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STUDIES IN GRAPH THEORY: 1. PARTIAL SQUARES OF TREES

2. FACTORIZATION OF GRAPHS

by

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Abstract

In this thesis three problems are considered -- the first two in Part I and the third one in Part II. The first problem we consider is the characterization of partial squares of trees. A uniqueness theorem of square roots is also proved with application to total graphs. Secondly, those line graphs which are clique graphs or partial tree squares are characterized. These special clique graphs are shown to have other interesting properties. Finally we have proved a generalized form of the following conjecture on factors: If $\langle d_i \rangle$ and $\langle d_i - k \rangle$, $1 \leq i \leq n$, are graphical sequences then there exists a graph with degree sequence $\langle d_i \rangle$ and containing a factor F which has k edges at each vertex. A similar conjecture for digraphs is also generalized and proved.

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INTRODUCTION

It is well known that an even (binary) matroid is graphic if and only if it has a 2-complete family of cocircuits. A natural question would be, "What graphs have a 2-complete family of fundamental cocircuits?" A family of fundamental cocircuits (cutsets) is defined as follows. Consider a basis B of the graphic matroid $M(G)$. Each edge e of G belonging to B defines a unique cutset $c(e)$ which intersects B at the element e alone. The cutsets $\{c(e): e \in B\}$ define what is known as a family of fundamental cutsets (with respect to B). It is observed that a graph G has a 2-complete family of fundamental cutsets if and only if G is a partial square of a forest.

In Chapter 1 we shall obtain a characterization of partial squares of trees. We also show that if a partial tree square is 'large' (in a suitable sense) with respect to the tree then the tree is determined uniquely from the partial square. In particular, the square of a tree T determines T uniquely. Harary [5] has obtained a characterization of graphs that are tree squares. (There is a mistake in his proof; although the theorem is possibly true. A counterexample seems almost impossible.) Our characterization of a tree square has a form similar to the one given (?) by Harary. Mukhopadhyay [8] has characterized squares of arbitrary graphs. His characterization requires existence of a family of complete subgraphs which satisfy certain intersection properties. It is not easy to use Mukhopadhaya's theorem for an efficient method to find if a given graph is a square graph because one has to examine all possible choices of complete subgraphs which are just too many in number.

Our characterization of partial squares of trees, and its generalizations to squares of graphs with girth > 6, do not suffer from such shortcomings. The characterizations involve only the 3-components and cliques of the graph.

As a special case we have shown that the total graph $T(G)$ of a graph G determines the graph G uniquely. This is similar to a well known result on line graphs which says that G is determined uniquely (with one exception) from its line graph. We note in passing that a planar partial tree square is 4-colorable.

In Chapter 2 explicit characterizations are obtained for those line graphs which are clique graphs, partial tree squares or both. Such a characterization was felt necessary because every partial tree square was observed to be a clique graph on the one hand and on the other hand the characterizations of line graphs, clique graphs and partial squares of trees all happen to have a very similar form -- the existence of a family of maximal subsets of vertices (3-components, cliques, etc.) with certain intersection properties.

In Part II we consider the problem of existence of graphs G with given degrees of the vertices and G containing a factor which is also specified by a degree sequence. Following generalized form of the k-factor conjecture (by A.R. Rao, S.B. Rao and B. Grunbaum) is proved.

If the sequences $\langle d_i \rangle$, $\langle d_i - k_i \rangle$, $i = 1,2,\ldots,n$ of non-negative integers d_i and d_i-k_i are graphical and $\langle k_i \rangle$ is almost regular (i.e., for some k, $k \leq k_i \leq k+1$, for all i) then there exists a graph G with degree sequence $\langle d_i \rangle$ whose i^{th} vertex has degree k_i in a factor $F \subseteq G$.

A corresponding theorem for directed graphs is also proved generalizing a conjecture by A.R. Rao and S.B. Rao. Examples show that our theorems are best possible in general. Several other variations are also considered.

Part I and Part II of the thesis can be read independently. The numbering of figures and theorems are independent in the two parts. The references for Part I appear at the end of Chapter 2. The references include only those works which are of primary importance.

Note to the reader. It is strongly urged that the reader make constant use of figures whenever possible. There are several little facts which become clear once a figure is drawn and are used in the proofs. We believe that giving formal explanations of all of them would make the reading of the thesis unpleasant.

PART I

Chapter 1

PARTIAL SQUARES OF TREES

1. Definitions and Introduction

A graph $G = (V(G), E(G))$ consists of a finite set of vertices $V(G)$ and a set $E(G)$ of unordered pairs (v_i, v_j), $v_i \neq v_j$, called edges of G. The square of G, denoted by G^2, has the vertex set $V(G)$ and (x,y) belongs to $E(G^2)$ if (x,y) is in $E(G)$ or there is a vertex z such that $(x,z),(z,y)$ are in $E(G)$. G is called a square root of G^2. The distance $d_G(x,y)$ between two vertices x,y is the length of a shortest path in G from x to y. Thus we can write $E(G^2) = \{(x,y): 1 \leq d_G(x,y) \leq 2\}$. The square of a disconnected graph is clearly the disjoint union of the squares of the components. We shall assume that all graphs are connected in the following and further, to avoid trivialities, $|V(G)| \geq 3$, unless otherwise mentioned. By $H \subseteq G$ we mean $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and say that H is a partial subgraph of G. If every edge of G having both end points in $V(H)$ is an edge of H then H is called an (induced) subgraph. We shall write in that case $H = G|V(H)$. Also $G-S$, $S \subseteq V(G)$ will denote the induced subgraph $G|V(G)-S$. A partial square H of G is defined by $G \subseteq H \subseteq G^2$. Note that the graphs G and H have same vertices. The complete graph K_n has n vertices and $\frac{n(n-1)}{2}$ edges so that any two vertices are adjacent. A graph G is called n-connected if it has at least n vertices and for any set of $n-1$ or less vertices S, $G-S$ is connected. This definition is slightly different from the one usually given. However, they coincide if G has more than n vertices. The only n connected graph on n vertices is the complete graph K_n. We say that a subset of vertices S is complete, n-connected (in general, P) if the graph $G|S$ is respectively complete, n-connected (has property P). A maximal complete set S is called a clique and a maximal n-connected set an n-component. We shall often identify the set S with the graph $G|S$ in case no confusion is likely. For general concepts in graph theory we refer the reader to Harary [5]. Few other definitions not included in the text are collected at the end of Part I. A cutset in a graph is a minimal set of edges removal of which from the graph increases the number of components.

The concept of a complete family of cutsets of a graph corresponds to that of a basis in a vector space. More precisely, a family $\mathcal{C}$ of cutsets is called complete if any cutset is a mod 2 sum of cutsets in $\mathcal{C}$ and $\mathcal{C}$ is minimal with respect to this property. $\mathcal{C}$ is called 2-complete if no edge belongs to more than two of the cutsets $\mathcal{C}$. These definitions can be made actually for an arbitrary matroid [10]. A family of fundamental cutsets (cocircuits) of an even matroid is complete.

THEOREM 1.1. A connected graph G has a 2-complete family of fundamental cutsets if and only if there is a tree T such that $T \subseteq G \subseteq T^2$.

Proof. Suppose G is a partial square of the tree T. The edges $E(T)$ form a basis of the graphic matroid on $E(G)$. An edge $e = (x,y)$ of T defines the cutset $c(e)$ as follows. $T - e$ has two components and the edges of G whose end points belong to different components constitutes the cutset $c(e)$. Clearly, the family $\{c(e): e \in E(T)\}$ is 2-complete. Conversely, suppose the family of fundamental cutsets with respect to the basis $B = E(T)$ is 2-complete. It is easy to see that for an edge (x,y) of G which is not in T, $d_T(x,y) = 2$. Thus G is a partial square of T.

A disconnected graph has a 2-complete family of fundamental cutsets if every component* of the graphic matroid $M(G)$ does. A component of $M(G)$ is nothing but a 2-component subgraph of G. Thus one can further assume that the graph G is 2-connected. To obtain a spanning tree T of G such that $T \subseteq G \subseteq T^2$ we shall first find corresponding trees for the 2-components of G and then 'attach' them in the same way as the 2-components themselves are attached. We state that as

<u>THEOREM 1.2</u>. A graph G is a partial tree square if and only if every 2-component of G is a partial tree square.

It is interesting to note the following equivalent conditions for a partial tree square to be 2-connected.

<u>PROPOSITION 1.3</u>. Let $T \subseteq G \subseteq T^2$. The following are equivalent.

i) G is 2-connected.

ii) $G|\Gamma_T(x)$ is connected for each vertex x.

iii) $G - T$ has two components.

iv) $G|N_G(x)$ is 2-connected.

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*Connected component -- i.e., an irreducible part of the graphic matroid in the sense of Whitney [12].

Proof. i) $\Leftrightarrow$ ii). x is an articulation vertex of G if and only if $G|\Gamma_T(x)$ is not connected.

ii) $\Rightarrow$ iii). $G - T$ has two components on the two color classes of the graph T. The components are obtained by 'attaching' the connected graphs $(G-T)|\Gamma_T(x)$ as x varies in each of the two parts.

iii) $\Rightarrow$ iv). If $G - T$ has two components then it is not hard to see that $\Gamma_T(x)$ is connected in G for all x. In other words $G|N_T(x)$ is 2-connected. If y is a vertex in $N_G(x) - N_T(x)$ then there is a vertex z in $\Gamma_T(x)$ such that $(z,y) \in E(T) \subseteq E(G)$. It follows that $N_G(x)$ is 2-connected in G.

iv) $\Rightarrow$ ii). This is easily shown by contradiction.

Condition iv) gives a simple method for obtaining the 2-components of G. The 2-components of $G|N_G(x)$ are contained in different 2-components of G. However such a property of a graph G does not imply that G is a partial tree square; for example, let G be the 'big' triangle (fig.11).

2. Characterization of Partial Squares of Trees

We proceed to obtain a n.s.c. for a graph to be a partial tree square. We shall assume that G is 2-connected.

LEMMA 2.1. The 3-components of T^2 are of the form $N_T(x_i)$, $d_T(x_i) \geq 2$. In particular, 3-components of a partial square of T are subsets of $N_T(x_i)$ containing x_i.

Proof. Easy.

It is important to remember that two 3-components in a graph can intersect at most at two vertices. A vertex x is called a <u>multiple point</u> if it belongs to 2 or more 3-components. Let G be a partial square of T and let the 3-components contained in $N_T(x_i)$ be labeled by C_{ij}. Then the following properties i), ii), iii) are easily verified where V_o is the set of nonterminal vertices of T.

i) $\cup C_{ij} = G$, i.e., $\cup V(C_{ij}) = V(G)$ and $\cup E(C_{ij}) = E(G)$.

ii) If C_{ij}, $C_{i'j'}$ intersect at one point, then there exists a sequence $s = \langle C_{ij}, C_{i''j_1}, \ldots, C_{i''j_k}, C_{i'j'} \rangle$ such that two consecutive C's in s meet at two vertices.* Moreover, $i = i''$ if $i = i'$.

iii) There exists a subset $V_o = \{x_i\colon 1 \le i \le n\}$ of multiple points such that a) $x_i \in C_{ij} \subseteq N_G(x_i)$ and $x_i \in C_{i'j'} \Rightarrow x_{i'} \in C_{ij}$ for some j, b) if $|C_{ij} \cap C_{i'j'}| = 2$ and $i \neq i'$ then the common points are in V_o.

Next we show that these three conditions are also sufficient.

<u>THEOREM 2.2</u>. If there exists a labeling $\{C_{ij}\}$ of the 3-components of G and a labeling $\{x_i\}$ for a subset V_o of multiple points such that the properties i), ii), iii) above hold, then G is a partial square of a tree.

<u>Proof</u>. Define the graph T by $V(T) = V(G)$, $E(T) = \{(y,x_i)\colon y \in C_{ij},\ y \neq x_i\}$. Certainly, $T \subseteq G \subseteq T^2$. We have to show that T is a tree, i.e., T has no cycle and T is connected. The

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*We say that the sequence $s = \langle s_o, s_1, \ldots, s_{k+1} \rangle$ joins the terms $s_o = C_{ij}$, $s_{k+1} = C_{i'j'}$ at the two ends.

proof is broken into three parts $1^0) - 3^0)$.

$1^0)$ First we show that if $i \neq i'$ and $|C_{ij} \cap C_{i'j'}| = 2$ then the points in the intersection are $x_i, x_{i'}$.

<u>Proof</u>. There are two cases to consider: $x_i \in C_{i'j'}$ and $x_i \notin C_{i'j'}$. Suppose $x_i \in C_{i'j'}$. By iii), there exists a j'' such that $x_{i'} \in C_{ij''}$. Without loss of generality let $j'' \neq j$. Let s be a sequence joining C_{ij}, $C_{ij''}$:

$$s = \langle C_{ij}, C_{ij_1}, \ldots, C_{ij_k}, C_{ij''} \rangle \quad .$$

(s is possibly of length 2). See fig. 1. Also let y be the other vertex in $C_{ij} \cap C_{i'j'}$. Since $(x_{i'}, y) \in E(G)$, $x_{i'} \in C_{ij''}$, $y \in C_{ij}$ we have $\cup s$ (= union of the sets in s) is 3-connected. This is a contradiction. Thus we consider the other case $x_i \notin C_{i'j'}$ and $x_{i'} \notin C_{ij}$. Let $C_{ij} \cap C_{i'j'} = \{x_k, x_m\}$.

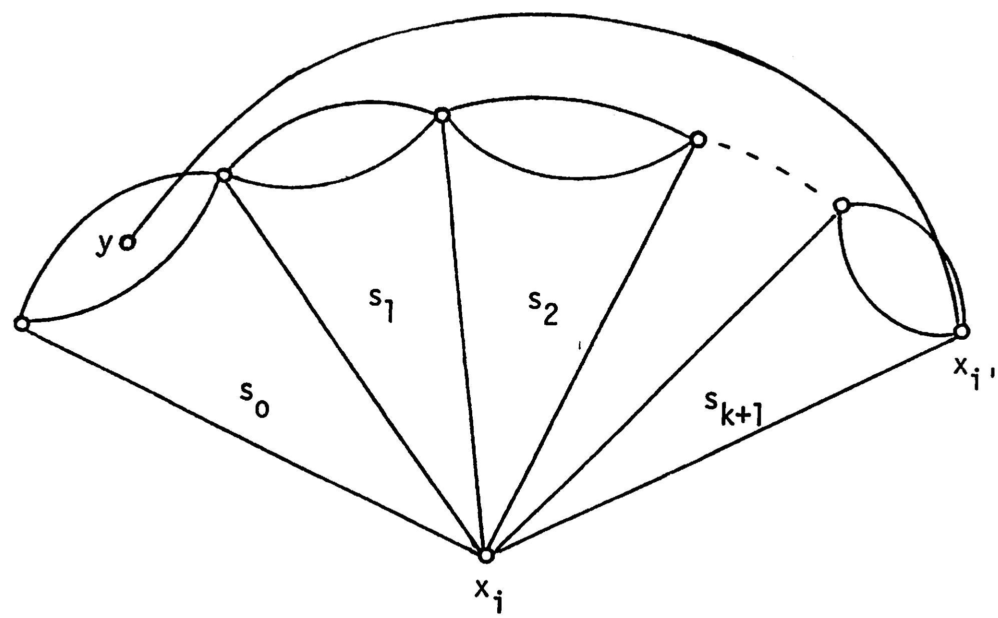


Fig. 1. The sequence of 3-components in $s = \langle s_0 = C_{ij}, s_1 = C_{ij_1}, \ldots, s_{k+1} = C_{ij''} \rangle$. Each egg shaped region is 2-connected and x_i is adjacent to all the vertices in that region.

We shall then have (x_i,x_k), (x_i,x_m), (x_i',x_k), (x_i',x_m) edges in T which form a cycle. As we show in 2^0 this is impossible. Thus $C_{ij} \cap C_{i'j'} = \{x_i,x_{i'}\}$.

2^0) We show that there does not exist a cycle in $T|V_o$.

<u>Proof</u> (by contradiction). Let $\xi = (x_1,x_2,\ldots,x_k,x_1)$ be a smallest cycle in $T|V_o$. $(x_i,x_{i+1}) \in E(T)$ implies, say, $x_{i+1} \in C_{ij_i}$. There is $C_{i+1,j_i'}$ containing x_i and then $\{x_i,x_{i+1}\} = C_{ij_i} \cap C_{i+1,j_i'}$. Let j_i, j_i' be chosen for all $1 \leq i \leq k$ (where $k+1$ is identified with $1 \bmod k$). Let s^i be a sequence joining $C_{ij_{i-1}'}$ and C_{ij_i} for $2 \leq i \leq k$ and s^1 joining $C_{1j_k'}$, C_{1j_1}. Also assume that the sequences s^i, $1 \leq i \leq k$ are so chosen that the total length of all s^i is minimum. Thus x_{i+1} belongs only to the last term in s^i. We show that $S = \cup(\cup s^i)$, union over $1 \leq i \leq k$, is 3-connected, a contradiction. Clearly, $x_j \notin \cup s^i$ unless x_i, x_j are consecutive vertices in ξ.

Choose two vertices x, y in S. If at most one of them is in cycle ξ we leave it to the reader to verify that $\underline{S} = S - \{x,y\}$ is connected; let us assume $x = x_1$, $y = x_t$. First let $t = 2$. $\underline{\cup s^1}$ is connected containing x_k and $\underline{\cup s^2}$ is connected containing x_3. Thus $\underline{S}$ is connected. Next, if $2 < t < k$ then $\underline{\cup s^1}$ (resp. $\underline{\cup s^t}$) is connected containing x_k, x_2 (resp. x_{t-1}, x_{t+1}) and $\underline{S}$ contains the paths $(x_2,x_3,\ldots,x_{t-1})$ and $(x_{t+1},x_{t+2},\ldots,x_k)$. Hence $\underline{S}$ is connected once again. This completes the proof.

It follows from 2^0 that, in fact, T has no cycle. If x is a multiple point and $x \notin V_o$ then necessarily $d_T(x) = 1$. Otherwise there are C_{ij}, $C_{i'j'}$ $(i \neq i')$ containing x and they must intersect at x only. Let i'' be as in condition (ii). Then it follows $x_{i''} \in C_{ij} \cap C_{i'j'}$, which means $x = x'' \in V_o$, a contradiction. A nonmultiple point of G has degree one in T. Thus any cycle of T is in fact a cycle of $T|V_o$ and there is none in the latter.

3^0) Finally, it remains to show that T is connected.

__Proof__. It is enough to show that $T|V_o$ is connected. Let $T|\{x_1,\dots,x_k\}$, $k < n$ be a component. Define the partial graph H by $E(H) = \bigcup_{i\leq k} E(C_{ij})$, $V(H) = \bigcup_{i\leq k} V(C_{ij})$. $x_i \notin V(H)$ for $i > k$. In particular, $V(H) \neq V(G)$. Choose an edge (x,y) of G such that only x belongs to $V(H)$. There exists $i \leq k$, $i' > k$ such that $x \in C_{ij}$, $(x,y) \in E(C_{i'j'})$. Since C_{ij}, $C_{i'j'}$ meet at one point, namely x, we can show as before that $x = x_{i''}$ and in fact $(x_i,x_{i''})$, $(x_{i''},x_{i'})$ are in $E(T)$ contradicting that $T|\{x_1,x_2,\dots,x_k\}$ is a component. Thus $k = n$ and T is a tree. The proof of the theorem is complete.

A particularly interesting special case of (2.2) is when there is just one 3-component C_i corresponding to each vertex $x_i \in V_o$. If G is a partial square of T and G has sufficiently many edges in the sense that $G|\Gamma_T(x_i)$ is 2-connected for all nonterminal vertices x_i of the tree T then we observe that $C_i = N_T(x_i)$ is a 3-component of G and V_o is precisely the set of multiple points of G. Conditions i), ii), iii) can now be rewritten as

<u>i</u>) $\cup C_i = G, \quad 1 \leq i \leq n$.

<u>ii</u>) If C_i, C_j intersect at one point then there exists C_k sharing two points with each of them.

<u>iii</u>) There is a 1-1 correspondence between the 3-components and the multiple points V_o, $x_i \leftrightarrow C_i$ such that $x_i \in C_i \subseteq N_G(x_i)$ and $x_i \in C_j \Leftrightarrow x_j \in C_i$.

These conditions are also sufficient for a graph G to be a partial square of a tree T such that $G|\Gamma_T(x)$ is 2-connected. This is our next theorem.

<u>THEOREM 2.3</u>. For G to be a partial square of a tree T such that $G|\Gamma_T(x)$ is 2-connected for all vertices x, $d_T(x) > 1$ it is both necessary and sufficient that <u>i</u>), <u>ii</u>), <u>iii</u>) above hold.

<u>Proof</u>. We need to prove sufficiency. Let T be the tree as constructed in (2.2). The edges of T are $E(T) = \{(y,x_i): y \in C_i, y \neq x_i\}$. V_o is the set of nonterminal vertices of T. Since C_i is 3-connected, $\Gamma_T(x_i) = C_i - x_i$ is 2-connected in G.

Yet another special case of (2.2) gives us a characterization of tree squares. If $G = T^2$ then each of the 3-components C_i is a clique of G. Thus the necessary part of the following corollary is easy.

<u>COROLLARY 2.4</u>. A graph G is a tree square if and only if there exists a 1-1 correspondence, $x_i \leftrightarrow C_i$, between the cliques $\{C_i\}$

and vertices $V_o = \{x_i\}$ belonging to two or more cliques, such that a), b) and c) or c') are true.

a) $|C_i \cap C_j| \leq 2$ and $|C_i \cap C_j| = 1$ implies there is a C_k sharing two points with each of them.

b) $x_i \in C_i$ and $x_i \in C_j \Leftrightarrow x_j \in C_i$.

c) C_i's are the 3-components of G, or

c') There are $|V_o|-1$ pairs of C_i which meet at two vertices.

Proof. In proving the sufficiency we need to consider only the combination a), b) and c'). We prove that $|C_i \cap C_j| = 2$ implies $x_i \in C_j$, i.e., $C_i \cap C_j = \{x_i, x_j\}$. If not, then $C_i \cap C_j = \{x_k, x_m\}$ and $C_k \cap C_m = \{x_i, x_j\}$ for some distinct i, j, k, m. Let C be a clique containing x_i, x_j, x_k, x_m. Then $|C_i \cap C| \geq 3$, a contradiction. Thus $C_i \cap C_j = \{x_i, x_j\}$. As one would expect we shall define a graph T by $E(T) = \{(y, x_i): y \in C_i, y \neq x_i\}$. It is not hard to prove that $T|V_o$ is connected. Condition c') then implies that $T|V_o$ has no cycle. It follows that T is a tree. G is clearly the square of T.

3. Uniqueness Theorems

Define the core of a tree T, $c(T)$, as the induced subgraph of T on the nonterminal vertices $\{x: d_T(x) \geq 2\}$. Since T is assumed to have at least three vertices the core is a tree (i.e., has at least one vertex). We prove that if a partial square of T satisfies the three conditions in (2.3) then tree T is determined uniquely except when $c(T)$ has at most two vertices. We shall let $t(x)$ denote the terminal vertices of T that are in $\Gamma_T(x)$.

THEOREM 3.1. Let G be a partial square of a tree. Suppose there do not exist two vertices in G which are adjacent to every other vertex. Then there is a unique tree T such that G is a partial square of T and $G|\Gamma_T(x)$ is 2-connected for vertices in $c(T)$.

Proof. The number of nonterminal vertices of T is the same as that of the 3-components of G. If $|V_o| \geq 2$, then (x_i,x_j) is an edge of $c(T)$ if and only if $x_i \in C_j$ or equivalently, there exists two 3-components (C_i,C_j) which meet at x_i, x_j. Suppose $c(T) = \{x_1,x_2\}$. If each of x_1, x_2 is adjacent to all other vertices of G then interchanging the labels of C_1, C_2 we shall obtain a different correspondence $x_i \leftrightarrow C_i$ and thus a different tree T_1 by the construction in (2.3). $T_1 \subseteq G \subseteq T_1^2$. T and T_1 are isomorphic (obtained by interchanging $t(x_1)$, $t(x_2)$). If G is 3-connected then T is a star. In these two cases we can determine tree T only up to isomorphism. (There is an isomorphism of order two between two such trees.) Now suppose that $|c(T)| \geq 3$. We show that the correspondence in 2.3(iii) is unique. For example let $x_{\sigma i} \leftrightarrow C_i$ be another such correspondence obtained from a tree T_1 (of which G is a partial square as in (3.1)) where σ is a permutation of $\{1,2,\ldots,n\}$. Let (x_1,x_2) be an edge in $c(T)$. Then $|C_1 \cap C_2| = 2$ implies that the intersection is equal to each of $\{x_1,x_2\}$ and $\{x_{\sigma 1},x_{\sigma 2}\}$. If (x_2,x_3) is another edge in $c(T)$ then $\{x_2,x_3\} = \{x_{\sigma 2},x_{\sigma 3}\}$. It follows that

$\sigma 2 = 2$ and thus $\sigma 1 = 1$, $\sigma 3 = 3$. One can repeat the argument and conclude that σ is identity. Hence $x_i \leftrightarrow C_i$ is unique. This means that $c(T)$ is identical with $c(T_1)$. The terminal vertices $t(x_i)$ are precisely the nonmultiple points in C_i. Thus $T = T_1$.

If a partial square of a tree does not satisfy the hypothesis of Theorem 3.1, then the tree T constructed in (2.2) is far from unique because there will be, in general, several many to one correspondence $c_{ij} \to x_i$. See Example 1.

<u>COROLLARY 3.2.</u> If G is a tree square and G has three or more cliques then there is a unique tree T such that $G = T^2$.

Before we close this section we would like to present some examples.

<u>Example 1</u>. Consider the tree T which is a path of length five, $(x_1,x_2,x_3,x_4,x_5,x_6)$. The edges in $E(T^2) - E(T)$ are shown by broken lines (Fig. 2).

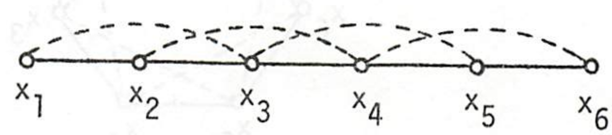


Fig. 2. A tree T and its square.

A correspondence $C_i \leftrightarrow x_i$ satisfying Harary's theorem* can be $\{x_1,x_2,x_3\} = C_2$, $\{x_2,x_3,x_4\} = C_4$, $\{x_3,x_4,x_5\} = C_3$, $\{x_4,x_5,x_6\} = C_5$. The tree, as constructed by Harary, is the path

*See Appendix.

$(x_1,x_2,x_4,x_3,x_5,x_6)$ which gives a different square than T^2. The tree square in Fig. 2 can be regarded also as a partial square of the following tree T'. However, $G|\Gamma_{T'}(x_3)$ is not biconnected. Thus it does not contradict our Theorem 3.1.

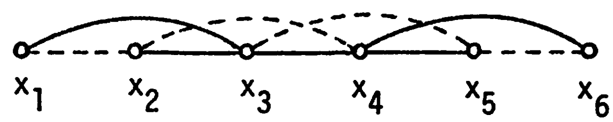


Fig. 3. Tree T' consisting of solid lines.

Example 2. Consider the graph G shown in solid lines (Fig. 4). The cliques of G^2 can be labeled as $C_i = \{x_{i-1},x_i,x_{i+1}\}$, $1 \leq i \leq 7$ where zero and 8, respectively, mean 7 and 1 (taken mod 7). Then the correspondence $C_i \leftrightarrow x_i$ satisfies conditions in Theorem II in [8].* But G^2 is not a tree square to be sure.

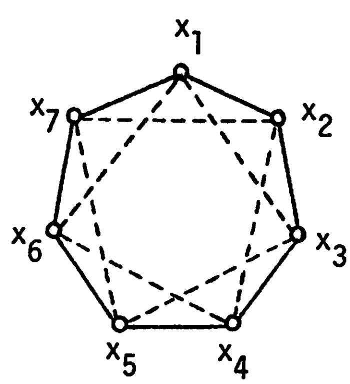


Fig. 4. Graph G (a 7-cycle) and G^2.

We shall see in section 4 that much of the theory developed in section 2 and section 3 will generalize for squares of graphs whose girth is at least 7.

*See Appendix.

4. Generalizations to Graphs of Girth > 6

We shall now generalize theorems in sections 2, 3 to squares of graphs which have finite girth > 6. (A tree can be regarded as a graph of infinite girth.) The reason that we can do so is that the cliques of the square of such a graph H are once again of the form $N_H(x)$. One of our theorems in this section will be the determination of H from its square uniquely which is a generalization of Corollary 3.2.

Throughout this section H will denote a graph of finite girth > 6 and we shall make repeated implicit use of that in the following.

LEMMA 4.1. For each nonterminal vertex x, $N_H(x)$ is a clique of H^2 and they are the only cliques.

Proof. That $N_H(x)$ is a clique is easy to verify. Now suppose S is a clique and x, y, z are three vertices in S. One can show easily that exactly two of the edges (x,y), (y,z), (z,x) are in H. Let $(x,y), (x,z) \in E(H)$ and u be another vertex in S. We show that $(x,u) \in E(H)$. If not, then considering the triples x, u, y and x, u, z we get $y, z \in \Gamma_H(u)$, which is impossible since girth of $H > 6$ and (x,z,u,y,x) is a 4-cycle of H. Thus $S = N_H(x)$.

COROLLARY 4.2. If S is a clique in a partial square of H then $S \subseteq N_H(x)$ for some x and x belongs to S. x is unique if $|S| \geq 3$.

Proof. x is the vertex such that $N_H(x)$ is a clique of H^2 containing S. If $|S| \geq 3$ then girth $H > 6$ implies x is unique.

If we call the clique $N_H(x_i)$ by C_i, then $|C_i \cap C_j| \leq 2$ with equality if and only if (x_i,x_j) is an edge of H. $\{C_i\}$ satisfy the three conditions in the following theorem.

THEOREM 4.3. A graph G is the square of a graph of finite girth > 6 if and only if there is an 1-1 correspondence between the cliques $\{C_i\}$ and vertices $V_o = \{x_i: 1 \leq i \leq n\}$ belonging to two or more cliques, $x_i \leftrightarrow C_i$, such that the following are true.

i) $|C_i \cap C_j| \leq 2$ and $|C_i \cap C_j| = 1$ implies there is C_k sharing two points with each of them.

ii) $x_i \in C_i$ and $x_i \in C_j \Leftrightarrow x_j \in C_i$.

iii) There exists at least one sequence $(C_i,C_j,\ldots,C_t,C_i)$ of which consecutive cliques meet at two points, and any such cyclic sequence has seven or more cliques in it.

Proof. Consider the graph H on $V(G)$ defined by the edges $E(H) = \{(x_i,y): y \in C_i,\ y \neq x_i\}$. We prove that H is a connected graph. It suffices to show that $H|V_o$ is connected. If not, let $H|\{x_1,x_2,\ldots,x_k\}$, $k < n$ be a component. Define $H' = \bigcup_{i \leq k} C_i$. x_j, $j > k$ is not a vertex of H'. Thus $V(H') \neq V(G)$, and G being connected there exists an edge $(x,y) \in E(G)$ such that only $x \in V(H')$. As in the proof of (2.2) a contradiction is reached from there. Suppose $x \in C_i$, $i \leq k$ and $(x,y) \in C_j$, $j > k$. Let $C_{i'}$ be such that $|C_i \cap C_{i'}| = 2 = |C_{i'} \cap C_j|$. Then $(x_i,x_{i'})$, $(x_{i'},x_j)$ are in $E(H)$, a contradiction. Thus H is connected. That $H^2 = G$ follows directly from the definition of H. The vertices $V(G) - V_o$ are terminal vertices in H. Condition iii) implies that any cycle contained in $V(G) - V_o$ is necessarily of length 7 or more and there is at least one of them.

We do not know at this moment how to characterize the partial squares of H. The set of 3-components of a partial square has no such property as those given in Theorem 2.2. Some basically different technique has to be developed to deal with partial squares of graphs having finite girth > 6. We prefer to have a characterization which uses families of maximal subsets of vertices with respect to some property P suitably defined (i.e., we want to avoid a theorem like the one given by Mukhopadhyay, see Appendix). We state this as an open problem.

Problem 1. Obtain characterization of partial squares of a graph of finite girth > 6.

On the other hand, the squares of graphs which have girth 4, 5, 6 seem to be difficult to characterize. However, if G is a subdivision graph, and hence has a girth > 5, then we can show that G^2 determines graph G uniquely, except in one case when there is an isomorphism of order two between two square roots of G^2. This will be the topic of Section 5. Our next open problem:

Problem 2. Characterize squares of graphs which have girth either 4 or 5[†] in terms of some maximal subsets.

The following example may illustrate the peculiarities of

[†]See footnote on page 20.

cliques in G^2 when G has girth 5.

Example 3. Consider the graph G which consists of a 5-cycle $(x_1,x_2,x_3,x_4,x_5,x_1)$ and three other edges (x_1,x_8), (x_3,x_6), (x_5,x_7) (and, of course, the vertices x_6, x_7, x_8). The graph G^2 has four cliques each of which is a K_6. They are $V(G) - S$ where S is $\{x_2,x_4\}$, $\{x_2,x_8\}$, $\{x_7,x_4\}$, $\{x_7,x_8\}$. Vertices x_1, x_3, x_5, x_6 belong to every clique of G^2.

It is worth noting that

PROPOSITION 4.4. Every graph is a partial square of a graph of girth ≥ 4.

Proof. It is clearly true for graphs having 3 or less vertices. The general case is proved by induction on the number of vertices. Suppose G has $n \geq 4$ vertices and the proposition is true for all graphs with $n-1$ vertices. Choose a vertex x of degree ≥ 2. Consider the graph $\underline{G} = (G-x) - \{(x_i,x_j) \in E(G)\colon x_i, x_j \in \Gamma_G(x)\}$. $\underline{G}$ has $n-1$ vertices and is possibly disconnected. Obtain a graph $\underline{F}$ such that $\underline{F} \subseteq \underline{G} \subseteq \underline{F}^2$ and girth $\underline{F} \geq 4$. Define F by $V(F) = V(G)$ and $E(F) = E(\underline{F}) \cup \{(x,x_i)\colon x_i \in \Gamma_G(x)\}$. F has girth ≥ 4 and $F \subseteq G$. Further $\Gamma_F(x) = \Gamma_G(x)$ implies that $G \subseteq F^2$.

The proof of Theorem 4.5 is almost the same as that of (3.1).

THEOREM 4.5 (Uniqueness). The graph H^2 determines the graph H uniquely.

Write yourself a characterization of H^2, girth $H = 6$, as in (4.3), using $N_H(x)$ is a clique of H^2 (see proof of Remark 4.5).

Proof. If possible let H, $\underline{H}$ be two graphs each of girth > 6 such that $H^2 = \underline{H}^2$. As in (3.1), it suffices to show that H, $\underline{H}$ have identical core subgraphs.* Vertices in $c(H)$, $c(\underline{H})$ are the same; they are the vertices in H^2 belonging to two or more cliques. Suppose $N_H(x_i) = C_i = N_{\underline{H}}(x_{\sigma i})$ where $x_{\sigma i} \leftrightarrow C_i$ satisfies i), ii), iii) in (4.3). Then one can show that for every edge (x_i,x_j) in $c(H)$, $\{x_i,x_j\} = \{x_{\sigma i},x_{\sigma j}\}$. Thus σ is either the identity permutation, when $H = \underline{H}$, or $c(H)$ has two vertices. The latter is impossible as $c(H)$ has finite girth. Thus $H = \underline{H}$.

REMARK 4.6. Suppose $G = H^2 = \underline{H}^2$ where H has girth > 6 (possibly infinite) and girth of $\underline{H}$ is at least 6. Then girth of $\underline{H}$ is also > 6. If H is a tree then so is $\underline{H}$.

Proof. If possible, suppose $\xi = (x_1,x_2,x_3,\ldots,x_6,x_1)$ is a 6-cycle in $\underline{H}$. First we show that $N_{\underline{H}}(x_1)$ is a clique in G. If not, let C be a clique containing it porperly and let y be a vertex in C which is not in $N_{\underline{H}}(x_1)$. There is a vertex z adjacent to x_1, y in $\underline{H}$. Also, either $(x_2,y) \in E(\underline{H})$†, or for some vertex u, (x_2,u), $(u,y) \in E(\underline{H})$. Each of them contradicts girth $\underline{H} \geq 6$. Thus $N_{\underline{H}}(x_1)$ is a clique.

The cliques $N_{\underline{H}}(x_i)$, $1 \leq i \leq 6$ violate the condition iii) in Theorem 4.3 if H has a finite girth. Thus G cannot be $\underline{H}^2$. If H is a tree we know that there cannot be a family of cliques as $\{N_{\underline{H}}(x_i)\}$. This proves the first part. Also, if H is a tree then $\underline{H}$ has no cycle, or $\underline{H}$ is a tree.

---

*The core $c(H)$ of H is the subgraph induced by the nonterminal vertices of H.

†Assume $x_2 \neq z$. Otherwise, consider x_6 instead of x_2.

Theorem 4.5 and Corollary 3.2 imply that $H = \underline{H}$ except when $c(H)$ as two vertices.

5. Application to Total Graphs

In his classical paper Whitney [11] proved that the line graph of a connected graph G determines the graph G uniquely up to an isomorphism except in one case when the line graph is a triangle. We shall show that a similar property holds true for total graphs, namely.

The total graph of a connected graph G determines the graph G uniquely up to an isomorphism.

The total graph $T(G)$ of a graph $G = (V,E)$ has vertex set $V \cup E$ and two vertices of $T(G)$ are adjacent if the corresponding elements of G are adjacent or incident (according to the nature of the elements) in G. From now onwards we shall say x, a are adjacent no matter whether one, both or none of them is a vertex. This will simplify the language considerably. A total graph is a special kind of square graph, namely, square of the subdivision graph. We define the subdivision graph $S(G)$ as follows. Graph $S(G)$ has vertices $V \cup E$ and the edges (x,e) where x in V, e is in E and e is incident with the vertex x in G. Behzad [1] has noted the following:

THEOREM 5.1 (Behzad). $T(G) = S(G)^2$.

Since the girth of $S(G)$ is at least 6 we have, in particular, a total graph is the square of a graph of girth 6 or more. The proof of (5.1) is easy. We leave it to the reader.

A triangle is a set of three mutually adjacent vertices. A triangle of $T(G)$ is even if every vertex of $T(G)$ is adjacent to an even number of vertices of the triangle. It is clear that the total graph of a disconnected graph is the disjoint union of the total graphs of the components. Thus we shall consider connected graphs only. To avoid trivialities we shall assume that the graph G has at least four vertices.

Let G_1 be a graph whose vertices and edges are labeled. Let G_2 be another graph whose vertices and edges are also labeled by the same set of labels in such a way that two labeled elements are adjacent in G_2 if and only if they are adjacent in G_1. This is the same as saying, $T(G_1) = T(G_2)$ (as labeled graphs). Observe that both G_1 and G_2 are connected.

THEOREM 5.2. G_1 is isomorphic to G_2.

We begin with three lemmas.

LEMMA 5.3. Let a, b, c be the vertices of an even triangle of $T(G)$. Then a, b, c consist of either the edges of a triangle of G or two vertices and the edge joining them in G.

Proof. Let a, b, c form an even triangle of $T(G)$. There are four typical arrangements of the elements a, b, c in G as shown below.

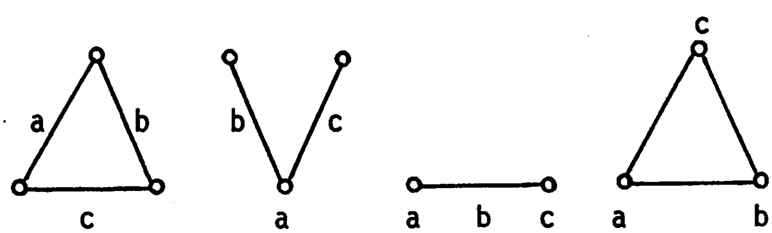


Fig. 5. The elements a,b,c in G such that they form a triangle of T(G).

Since G has four or more vertices the 2nd and 4th possibilities are excluded. Hence the lemma is proved.

<u>LEMMA 5.4</u>. Let 1 and 2 (resp. a) be vertices of G_1 (resp. G_2) and a (resp. 2) the edge joining them as in Fig. 6. Then G_1 and G_2 are complete graphs of the same order.

<u>Proof</u>. Let $k \neq 1$ be a vertex of G_1 adjacent to 2 and let $c = (2,k)$. Since c is adjacent to 2 and a in G_1, c must be an edge at a in G_2. But k being not adjacent to a in G_1, the other end vertex of c in G_2 is different from k; let it be b. Also in G_1, k is adjacent to 2 and c but not to a. It follows that $k = (1,b)$ in G_2. Hence k is

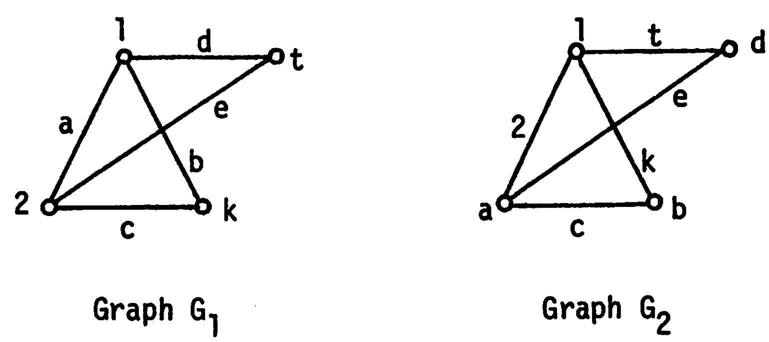


Fig. 6.

adjacent to 1 in G_1; b being adjacent to 1, k, c in G_2 we have $b = (1,k)$ in G_1. Thus, in G_1, every vertex adjacent to 2 is also adjacent to 1.

On the other hand, let $t \neq 2$ be a vertex of G_1 joined to 1 by an edge $d = (1,t)$. Since d is adjacent to 1 and a but not to 2 in G_1, d is a vertex of G_2. It is not hard to see that the edge $(1,d)$ in G_2 has label t. Also let $e = (a,d)$ in G_2. It follows that $e = (2,t)$ in G_1.

In each of G_1, G_2 one of the elements 2, a is an edge and the other is a vertex. Thus any deduction from G_1 to G_2 holds true as well from G_2 to G_1. Also observe the "interchanges" of the pairs $\{2,a\}$, $\{k,b\}$, $\{t,d\}$ in G_1 and G_2. The edges e, c of G_1 are also edges of G_2 between corresponding vertices. To complete the proof of the lemma it remains to apply the same argument back and forth repeatedly between G_1 and G_2 and use connectivity of the two graphs.

<u>LEMMA 5.5</u>. In G_1, G_2 let 1 and 2 be vertices and $a = (1,2)$. Then G_1, G_2 are isomorphic graphs.

<u>Proof</u>. Let $k \neq 1$ be a vertex of G_1 joined to 2 by an edge b (Fig. 7). Since b is adjacent to a and 2 in G_1, b is an edge at 2 in G_2. Moreover k being adjacent to b and 2 but not to a in G_1, it follows that in G_2, k is the other end vertex of b. Repeat the argument and use the connectivity of G_1. It follows that G_1, G_2 are isomorphic.

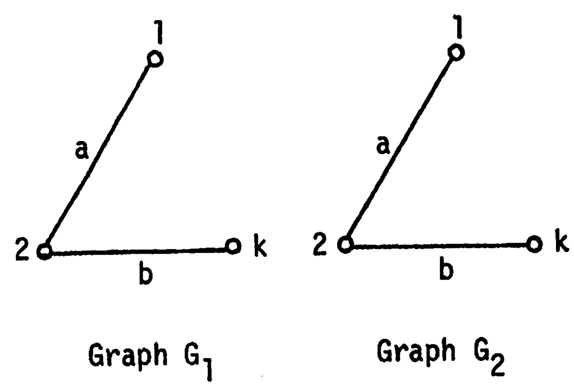


Fig. 7.

Proof of Theorem 5.2. Suppose first that one of the two graphs, say G_1, has a triangle $\{1,2,3\}$ and a, b, c are its edges: a = (1,2), b = (1,3), c = (2,3). The vertices a, b, c form an even triangle of $T(G_1)$. In view of Lemma 5.3 there are two cases.

Case 1. a, b, c are edges of a triangle in G_2.

Now the element 1 can be a vertex or an edge of G_2. Let 1 be a vertex. Since 1 is adjacent to a, b in G_1, 1 has to be the vertex adjacent to a, b in G_2. The adjacencies from G_1 imply that in G_2, 2 is the common vertex of a, c. By Lemma 5.5, G_1 and G_2 are isomorphic. On the other hand, if 1 is an edge of G_2 let d be the vertex of G_2 common to 1, a and b. Evidently, d is different from 2,3. Now one can easily see that in G_1, d is an edge at 1. Further, if d = (1,4) in G_1 then 1 = (d,4) in G_2 since in G_1, 4 is adjacent to 1, d and not to a. Applying Lemma 5.4 to 4, 1, d we get the two graphs are isomorphic.

Case 2. In G_2 one of a, b, c is an edge and the other two are its end vertices.

Without loss of generality, let b be the edge b = (a,c). Since 2 is adjacent to a and c but not to b in G_1, it follows that 2 is a vertex of G_2 adjacent to a and c. Also, 1 being adjacent to a, b and 2 in G_1, 1 = (a,2) in G_2. Lemma 5.4 applies to the triplet 2, a, 1. We conclude that G_1 is isomorphic to G_2. This completes the proof when G_1 has a triangle.

Suppose now that G_1, G_2 have no triangle. We have

$$S(G_1)^2 = T(G_1) = T(G_2) = S(G_2)^2$$

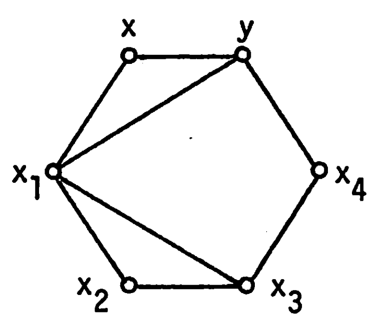
where $S(G)$ denotes the subdivision graph of G. The graphs $S(G_1)$, $S(G_2)$ have girth at least 8, and thus they are equal by (4.5). However, it is easy to see that G_1, G_2 are determined uniquely up to isomorphism from their subdivision graphs. (The isomorphism is nontrivial only when G_1, G_2 are cycles.) Thus they are isomorphic.

The theorem is proved.

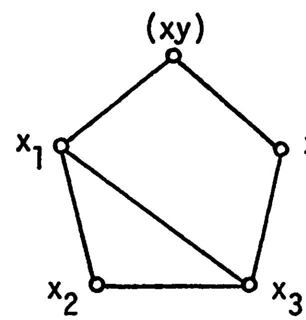
Behzad has noted that the total graph of K_n is the line graph of K_{n+1}. It is not hard to see that $T(K_n)$, $n \geq 1$ are the only total graphs that are also line graphs. Thus if a total graph G is a line graph then cliques of G are of two sizes, 3 and $n \geq 3$. There are $\binom{n+1}{3}$ cliques of the first kind and (n+1) of the second kind. A graph G which fails to be a line graph is a total graph if G is the total graph of the graph H constructed from the cliques of G as in Theorem 4.3. Note that we do not need to know the correspondence $x_i \leftrightarrow C_i$ before hand. An edge (x_i, x_j) is being taken in H provided two cliques intersect at these two points. A vertex y other than x_i's is joined by an edge (y, x_i) if and only if there is a clique containing both y and x_i.

6. A 4-color Theorem

In this section we shall obtain a four-color theorem for a class of planar graphs. Let $r_1, r_2, \ldots$ denote the colors. A (vertex) coloring of a graph G is an assignment of colors $x \leftrightarrow r(x)$ to the vertices of G such that $r(x)$ is different from $r(y)$ whenever x, y are adjacent in G. The <u>chromatic number</u> $\chi(G)$ is the minimum number of colors needed to color G. An elementary contraction of a graph is defined as follows. In G identify two adjacent vertices x, y to a single vertex (xy). A vertex z is adjacent to (xy) if z is adjacent to at least one of x, y. Other edges (z,z') remain as they are. This is illustrated in Fig. 8 below.



Graph G



x, y are identified to a single vertex (xy)

Fig. 8. An example of contraction.

A graph obtained by successive elementary contractions of G is called a <u>contraction</u> of G. A famous problem in the theory of graph colorings is the following:

<u>HADWIGER'S CONJECTURE</u>. If the chromatic number of a graph G is

n, then G can be contracted to a complete graph on n vertices.

The conjecture is known to be true for $n \leq 4$. The cases $n = 2, 3$ are trivial and the case $n = 4$ has been proved by Dirac [3]. It is also known that if the conjecture is true for $n = 5$, then every planar graph is colorable with four colors.

Let us assume that G is a partial square of a tree T. It is easy to see that $\chi(G) = \max \chi(G|N_T(x)) = 1 + \max \chi(G|\Gamma_T(x))$ where the maxima is taken over all vertices x.

LEMMA 6.1. If $\chi(G) = n \leq 5$, then the graph G can be contracted to K_n.

Proof. First observe that if there is a contraction of $G|\Gamma_T(x)$ to K_p then G can be contracted to K_{p+1}. The elementary contractions which take $G|\Gamma_T(x)$ to K_p will take $G|N_T(x)$ to K_{p+1}. The lemma follows easily from the theorem of Dirac.

THEOREM 6.2. A planar graph is 4-colorable (i.e., $\chi \leq 4$) if it has a 2-complete family of fundamental cocircuits.

Proof. Apply Theorem 1.1 and Lemma 6.1.

The following is a n.s.c. for a partial tree square to be planar. A graph G is called outer planar if there exists a planar embedding of G such that all vertices are on the exterior face.

THEOREM 6.3. A partial tree square, $T \subseteq G \subseteq T^2$, is planar if and only if $G|\Gamma_T(x)$ is outer planar for every vertex x.

Proof. The necessity is clear since $G|N_T(x)$ is planar. To prove the sufficiency we note that $G|\Gamma_T(x)$ outer planar implies $G|N_T(x)$ is planar. A planar representation of G is obtained as follows. Take a planar representation of $G|N_T(x)$. For $y \in \Gamma_T(x)$, choose a face of $G|N_T(x)$ with (x,y) on its boundary. Then 'insert' in that face a planar representation of $G|N_T(y)$ in which (x,y) is on the boundary of the exterior face (see Fig. 9). Continue in this way until all of G have been drawn.

Example 4.

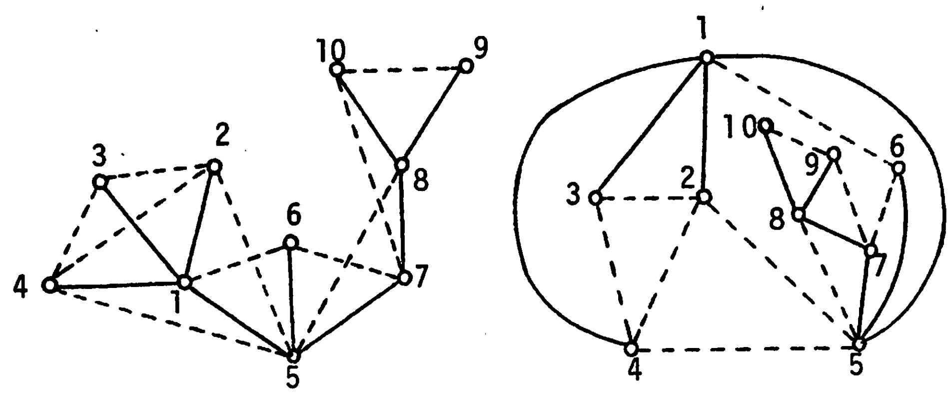


Fig. 9. A planar partial tree square and its planar embedding.

In particular, T^2 is planar if and only if the degree of each vertex is at most 3.

Appendix.

Some of the previous works on squares of trees and graphs in general are collected here.

Harary [5] states the following theorem characterizing tree squares. We state it in our terminology.

<u>THEOREM</u> (Harary). A connected graph G is a tree square if and only if there exists a 1-1 correspondence $x_i \leftrightarrow C_i$ between the cliques of G and vertices $\{x_i : 1 \leq i \leq n\}$ belonging to two or more cliques such that the following are true.

i) $|C_i| \geq 3$

ii) $|C_i \cap C_j| \leq 2$; if the intersection has one point then for some C_k, $C_k \cap C_i$, $C_k \cap C_j$ has two points each.

iii) There are as many C_j's containing x_i as there are x_j's in C_i.

iv) There are $(n-1)$ pairs of cliques which meet at two points.

On page 646, line 7, [5] Harary says "It is clear that the tree T constructed from the algorithm is a tree square root." We feel that this is not at all clear. His algorithm is given below.

(x_i, x_j) is an edge of T if $|C_i \cap C_j| = 2$. Other than those, T has the edges (y, x_i) where y is in C_i and $y \neq x_i$.

The condition iii) in Harary's theorem seems to be very strong. It is precisely this requirement that makes the construction of a counter example to the theorem very unlikely.

The following theorems were obtained by A. Mukhopadhyay [8]. Theorem I characterizes the square of an arbitrary graph. There are practical difficulties in applying the theorem since one has to consider all complete subgraphs.

<u>THEOREM I</u> (Mukhopadhyay). A connected graph G with n vertices $v_1, v_2, \ldots, v_n$ is a square graph if and only if some set of n

complete subgraphs of G whose union is G can be labeled $C_1, C_2, \ldots, C_n$ so that, for all $i, j = 1,2,\ldots,n$ the following conditions hold:

i) $v_i \in C_i$

ii) $v_i \in C_j$ if and only if $v_j \in C_i$.

For tree squares, he states

<u>THEOREM II</u> (Mukhopadhyay). A graph G with n vertices is a tree square if and only if G has p cliques and q vertices belonging to only one clique of G such that

i) $n = p + q$

ii) There exists a labeling C_i of cliques such that the vertices x_i belonging to two or more cliques has the properties

a) $x_i \in C_i$

b) $x_i \in C_j \leftrightarrow x_j \in C_i$.

Theorem II has been shown to be wrong in Example 2.

Chapter 2

CLIQUE GRAPHS, LINE GRAPHS AND SQUARE GRAPHS

In this chapter we shall consider three classes of graphs: clique graphs, line graphs and graphs that are partial squares of a tree. In Chapter 1, we have obtained a characterization of graphs which are partial squares of trees. Clique graphs have been characterized by Roberts and Spencer [9]. A common feature of these characterizations is the existence of a family of subsets of vertices of the graph with certain intersection properties. For partial tree squares the subsets are 3-components of the graph and for clique graphs each subset is a complete set of vertices. We show that every partial tree square is a clique graph. The line graphs become important in our discussion because many line graphs happen to be clique graphs as well. We have obtained a n.s.c. for a line graph to be a clique graph. We show that the line graph of H is a clique graph if and only if H does not contain a triangle all of whose vertices have degree 3 or more. An explicit characterization is obtained for the graphs H whose line graph is a partial tree square. Such graphs H are quite easily described and are closely related to trees. In the last section we note that some products of clique graphs are also clique graphs.

1. Definitions

Let G be any graph. The line graph $L(G)$ of G is defined as follows. Corresponding to each edge $a = (x_i, x_j)$ of

G there is a vertex of $L(G)$ (which is written as $a = v(x_i,x_j)$), and two of the vertices are adjacent if the corresponding edges of G are incident with a common vertex. The line graph $L(G)$ gives us the incidence relation among the lines of G. Similarly, in order to study the intersection pattern of the cliques of G one defines the clique graph of G. The vertices of the <u>clique graph</u> $K(G)$* represent cliques of G and two vertices K, K_2 in $K(G)$ are adjacent if the corresponding cliques of G intersect. If G has girth ≥ 4, then the cliques of G are nothing but the lines of G and the clique graph $K(G)$ coincides with the line graph $L(G)$. We shall obtain characterizations of those line graphs which are clique graphs and those which are partial squares of a tree.

The fundamental problem of characterizing clique graphs themselves is solved by Fred Roberts and Joel Spencer [9]. Our results are obtained as applications of their theorem.

Let $\mathcal{F}$ be a family of nonempty subsets. We say $\mathcal{F}$ has property $\underline{P}$ (or F is a $\underline{P}$ family) if each pair of sets in F has a nonempty intersection. F is said to have property $\mathcal{I}$ if for every $\underline{P}$ subfamily $F' \subseteq F$, $\cap F' \neq \emptyset$. In particular, $\cap F \neq \emptyset$ implies F is an $\mathcal{I}$-family. In the following $\mathcal{F}$ will consist of complete sets of vertices. A triangle will be often denoted by $\Delta = \{x_i,x_j,x_k\}$.

<u>THEOREM 1.1</u> (Roberts and Spencer). A graph H is a clique graph if and only if there is a collection $\mathcal{K}$ of complete subgraphs of

*We regard L and K as operators on the class $\mathcal{G}$ of 'finite' connected graphs into themselves. $L(\mathcal{G})$ is the set of line graphs and $K(\mathcal{G})$ the set of clique graphs.

H which satisfies the following two properties:

i) K covers all the edges, i.e., if $(x,y) \in E(H)$ then some element of K contains vertices x, y.

ii) K satisfies property I.

Using Theorem 1.1 one can easily see that all the nine forbidden subgraphs in a line graph (see page 75, Harary [6]) are clique graphs. For example, the graph G_3 is union of two copies of K_4 having the three points on the horizontal line in common. This simply means that a graph G whose clique graph is one of these nine forbidden subgraphs has girth 3. The reason that led us to a joint study of the line graphs, the tree-square graphs, and the clique graphs is that they are all characterized by the existence of a family of complete subgraphs (which are often cliques) with some sort of intersection properties.

In Section 2 we shall determine the intersection $L(G) \cap K(G)$ of the family of line graphs with the family of clique graphs. Section 3 will be devoted to the consideration of the squares of trees and other graphs. Finally in the last section we discuss some products of clique graphs. Clearly, a graph is a clique graph if and only if each of its (connected) components is a clique graph. As before, all graphs will be assumed connected unless otherwise mentioned. To avoid trivialities we shall often assume, without explicit mention, that graphs have four or more vertices.

2. Characterization of Clique Graphs of the Form L(H)

We have noted earlier that many line graphs are also clique graphs. A complete characterization of such line graphs is given by

THEOREM 2.1. $L(H)$ is a clique graph if and only if H does not contain a triangle whose vertices are of degree ≥ 3.

In particular, if $L(H)$ is a clique graph then $H \not\supseteq K_4$.

We prove the following lemma as a preparation to Theorem 2.1.

LEMMA 2.2. Let there be a triangle in H whose vertices have degrees ≥ 3. Then $L(H)$ is not a clique graph.

Proof. We shall let a, b, c denote the edges of such a triangle and a', b', c' denote three other edges, one at each vertex of the triangle, as in Fig. 10.

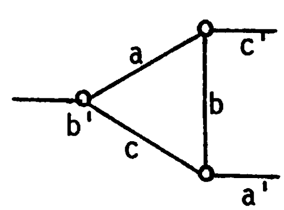


Fig. 10. A triangle in H whose vertices have degree ≥ 3.

Let $\mathcal{F} = \{L_i\}$ be a family of complete subgraphs of $L(H)$ such that $\cup \mathcal{F} = L(H)$ and, if possible, let $\mathcal{F}$ have property $\mathcal{I}$.

If $\{a,b,c,'\} \not\subseteq L_i$ for all i then let $\{a,b\} \subseteq L_1$, $\{b,c'\} \subseteq L_2$, $\{c',a\} \subseteq L_3$; $\{L_1,L_2,L_3\}$ is a $\mathcal{P}$ subfamily of $\mathcal{F}$ which has empty intersection. Thus $\mathcal{F}$ is not $\mathcal{I}$. Thus there exists an L_i in $\mathcal{F}$ containing $\{a,b,c'\}$. Similarly let $L_j \supseteq \{b,c,a'\}$, $L_k \supseteq \{c,a,b'\}$ $(i,j,k$ distinct). But then

$\{L_i, L_j, L_k\}$ is a $\underline{P}$ subfamily having an empty intersection. The contradiction shows that F is not I and hence $L(H)$ is not a clique graph.

<u>Proof of Theorem 2.1</u>. The lemma proves the 'only if' part. We prove now sufficiency. Let H be a graph satisfying the hypothesis of the theorem. If H has no triangle, we have seen that $L(H) = K(H)$ is trivially a clique graph. Next, let H have a triangle. For each triangle Δ in H, let $\delta(\Delta)$ be a vertex in Δ which is of degree two. δ is an 1-1 map from triangles of H into the vertices of H. Define the following cliques in $L(H)$. For $x_i = \delta(\Delta)$, $L_i = \{$the edges of the triangle $\Delta\}$, otherwise $L_j = \{$set of edges incident with vertex $x_j\}$ if $d_G(x_j) \geq 3$ or $d_G(x_j) = 2$ and x_j does not belong to any triangle. L_i is a complete set in $L(H)$. Clearly, $\cup L_i = L(H)$. Let P be a subfamily of $\{L_i\}$ with the intersection property $\underline{P}$. If $L_i \in P$ where $x_i = \delta(\Delta)$ and $\Delta = \{x_i, x_j, x_k\}$ then $P \subseteq \{L_i, L_j, L_k\}$ and vertex $v(x_j, x_k)$ of $L(H)$ is in $\cap P$. On the other hand if P does not contain any $L_{\delta(\Delta)}$ then it is not hard to see that P contains only two L_i's and $\cap P \neq \emptyset$. By Theorem 1.1 $L(H)$ is a clique graph.

<u>COROLLARY 2.3</u>. $L(H)$ is a clique graph if and only if the family of cliques of $L(H)$ has property I.

Compare Corollary 2.3 with the following theorem of Roberts and Spencer [9]. The <u>clique number</u> of a graph is the maximum number of vertices in a clique. The clique number of $L(H)$ can be, certainly, arbitrarily large.

THEOREM 2.4 (Roberts and Spencer). If the clique number of a graph G is < 4, then G is a clique graph if and only if the cliques of G have property $\mathcal{I}$.

The following is a simple corollary to Theorem 2.1.

COROLLARY 2.5. If L(H) is a clique graph, then all induced subgraphs of L(H) are also clique graphs.

It is not true in general that the induced subgraphs of a clique graph are also clique graphs. See Example 5. The converse of Corollary 2.5 is also false, i.e., a graph and all its induced subgraphs may be clique graphs and still the graph itself may not be a line graph. For example, consider the wheel graph on 6 vertices shown in Fig. 12.

Example 5. Consider the graph G shown in Fig. 11. G is a clique graph since all cliques of G contain the vertex x. G - x is not a clique graph by (2.1). However, all proper induced subgraphs of G - x are clique graphs since they are line graphs satisfying Theorem 2.1.

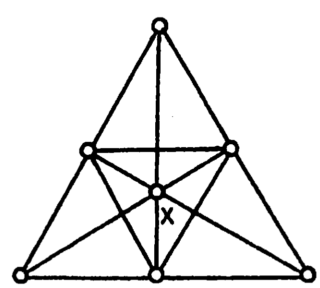


Fig. 11. The graph G. The graph G-x is called the big triangle.

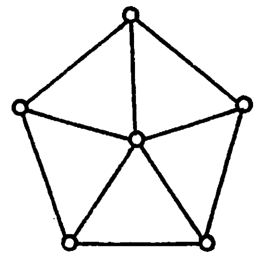


Fig. 12. The wheel graph on 6 vertices.

3. Partial Square Graphs and Clique Graphs

Another large class of clique graphs are partial squares of graphs G where girth of G is at least 7. The partial square graphs in this class share some of the important properties of the clique graphs of the form L(H). For example, it is true that the cliques of a partial square of G have property I. Also induced subgraphs of G^2 are clique graphs. One is thereby naturally inclined to ask "Which of these partial square graphs are line graphs?" The answer is given in Theorems 3.5, 3.8.

We recall that a graph H is called a partial square (of G) if for some graph G with the same vertices as those of H, one has $G \subseteq H \subseteq G^2$. The definition is not very useful unless one restricts the graph G in some way since every graph is a partial square of itself. (In fact, every graph is a partial square of a graph of girth ≥ 4. See (4.4), Chapter 1.) In the study of clique graphs, the natural restriction seems to require that G be of girth 7 or more. A tree has girth ∞ by definition.

THEOREM 3.1. If G is a connected graph and G has girth ≥ 7 then every graph H, $G \subseteq H \subseteq G^2$, is a clique graph and the cliques of H have property I.

Proof. The theorem is trivial for $|V(G)| \leq 3$. To prove the general case we recall Lemma 4.1 in Chapter 1. The cliques of G^2 were shown to be of the form $N_G(x)$ where degree of x is 2 or more. Thus cliques of H are subsets of the form $L \subseteq N_G(x)$ and $x \in L$.* The cliques contained in $N_G(x)$ are written as $L_x, L_x', L_x'', \ldots$. We show that the set of cliques of H, K, has property I. Let P be a $\underline{P}$-subfamily of K. If for some x, all cliques in P are of the form $L_x^{(i)}$ then $x \in \cap P$. So let there be cliques L_x, L_y in P such that they are not contained in the same neighborhood. To simplify the argument let each of them have three or more vertices. We have $1 \leq d_G(x,y) \leq 2$ since $L_x \cap L_y \neq \emptyset$.

Case 1. For any L_t, L_z in P, $d_G(t,z) \leq 1$.

Then $d_G(x,y) = 1$ and $L_x \cap L_y = \{x,y\}$. If L_z' is any clique in P then $z = x$ or y; otherwise $\{x,y,z\}$ is a triangle in G. If $y \in L_x'$ for all $L_x' \in P$ then $y \in \cap P$. On the other hand if there exists an L_x' not containing y then x belongs to every clique in P.

Case 2. $d_G(x,y) = 2$.

Then $L_x \cap L_y = z$ where $z = N_G(x) \cap N_G(y)$. Let L_t be a third clique in P. We show that $z \in L_t$ proving $\cap P \neq \emptyset$. If

*There is a slight ambiguity as far as x is concerned when L_x is a K_2 graph. But this should not create any serious difficulty in following the proof.

$L_t \subseteq N_G(x)$, then $z \in L_t$ because $L_t \cap L_y \neq \emptyset$. Similarly, $z \in L_t$ for $L_t \subseteq N_G(y)$. Now suppose that $L_t \not\subseteq N_G(x), N_G(y)$ and $u \in L_t \cap L_x$. Then $L_t \cap L_y \neq \emptyset$ implies that G contains a cycle containing u and the length of the cycle is no more than 6 (Fig. 13) which is impossible. Thus the theorem is proved.

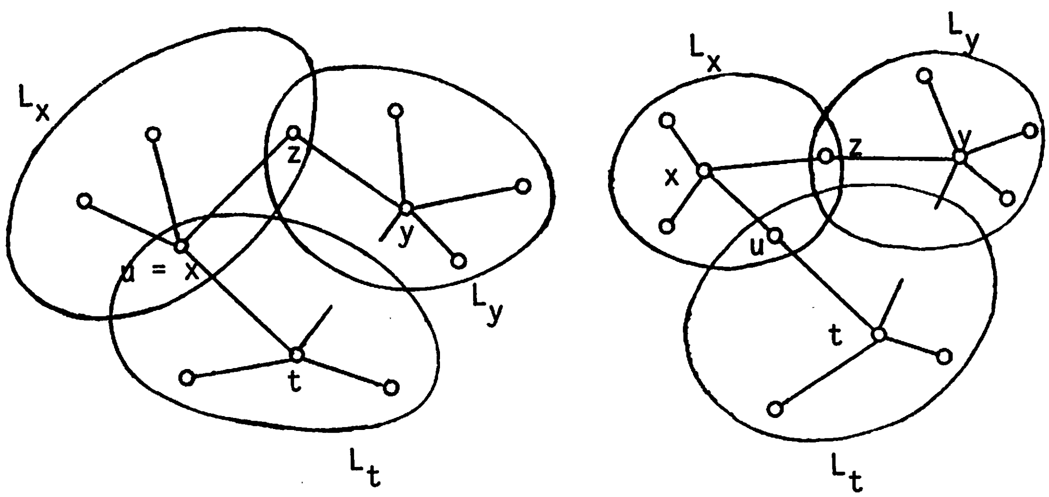


Fig. 13. Mutually intersecting cliques L_x, L_y, L_t in H. The edges shown are in G.

Theorem 3.1 is not true if girth of G $\leq$ 6. The big triangle is a partial square of a 6-cycle and it is not a clique graph. Also see Theorem 4.1. Compare Corollary 3.2 with Corollaries 2.3 and 2.5.

<u>COROLLARY 3.2</u>. Let G be a graph of girth ≥ 7 and $S \subseteq V(G)$. Then $G^2 - S$ is a clique graph.

<u>Proof</u>. Consider the family K of nonempty members of $\{N'_G(x) = N_G(x) - S: x \in V(G)\}$. It is clear that K consists of cliques of $G^2 - S$. Because, a clique K of $G^2 - S$ is contained in a clique

$N_G(x)$ of G^2 and therefore $K = N_G(x) - S$. Let P be a $\underline{P}$-subfamily of K and $|P| \geq 2$. We can assume without loss of generality that $|N_G'(x)| \geq 2$ for cliques in P. Let $P' = \{N_G'(x) \in P: x \notin S\}$. Also consider the following partial square of G, $H = G^2 - \{$edges in $E(G^2) - E(G)$ which are incident with a vertex in $S\}$ and let P'' be the family $\{N_G'(x) \cup x: N_G'(x) \in P - P'\}$. Note that for $N_G'(x) \in P - P'$, the vertex x is determined unambiguously; this is not necessarily so for $N_G'(x) \in P'$ unless $|N_G'(x)| \geq 3$.

Since cliques of H are subsets of neighborhoods $N_G(x)$, it follows that $P' \cup P''$ is a family of cliques in H and it satisfies the intersection property $\underline{P}$. We have $\cap P = (\cap P') \cap (\cap P'')$ since the right hand side is a subset of $V(G) - S$ whether or not $P' = \emptyset$. Therefore $\cap P \neq \emptyset$ by Theorem 3.1 applied to the graph H.

<u>REMARK</u>. Corollary 3.2 cannot be generalized to partial squares of G, i.e., an induced subgraph of a partial square of G may not be a clique graph. Nevertheless, if $G \subseteq H \subseteq G^2$ and $H|N_G(x)$ is complete then $H - x$ is a clique graph. If $N_G(x)$ is not complete in H, then the conclusion is again false. For example, consider the graph G in Fig. 11. It can be regarded as a partial square of a star 'rooted' at the vertex x. $G - x$ is not a clique graph.

We proceed to the characterization of line graphs such that $G \subseteq L(H) \subseteq G^2$. We shall study the problem in two steps -- G is a tree (girth $= \infty$) and graph G has a finite girth ≥ 7. The cases girth 5, 6 are undecided and remain open. Throughout the rest of

Section 3, $L(H)$ is assumed to be a clique graph. For our purpose a graph G which is a partial square of a tree T, $T \subseteq G \subseteq T^2$, is best described as edge disjoint union of two subgraphs: T and $G-T$. The graph $G-T$ is a disjoint union of two induced subgraphs, one on each of the color classes of T. Each of the induced subgraphs in turn consists of a bunch of smaller subgraphs $G|\Gamma_T(x)$ (possibly disconnected) "hinged" at single vertices. See Fig. 14 below.

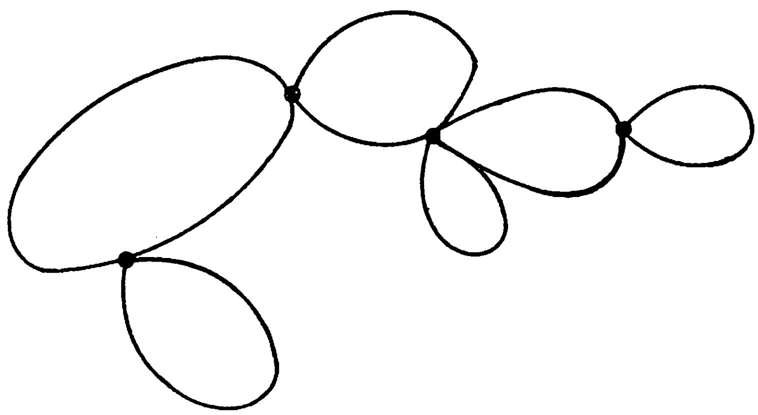


Fig. 14. Schematic view of $G|\{$a color class of $T\}$. Each of the circled regions is of the form $G|\Gamma_T(x)$ which is a complete graph in case $G = T^2$.

It is interesting to note the structure of an arbitrary cycle of a partial square of T or, for that matter, a tree square.

<u>PROPOSITION 3.3</u>. A cycle of T^2, $C = (x_1,x_2,\ldots,x_n,x_1)$ which is not contained in any $\Gamma_T(x)$ has exactly two edges of T and consists of consecutive sets of vertices from $\Gamma_T(.)$'s as indicated in Fig. 15.

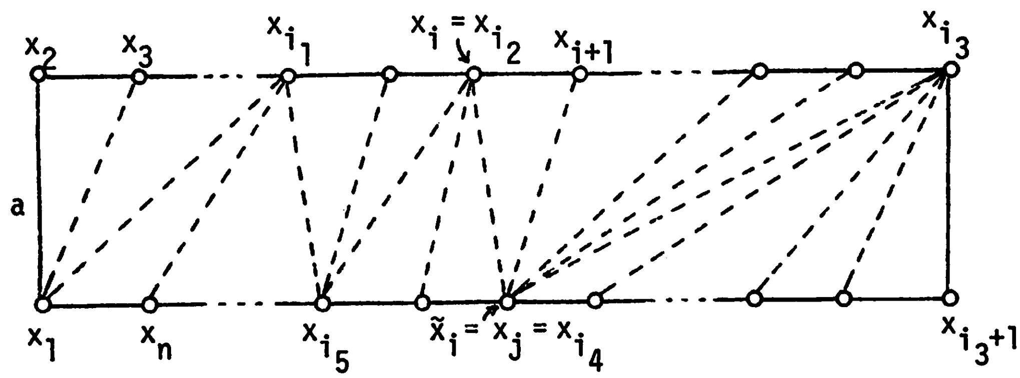


Fig. 15. A cycle C of T^2. Edges a, b belong to tree T. Chords of C are shown as broken lines and they are edges of T. $\Gamma_T(x_1) \supseteq \{x_t\colon 2 \le t \le i_1\}$, $\Gamma_T(x_{i_5}) \supseteq \{x_t\colon i_1 \le t \le i_2\}$, $\Gamma_T(x_{i_4}) \supseteq \{x_t\colon i_2 \le t \le i_3\}$, $\Gamma_T(x_{i_3}) = \{x_t\colon \{i_3+1 \le t \le i_4\}$, $\Gamma_T(x_{i_2}) = \{x_t\colon i_4 \le t \le i_5\}$ etc.

Proof. We prove it by induction on the number of vertices n in the cycle. The proposition is trivial for $n = 3$. Suppose it is true for all cycles of length $< n$ and C is a cycle of length $n \ge 4$. If C contains no edge of T then $C \subseteq \Gamma_T(x)$ for some x, contrary to the assumption. So let $(x_1,x_2) \in T$.

Case 1. $(x_2,x_3) \in T$. Then $(x_1,x_3) \in T^2$. Consider the cycle $C' = (x_1,x_3,x_4,\ldots,x_1)$. C' is easily seen to be contained in $\Gamma_T(x_2)$,* and the proposition is proved.

Case 2. No two consecutive edges of C are in T. For every edge (x_i,x_{i+1}) $(i+1 = 1$ for $i = n)$ of C which is not in T there exists a vertex $\tilde{x}_i$ such that $x_i, x_{i+1} \in \Gamma_T(\tilde{x}_i)$. We claim that $\tilde{x}_i = x_j$ for some i, j; otherwise T should contain a

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* $\Gamma_T(x_2)$ is a 2-component of $T^2 - x_2$ containing x_1, x_3.

cycle contained in the set $\{x_i\} \cup \{\tilde{x}_i\}$. Now one of the following two arcs of C:(a) x_i to x_j not containing x_{i+1} (b) x_{i+1} to x_j not containing x_i, is of length ≥ 2. Use the induction hypothesis for each of the cycles (if it is one): arc(a) $\cup$ (x_i, x_j), arc(b) $\cup$ (x_j, x_{i+1}). The result follows immediately.

COROLLARY 3.4. A cycle C of a partial square of T which has no chord is of length 3 or is contained in $\Gamma_T(x)$ so that it is the rim of a wheel with center at x.

Now we are in a position to prove one of our important theorems on line graphs. We make the following

DEFINITION. A *thick tree* is either a tree or it is a graph in which 1) the only cycles without a chord are triangles, 2) two triangles have two or no vertices in common and 3) each triangle has a vertex of degree 2.

A typical method for constructing a thick tree can be described as follows. Take a tree T and select a set of nonadjacent edges $E_o = \{e = (x,y)\}$. Corresponding to each $e \in E_o$, add a finite number of vertices $\{e_i\}$, all distinct, and put the edges $\{(e_i,x),(e_i,y)\}$ to make (x,y) "thick". The result is a thick tree. The tree T itself is called a skeleton of the thick tree so constructed. An edge $e \in E_o$ is called a thick edge of thickness $|\{e_i\}| + 1$. It is easy to see that the skeleton of a thick tree is determined uniquely up to isomorphism.

THEOREM 3.5. A line graph $L(H)$ is a partial square of a tree if and only if H is a thick tree.

Proof. "only if" part. Assume $L(H)$ is a partial square of tree T. Let C be a cycle in H having $k \geq 4$ edges. The induced graph $L(C) \subseteq L(H)$ is a cycle which has no chord. By (3.4), $L(C)$ is the rim of a k-wheel. For $k = 5$, and $k \geq 6$ this implies respectively G_9 and G_1 (page 75, [6]) are induced subgraphs of $L(H)$ which is impossible. If $k = 4$, then the center of the 4-wheel corresponds in H to a chord of cycle C. Also by (2.1) every triangle of H has a vertex of degree 2. It remains to show that two triangles in H do not intersect at a single vertex. Let Δ_1, Δ_2 be two triangles and $\Delta_1 \cap \Delta_2 = x$. Then $L(\Delta_1)$, $L(\Delta_2)$ and the induced subgraph L_x = set of edges incident with x are cliques of $L(H)$ and $L(\Delta_1) \cap L(\Delta_2) = \emptyset$, $|L(\Delta_1) \cap L_x| = |L(\Delta_2) \cap L_x| = 2$. However this is impossible in a partial tree square (since each clique is a subset of $N_T(x_i)$).

"If" part. This is a special case of Theorem 3.8. We observe that the thick edges each taken with two opposite orientations constitute a d-matching of H that satisfies conditions i), ii) of Theorem 3.8 with 7 (see Remark 3.9) replaced by $|E(H)| + 1$. Therefore $L(H)$ is a partial tree square.

Each of the graphs H in Fig. 16 has the line graph $L(H)$ which is equal to a tree square. Corollary 3.6 shows that they are the only graphs with this property.

COROLLARY 3.6. $L(H)$ is a tree square if and only if H is one of the graphs in Fig. 16.

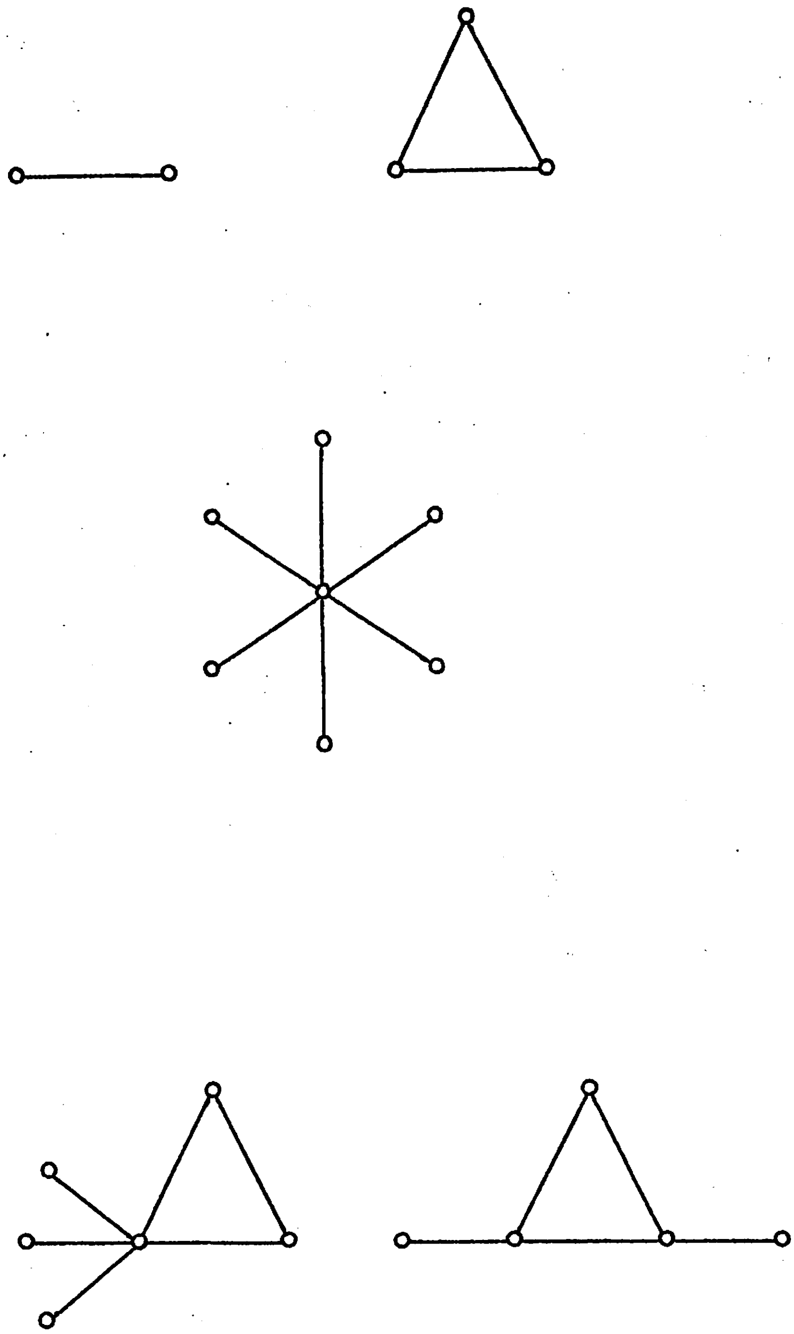


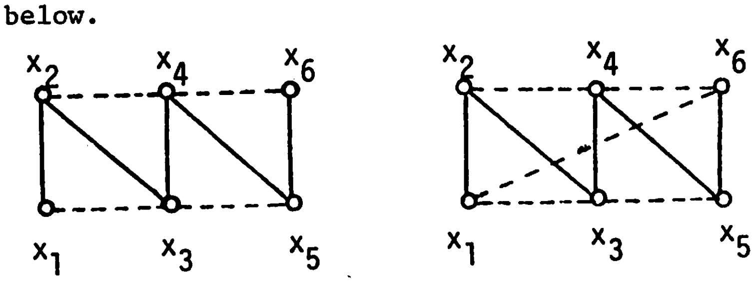
Fig. 16. Two graphs of diameter one, two classes of graphs of diameter 2 -- star and thick star with one edge of thickness 2, and one graph of diameter 3.

Proof. $L(H) = T^2$ implies that H does not contain a four cycle since the only chordless cycle of T^2 is a 3-cycle. We first show that a tree skeleton T_1 of H has diameter ≤ 3. Suppose not and (x_1,x_2,x_3,x_4,x_5) is a path in T_1. We recall that L_x denotes the induced subgraph of $L(H)$ on the edges of H at vertex x. Since (x_1,x_3), (x_2,x_4) are not in H, L_{x_2}, L_{x_2} are cliques of $L(H)$ and $|L_{x_2} \cap L_{x_3}| = 1$. But $L(H)$ being a tree square there is a clique of $L(H)$ that meets each of L_{x_2}, L_{x_3} at two vertices. Such a clique must be the edges of a triangle of H containing x_2, x_3. Therefore (x_2,x_3) is a thick edge. Similarly, (x_3,x_4) is a thick edge which is impossible. Thus diameter of $T_1 \leq 3$. In case of equality we have noted that the "middle" edge of a path of length 3 is a thick edge. It is easy to verify that degrees of x_2, x_3 are necessarily 3 in that case.

For the rest of the section 3, G will denote a graph of finite girth ≥ 7. Unlike the tree case a line graph is never equal to G^2. $L(H)$ is equal to G if H is a cycle of length ≥ 7, and only then.

PROPOSITION 3.7. G^2 is not a line graph.

Proof. Let $P = (x_1,x_2,\ldots,x_6)$ be an arc of a cycle in G of smallest length. The possible induced graphs $G^2|\{x_1,x_2,\ldots,x_6\}$ are shown below.



(i) (ii)

Fig. 17

The graph in Fig. 17(i) arises if there does not exist a vertex x_7 adjacent to x_1 and x_6; this graph is G_8.[†] The graph in Fig. 17(ii) contains G_2 (remove x_5).[†] Thus $G^2 \notin L(G)$.

The following concepts are essential for our next theorem. An edge of H with a specified direction (orientation) is called an <u>arc of</u> H. $(x,y)^-$ stands for the arc: from x to y.

<u>DEFINITION</u>. A subset M of the arcs of a graph H is called a <u>d-matching</u>[*] if the common terminal vertex of two arcs in M has degree 2 in H.

A d-matching may contain two arcs obtained by opposite orientations of the same edge. If a d-matching is symmetric (i.e., $(x,y)^- \in M$ implies $(y,x)^- \in M$) then the edges $\{(x,y): (x,y)^- \in M\}$ constitute a matching except that for a vertex of degree 2 both the edges adjacent to it may occur in the matching. We say arc $(x,y)^-$ <u>covers</u> y and y is an <u>unexposed</u> vertex. A vertex that is not unexposed is called <u>exposed</u>. An arc $(x,y)^-$ is called an <u>escape-arc</u> for a cycle C of H if y is a vertex of C and x is not.

The basic theorem can now be stated in terms of d-matching. In the sufficiency part of the theorem we do not actually assume that L(H) is a clique graph. It follows from Theorem 3.1.

<u>THEOREM 3.8</u>. A line graph L(H) is a partial square of a graph G of finite girth ≥ 7 if and only if the graph H has a d-matching

[†] See page 75, [6].

[*] d for directed.

M such that the following are true.

i) At most two arcs of M belong to each triangle of H and they cover the vertices of the triangle that have degree ≥ 3.

ii) A chordless n-cycle of H, $4 \leq n < 7$, has at least $7 - n$ escape-arcs in M.

<u>Proof</u>. "Sufficiency". Suppose that graph H has a d-matching satisfying 3.8(i), (ii) and $|V(H)| \geq 4$.

It is rather immediate from 3.8(i) that not all vertices of a triangle of H have degree ≥ 3 and two triangles do not meet at exactly one vertex. Also note that if there are two triangles $\Delta = \{x,y,z\}$, $\Delta' = \{x',y,z\}$, then necessarily $(y,z)^-$, $(z,y)^- \in M$. Enlarge the d-matching M to $M_o = M \cup M'$ where M' consists of a set of arcs $\{(.,x)^-\}$[†] chosen arbitrarily, one at each exposed vertex x, $d(x) \geq 2$ where either x is a vertex not belonging to a triangle or $d(x) = 2$ and M has only one arc in the triangle containing x (if any[*]). We construct a graph G on the vertices of L(H) as follows. Take an unexposed vertex x and let $(a,x)^-$ be the arc in M_o. Join the vertex $v(a,x)$ (in G) to all the vertices of the form $v(.,x)$. We show first that $G \subseteq L(H) \subseteq G^2$. That $G \subseteq L(H)$ is trivial from construction. Let (y,x), (x,z) be two edges of H. If $d(x) \geq 3$ and $(y,x)^-$, $(z,x)^- \notin M_o$, there is an arc $(a,x)^- \in M_o$ and hence $v(y,x)$, $v(z,x)$ are adjacent in G^2. The other nontrivial case is when $d(x) = 2$ and $(y,x)^-$, $(z,x)^- \notin M_o$; but then $\Delta = \{x,y,z\}$ must be a triangle. Two cases arise.

[†] $(.,x)^-$ means some arc into x.

[*] Such a triangle Δ has exactly one vertex of degree ≥ 3. Do not choose an arc $(.,x)^-$ for each of the two vertices of degree 2 in Δ, just for one of them.

Case 1. $d(y)$ and $d(z) \geq 3$. One of the following pairs of arcs is in M_o: $(x,y)^-$, $(x,z)^-$; $(x,y)^-$, $(y,z)^-$; $(x,z)^-$, $(z,y)^-$; $(y,z)^-$, $(z,y)^-$. Therefore $(v(y,x),v(x,z)) \in E(G^2)$.

Case 2. Say, $d(y) = 2$, $d(z) \geq 3$. We leave the verification to the reader.

We show that girth of $G \geq 7$. Let $\xi = (e_1,e_2,\ldots,e_k,e_1)$ be a smallest cycle of G. (k is necessarily ≥ 4). Consider the partial subgraph $\underline{H}$ of H induced by the edges $\{e_1,e_2,\ldots,e_k\}$ of H. Let $\underline{M} = \{(x,y)^- \in M_o: (x,y) \in E(\underline{H})\}$ and $\underline{V}$ be $\underline{M}$-unexposed vertices. For $(x,y)^- \in \underline{M}$ it is easily seen that $2 \leq d_{\underline{H}}(y) \leq 3$. If $d_{\underline{H}}(y) = 3$ then the three edges of $\underline{H}$ at y are consecutive: e_{i-1}, $e_i = (x,y)$, e_{i+1}; furthermore vertex x is $\underline{M}$-exposed and the other end vertices of the edges e_{i-1}, e_{i+1} are in $\underline{V}$. For $d_{\underline{H}}(y) = 2$ we leave it to the reader to verify that the two vertices that y is adjacent to in $\underline{H}$ are in $\underline{V}$ and the two edges of $\underline{H}$ at y are e_i, e_{i+1} (or e_i, e_{i-1}). Thus $\underline{H}|\underline{V}$ has all vertices of degree 2 and it should not be difficult to see that $\underline{H}|\underline{V}$ is a cycle, say, $C = (x_0,x_1,\ldots,x_t,x_0)$. Also note that $x_i \in C$ and $d_{\underline{H}}(x_i) = 2$ are covered by arcs of $\underline{M}$ belonging to C. It is not hard to see that C has four or more edges (use 3.8(i)). We claim that C has no chord in H. If possible, let $e = (x_0,x_s)$ be a chord and s is minimum. Suppose $s = 2$. Since $e \notin E(\underline{H})$, considering $\Delta = \{x_0,x_1,x_2\}$ we have $(x_1,x_0)^-$, $(x_1,x_2)^- \in M$ and $d_H(x_1) = 2$. But this being impossible, because x_1 is unexposed, $s \geq 3$. Suppose $d_{\underline{H}}(x_0) = 3$, $(y,x_0)^- \in \underline{M}$; then ξ is not a smallest cycle since the arc

$v(y,x_0), v(x_0,x_1), \ldots, v(x_s,x_{s+1})$ in ξ can be replaced by one of the following,

$$v(y,x_0), e, v(x_{s-1},x_s), v(x_s,x_{s+1}) \quad \text{if} \quad (x_{s-1},x_s)^- \in \underline{M}$$

$$v(y,x_0), e, v(x_s,x_{s+1}) \quad \text{if} \quad (x_{s+1},x_s)^- \in \underline{M}$$

$$v(y,x_0), e, v(z,x_s), v(x_s,x_{s+1}) \quad \text{if} \quad d_{\underline{H}}(x_s) = 3 \ \text{and} \ (z,x_s)^- \in \underline{M}$$

Thus $d_{\underline{H}}(x_0) = 2$ and similarly $d_{\underline{H}}(x_s) = 2$. Further reasoning in a similar spirit (if $d_{\underline{H}}(x_0) = 2$ etc.) shows that $s \not> 3$ (hence $s = 3$) and $(x_1,x_0)^-$, $(x_2,x_3)^- \in \underline{M}$ and $d_{\underline{H}}(x_1) = 2 = d_{\underline{H}}(x_2)$. But this contradicts 3.8(ii) for the 4-cycle of H (x_0,x_1,x_2,x_3,x_0). Thus C has no chord. Now the vertices $x_i \in V(C)$, $d_{\underline{H}}(x_i) = 2$ are covered by arcs of M_o belonging to C. Therefore C has at most $k-t$ escape-arcs in M_o. By 3.8(ii), $7-t \leq (k-t)$, or $7 \leq k$.

"Necessity". Let the line graph $L(H)$ be a partial square of a graph G of finite girth ≥ 7. We construct a d-matching M going from vertices to vertices of H as follows. Notice that $L(H)$ being a clique graph, by (2.1), every triangle of H has a vertex of degree 2. Also two triangles do not meet at exactly one point.[†] If $d_H(x) \geq 3$, or $= 2$ and x does not belong to a triangle of H, then $L_x = \{$set of edges at $x\}$ is a clique in $L(H)$. There exists an edge $e_x \in L_x$ such that $N_G(e_x) \supseteq L_x$ by (4.2), Chapter 1; e_x is unique if degree of $x \geq 3$. Orient the edge e_x into the vertex x. Let $d_H(x) = 2$ and x belong to the triangle $\Delta = \{x,y,z\}$ in H and let $d_H(z) \geq 3$. Since the

[†]As in proof of (3.5), the intersection pattern of the cliques $L(\Delta_1)$, $L(\Delta_2)$, L_x is an impossibility in a partial square of a graph of finite girth ≥ 7 as it is for infinite girth (i.e., a tree).

vertices $L_\Delta = \{v(x,y), v(y,z), v(z,x)\}$ form a clique of $L(H)$ and girth $G \geq 7$ we have that $e_z = (y,z)^-$ or $(x,z)^-$ and e_z covers z (prove it by contradiction). If $d_H(y) \geq 3$ we do nothing for x. Observe that in case there is another triangle $\Delta' = \{x', y, z\}$ then e_z is also one of $(x',z)^-$, $(y,z)^-$; hence $e_z = (y,z)^-$ and similarly $e_y = (z,y)^-$. Finally, if we had $d_H(y) = 2$ choose only one of the arcs $(x,y)^-$, $(y,x)^-$ in M for the pair of vertices x, y. An M-exposed vertex is either of degree 1 or of degree 2 belonging to a triangle.

We show that M satisfies properties 3.8(i), (ii). That M is a d-matching is clear from construction as also the property 3.8(i). Property 3.8(ii) follows because girth G is, by hypothesis, at least 7.

<u>REMARK 3.9</u>. In the proof of sufficiency in (3.8) we can replace 7 by an arbitrary number $g \geq 5$ and get a very general theorem. However, number 7 plays an important role throughout the necessary part, so much so that it cannot be replaced by 5 or 6.

<u>Example 6</u>. In the graph H (Fig. 18) there is no symmetric d-matching M with properties 3.8(i), (ii). Consider the cycle $(x_1,x_2,x_3,x_4,x_5,x_1)$. Edges 6, 7 must be in M if M is symmetric. But then there is no way to help the cycle $(x_1,x_5,x_4,x_8,x_6,x_1)$. A d-matching M is indicated by arrows on the arcs of M.

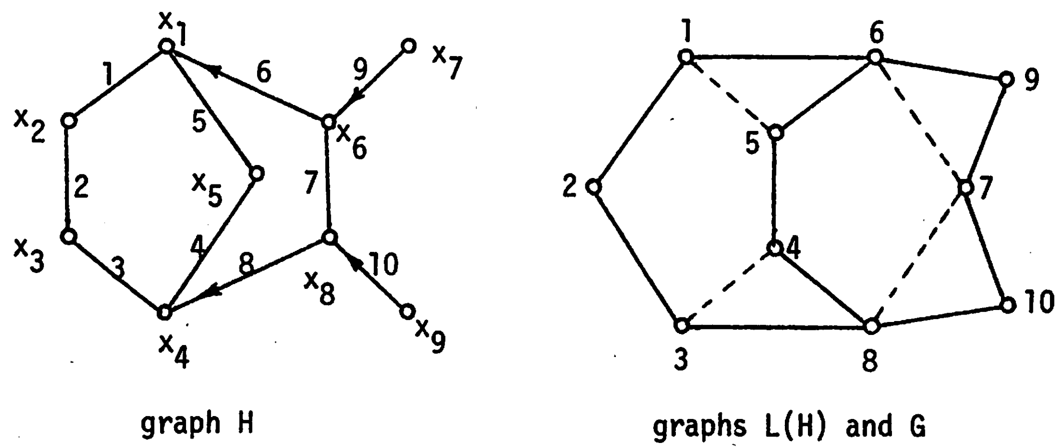


Fig. 18. L(H) is a clique graph. Corresponding to the d-matching M, the edges of G are shown in bold lines.

4. Miscellaneous Theorems on Clique Graphs

An example of a non-clique graph:

THEOREM 4.1. If G is a graph of girth 3 then the total graph T(G) is not a clique graph.

Proof. Let $\Delta = \{x_1,x_2,x_3\}$ be a triangle of G and $a = (x_1,x_2)$, $b = (x_2,x_3)$, $c = (x_3,x_1)$. The triangles $\Delta_1 = \{a,x_1,x_2\}$, $\Delta_2 = \{b,x_2,x_3\}$, $\Delta_3 = \{c,x_1,x_3\}$ are cliques of T(G). If possible let $\{L_i\}$ be a cover by complete subgraphs of T(G) and $\{L_i\}$ be $\mathcal{I}$. Then the edges (x_1,x_2), (x_1,a), (x_2,a) of T(G) must be in the same L_i for some i and thus $L_i = \{x_1,x_2,a\}$. Let $\Delta_i = L_i$, $i = 1,2,3$. But then $\{L_i\}$, $1 \leq i \leq 3$ is $\mathcal{P}$-family but not $\mathcal{I}$, a contradiction. Thus T(G) is not a clique graph.

However, there are square graphs G^2, girth $G = 6$, which are clique graphs.

Example 7. Let G be the graph induced by the solid edges in

Fig. 19. The graph G^2 is the union of 6 cliques $N_G(x_i)$, $1 \leq i \leq 6$ each of which is K_4. The cliques of G^2 have property I.

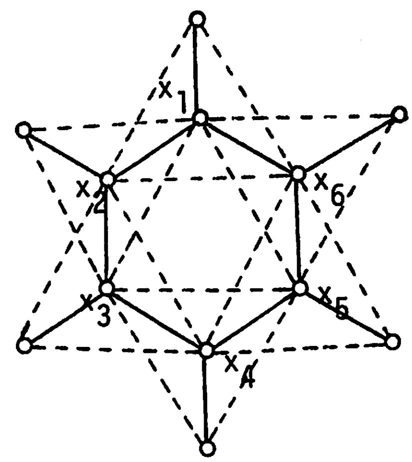


Fig. 19. A clique graph

Theorem 4.2 characterizes a graph G whose clique graph is bipartite. Since a graph of girth ≥ 4 is always a clique graph, all bipartite graphs are in $K(\mathcal{G})$.

<u>THEOREM 4.2</u>. The clique graph $K(G)$ is bipartite if and only if G satisfies the conditions:

i) A vertex is in at most two cliques of G.

ii) Every odd cycle of length ≥ 5 has a chord.

<u>Proof</u>. Necessity. Suppose $C = (x_1, x_2, \ldots, x_k, x_1)$, $k \geq 5$ is an odd cycle and has no chord. Let $K_{i,i+1}$, $1 \leq i \leq k$ $(k+1 \equiv 1 \bmod k)$ be cliques of G containing $\{x_i, x_{i+1}\}$. They are all distinct. The vertices $K_{i,i+1}$, $i \leq k$ of $K(G)$ form an odd cycle (taken in that order). Trivially, a vertex should not belong

to three cliques of G.

Sufficiency. Let $C = (K_1, K_2, \ldots, K_k, K_1)$ be a smallest odd cycle in $K(G)$. C has no chord. If possible, let $k \geq 5$. Choose a vertex of G, $x_i \in K_i \cap K_{i+1}$ for each $1 \leq i \leq k$. Then $C' = (x_1, x_2, \ldots, x_k, x_1)$ is a cycle in G and fortunately has no chord either. This is easy to see. Thus we contradict (ii) and hence $k = 3$. We shall now prove that $K_1 \cap K_2 \cap K_3$ is non empty which is impossible and the theorem will be proved. If not, x_1, x_2, x_3 (as above) are distinct and form a triangle. Let K be any clique containing the triangle. If $K = K_1$ then $x_2 \in K_1 \cap K_2 \cap K_3$; otherwise $x_2 \in K \cap K_2 \cap K_3$.

We close this chapter with two theorems on product graphs. The product graphs $G \odot H$, $G \circledast H$ are defined as follows. They have vertices $V(G) \times V(H)$ while the edges are respectively

$$E(G \odot H) = \{((x,y),(x',y'): (x,x') \in E(G),\ (y,y') \in E(H)\}$$

$$E(G \circledast H) = \{((x,y),(x,y')): (y,y') \in E(H),\ x \in V(G)\}$$
$$\cup \{(x,y),(x',y): (x,x') \in E(G),\ y \in V(H)\} \cup E(G \odot H).$$

<u>THEOREM 4.3</u>. If G, H are clique graphs then the Kronecker product $G \odot H$ is a clique graph.

<u>Proof</u>. Let $\{G_i: 1 \leq i \leq m\}$, $\{H_j: 1 \leq j \leq n\}$ be two families of complete subgraphs of G and H respectively which satisfy I and $\cup G_i = G$, $\cup H_j = H$ as in (1.1). Define $(x_t, H_j) = \{(x_t, y_k): y_k \in H_j\}$ and $(G_i, y_s) = \{(x_k, y_s): x_k \in G_i\}$ for all $x_t \in V(G)$, $y_s \in V(H)$; they are complete subgraphs of $G \odot H$

and their union is all of $G \odot H$. Let $P \subseteq \{(x_t,H_j),(G_i,y_s)\}$ be a $\underline{P}$-family. (x_t,H_j), $(x_{t'},H_{j'}) \in P$ implies $x_t = x_{t'}$ and $H_j \cap H_{j'} \neq \emptyset$. Thus P can be written as

$$P = \{(x_t,H_{j_1}),(x_t,H_{j_2}),\ldots,(x_t,H_{j_k})\}$$
$$\cup \{(G_{i_1},y_s),(G_{i_2},y_s),\ldots,(G_{i_p},y_s)\} \quad .$$

But $(x_t,H_{j_1}) \cap (G_{i_1},y_s) \neq \emptyset$ implies $x_t \in G_{i_1}$, $y_s \in H_{j_1}$. We conclude $(x_t,y_s) \in \cap P$. Thus $G \odot H$ is a clique graph.

<u>THEOREM 4.4</u>. $K(G \circledast H) = K(G) \circledast K(H)$.

In particular, $\circledast$-product of clique graphs is a clique graph.

<u>Proof</u>. The cliques of $G \circledast H$ are of the form $C \times D$ where C, D are cliques of G and H respectively. If C, D are cliques, clearly $C \times D$ is a complete subgraph. On the other hand, if $S \subseteq V(G) \times V(H)$ a clique then S_1, S_2 are complete where S_1 (S_2) is projection of S on $V(G)$ (resp. $V(H)$). Since $S \subseteq S_1 \times S_2$ we have $S = S_1 \times S_2$. It follows that $C \times D$ is a clique.

Finally $C \times D \cap C' \times D' \neq \emptyset$ if and only if $C \cap C' \neq \emptyset$, $D \cap D' \neq \emptyset$. This is the same as saying $(C,C') \in E(K(G))$ and $D = D'$, or $(D,D') \in (K(H))$ and $C = C'$, or (C,C'), (D,D') are edges of $K(G)$, $K(H)$. This proves the theorem.

Notations.

Arc: If $\xi = (x_0, x_1, \ldots, x_{k-1}, x_0)$ is a cycle then the path $(x_0, x_1, \ldots, x_i)$, $i \leq k-1$ is called an arc of ξ.

Articulation vertex: It is a vertex x such that $G - x$ has more components than G.

Big triangle: See Fig. 11, page 38.

Core: The point induced subgraph by the nonterminal points.

$c(G)$: Core of G.

Component: 1-component

Connected: If G is 1-connected then we simply say that it is connected.

Cutset: A minimal set of edges whose removal from the graph increases the number of components. This is the same as a cocircuit of the graphic matroid.

Chord: An edge (x_i, x_j), $|i-j| \neq 1 \pmod{k}$, is called a chord of cycle $(x_0, x_1, \ldots, x_{k-1}, x_0)$.

Cycle $(x_0, \ldots, x_{k-1}, x_0)$: It is a path whose first and last vertices are the same. A cycle of length n is called an n-cycle.

d-matching: See page 49.

$d_G(x)$: Degree of vertex x in G, $d_G(x) = |\Gamma_G(x)|$.

$d_G(x,y)$: Distance of vertices x and y in graph G.

G: $G = (V(G), E(G))$ is a finite graph without multiple edges and loops.

$G \subseteq H$: G is a partial subgraph of H, i.e., $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

$G|S$: For a subset of vertices S, it denotes the point induced subgraph of G on S.

$\Gamma_G(x)$: Set of vertices adjacent to x in G.

G^2: Square of G. $V(G^2) = V(G)$, $E(G^2) = \{(x,y): d_G(x,y) \leq 2\}$.

$G \odot H$: See page 56.

$G \circledast H$: See page 56.

Girth: Length of the shortest cycle. Girth of a tree is by definition ∞.

$H \cup G$: It is the graph whose vertices are $V(H) \cup V(G)$ and edges $E(H) \cup E(G)$.

$H - G$: It is defined for $H \supseteq G$ and $H - G$ has vertices $V(H)$ and edges $E(H) - E(G)$.

K_n: The complete graph on n vertices.

$M(G)$: The graphic matroid on $E(G)$. Circuits of $M(G)$ are the cycles of G.

Multiple point: A vertex belonging to two or more 3-components.

$N_G(x)$: $x \cup \Gamma_G(x)$, called neighborhood of x.

$\underline{P}$: See page 34.

Partial square: G is partial square of H if $H \subseteq G \subseteq H^2$.

Path$(x_0,x_1,\ldots,x_k)$: Except in Part II, we always take the vertices $x_0,\ldots,x_k$ distinct unless $x_0 = x_k$ when the path becomes a cycle. The length of the path is the number of edges in it, k (namely, (x_i,x_{i+1}), $0 \leq i \leq k-1$). In Part II, a path is indicated by the sequence of edges (x_i,x_{i+1}).

Property I: See page 34.

T: A tree T is a connected graph having no cycle.

Terminal point: A vertex x of degree one.

$t(x)$: Terminal vertices adjacent to x.

Thick tree: See page 45.

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PART II: Factorization of Graphs

THE k-FACTOR CONJECTURE IS TRUE

We shall show that if sequences $\langle d_i \rangle$, $\langle d_i - k \rangle$ are graphical then there exists a graph G with degrees d_i which has a factor with k lines at each vertex. Similar results have been obtained for digraphs. Also some other related problems are discussed.

To every graph G whose vertices are labelled v_i, $1 \leq i \leq n$, one can associate the degree sequence $\langle d_i \rangle$ where d_i = degree of the vertex v_i. G is called a representing graph for $\langle d_i \rangle$ and the sequence itself is called graphical. The degree sequence is an 'invariant' of a graph. It is a rather 'weak' invariant and there is almost always more than one graph with the same degree sequence [2]. This 'incompleteness' of degree sequences allows one to raise many existence problems about representing graphs. In that regard, the following conjecture was made by A.R. Rao and S.B. Rao [8] and also by B. Grünbaum.

If $\langle d_i \rangle$, $\langle d_i - k \rangle$ are graphical sequences then there exists a graph G with degrees $\langle d_i \rangle$ which has a factor with k lines at each vertex.

Also a similar conjecture for digraphs was made by A.R. Rao and S.B. Rao [8]. We shall prove a generalized version for each of them, first for graphs and then for digraphs. The ideas involved in the two cases are very similar. The concept of an alternating path will be important throughout.

1. UNDIRECTED GRAPHS

We shall assume that graphs have no multiple lines and loops.* All graphs are drawn on a fixed set of vertices $V = \{v_1, v_2, \ldots, v_n\}$. Therefore it is convenient to identify a graph G with the subset of unordered pairs $\{(v_i, v_j)\}$, where (v_i, v_j) are lines of G. A sequence (of n nonnegative integers) $\langle d_i : 1 \leq i \leq n \rangle$ is called <u>graphical</u> if there exists a graph G with degree of v_i being equal to d_i for all i. We say that G is a representing graph of $\langle d_i \rangle$. For a given sequence $\langle k_i : 0 \leq k_i \leq d_i \rangle$ a subgraph $F \subseteq G$ is called a <u>subfactor</u> if F has at most k_i lines at v_i. Call v_i a saturated vertex (with respect to F) if F has exactly k_i lines at v_i. We shall denote by $S = S(F)$ the set of saturated vertices. F is called a <u>factor</u> if $S = V$. If $k_i = k$, $1 \leq i \leq n$, F is called a k-factor. We often consider two graphs G, H simultaneously. To distinguish their lines we shall put colors on the lines (v_i, v_j) as follows: lines of G (resp. H) not in H (resp. G) are colored red (resp. blue), the lines common to G and H are colored green and all other lines are colored white. We shall write r = red, b = blue, g = green, w = white and $c(v_i, v_j)$ for the color of line (v_i, v_j). A few other notations like $g = r+b$, $r = g-b$, $w = b-b$, etc. will be useful. We shall let $E_c(v_i)$ denote the set of lines at vertex v_i with color c, $c = r,b,g$, and $E_c = \cup E_c(v_i)$, union over all v_i. Clearly $|E_r(v_i)| + |E_g(v_i)|$,

*We shall use the word line for edge. A loop is an edge whose endpoints are equal.

$|E_b(v_i)| + |E_g(v_i)|$ are respectively the degrees of v_i in G and H. Finally, an <u>alternating path</u> $P = (x_0,x_1),(x_1,x_2),(x_2,x_3),\ldots$ is a path whose lines are distinct and $c(x_i,x_{i+1}) = r$ or b according as i even or odd.

<u>THEOREM 1.1</u>. Let $\langle d_i\rangle$, $\langle d_i-k_i\rangle$ be two graphical sequences such that for some $k \geq 0$, $k \leq k_i \leq k+1$ for $1 \leq i \leq n$. Then there is a graph G with degree sequence $\langle d_i\rangle$ and having a $\langle k_i\rangle$-factor.

<u>Proof</u>.

Consider two graphs G', H' with degree sequences $\langle d_i\rangle$, $\langle d_i-k_i\rangle$ respectively and the associated coloring of the lines (v_i,v_j) in white, red, blue and green. Clearly, $|E_r(v_i)| = |E_b(v_i)| + k_i$ for all vertex v_i. Let $F' \subseteq E_r$* be a $\langle k_i\rangle$-subfactor; F' is possibly empty. Suppose that the graphs G, H and a subfactor F are so chosen that $|F| + |E_g|$ has maximum value among all possible choices of G', H', F'. If all vertices v_i are saturated in F we are done. We shall show that it is indeed so. This is accomplished in several steps. Let $S = S(F)$ and assume that $S \neq V$.

1^0) If x_0, x_1, x_2, x_3 are four distinct vertices such that $c(x_0,x_1) = b = c(x_2,x_3)$, $c(x_1,x_2) = r$ and $(x_1,x_2) \notin F$ then $c(x_0,x_3) = b$.

<u>Proof</u>. If $c(x_0,x_3) = r$ or g then changing the colors $c(x_0,x_1)$, $c(x_2,x_3)$ from b to $g = b+r$, $c(x_1,x_2)$ to w and $c(x_0,x_3)$

*All subfactors will be a subset of red lines.

to $c(x_0,x_3) - r$ we increase $|E_g|$ by two or one according as $c(x_0,x_3) = r$ or g. In the worst case, when $(x_0,x_3) \in F$ we form the new subfactor $F - (x_0,x_3)$. In any case, $|F| + |E_g|$ has been increased, a contradiction.

If $c(x_0,x_3) = w$ then change each of (x_0,x_1), (x_2,x_3) to a white line while adding blue to $c(x_1,x_2)$ and $c(x_0,x_3)$. The result is an increase in $|E_g|$ without changing F. Thus $c(x_0,x_3) = b$. Note that the changes in colors did not disturb the equations $|E_r(v_i)| + |E_g(v_i)| = d_i$, $|E_b(v_i)| + |E_g(v_i)| = d_i - k_i$, $1 \leq i \leq n$. This will always be the case in all recolorings.

2^0) Let $(x_0,x_1),(x_1,x_2),\ldots,(x_{2t},x_{2t+1})$, $t \geq 1$, $x_0 \neq x_{2t+1}$ be an alternating path P which is line disjoint with F and $(x_0,x_{2t+1}) \notin P$. Then $c(x_0,x_{2t+1}) = r$.

<u>Proof</u>. Suppose $c(x_0,x_{2t+1}) \neq r$. We show as in 1^0 that by suitable recoloring of the lines of P and the line (x_0,x_{2t+1}) we can increase $|E_g|$, F remaining unchanged. For example, if $c(x_0,x_{2t+1}) = b$ or g then change the color of all red lines of P to green by adding blue to them, change the color of all blue lines of P to white and $c(x_0,x_{2t+1})$ to $c(x_0,x_{2t+1}) - b$. If $c(x_0,x_{2t+1}) = w$ then change it to red, $c(x_{2i},x_{2i+1})$ to white for $0 \leq i \leq t$ and $c(x_{2i+1},x_{2i+2})$ to green for $0 \leq i \leq t-1$.

Next, observe that for each vertex $v_i \in V - S$, there are at least $1 + |E_b(v_i)|$ red lines not in F which are incident with

v_i whereas for $v_i \in S$, $|E_r(v_i) - F| = |E_b(v_i)|$. This is straight forward from the definition of S. Also note that a red line with both end points in $V-S$ is necessarily in F. (Otherwise we can add it to F!) Choose a vertex x_0 in $V-S$ and a red line $(x_0,x_1) \notin F$; $x_1 \in S$. There is a blue line, say (x_1,x_2) and thus a red line $(x_2,x_3) \notin F$ (x_3 is possibly same as x_0) and a blue line (x_3,x_4) if $x_3 \notin V-S$. One can proceed in this way and get an alternating path (line) disjoint with F and terminating at a vertex in $V-S$. Let $P = (x_0,x_1),(x_1,x_2),\ldots,(x_{2t},x_{2t+1})$, be an alternating path with smallest number of lines among all the alternating paths from x_0 terminating in $V-S$ and being disjoint with F. It is shown in part 6^0 that $t = 1$ or 2; moreover, if $t = 2$ we have $x_1 = x_4$ (see fig. 1). For $t = 1$, there are two possibilities: $x_0 \neq x_3$ (fig. 2) and $x_0 = x_3$. The case $x_0 = x_3$ will be taken up in 4^0 and 5^0 while 3^0 deals with the other cases.

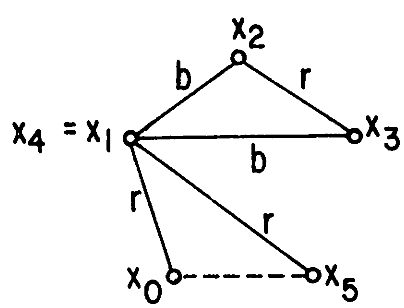

Fig. 1. $t = 2$. The broken line is in F.

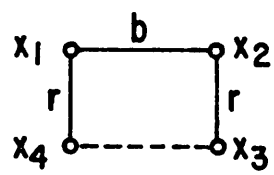

Fig. 2. $t = 1$ and $x_0 \neq x_3$.

3^0) Each of the following gives a contradiction. The alternating path P has 1) 5 lines, 2) three lines and $x_0 \neq x_3$.

Proof. Let us write $y = x_5$ or x_3 according as we are in 1) or 2). By 2^0, $c(x_0,y) = r$ and x_0, y being in $V-S$, $(x_0,y) \in F$. Thus $k(y) \geq 2$ (where $k(y) = k_i$ if $y = v_i$) and therefore by the hypothesis of the theorem $k(x_1) \geq 1$. Let $(x_1,u) \in F$. Note that in the case 2) we can assume that $u \neq y = x_3$ because otherwise $k(y)$ is in fact ≥ 3 and thus $k(x_1)$ being at least 2 we can find a vertex $v \neq x_3$, such that $(v,x_1) \in F$. In case 1) obviously $u \neq y$. Let us define $F' = F - (u,x_1) + (x_1,x_0)$; $|F'| = |F|$. Consider the path Q from u to y obtained by replacing (x_0,x_1) in P with (u,x_1). Since u and y are unsaturated with respect to F' and Q is disjoint from F', by 2^0, $(u,y) \in F'$ and hence $(u,y) \in F$. We can now say that $k(y) \geq 3$ and obtain another vertex $u' \neq u$, such that u' is not incident with any line of P and $(u',x_1) \in F$. Repeating the same argument again and again we obtain $k(y)$ is arbitrarily large which is certainly impossible.

4^0) $t = 1$ and $x_0 = x_3$. Then there is a blue line at x_0.

Proof. Suppose not. Then $k(x_0) = |E_r(x_0)| \geq 2$, and therefore $k(x_1) \geq 1$. Obtain a vertex u such that $(u,x_1) \in F$. Define, $F' = F - (u,x_1) + (x_0,x_1)$ as before and consider the path (u,x_1), (x_1,x_2), (x_2,x_0). But then we are back in 3^0 which is just shown not possible.

5^0) $t = 1$ and $x_0 = x_3$. There cannot be a blue line at x_0.

Proof. Suppose there is a blue line at $x_3 = x_0$, say (x_3,x_4). Consider all possible alternating paths Q from x_0 to some point of $V-S$ which contains P properly as an initial part and disjoint with F. Paths Q exist because for all points $v_i \in S$, at v_i there are as many red lines not belonging to $F \cup P$ as there are blue lines not in $F \cup P$. No such path Q 'returns' to the vertex x_0 'after' x_3.* Let $P' = P \cup P_0$ have smallest number of lines among all Q. We show that P_0 has two lines only.

By 1^0 and proper choices of three lines one can show that $c(x_1,x_4) = c(x_2,x_4) = b$. If P_0 has three or more lines let (x_3,x_4), (x_4,x_5), (x_5,x_6) be the first three lines. $x_1,x_2,\ldots,x_5$ are all distinct. $c(x_4,x_5) = r$ implies, by 2^0, $c(x_5,x_i) = r$, $1 \leq i \leq 3$. Thus x_6 is different from x_i, $1 \leq i \leq 5$. But $c(x_3,x_6) = b$ (by 1^0) and hence $(x_3,x_6) \notin P \cup P_0$ implies we can replace the subsequence (x_3,x_4), (x_4,x_5), (x_5,x_6) in P_0 by (x_3,x_6) and we get a shorter alternating path contradicting

*Because it will imply the existence of an even cycle C disjoint with F and whose lines are alternately blue and red. But then one can increase $|E_g|$ ($|F|$ remaining fixed) by making the red lines of C white and blue lines of C green.

minimality of P_0. Thus $P_0 = (x_3,x_4), (x_4,x_5)$. (see fig. 3.) Now consider the path $P'' = (x_0,x_1), (x_1,x_4), (x_4,x_5)$. P'' is line disjoint with F. However, existence of such an alternating path is shown to be impossible in 3^0.

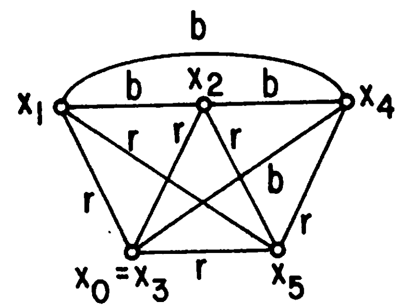

Fig. 3. x_0, x_5 are in V-S.

6^0) The shortest alternating path $P = (x_0,x_1),(x_1,x_2),\ldots,(x_{2t},x_{2t+1})$, starting at x_0 and terminating at a vertex in V-S, has at most 5 lines. P is line disjoint with F.

<u>Proof</u>. It is useful to regard (for the moment) the lines of P being oriented in the direction from x_i to x_{i+1}, $0 \leq i \leq 2t$; we write them as arc $(x_i,x_{i+1})^-$. Then at a vertex v_i, there is at most one line, of each color, directed from and one line directed into v_i that belong to P. For example, if there are two blue lines directed from v_i one of them precedes the other as one traverses P. But this implies that P 'enters' v_i with a red line after it had left v_i by the first blue line. In other words there is an even cycle whose lines are alternately blue and red. As we have seen earlier this would imply that $|F| + |E_g|$ is not maximum, contrary to the assumption. Suppose P has five or more lines.

If possible, let there be three consecutive lines of P as follows: $c(x_i,x_{i+1}) = c(x_{i+2},x_{i+3}) = b$, $c(x_{i+1},x_{i+2}) = r$ and $x_i \neq x_{i+3}$. By 1^0, $c(x_i,x_{i+3}) = b$; (x_i,x_{i+3}) must be a line of P (otherwise we can replace the sequence $(x_i,x_{i+1}),\ldots,(x_{i+2},x_{i+3})$ by (x_i,x_{i+3})). Further, the line (x_{i+3},x_i) is oriented into x_i (fig. 4).

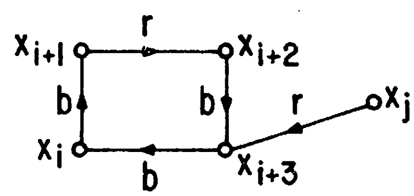

Fig. 4.

Let $(x_{j-1},x_j)^-$, $(x_j,x_{j+1} = x_{i+3})^-$ be the two lines of P immediately preceding $(x_{i+3},x_i)^-$; they are respectively blue and red and the blue line exists if $x_j \neq x_0$. Observe that $x_j \neq x_p$: $i \leq p \leq i+3$, and $x_{j-1} \neq x_i$. By 1^0, $c(x_i,x_{j-1}) = b$, and it cannot be in P. However, this contradicts the minimality of P (as the sequence $(x_{j-1},x_j)^-$, $(x_j,x_{i+3})^-$, $(x_{i+3},x_i)^-$ can be replaced by $(x_{j-1},x_i)^-$). Thus $x_j = x_0$. Similarly, the red line following $(x_{i+3},x_i)^-$ in P must be (x_i,x_{2t+1}). But then (x_0,x_{i+3}), (x_{i+3},x_i), (x_i,x_{2t+1}) is an alternating path disjoint with F, a contradiction. Thus $x_i = x_{i+3}$. Then the path P can be written as

$$P = (x_0,x_1),(x_1,x_2),(x_2,x_3),(x_3,x_1),(x_1,x_5),\ldots,(x_{2t},x_{2t+1}) .$$

If $t \geq 3$, then the 6th line (x_5,x_3) is blue and we can replace the first six arcs by $(x_0,x_1)^-$, $(x_1,x_3)^-$ to obtain an alternating path with less lines than P. Thus $t \leq 2$ and in case P has five lines it conforms to the description that $x_1 = x_4$.

The contradictions in 3^0 together with 4^0, 5^0 show that $V-S \neq \phi$ is impossible. Therefore $V = S$ and we have proved the theorem.

<u>Remark</u>. If there exists a graph with degree sequence $\langle d_i \rangle$ and containing a $\langle k_i \rangle$- factor then, trivially, $\langle d_i - k_i \rangle$ is graphical.

The following examples show that the theorem is not true if two k_i's differ by two or more.

<u>Example 1</u>. Let $\langle d_i \rangle = \langle 5,5,4,3,3,2 \rangle$ and $\langle k_i \rangle = \langle 3,3,1,3,3,1 \rangle$. Each of $\langle d_i \rangle$, $\langle k_i \rangle$, $\langle d_i - k_i \rangle = \langle 2,2,3,0,0,1 \rangle$ is a graphical sequence. If Theorem 1.1 was true for these $\langle d_i \rangle$, $\langle k_i \rangle$ then it would be possible to find a graph G with degree sequence $\langle d_i \rangle$ that contains the unique graph H (fig. 5) with degree sequence $\langle d_i - k_i \rangle$. Unfortunately, there is no such G, as can be seen easily.

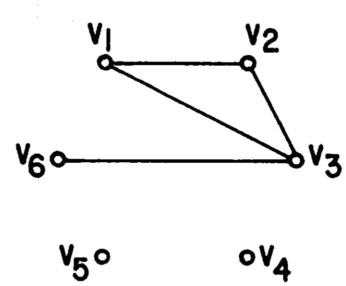

Fig. 5. The graph H with degree sequence $\langle d_i - k_i \rangle$.

Example 2. The sequences $\langle d_i\rangle = \langle 4,3,2,2,1\rangle$, $\langle k_i\rangle = \langle 3,1,2,2,0\rangle$, $\langle d_i - k_i\rangle = \langle 1,2,0,0,1\rangle$ are graphical. But there is no graph G whose degree sequence is $\langle d_i\rangle$ and which contains the graph F with degrees $\langle k_i\rangle$.

COROLLARY 1.2. If $\langle d_i\rangle$ is graphical then there is a graph G with degree sequence $\langle d_i\rangle$ and having a k-factor if and only if $\langle d_i - k\rangle$ is graphical.

A result of A.R. Rao and S.B. Rao [7,8] on connected factors implies rather immediately

COROLLARY 1.3. There exists a graph with degree sequence $\langle d_i\rangle$ and containing a Hamiltonian cycle if and only if $\langle d_i\rangle$, $\langle d_i - 2\rangle$ are graphical and for all $p < \frac{n}{2}$, $\sum_{i=1}^{p} d_i^* < p(n-p-1) + \sum_{n-p+1}^{n} d_i^*$ where $\langle d_i^*\rangle$ is a rearrangement of $\langle d_i\rangle$ into a non-increasing sequence.

COROLLARY 1.4. If $\langle d_i\rangle$, $\langle k_i\rangle$ are arbitrary graphical sequences such that $k \le d_i - k_i \le k+1$ for some k then there exists a graph G with degree sequence $\langle d_i\rangle$ and containing a $\langle k_i\rangle$– factor.

Proof. Interchange the role of $\langle k_i\rangle$ and $\langle d_i - k_i\rangle$ in Theorem 1.1. We have a graph G with degree sequence $\langle d_i\rangle$ which contains a factor F having $d_i - k_i$ lines at vertex v_i. The lines G–F form the required factor (see examples 1 and 2).

COROLLARY 1.5. If there exists a graph with degree sequence $\langle d_i\rangle$ and containing a k-factor, then for $0 < \ell < k$, $\ell \equiv k \pmod 2$ if n is odd, there is graph with degree sequence $\langle d_i\rangle$ and containing an ℓ-factor.

Proof. One simply notes that under the hypothesis of the corollary $\langle d_i-\ell\rangle$ is graphical. This follows (after some symmetrizations as shown by A.R. Rao) from a theorem of Fulkerson [1] on the existence of (0,1) matrices with given row sums and column sums and zero diagonal elements.

The case $\ell \equiv k \pmod 2$ for arbitrary n was obtained earlier in [7,8].

The next theorem gives a n.s.c. for existence of graphs G with given degree sequences and containing a given graph F. It happens that we have to assume a lot more than before in order that $G \supseteq F$. But at the same time such extra assumptions allow more flexibility, though not as much as one would wish, on the choice of F than those given by the degree sequences $\langle k_i\rangle$, $k \le k_i \le k+1$. This time the proof is by induction.

THEOREM 1.6. Let F be a graph not containing an induced subgraph isomorphic to the graph in fig. 6. There exists a graph G with degree sequence $\langle d_i\rangle$ and G containing the graph F if and only if for every graph $F' \subseteq F$ we have $\langle d_i-k_i\rangle$ is graphical where $\langle k_i\rangle$ is the degree sequence of F'.

Proof. The necessity is trivial. We prove 'if' part by induction on $|F|$. If $|F| = 1$ then (1.6) is same as (1.1). Let $|F| = m \ge 2$ and let the theorem be true for all graphs F having m-1 or less lines. Let (x,y) be a line in F and $F_0 = F-(x,y)$. F_0 does not contain a copy of the subgraph in fig. 6. Thus there exists a graph $G \supseteq F_0$ with degree sequence $\langle d_i \rangle$, and let (x,y) not be in any such G. Also there exists a graph $H \supseteq F_0$ which has d_i lines at all vertices $v_i \neq x, y$ where it has only d_i-1 lines. Consider the pair of graphs G, $H-F_0$ and associated coloring of the lines (v_i,v_j) in r,b,g,w. Suppose that G, H are so chosen that $|E_g|$ is maximum $(E_g \cap F_0 = \phi)$. It is easy to see that there is an alternating path from x to y which is line disjoint with F_0 and has one more red lines than blue lines; let P be one such path with minimum number of lines. If $(x,y) \notin P$ then as in 2^0 of (1.1) one can perform recoloring on $P \cup (x,y)$ so that (x,y) becomes green or red and we have proved the theorem. Let us therefore assume that $(x,y) = (x_i,x_{i+1})^*$ and $P = (x = x_0,x_1)$, $(x_1,x_2),\ldots,(x_i = x,x_{i+1} = y),(x_{i+1},x_{i+2}),\ldots,(x_{2t},x_{2t+1} = y)$; $c(x_i,x_{i+1}) = b$. Observe that P 'arrives' at y only at x_{i+1} and x_{2t+1}. Consider the subpath P' of P: from (x_1,x_2) to (x_i,x_{i+1}). If $c(x_1,x_{i+1})$ is blue then we can form alternating path $P'' = (x_0,x_1),(x_1,x_{i+1}),(x_{i+1},x_{i+2}),\ldots,(x_{2t},x_{2t+1})$ not containing (x,y). If $c(x_1,x_{i+1}) = w,g$, or $c(x_1,x_{i+1}) = r$ and $(x_1,x_{i+1}) \notin F_0$ then one can increase $|E_g|$ by a recoloring of $P' \cup (x_1,x_{i+1})$. Thus $(x_1,x_{i+1}) \in F_0$. Considering the path $(x_i,x_{i-1}),(x_{i-1},x_{i-2}),\ldots,(x_1,x_0),(x_0,x_{i+1}),(x_{i+1},x_{i+2}),\ldots,$ (x_{2t},x_{2t+1}) one can show that $(x_{i-1},x_{i+1}) \in F_0$. Similarly (x_i,x_{i+2}), $(x_i,x_{2t}) \in F_0$. But $x_1 \neq x_{i-1}$, $x_{i+2} \neq x_{2t}$ implies we have F contains the subgraph shown in fig. 5, a contradiction.

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*(x,y) cannot be (x_{i+1},x_i). Also note that $(x_1,x_{i+1}) \notin P$.

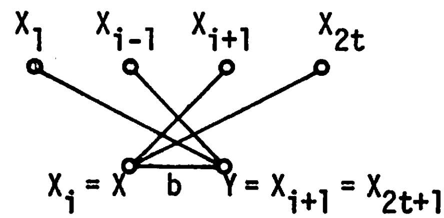


Fig. 6. The excluded subgraph in F.

This proves the theorem.

Following example shows that Theorem 1.6 may not be true if F does contain the graph in fig. 6.

<u>Example 3</u>. Let $\langle d_i \rangle = \langle 4,4,4,4,4,4 \rangle$ and F be $K_{3,3}$. F has 9 edges and thus 2^9 subgraphs. Since d_i's are same it is enough to check that $\langle d_i - k_i \rangle$ is graphical for different isomorphic subgraphs $F' \subseteq F$. For $|F'| \leq 5$ they are shown in table 1. There is no graph $G \supseteq F$ with degree sequence $\langle d_i \rangle$. You may note, however, that there is a graph G which contains all but one edge of $K_{3,3}$ (fig. 7).

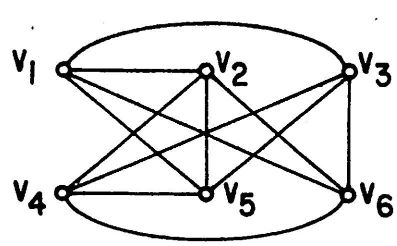
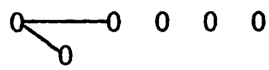
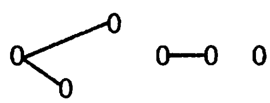
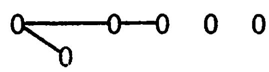
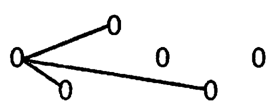
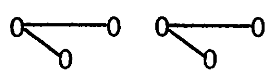
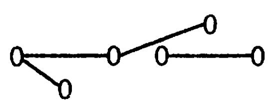
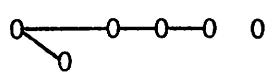


Fig. 7. Graph G containing all but one edge of F.

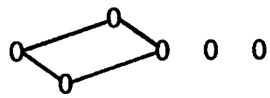
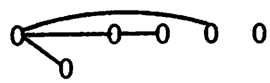
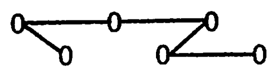
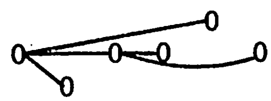
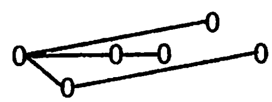
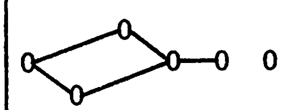
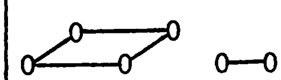
Table 1.

$\lvert F'\rvert$	isomorphic type* of $F' \subseteq F$	$\langle d_i - k_i \rangle$	representing graph
0	0 0 0 0 0 0	$\langle 4,4,4,4,4,4 \rangle$	G in fig. 6.
1	0—0 0 0 0 0	$\langle 3,3,4,4,4,4 \rangle$	$G_1 = G - (v_1,v_2)$
2	0—0 0—0 0 0	$\langle 3,3,3,3,4,4 \rangle$	$G_2 = G - \{(v_1,v_2),(v_3,v_4)\}$
		$\langle 2,3,3,4,4,4 \rangle$	$G_3 = G - \{(v_1,v_2),(v_1,v_3)\}$
3	0—0 0—0 0—0	$\langle 3,3,3,3,3,3 \rangle$	$G_4 = G - \{(v_1,v_3),(v_4,v_6),(v_2,v_5)\}$
		$\langle 2,3,3,3,3,4 \rangle$	$G_5 = G_3 - (v_4,v_5)$
		$\langle 2,3,2,3,4,4 \rangle$	$G_6 = G_3 - (v_3,v_4)$
		$\langle 1,3,3,4,3,4 \rangle$	$G_7 = G_3 - (v_1,v_5)$
4		$\langle 2,3,3,2,3,3 \rangle$	$G_8 = G_5 - (v_4,v_6)$
		$\langle 2,3,2,3,3,3 \rangle$	$G_9 \approx G_8$†
		$\langle 3,2,2,2,3,4 \rangle$	$G_{10} = G_6 - (v_4,v_5)$

*The vertices can be thought of as $v_1,v_2,\ldots,v_6$ in that order.

†$\approx$ stands for isomorphism.

Table 1 (continued)

$\|F'\|$	isomorphic type of $F' \subseteq F$	$\langle d_i - k_i \rangle$	representing graph
		$\langle 2,2,2,2,4,4 \rangle$	$G_{11} = G_6 - (v_2,v_4)$
		$\langle 1,3,2,3,3,4 \rangle$	$G_{12} = G_7 - (v_4,v_5)$
5		$\langle 2,3,2,2,2,3 \rangle$	$G_{13} = G_9 - (v_4,v_5)$
		$\langle 1,3,1,3,3,3 \rangle$	$G_{14} = G_{12} - (v_3,v_6)$
		$\langle 1,2,2,3,3,3 \rangle$	$G_{15} = G_{12} - (v_2,v_6)$
		$\langle 2,2,2,1,3,4 \rangle$	$G_{16} = G_{11} - (v_4,v_5)$
		$\langle 2,2,2,3,2,3 \rangle$	$G_{17} \approx G_{13}$

For $|F'| \geq 6$ the subgraphs are obtained by removing a subgraph of $9 - |F'|$ lines from F. There are 8 of them. The sequences $\langle d_i - k_i \rangle$ are listed below. They are all graphical as can be checked easily.

$\|F'\|$	$\langle d_i - k_i \rangle$
9	$\langle 1,1,1,1,1,1 \rangle$
8	$\langle 2,2,1,1,1,1 \rangle$
7	$\langle 2,2,2,2,1,1 \rangle$
	$\langle 3,2,2,1,1,1 \rangle$
6	$\langle 2,2,2,2,2,2 \rangle$
	$\langle 2,3,2,2,2,1 \rangle$
	$\langle 2,3,3,2,1,1 \rangle$
	$\langle 2,4,2,2,1,1 \rangle$

COROLLARY 1.7. There exists a graph G with degree sequence $\langle d_i \rangle$ and disjoint from F if and only if $\langle d_i+k_i \rangle$ are graphical where $\langle k_i \rangle$ is degree sequence of an arbitrary graph $F' \subseteq F$.

Proof. Note that $\langle d_i+k_i \rangle$ is graphical if and only if $\langle (n-1-d_i)-k_i \rangle$ is graphical. Rest is easy.

If $F = \{(v_1,v_2),(v_3,v_4),(v_5,v_6)\}$ and $\langle d_i \rangle = \langle 2,2,1,1,3,3 \rangle$ then the sequence $\langle d_i-k_i \rangle$ is graphical for all $F' \subseteq F$ except when $F' = \{(v_1,v_2),(v_3,v_4)\}$. And there is no graph G containing F with degree sequence $\langle d_i \rangle$.

COROLLARY 1.8. Let $\langle d_i \rangle$, $\langle k_i-d_i \rangle$ be two graphical sequences where $k \leq k_i \leq k+1$, $1 \leq i \leq n$ and $d_i \leq k_i \leq n-1$. Then there are disjoint graphs with degree sequences $\langle d_i \rangle$, $\langle k_i-d_i \rangle$.

Proof. There exists a graph G with degree sequences $\langle (n-1)-d_i \rangle$ and G containing a $\langle (n-1)-k_i \rangle$-factor F. The graphs K_n-G and $G-F$ satisfy the corollary (K_n is the complete graph).

The proof of Theorem 1.1 (and also 1.6, 2.1, 2.3) in fact yields an algorithm to obtain a graph G with degree sequence $\langle d_i \rangle$ containing a $\langle k_i \rangle$-factor (resp. required graphs and digraphs). To start with choose arbitrary graphs G, H with degree sequences $\langle d_i \rangle$, $\langle d_i-k_i \rangle$ respectively and a subfactor F. One constructs an alternating path $P = (x_0,x_1),(x_1,x_2),\ldots$ from a vertex $x_o \in V-S$, and then increases $|F|+|E_g|$ by recoloring some of the lines (v_i,v_j) as indicated in $1^0) - 6^0)$ until $|F|+|E_g| = \frac{1}{2}\Sigma d_i$.

2. DIRECTED GRAPHS

We shall assume that digraphs have no multiple arcs and loops and all digraphs are drawn on vertices $v_1,\dots,v_n$. A pair of arcs (v_i,v_j), (v_j,v_i) is however possible. Note that an arc from v_i to v_j is written as (v_i,v_j). Given a sequence of ordered pairs of non negative integers, $\langle(d_i^+,d_i^-)\rangle$, we say it is graphical if there exists a digraph $\vec{G}$ with outdegree and indegree of vertex v_i being equal to respectively d_i^+ and d_i^-. We say $\vec{G}$ has degree sequence $\langle(d_i^+,d_i^-)\rangle$. We shall identify $\vec{G}$ with the set of arcs in $\vec{G}$. A n.s.c. for a sequence $\langle(d_i^+,d_i^-)\rangle$ to be graphical is obtained by Fulkerson [1]. Most of the terminology introduced for graphs in §1 has a natural extension to digraphs. A subdigraph $\vec{F} \subseteq \vec{G}$ is called a subfactor with respect to $\langle(k_i^+,k_i^-)\rangle$ if $\vec{F}$ has at most k_i^+ arcs from v_i and k_i^- arcs into v_i. We shall let $S^+ = S^+(\vec{F})$ $(S^- = S^-(\vec{F}))$ denote the vertices v_i having k_i^+ (respectively k_i^-) arcs from (into) v_i in $\vec{F}$. A vertex in S^+ (S^-) is called outer (inner) saturated. A vertex that is both outer and inner saturated is simply called saturated and $S = S^+ \cap S^-$ is the set of saturated vertices. $\vec{F}$ is called a factor if $S = V$. Notations $E_c^+(v_i)$, $E_c^-(v_i)$ will be used with their obvious meanings. For example, $E_c^+(v_i)$ is the set of red arcs from vertex v_i. $E_c = \cup E_c^+(v_i) = \cup E_c^-(v_i)$.

We shall prove the following Theorem.

THEOREM 2.1. Let the sequence $d = \langle(d_i^+,d_i^-)\rangle$ be graphical and let $\langle(k_i^+,k_i^-)\rangle$ be a sequence such that for some $k \geq 0$, $k = k_i^-$ (or for that matter $k = k_i^+$), $1 \leq i \leq n$. Then there exists a digraph

$\vec{G}$ with degree sequence $\langle(d_i^+,d_i^-)\rangle$ containing a $\langle(k_i^+,k_i^-)\rangle$-factor if and only if the sequence $\langle(d_i^+-k_i^+,d_i^--k_i^-)\rangle = d'$ is graphical.

That the sequence $\langle(d_i^+-k_i^+,d_i^--k_i^-)\rangle$ be graphical is clearly necessary. The theorem says that it is also sufficient. The special case $k_i^+ = k = k_i^-$, $1 \le i \le n$ was conjectured by A.R. Rao and S.B. Rao along with their conjecture on undirected graphs (see §1). The proof of (2.1) is a modification of that of (1.1) to accomodate arcs instead of lines.

Proof of Theorem 2.1. Let $\vec{G}$ and $\vec{H}$ be representing digraphs for the sequences d and d' respectively. Consider the coloring of arcs (v_i,v_j) in r, b, g and w as before, namely, $c(v_i,v_j) = r$ if $(v_i,v_j) \in \vec{G}\backslash\vec{H}$, b if $(v_i,v_j) \in \vec{H}\backslash\vec{G}$, etc. One has $|E_r^+(v_i)| + |E_g^+(v_i)| = d_i^+$, $|E_b^+(v_i)| + |E_g^+(v_i)| = d_i^+-k_i^+$ and similar equations for indegrees. We choose a subfactor $\vec{F} \subseteq E_r$. Let us assume that $\vec{G}$, $\vec{H}$, $\vec{F}$ have been so chosen that $|E_g| + |\vec{F}|$ has maximum value. We show that $\vec{F}$ is a factor. For brevity let $k = k_i^-$, $1 \le i \le n$ and let $S \ne V$. We observe that i) $S^+ \ne V \ne S^-$; ii) $|E_r^+(v_i) - \vec{F}| \ge |E_b^+(v_i)|$, $|E_r^-(v_i) - \vec{F}| \ge |E_b^-(v_i)|$ for all v_i and the equality holds precisely for $v_i \in S^+$ and $v_i \in S^-$ respectively; iii) There do not exist distinct arcs $(y_0,y_1),(y_2,y_1),(y_2,y_3),\ldots,(y_{2t},y_{2t+1}),(y_0,y_{2t+1})$ whose colors are red and blue alternately in that order such that all the red arcs are in $E_r - \vec{F}$. Property iii) is almost trivial. One can change $c(y_{2m},y_{2m+1})$ to white and $c(y_{2m+2},y_{2m+1})$ to green for $0 \le m \le t$ $(y_{2t+2} = y_o)$ and increase $|E_g|$, keeping $|\vec{F}|$ unchanged, contradicting that $|E_g| + |\vec{F}|$ was maximum. The sequence $(y_0,y_1),(y_2,y_1),(y_2,y_3),\ldots,(y_{2t},y_{2t+1})$ is said to constitute an <u>alternating chain</u>. Note that the vertices $y_0,y_1,\ldots,y_{2t+1}$ need not be distinct.

Take a vertex $x_0 \in V-S^+$ and let (x_0,x_1) be a red arc not in $\vec{F}$. x_1 is necessarily in S^- and let (x_2,x_1) be a blue arc; there exists a red arc (x_2,x_3) not in $\vec{F}$. We can continue in this way to build an alternating chain P until it 'terminates' at a vertex in $V-S^-$. Let $P = (x_0,x_1),(x_2,x_1),(x_2,x_3),\ldots,$ (x_{2t},x_{2t+1}), $x_{2t+1} \in V-S^-$.

We show that $c(x_0,x_{2t+1}) = r$ if $x_0 \neq x_{2t+1}$ and thus $(x_0,x_{2t+1}) \in \vec{F}$. This is easy once we show that $(x_0,x_{2t+1}) \notin P$. The proof is similar to the one in 2^0, Theorem 1.1. Suppose (x_0,x_{2t+1}) is in P. If $c(x_0,x_{2t+1}) = b$ then P would contain a closed alternating chain as in iii) because in P blue arcs are traversed in opposite direction. If $c(x_0,x_{2t+1}) = r$ and $(x_0,x_{2t+1}) = (x_i,x_{i+1})$, then $(x_0,x_1),\ldots,(x_i,x_{i-1})$ is a closed alternating chain of even length. Thus we conclude that $(x_0,x_{2t+1}) \notin P$.

Now we prove by contradiction that an alternating chain P does not exsit.

Case I.* $|P| = 3$, $x_0 = x_3$. Since $k^-(x_1) = k^-(x_0) \geq 1$ there exists $x_4 \neq x_0$, x_1, x_2 such that $(x_4,x_1) \in \vec{F}$. Consider $\vec{F}' = \vec{F} - (x_4,x_1) + (x_0,x_1)$; $|\vec{F}'| = |\vec{F}|$. The chain

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*Compare the corresponding situation in Theorem 1.1.

$P' = (x_4,x_1),(x_2,x_1),(x_2,x_3)$ implies $(x_4,x_3) \in \vec{F}'$ because $x_4 \in V-S^+(\vec{F}')$, $x_3 \in V-S^-(\vec{F}')$. Hence $(x_4,x_3) \in \vec{F}$ and $k^-(x_0) \geq 2$ which implies there exists $x_5 \neq x_i$, $0 \leq i \leq 4$, such that $(x_5,x_1) \in \vec{F}$. We can prove, as before, that $(x_5,x_3) \in \vec{F}$ and so on, finally obtaining $k^-(x_0)$ is arbitrarily large. This is impossible.

Case II. $|P| = 3$, $x_0 \neq x_3$. Same as above as long as $x_4,x_5,\ldots$ remain different from x_3. If $x_i = x_3$, for some $i \geq 4$, then with respect to $\vec{F}' = \vec{F} - (x_i,x_1) + (x_0,x_1)$ and $P' = (x_i,x_1),(x_2,x_1),(x_2,x_3)$ we are in Case I.

Case III. $|P| = 2t+1$, $t \geq 2$, $x_0 = x_{2t+1}$. As in Case I there exists $(x_{2t+2},x_1) \in \vec{F}$ and let $\vec{F}'$, P' be defined as $\vec{F}' = \vec{F} - (x_{2t+2},x_1) + (x_0,x_1)$, $P' = P + (x_{2t+2},x_1) - (x_0,x_1)$. P' is an alternating chain and $(x_{2t+2},x_{2t+1}) \notin P'$. It follows that $(x_{2t+2},x_{2t+1}) \in \vec{F}'$ and thus it belongs to $\vec{F}$ and $k^-(x_{2t+1}) \geq 2$. A contradiction is ahead as in Case I.

Case IV. $|P| = 2t+1$, $t \geq 2$, $x_0 \neq x_{2t+1}$. It can be reduced to Case III or a contradiction otherwise (as in Case II).

<u>COROLLARY 2.2</u>. Let $\langle(d_i^+,d_i^-)\rangle$ be graphical. Suppose $\langle(k_i^+,k_i^-)\rangle$ is graphical and $d_i^+ \geq k_i^+$, $d_i^- \geq k_i^-$, $1 \leq i \leq n$ and either $\langle d_i^+-k_i^+\rangle$ or $\langle d_i^--k_i^-\rangle$ is a sequence of constant terms. Then a digraph $\vec{G}$ with degree sequence $\langle(d_i^+,d_i^-)\rangle$ and a $\langle(k_i^+,k_i^-)\rangle$-factor exists.

Example 4. In the following we have the sequences d, k, d-k all graphical and yet there is no graph with degree sequence d and having $\langle k_i^+, k_i^- \rangle$-factor. The sequence $\langle k_i^+ \rangle$ vary only by 1.

$$d = \langle (4,4),(3,3),(2,2),(2,2),(1,1) \rangle$$

$$k = \langle (1,1),(1,2),(1,0),(0,0),(1,1) \rangle$$

$$d\text{-}k = \langle (3,3),(2,1),(1,2),(2,2),(0,0) \rangle$$

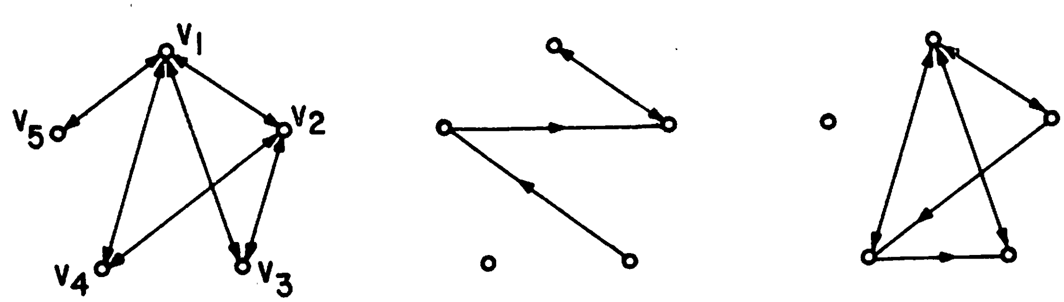

Fig. 8. Digraphs with degree sequences d, k, d-k respectively. There is a unique digraph with degree sequence d-k.

Note. In contrast with the undirected case the shortest alternating path in directed case can be of arbitrary length and thus we don't use them in the proof.

Corresponding to Theorem 1.6, we have

THEOREM 2.3. Let $\vec{F}$ be a given digraph. There exists a digraph $\vec{G}$ with degree sequence $\langle (d_i^+, d_i^-) \rangle$ and $\vec{G}$ containing $\vec{F}$, if and only if for every subdigraph $\vec{F}' \subseteq \vec{F}$ the sequence $\langle (d_i^+ - k_i^+, d_i^- - k_i^-) \rangle$ is graphical where $\langle (k_i^+, k_i^-) \rangle$ is degree sequence of $\vec{F}'$.

Proof. The necessity is trivial. We shall prove sufficiency by induction on the number of arcs in $\vec{F}$.

1^0) Let $\vec{F} = (x,y)$ and $\langle(k_i^+,k_i^-)\rangle$ be the degree sequence of $\vec{F}$. Suppose there is no digraph $\vec{G} \supseteq \vec{F}$. We shall obtain a contradiction. Consider digraphs $\vec{G}$, $\vec{H}$ with degree sequences $\langle(d_i^+,d_i^-)\rangle$, $\langle(d_i^+-k_i^+,d_i^--k_i^-)\rangle$, respectively, such that $|E_g|$ is maximum in the corresponding coloring; $c(x,y) = b$ or w. It is easy to see that there is an alternating chain P from x to y such that $P \subseteq E_r \cup E_b$, $(x,y) \notin P$ (because that would imply P traverses back to x by an even cycle whose arcs are alternately red and blue and this in turn implies that $|E_g|$ is not maximum). But then $C = P \cup (x,y)$ is an even cycle. Perform a suitable recoloring of C such that $c(x,y) = g$ or r. We obtain a digraph $\vec{G} \supseteq \vec{F}$.

2^0) Suppose the theorem is true for all digraphs with $m-1$ or less arcs and $\vec{F}$ has m arcs. Let (x,y) be an arc of $\vec{F}$; write $\vec{F}_0 = \vec{F} - (x,y)$. Thus there are digraphs $\vec{G}$, $\vec{H}$ containing $\vec{F}_0$ with degree sequence $\langle(d_i^+,d_i^-)\rangle$ for $\vec{G}$ and for $\vec{H}$, x (y) has outdegree (indegree) one less than that in $\vec{G}$. Consider a pair of $\vec{G}$, $\vec{H}$ such that in the associated coloring of $\vec{G}$, $\vec{H} - \vec{F}_0$, $|E_g|$ is maximum. Without loss of generality we can assume that $c(x,y) \neq r, g$. There exists an alternating chain $P \subseteq (E_r - \vec{F}_0) \cup E_b$ from x to y since for every vertex v_i, $|E_r^+(v_i) - \vec{F}_0| \geq |E_b^+(v_i)|$, $|E_r^-(v_i) - \vec{F}_0| \geq |E_b^-(v_i)|$ with strict inequality respectively for x and y. As before $(x,y) \notin P$. There exists a recoloring of the even cycle $C = P \cup (x,y)$ such that $c(x,y) = r$

or g. Then we have a digraph $\vec{G} \supseteq \vec{F}$.

For digraphs one can state and prove theorems as in (1.7), (1.8). For example the following is true.

<u>COROLLARY 2.4</u>. Suppose $\langle (d_i^+, d_i^-) \rangle$, $\langle (k_i^+ - d_i^+, k_i^- - d_i^-) \rangle$ are graphical sequences where k_i^+ (or k_i^-) are same for all i and k_i^+, $k_i^- \leq n-1$. Then there are disjoint representing digraphs for them.

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