theta\}.$$

Proof: Note that $x = (x_1,x_2) \in \mathcal{S}$ if and only if

$$\begin{bmatrix} -F_{vv} \\ I \end{bmatrix} x_1 \in \mathcal{B}\begin{pmatrix} \hat{i}_{v\ell} \\ \hat{i}_{vt} \end{pmatrix}, \quad F_{vc}x_1 = \theta \tag{6a}$$

and

$$\begin{bmatrix} I \\ F_{cc}^T \end{bmatrix} x_2 \in \mathcal{B}\begin{pmatrix} \hat{v}_{c\ell} \\ \hat{v}_{ct} \end{pmatrix}, \quad F_{vc}^T x_2 = \theta \tag{6b}$$

We are going to show that if $x = (x_1,x_2) \in \mathcal{S}$, then $x_1 = \theta$ and $x_2 = \theta$. Recall that $x_1 = v_{vt}$ and $x_2 = i_{c\ell}$.

(1) Let x_2 be a vector satisfying (6b), if we let (θ,x_2) to be the link currents of $\mathcal{N}$, then KCL requires that the branch-current vector i to be

$$i = \begin{bmatrix} I & 0 \\ 0 & I \\ F_{vv}^T & F_{vc}^T \\ 0 & F_{cc}^T \end{bmatrix} \begin{bmatrix} \theta \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x_2 \\ F_{vc}^T x_2 \\ F_{cc}^T x_2 \end{bmatrix} \tag{7}$$

Interpreting (6b) in (7), it demands that the branch-currents in all v.c. resistors, and type U c.c. resistors be zero; and the direction of the actual current flow through any type H c.c. resistor is identical to the preassigned reference direction. We are going to show that these conditions, together with condition (i) of Theorem 1, will force all the branch-currents to be zero. As far as KCL is concerned, those v.c. and type U c.c. resistors with zero currents can be removed. It follows from Comment (d) (or the Fact in Sec. VIII) that condition (i) of Theorem 1 implies that in the remaining network for each type H c.c. resistor b there is a cutset containing b, of similarly-directed type H c.c resistors. Since the actual current flows in these resistors are the same as their reference directions, KCL requires that all branch-currents in type H c.c. resistors be zero. Next remove all type H resistors. In the remaining network which is made of only type B c.c. resistors, there is no loop by condition (i), hence all currents in type B c.c. resistors are also zero. Therefore, $x_2 = \theta$.

(2) Dually one can show that condition (ii) implies that any vector x_1 satisfying (6a) must be zero. ¤

Remark: Lemma 1 and Lemma 2 remain valid if we replace "if" by "if and only if".

Proof of Theorem 1

$\Rightarrow$ By contradiction. Clearly if there is a loop (resp. cutset) of type B c.c. (resp. v.c.) resistors or a loop (resp. cutset) of type H and type B c.c. (resp. v.c.) resistors, in which all type H c.c. (resp. v.c.) resistors are similarly directed, then for some input vector (u,y), KVL (resp. KCL) could not be satisfied.

$\Leftarrow$ We are going to show that for all u, $G(\cdot,u)$ is a homeomorphism from $\mathbb{R}^m$ onto $\mathbb{R}^m$.

(1) Claim: $G(\cdot,u)$ is injective. Suppose not; then for some u, there exist $x \neq \bar{x}$ such that $G(x,u) = G(\bar{x},u)$. Thus

$$C[\mathcal{F}(C^T x+u) - \mathcal{F}(C^T\bar{x}+u)] + S(x-\bar{x}) = \theta$$

Premultiply by $[x-\bar{x}]^T$, obtain

$$\langle C^T(x-\bar{x}) | \mathcal{F}(C^T x+u) - \mathcal{F}(C^T\bar{x}+u) \rangle = \theta$$

But each component of $\mathcal{F}$ is strictly increasing, hence we reach contradiction.

(2) Claim: $G(\cdot,u)$ is surjective. For a fixed u, given any $\varepsilon > 0$, let us construct, following Lemma A1, [App. II] for each resistor characteristic f_i in $\mathcal{F}$, a sequence of strictly increasing C^1 functions $\{f_i^k\}_{k=1}^{\infty}$ such that

$$|f_i(\rho) - f_i^k(\rho)| < \frac{\varepsilon}{kr\|C\|} \qquad \forall\rho \in \mathbb{R}$$

Let $$\mathcal{F}_k = \begin{pmatrix} f_1^k \\ \vdots \\ f_r^k \end{pmatrix}$$

and $$G_k(x,u) = C\mathcal{F}_k(C^T x+u) + Sx$$

Thus $$\|\mathcal{F}_k(\zeta) - \mathcal{F}(\zeta)\| < \frac{\varepsilon}{k\|C\|} \qquad \forall\zeta \in \mathbb{R}^r,$$

hence $$\|G_k(x,u) - G(x,u)\| < \frac{\varepsilon}{k} \qquad \forall x \in \mathbb{R}^m,\ k = 1,2,3\ldots$$

Note that Lemma 1 and Lemma 2 assert that the conditions (i) and (ii) imply that $G_k(\cdot,u)$ is a diffeomorphism (hence, homeomorphism) for each k, it then follows from Lemma A2 [App. II] that $G(\cdot,u)$ is surjective.

(3) Claim: $G(\cdot,u)$ is a homeomorphism. Brouwer's Domain Invariance Theorem [8,pp XXIX 1-2] states that a bijective continuous function is a homeomorphism, hence $G(\cdot,u)$ is a homeomorphism $\forall u \in \mathbb{R}^r$. It then follows from the Global Implicit Function Theorem that there is a unique continuous function Φ: $\mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying $G(\Phi(u,y),u) = y$, $\forall u \in \mathbb{R}^r$, $\forall y \in \mathbb{R}^m$. Once $x = (v_{vt}, i_{c\ell})$ is known, all (v,i) will be given by simple substitution into Kirchhoff laws (2a) and resistor characterisitcs (1). Finally the dependence of (v,i) on (u,y) is continuous since (1) and (2a) are all continuous maps. $\square$

IV. Increasing Resistive Networks

In this section we allow the resistors to be increasing, but not necessarily strictly increasing. The conditions (i) and (ii) of Theorem 1 are also the necessary and sufficient conditions for the existence of solutions for increasing resistive networks (Theorem 2). With some additional restriction on the topology of the network it is shown in Theorem 3 that these conditions are again necessary and sufficient for the existence and uniqueness of solutions for a general class of increasing resistive networks.

Theorem 2 (Existence)

Let $\mathcal{N}$ be a finite network made of increasing, continuous, and uncoupled resistors. Then for all independent voltage sources with pliers entries and for all independent current sources with soldering-iron entries, the network $\mathcal{N}$ has at least one solution if and only if conditions (i) and (ii) of Theorem 1 hold.

Proof: $\Rightarrow$ Same as Proof of Theorem 1.

$\Leftarrow$ Note that the function f in Lemma A1 is only required to be increasing, hence in the proof of Theorem 1 the part that $G(\cdot,u)$ is surjective for all u applies to here too. However in the present case, we cannot guarantee uniqueness nor continuous dependence. ◻

<u>Theorem 3</u>. (Existence, uniqueness, and continuous dependence)

Let $\mathcal{N}$ be a finite network made of <u>increasing</u>, continuous, and uncoupled resistors. Suppose that $\mathcal{N}$ satisfies conditions (U_ℓ) and (U_c):

(U_ℓ) every loop made of c.c. resistors contains at least one strictly increasing resistor;

(U_c) every cutset made of v.c. resistors contains at least one strictly increasing resistor.

Under these conditions, for all independent voltage sources with pliers entries and for all independent current sources with soldering-iron entries, the network has one and only one solution and this solution depends continuously on the inputs if and only if conditions (i) and (ii) of Theorem 1 hold.

<u>Remarks</u>: (a) Physically condition (U_c) can be explained as follows. Suppose there is a cutset of v.c. resistors in which none of the re-

sistors is strictly increasing. By choosing appropriate voltage sources, we can place the operating point of each resistor in the cutset to be in the interior of an interval where its characteristic is constant. If we change the branch-voltages in the cutset by the same sufficiently small amount Δv so that the corresponding current in each resistor remains the same, then we have another solution which is identical to the preceding one except for the branch-voltages of the cutset which differ by Δv. Dually for condition (U_ℓ). Therefore, only when conditions (U_ℓ) and (U_c) are satisfied, one can expect for uniqueness.

(b) If there exists a tree for the network such that all its tree branches are c.c. and all its links are v.c., then clearly conditions (U_ℓ) and (U_c), as well as conditions (i) and (ii) of Theorem 1 are satisfied. Hence the result of Desoer and Katzenelson [2; Theorem I] is a special case of Theorem 3.

<u>Proof of Theorem 3</u>

We need only to show that $G(\cdot,u)$ is injective for all $u \in \mathbb{R}^r$. By contradiction. Suppose not, there is an input (u,y) for which $G(x,u) = G(\bar{x},u) = y$ and $x \neq \bar{x}$. Now x and $\bar{x}$ each specifies a unique set of branch-voltages and branch currents, so $(v,i) \neq (\bar{v},\bar{i})$, but they satisfy Kirchhoff laws and the branch characteristics. By Tellegen Theorem, $(v-\bar{v})^T(i-\bar{i}) = 0$, i.e., $\sum_{k=1}^{r} \Delta v_k \Delta i_k = 0$. Since resistor characteristics are increasing, $\Delta v_k \Delta i_k = 0$ for $k = 1,2,\ldots r$. Therefore (a) along every loop in which $\Delta i_i \neq 0$, all $\Delta v_i = 0$ and (b) for every cutset in which $\Delta v_j \neq 0$, all $\Delta i_j = 0$. In case (a) the loop can not contain a v.c. resistor because for v.c.

resistors, $\Delta i_k \neq 0$ implies $\Delta v_k \neq 0$, so (a) can only happen if all resistors in the loop are c.c.. However, condition (U_ℓ) requires in such a loop there is a strictly increasing resistor for which $\Delta i_k \neq 0$ implies $\Delta v_k \neq 0$. Hence $\Delta i_k = 0$, for all $k = 1,\ldots r$. Dually for case (b), hence $\Delta v_k = 0$, for $k = 1,\ldots,r$. This contradicts $x \neq \bar{x}$. ◻

Remark: In Theorem 3, if all resistor characteristics are C^1, then the dependence of the unique solution on the inputs is C^1. Note that, by comparing with Lemmas 1 and 2, all we need to show in this case is that $D_1G(x,u)$ is still nonsingular for all (x,u). Suppose $D_1G(x,u)$ were singular for some $(\underline{x},\underline{u})$, hence there would exist a $\zeta \neq \theta$ such that

$$C(D\mathcal{F}(C^T\underline{x}+\underline{u}))\ C^T\ \zeta + C\zeta = \theta.$$

Now, let us consider the small-signal equivalent circuit $\mathcal{N}_s$ of $\mathcal{N}$ at $(\underline{x},\underline{u})$. The Kirchhoff laws for $\mathcal{N}_s$ are expressed by

$$C(D\mathcal{F}(C^T\underline{x}+\underline{u}))\ C^T\ \Delta x + C\Delta x = \theta$$

where $\Delta x = (\Delta v_{vt}, \Delta i_{c\ell})$. However, we have just shown that conditions (U_ℓ) and (U_c) imply that $\Delta x = \theta$, hence $\zeta = \theta$ and we reach a contradiction.

V. Eventually Increasing Resistive Networks

Sandberg and Willson[6] have developed a technique whereby the existence of solutions can be asserted, given only the asymptotic behavior of the characteristics. Applying their technique, we have the following Corollary.

Corollary 1. (Existence)

Let $\mathcal{N}$ be a finite network made of eventually strictly increasing (resp. eventually increasing), continuous, and uncoupled resistors. Suppose $\mathcal{N}$ satisfies conditions (i) and (ii) of Theorem 1 (resp. conditions (i) and (ii) of Theorem 1 and conditions (U_ℓ) and (U_c) of Theorem 3). Then for all independent voltage sources with pliers entries and for all independent current sources with soldering-iron entries, the network $\mathcal{N}$ has at least one solution.

Proof: Since there is a $M > 0$ such that for $k = 1,2,\ldots r$, $f_k(z_k)$ is (strictly) increasing on $|z_k| > M$. Let us define

$$g_k(z_k) = f_k(z_k) \qquad |z_k| > M$$

$$g_k(z_k) = \frac{f(M) - f(-M)}{2}\,\frac{z_k}{M} + \frac{f(M) + f(-M)}{2} \qquad |z_k| \leq M.$$

Let
$$\mathcal{G}(z) = [g_1(z_1),\ldots g_r(z_r)]^T$$

$$U(x,u) = C\,\mathcal{G}(C^Tx+u) + Sx$$

$$V(x,u) = C[\mathcal{F}(C^Tx+u) - \mathcal{G}(C^Tx+u)]$$

Consider a given u. Let $z \triangleq C^Tx+u$, $z \in \mathbb{R}^r$. Note that $U(\cdot,u)$ is a homeomorphism from $\mathbb{R}^m$ onto $\mathbb{R}^m$, by Theorem 1 (resp. Theorem 3), and $V(\cdot,u)$: $\mathbb{R}^m \to \mathbb{R}^m$ is a continuous map. Now $\|V(x,u)\| \leq \|C\|\ \|\mathcal{F}(z) - \mathcal{G}(z)\|$. Since by construction, each component of $[\mathcal{F}(z) - \mathcal{G}(z)]$ satisfies for $k = 1,\ldots r$,

$$|f_k(z_k) - g_k(z_k)| \leq \max_{|z_k|\leq M} |f_k(z_k) - g_k(z_k)| \triangleq \alpha_k$$
$$\forall z_k \in \mathbb{R}$$

Hence $\|V(x,u)\| \leq \|C\|\cdot\|\alpha\|$, $\forall x \in \mathbb{R}^m$ where $\alpha = (\alpha_1\ldots\alpha_r)^T$. Since $U(\cdot,u)$ is a homeomorphism, by Global Inverse Function Theorem[9,10], given any $N > 0$

$\exists$ k such that $\|x\| > k \Rightarrow \|U(x,u)\| \geq N$. Set $N = 2\|C\|\cdot\|\alpha\|$, we have

$$\|V(x,u)\| \leq \frac{1}{2}\|U(x,u)\| \text{ whenever } \|x\| > k.$$

Then apply a theorem of Sandberg and Willson [App. I] the corollary is thus proved. ◻

VI. Boundedness

For resistive networks, the next basic question, besides the existence and uniqueness of solutions, is the dependence of solutions on the inputs. In the existence and uniqueness Theorems 1 and 3, the results state that the solutions depend continuously on the inputs. We would like to know the dependence of solutions on the inputs for the existence Theorem 2 and Corollary 1. Theorem 4 below asserts that in those cases bounded inputs produce bounded solutions. Not all resistive networks which have continuous characteristics and which have a solution for all inputs have this bounded-input bounded-solution property. Consider a c.c. resistor with the characteristic $v = i \sin^2 i$ connected to an independent voltage source e: for each $e \in \mathbb{R}$, there are infinitely many solutions larger than any prescribed number.

In Theorem 4, we do <u>not</u> require that resistor characteristics be increasing, nor even eventually increasing.

<u>Theorem 4</u>. Let $\mathcal{N}$ be a finite network made of continuous, and uncoupled resistors. Each resistor is required to be either of type U, or type H, or type B. Suppose that conditions (i) and (ii) of Theorem 1 hold. Suppose that for some independent voltage sources connected with pliers entries and for some independent current-sources with soldering-iron entries, the network has solutions. Under these conditions, if for some

$B_1 < \infty$, the inputs satisfy $|e_k| < B_1$ and $|j_k| < B_1$ for all k, then there exists a $B_2 < \infty$ such that all network solutions satisfy $|v_k| < B_2$ and $|i_k| < B_2$, for all k.[3]

Proof: Consider the network equation

$$G(x,u) = C\,\mathcal{F}(C^T x+u) + Sx = y.$$

An equivalent statement of the conclusion of the theorem is that if for some $\bar{B}_1 < \infty$, $\|(u,y)\| \leq \bar{B}_1$ then there exists a $\bar{B}_2 < \infty$ such that $\|x\| \leq \bar{B}_2$. Clearly this is true if $\|x\| \to \infty$ implies $\|(u,y)\| \to \infty$. We are going to show this by contradiction. Let $\{x^i\}$ be a sequence with $\|x^i\| \to \infty$, let $\{u^i\}$ and $\{y^i\}$ be two corresponding sequences such that $C\,\mathcal{F}(C^T x^i+u^i) + Sx^i = y^i$ is satisfied, and suppose $\{(u^i,y^i)\}$ is bounded; hence in particular $\{u^i\}$ is bounded. Lemma 2 states that if conditions (i) and (ii) of Theorem 1 hold then the assumption $\{x|C^T x \in \mathcal{B}(\mathcal{F})$ and $x \in \mathcal{N}(S)\} = \{\theta\}$, of Lemma 1 holds. Note that in part (2) of the proof of Lemma 1 we have shown that for any fixed u, $\|x\| \to \infty \Rightarrow \|G(x,u)\| = \|y\| \to \infty$. Observe that in that proof in fact we only require (i) u remains bounded, (2) each resistor is either of type U, or type H, or type B. Hence if $\|x^i\| \to \infty$ and $\|u^i\|$ remains bounded, then we must have $\|y^i\| \to \infty$. Thus a contradiction is reached. ∎

VII. Monotone RLC Networks

The natural framework for considering general nonlinear networks is provided by the differentiable manifold formulation.[11,12] However, in

[3] Here we use e_k (resp. j_k) to denote the magnitude of an independent voltage (resp. current) source; v_k (resp. i_k) to denote the branch-voltage (resp. branch-current) of a resistor.

many special cases, the manifold of configuration space is diffeomorphic to a linear vector space, such networks can then be characterized by a differential equation in normal form.

An RLC network can be considered as a connection of three one-element-kind subnetworks. Therefore, our results on resistive networks (in fact, on one-element-kind networks) leads directly to the Theorem 5 below, which considers increasing RLC networks. Let us first define the class of inductors and capacitors under consideration.

We define a capacitor (resp. inductor) as a two-terminal element that is characterized by a C^1 function f which maps the real line $\mathbb{R}$ into itself and $\sigma = f(\rho)$, where either ρ is the branch-voltage (resp. flux) and σ the stored charge (resp. branch-current), or vice versa. If ρ is the branch-voltage (resp. flux) we say that the capacitor (resp. inductor) is voltage-controlled (resp. flux-controlled), abbreviated v.c. (resp. ϕ.c.); on the other hand, if ρ is its stored charge (resp. branch-current), we say that it is charge-controlled (resp. current-controlled), abbreviated q.c. (resp. c.c.). We define increasing (resp. strictly increasing, eventually increasing, eventually strictly increasing, type U, type H, type B) capacitor or inductor according to its characteristic, as was done for resistors.

Theorem 5. (State equations for monotone RLC networks)

Let $\mathcal{N}$ be a finite network made of increasing, time-varying, uncoupled resistors, inductors, and capacitors. Thus all characteristics have the form $\sigma = f(\rho,t)$ and we assume that f is C^1 both in ρ and t. Let us derive from $\mathcal{N}$ three subnetworks:

$\mathcal{N}_L$ (inductive subnetwork): replace by short-circuits all elements, except inductors, of $\mathcal{N}$.

$\mathcal{N}_C$ (capacitive subnetwork): remove all elements, except capacitors, of $\mathcal{N}$.

$\mathcal{N}_R$ (resistive subnetwork): replace by short-circuits all capacitors and remove all inductors, of $\mathcal{N}$.

Suppose that in $\mathcal{N}_L$ (resp. $\mathcal{N}_C$, $\mathcal{N}_R$), the conditions (a)-(d) are satisfied:

(a) every loop[5] made of c.c. inductors (resp. q.c. capacitors, c.c. resistors) contains at least one which is strictly increasing;

(b) every cutset[6] made of ϕ.c. inductors (resp. v.c. capacitors, v.c. resistors) contains at least one which is strictly increasing;

(c) every loop made of c.c. inductors (resp. q.c. capacitors, c.c. resistors) either contains at least one type U inductor (resp. capacitor, resistor) or if not, then it contains at least two type H inductors (resp. capacitors, resistors) and not all such type H inductors (resp. capacitors, resistors) are similarly directed.

(d) every cutset made of ϕ.c. inductors (resp. v.c. capacitors, v.c. resistors) either contains at least one type U inductor (resp. capacitor, resistor) or if not, then it contains at least two type H inductors (resp. capacitors, resistors) and not all such type H inductors (resp. capacitors, resistors) are similarly directed. Suppose that independent sources

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[5] Self-loop is regarded as a loop.

[6] A cutset may contain only a single branch, in which case, we call it an "open branch".

are all regulated functions [13, p. 145] of time. Under these conditions, for all independent voltage sources with pliers entries and for all independent current sources with soldering-iron entries, and given any initial time t_0 and any initial conditions, the network $\mathcal{N}$ has one and only one solution on some nonvanishing interval $[t_0,t_\alpha)$.

Proof: First pick a normal tree and let the subscripts S,R,L (resp. C,G,Γ) correspond to link (resp. tree-branch) capacitors, resistors, and inductors; so that the fundamental loop matrix takes the form:

$$\begin{bmatrix} I & 0 & 0 & F_{SC} & 0 & 0 \\ 0 & I & 0 & F_{RC} & F_{RG} & 0 \\ 0 & 0 & I & F_{LC} & F_{LG} & F_{L\Gamma} \end{bmatrix}$$

Define a set of state variables as:

$$q = q_C - F_{SC}^T q_S$$

$$\phi = \phi_L + F_{L\Gamma} \phi_\Gamma$$

If the following three sets of equations (L), (C), and (R) possess unique solutions for i_L, v_c, i_R, and v_G in terms of q, ϕ, and t; i.e., $i_L = \tilde{i}_L(\phi,t)$, $v_c = \tilde{v}_c(q,t)$, $i_R = \tilde{i}_R(\tilde{v}_c(q,t), \tilde{i}_L(\phi,t), t)$, and $v_G = \tilde{v}_G(\tilde{v}_c(q,t), \tilde{i}_L(\phi,t), t)$, then the network $\mathcal{N}$ is characterized by differential equations in (q,ϕ), [See ref. 17, p. 61-65], namely

$$\dot{q} = F_{RC}^T \tilde{i}_R(\tilde{v}_C(q,t), \tilde{i}_L(\phi,t),t) + F_{LC}^T \tilde{i}_L(\phi,t) + j_C(t)$$

$$\dot{\phi} = - F_{LG} \tilde{v}_G(\tilde{v}_C(q,t), \tilde{i}_L(\phi,t),t) - F_{LC} \tilde{v}_C(q,t) + e_L(t).$$

$$\begin{cases} [I \;\; F_{L\Gamma}] \begin{bmatrix} \phi_L \\ \phi_\Gamma \end{bmatrix} = \phi \\ [-F_{L\Gamma}^T \;\; I] \begin{bmatrix} i_L \\ i_\Gamma \end{bmatrix} = j_L(t) \\ f_L(\phi_L,\phi_\Gamma,i_L,i_\Gamma,t) = \theta \end{cases} \qquad \text{(L)}$$

$$\begin{cases} [I \;\; F_{SC}] \begin{bmatrix} v_S \\ v_C \end{bmatrix} = e_S(t) \\ [-F_{SC}^T \;\; I] \begin{bmatrix} q_S \\ q_C \end{bmatrix} = q \\ f_C(v_S,v_C,q_S,q_C,t) = \theta \end{cases} \qquad \text{(C)}$$

$$\begin{cases} [I \quad F_{RG}] \begin{bmatrix} v_R \\ v_G \end{bmatrix} = F_{RC} v_C + e_R(t) \\ [-F_{RG}^T \quad I] \begin{bmatrix} i_R \\ i_G \end{bmatrix} = F_{RL}^T \, i_L + j_G(t) \\ f_R(v_R, v_G, i_R, i_G, t) = \theta \end{cases} \tag{R}$$

where we use f_L (resp. f_C, f_R) to denote inductor (resp. capacitor, resistor) characteristics. Note that if we consider the right hand sides as inputs, the first two sets of equations of (L) (resp. (C), (R)) are precisely KVL and KCL for $\mathcal{N}_L$ (resp $\mathcal{N}_C$, $\mathcal{N}_R$). Therefore, by the Remark following the proof of Theorem 3,[7] conditions (a)-(d) on $\mathcal{N}_L$ (resp $\mathcal{N}_C$) imply that i_L (resp v_c) is uniquely determined by $(\phi, j_L(t), t)$ (resp $(q, e_s(t), t)$)[8] and the dependence is C^1. Thus, $i_L = \tilde{i}_L(\phi,t)$ (resp $v_c = \tilde{v}_c(q,t)$), where $\tilde{i}_L$ (resp $\tilde{v}_c$) is C^1 in ϕ (resp q). Moreover, since $j_L(t)$ (resp $e_s(t)$) is a regulated function, $\tilde{i}_L$ (resp $\tilde{v}_c$) is a regulated function of t. Conditions (a)-(d) on $\mathcal{N}_R$ imply that $i_R = \tilde{i}_R(v_c, i_L, t)$ and $v_G = \tilde{v}_G(v_c, i_L, t)$ where $\tilde{i}_R$ and $\tilde{v}_G$ are C^1 in (v_c, i_L) and regulated in t. Hence $i_R = \tilde{i}_R(\tilde{v}_c(q,t), \tilde{i}_L(\phi,t), t) \triangleq \bar{i}_R(q,\phi,t)$ and $v_G = \tilde{v}_G(\tilde{v}_c(q,t), \tilde{i}_L(\phi,t), t) \triangleq \bar{v}_G(q,\phi,t)$ where $\bar{i}_R$ and $\bar{v}_G$ are C^1 in (q,ϕ) and regulated in t. The Theorem then follows from the Fundamental Theorem of differential euqations [13, pp. 285-289].

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[7] In the time-varying case, we will have $C\mathcal{F}(C^T x+u,t) + Sx = y$. By assumption $\mathcal{F}$ is C^1 in t. In applying Global Implicit Function Theorem, consider $G(x,u,t) : \mathbb{R}^m \times \mathbb{R}^{r+1} \to \mathbb{R}^m$.

[8] Note that the solutions exist even if ϕ.c. self-loops (resp. c.c. open-branches) of $\mathcal{N}_L$ (resp. $\mathcal{N}_C$) are not increasing; and v.c. self-loops and c.c. open-branches of $\mathcal{N}_R$ are not increasing.

Remark: In comparison with the result of Desoer and Katzenelson [2, Theorem IV], we note that their circuit-theoretic conditions are sufficient for conditions (a)-(d); we require that characteristics be differentiable, however.

VIII. Algorithms

We propose two efficient algorithms for checking conditions (i) and (ii) of Theorem 1. First, we present an immediate consequence of the Colored Arc Lemma.

Fact. Condition (i) (resp. (ii)) of Theorem 1 holds if and only if, after removing all v.c. and type U c.c. resistors (resp. replacing by short-circuits all c.c. and type U v.c. resistors),

(A) every loop (resp. cutset) contains at least one type H c.c. (resp. v.c.) resistor.

(B) for every type H c.c. (resp. v.c.) resistor b, there is a similarly-directed cutset (resp. loop) of type H c.c. (resp. v.c.) resitors, containing b.

Algorithm 1. (For checking condition (i))

Step 1) Remove all v.c. and type U c.c. resistors from $\mathcal{N}$.

Remove all "open branches".

Call the resultant network $\mathcal{N}_0$, set $i = 0$.

Step 2) If $\mathcal{N}_i$ has a type H c.c. resistor go to (4), else go to (3).

Step 3) If $\mathcal{N}_i$ has a loop, output: <u>condition (i) is not satisfied</u>.

(There is a loop in $\mathcal{N}$ of type B c.c. resistors)

else output: <u>condition (i) holds</u>.

(This conclusion follows in view of the foregoing Fact.)

Step 4) Pick a type H c.c. resistor b, directed from, say, node n_0 to node n_1.

Set $V_1 = \{n_1\}$, set $k = 1$.

Step 5) If there is a type H c.c. resistor directed from some node in V_k to a node t not in V_k, go to (6),

otherwise if there is a type B c.c. resistor connecting some node in V_k and a node t not in V_k, go to (6),

else go to (7).

Step 6) If $t = n_0$, output: <u>condition (i) is not satisfied</u>.

(There is a loop of type H and type B c.c. resistors, in which all type H resistors are similarly directed)

else set $V_{k+1} = V_k \cup \{t\}$, set $k = k+1$, go to (5).

Step 7) Remove all resistors which have only one terminal node in V_k.

Call the resultant network $\mathcal{N}_{i+1}$, set $i = i+1$, go to (2).

(There is a cutset, in $\mathcal{N}_0$, containing b, of similarly-directed type H c.c. resistors. We may remove them from further consideration).

<u>Algorithm 2</u>. (For checking condition (ii))

Step 1) Replace all c.c. resistors and type U v.c. resistors in $\mathcal{N}$ by short-circuits and identify any two nodes connected by a short-circuit.

Remove all self loops.

Call the resultant network $\mathcal{N}_0$, set $i = 0$.

Step 2) If $\mathcal{N}_i$ has a type H v.c. resistor, go to (4); else go to (3).

Step 3) If $\mathcal{N}_i$ has a type B v.c. resistor, output:condition (ii) is not satisfied.

(There is a cutset in $\mathcal{N}$ of type B v.c. resistors)

else output:condition (ii) holds.

(This conclusion follows in view of the foregoing Fact.)

Step 4) Pick a type H v.c. resistor b, directed from, say node n_0 to node n_i; set $k = 1$.

Step 5) If there is a type H v.c. resistor directed from node n_k to some node t, go to (6)

else output:condition (ii) is not satisfied.

(There is a type H v.c. resistor which is not in a loop, in $\mathcal{N}_0$, of similarly-directed type H v.c. resistors, this violates condition (B)).

Step 6) If $t \neq n_j$ for $0 \leq j < k$, set $n_{k+1} = t$, $k = k+1$ go to (5); else go to (7).

Step 7) Identify node n_j, n_{j+1}, ... n_k, t and remove all self-loops.

(There is a loop in $\mathcal{N}_0$ of similarly-directed type H v.c. resistors. We may disregard them from further consideration, i.e., shrink them down into a node).

If $j > 1$, set $k = j$ and go to (5);

else call the resultant network $\mathcal{N}_{i+1}$, set $i = i+1$, go to (2).

(The loop contains b, we have to start again).

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Appendix I

Palais Global Implicit Function Theorem[9,14]

Let $G\colon \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m$ be continuous (resp. C^k, $1 \leq k \leq \infty$). Given $G(x,u) = y$ there exists a unique continuous (resp. C^k) function $\Phi\colon \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m$ such that $G(\Phi(y,u),u) = y$ $\forall u \in \mathbb{R}^r$, $\forall y \in \mathbb{R}^m$ if and only if

(i) $G(\cdot,u)$ is a local homeomorphism (resp. local diffeomorphism[9]) from $\mathbb{R}^m$ onto $\mathbb{R}^m$, $\forall x \in \mathbb{R}^m$, $\forall u \in \mathbb{R}^r$.

(ii) $\forall$ fixed $u \in \mathbb{R}^r$, $\|G(x,u)\| \to \infty$ whenever $\|x\| \to \infty$.

Minty's Colored Arc Lemma [15,16]

Let $\mathcal{N}$ be a directed graph whose branches are partitioned into three sets (or colored with three colors) A, B, and C, and let $b \in B$. Then there exists one and only one of the following:

(i) There is a loop, containing b, of branches in A and B only, in which all branches of B are similarly directed.

(ii) There is a cutset, containing b, of branches in B and C only, in which all branches of B are similarly directed.

Sandberg and Willson's Theorem[6]

Let U be a homeomorphism from $\mathbb{R}^m$ onto $\mathbb{R}^m$, and let V be a continuous map from $\mathbb{R}^m$ into $\mathbb{R}^m$ with the property that there exist real numbers $0 < c < 1$ and $M > 0$ such that for all $\|x\| > M$,

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[9] By inverse function theorem, $G(\cdot,u)$ is a local diffeomorphism from $\mathbb{R}^m$ onto $\mathbb{R}^m$ $\forall x \in \mathbb{R}^m$, $\forall u \in \mathbb{R}^r$ if and only if $D_1G(x,u)$ is nonsingular $\forall x \in \mathbb{R}^m$, $\forall u \in \mathbb{R}^r$.

$$\|V(x)\| \leq c\|U(x)\|$$

Then for each $y \in \mathbb{R}^m$ there exists at least one $x \in \mathbb{R}^m$ such that $U(x) + V(x) = y$.

Appendix II

Lemma A1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous and increasing function. Then given any $\varepsilon > 0$ there exists a strictly increasing C^1 function $f^\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that

$$|f(\rho) - f^\varepsilon(\rho)| < \varepsilon \qquad \forall\, \rho \in \mathbb{R}$$

Proof: By construction. First assume that f is continuous and strictly increasing. Consider the compact interval $I_n = [n,n+1]$, n is an integer. f is uniformly continuous on I_n, hence $\exists\ \delta_n$ such that $|\rho_1 - \rho_2| < \delta_n \Rightarrow |f(\rho_1) - f(\rho_2)| < \frac{\varepsilon}{4}\ \forall\ \rho_1, \rho_2 \in I_n$. Without loss of generality, we may take δ_n to be the inverse of a positive integer. Now construct a piecewise linear function $h(\rho)$ on I_n such that $h(\rho) = f(\rho)$ for $\rho = n, n + \delta_n, \ldots n+1$, and linear between any two consecutive points. Repeating the construction for all n, we obtain a piecewise linear, strictly increasing function $h\colon \mathbb{R} \to \mathbb{R}$ and $|h(\rho) - f(\rho)| \leq \frac{\varepsilon}{4}\ \forall \rho \in \mathbb{R}$. Now we round off the corners of h by circular arcs with radius $1/(8\delta_n)$ for corners inside I_n and $(8 \min\{\delta_n,\delta_{n+1}\})^{-1}$ for corners at n, the result is a strictly increasing C^1 function f^ε such that $|f(\rho) - f^\varepsilon(\rho)| < \frac{\varepsilon}{2}\ \forall \rho \in \mathbb{R}$.

Suppose now that f is only increasing. Then the piecewise linear approximation h constructed above may have line segments with zero slope. So it needs modification to make h a piecewise linear, strictly increasing function h_1.

case 1) $h(\rho) = c$ on $[\alpha,\beta]$. Let ρ_1, ρ_2 be the points where $f(\rho_1) = c - \frac{\varepsilon}{4}$ and $f(\rho_2) = c + \frac{\varepsilon}{4}$.[10] Let $h_1(\rho)$ on $[\rho_1,\rho_2]$ be the straight line connecting $(\rho_1, c - \frac{\varepsilon}{4})$ and $(\rho_2, c + \frac{\varepsilon}{4})$.

[10] If such points can not be found, reduce $\frac{\varepsilon}{4}$ to $\frac{\varepsilon}{8}$, and so on.

case 2) $h(\rho) = c$ on $[\alpha,\infty)$. Let ρ_1 be the point where $h(\rho_1) = c - \frac{\varepsilon}{4}$. Let $h_1(\rho)$ on $[\rho_1,\infty)$ to be $h_1(\rho) = c - \frac{\varepsilon}{4} e^{-\lambda(\rho-\rho_1)}$, where $\lambda = \frac{4}{\varepsilon} h'(\rho_1)$.

case 3) $h(\rho) = c$ on $(-\infty,\infty)$. Let $h_1(\rho) = \frac{\varepsilon}{4} \tanh \rho + c$.

Clearly $|h_1(\rho) - f(\rho)| < \frac{\varepsilon}{2}$ $\forall \rho \in \mathbb{R}$. Round off corners of h_1, we obtain f^ε and $|f^\varepsilon(\rho) - f(\rho)| < \varepsilon$ $\forall \rho \in \mathbb{R}$.

Lemma A2. Let $\{\mathcal{F}_k\}$ be a sequence of homeomorphisms from $\mathbb{R}^n$ onto $\mathbb{R}^n$, and $\mathcal{F}$: $\mathbb{R}^n \to \mathbb{R}^n$ is continuous. If for a given $\varepsilon > 0$, $\|\mathcal{F}_k(x) - \mathcal{F}(x)\| < \frac{\varepsilon}{k}$, $\forall x \in \mathbb{R}^n$, for $k = 1,2,\cdots$. Then $\mathcal{F}$ is a surjective map and the inverse image under $\mathcal{F}$ of any bounded set is a bounded set.

Proof: (1) First we show that $\mathcal{F}^{-1}(B)$ is bounded whenever $B \subseteq \mathbb{R}^n$ is bounded. Suppose not, then there exists an unbounded sequence $\{\xi_i\} \to \infty$ and $\|\mathcal{F}(\xi_i)\| < M$ for $i = 1,2,\cdots$. Since $\|\mathcal{F}_k(\xi_i) - \mathcal{F}(\xi_i)\| < \frac{\varepsilon}{k}$ $\forall i$. $\|\mathcal{F}_k(\xi_i)\| < \|\mathcal{F}(\xi_i)\| + \frac{\varepsilon}{k} < M + \frac{\varepsilon}{k}$. But $\mathcal{F}_k$ is a homeomorphism, from Global Inverse Function Theorem, $\|\xi_i\| \to \infty \Rightarrow \|\mathcal{F}_k(\xi_i)\| \to \infty$. We reach contradiction.

(2) We will show that given any $y \in \mathbb{R}^n$, there exists an $x \in \mathbb{R}^n$ such that $\mathcal{F}(x) = y$. Let x_k be the points satisfying $\mathcal{F}_k(x_k) = y$. Consider the sequence $\{x_k\}$. We claim that it is bounded. Since for all positive integer k, $\|\mathcal{F}(x_k) - \mathcal{F}_k(x_k)\| < \frac{\varepsilon}{k}$, so $\|\mathcal{F}(x_k)\| < \|y\| + \frac{\varepsilon}{k}$. Hence $\{x_k\}$ is in the inverse image under $\mathcal{F}$ of a bounded set, thus it is bounded by (1). Therefore, $\{x_k\}$ has a convergent subsequence, $\{x_{k_i}\}$, say converging to $\bar{x}$. Now we claim $f(\bar{x}) = y$. Indeed,

$$\|y - \mathcal{F}(\bar{x})\| = \|\mathcal{F}_{k_i}(x_{k_i}) - \mathcal{F}(\bar{x})\|$$

$$\leq \|\mathcal{F}_{k_i}(x_{k_i}) - \mathcal{F}(x_{k_i})\| + \|\mathcal{F}(x_{k_i}) - \mathcal{F}(\bar{x})\|$$

$$\leq \frac{\varepsilon}{k_i} + \|\mathcal{F}(x_{k_i}) - \mathcal{F}(\bar{x})\|$$

Since $\mathcal{F}$ is continuous, and $x_{k_i} \to \bar{x}$ as $k_i \to \infty$. The right-hand side can be made as small as we please by picking k_i large enough, hence $\|y - \mathcal{F}(\bar{x})\| = 0$, i.e., $y = \mathcal{F}(\bar{x})$. ¤

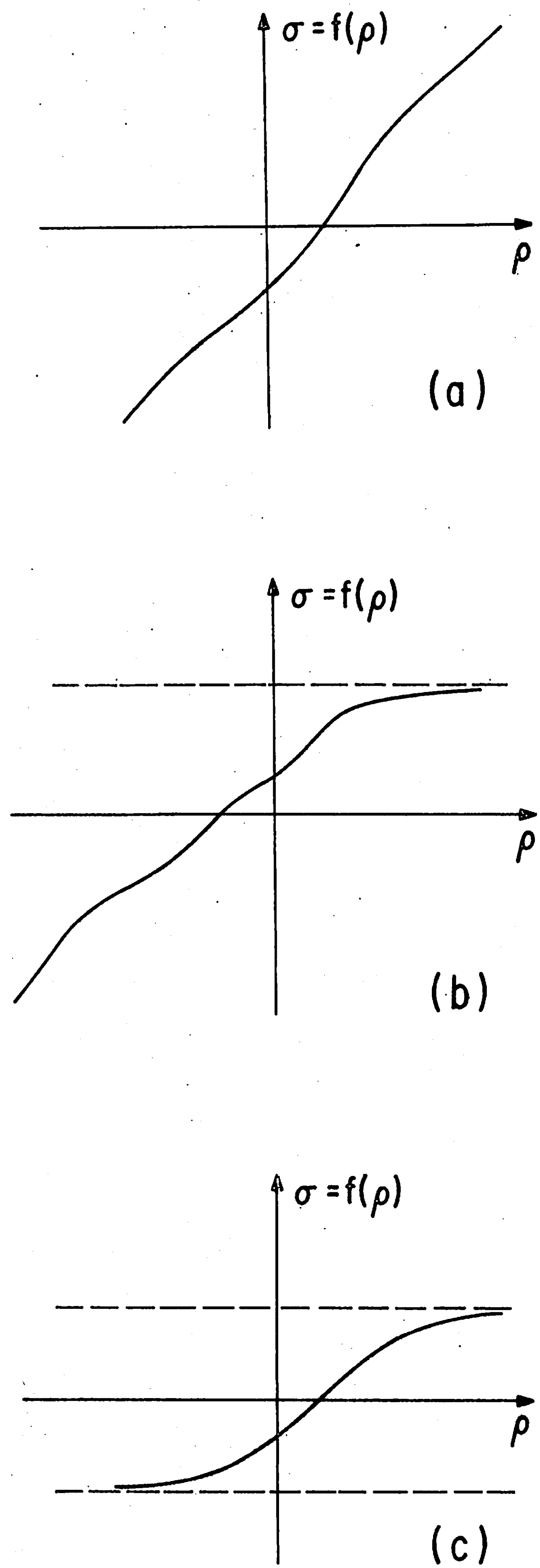
σ = f(ρ)
ρ
(a)
σ = f(ρ)
ρ
(b)
σ = f(ρ)
ρ
(c)

FIGURE CAPTIONS

(a) Type U resistor: $f(\rho)$ unbounded as ρ grows without bound.

(b) Type H resistor: $f(\rho)$ bounded on a half real line.

(c) Type B resistor: $f(\rho)$ bounded on the whole real line.