A MATROID GENERALIZATION OF A

THEOREM OF MENDELSOHN AND DULMAGE

by

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Abstract

A matroid generalization is given to a theorem of Mendelsohn and
Dulmage concerning assignments in bipartite graphs. The generalized
theorem has applications in optimization theory and provides a simple
proof of a theorem of Nash-Williams.

1. A Theorem of Mendelsohn and Dulmage

A theorem of Mendelsohn and Dulmage [3] was originally proved for (0,1) matrices. However, it can be regarded as a theorem on bipartite matchings and hence it is a special case (finite) of Banach's mapping theorem [1]. We first give a simple proof of this theorem using the concept of a matching.

Theorem (1.1) (Mendelsohn-Dulmage). Let $G(X,Y)$ be a bipartite graph, the two parts being X and Y. Let S, T be subsets of X,Y respectively such that each set has an assignment into the other part. Then there is an assignment between two subsets of X,Y which contain both S and T.

Proof. Let $f: S \to Y$ and $g: T \to X$ be the assignments. Write $F = \{(x,f(x)): x \in S\}$, $H = \{(g(y),y): y \in T\}$. F and H are matchings in G and they cover respectively S and T. Form the symmetric difference $F \triangle H$. It consists of the five types of cycles and chains shown in Figure 1. (Some of these chains could be infinite in one or both directions if G is an infinite graph). In each case above we can select a matching set of edges $M \subseteq F \triangle H$ such that it covers all the vertices of $X \cup Y$ covered by $F \triangle H$. Then $M \cup (F \cap H)$ is a matching that covers $S \cup T$.

Remark. Banach's mapping theorem is same as Theorem (1.1) for arbitrary $|X \cup Y|$, possibly infinite. For other variations of Banach's theorem see [4].

We proceed to prove our main theorem We shall write a matroid as $M = (E, \mathcal{Q})$ where E is the set of elements and $\mathcal{Q}$ is the family of

independent subsets of E.

Theorem 1.2. Let M_1, M_2 be two matroids on E and I_1, I_2 each be independent in both matroids. Then there exists a set $I \subseteq I_1 \cup I_2$ independent in M_1, M_2 such that

$$sp^i(I) \supseteq sp^i(I_i), \quad i=1,2$$

where $sp^i(\ )$ stands for span in matroid M_i.

To see how these two theorems are related we take E = set of edges of the graph G(X,Y) and consider the natural partitions of E defined as follows. $P_1 = \{E_x: \ x \in X\}$, $P_2 = \{E_y: \ y \in Y\}$ where E_v = set of edges incident with vertex v. Let M_i, i=1,2 be the partition matroids corresponding to P_i, i.e., a set I is independent in M_i if and only if $|I \cap E_x| \leq 1$ for all x (resp. $|I \cap E_y| \leq 1$ for all y). Then

$\quad I_1$ = the edges of the assignment of S

$\quad I_2$ = the edges of the assignment of T

$\quad I$ = the edges of the assignment given by the theorem.

Proof of Theorem 1.2. If $sp^2(I_1) \supseteq I_2$ there is nothing to prove. Let $e \in I_2 \backslash sp^2(I_1)$; $I_1 + e$ is in $\mathcal{J}_2$. If $I_1 + e$ belongs to $\mathcal{J}_1$ let $I_1' = I_1 + e$. Otherwise, there exists a M_1-circuit C such that $e \in C \subseteq I_1 + e$. Now $C - e \nsubseteq I_1 \cap I_2$ since I_2 is in $\mathcal{J}_1$. Choose $e' \in C \cap (I_1 \backslash I_2)$ and define $I_1' = I_1 - e' + e$. We have $I_1' \in \mathcal{J}_1$ and $sp^1(I_1') = sp^1(I_1)$ and also I_1' is trivially independent in M_2. However, $|I_1' \cap I_2| > |I_1 \cap I_2|$.

Thus we can apply the same procedure to define $I_1^{(k)}$, k=1,2,.. such that $sp^1(I_1^{(k)}) = sp^1(I_1)$, $I_1^{(k)} \in \vartheta_1 \cap \vartheta_2$ until $sp^2(I_1^{(k)}) \supseteq I_2$. Then $I = I^{(k)}$ proves the theorem.

2. Application to Optimization.

Suppose θ_1 and θ_2 are two different criteria of optimality, such that

$$sp^i(A) \supseteq sp^i(B)$$

implies

$$A \geq B \quad (\theta_i), \quad i=1,2,$$

i.e. A is to be preferred to B with respect to θ_i. Let I_1, I_2 be sets in the family $\vartheta_1 \cap \vartheta_2$ which are maximal with respect to θ_1, θ_2 respectively. Then by Theorem 1.2 there exists a set $I \in \vartheta_1 \cap \vartheta_2$ which is maximal with respect to both θ_1 and θ_2.

Specifically, let X, Y in the bipartite graph G(X,Y) represent men and jobs to be matched, where the edges denote the compatibility relation. Suppose θ_1 is a union-determined criteria of optimality based on seniority of men and θ_2 is a management-determined criterion of optimality based on priority of jobs. Let I_1 be a subset of edges representing a union-optimal assignment of men to jobs, possibly as determined in Gale [2], and let I_2 be a management-optimal subset. Then by Theorem 1.2 (or 1.1) there exists an assignment $I \subseteq I_1 \cup I_2$ which is simultaneously union-optimal and management-optimal.

3. Proof of a Theorem of Nash-Williams.

Theorem 1.2 provides a simple and direct proof of a theorem of Nash-Williams [5].

Theorem 3.1

Let $M_1 = (E, \vartheta_1)$ be a matroid and $h: E \to E_0$ be a mapping of E into E_0. Then $M_0 = (E_0, \vartheta_0)$ is a matroid, where

$$\vartheta_0 = \{I_0 \subseteq E_0: \text{ for some } I_1 \in \vartheta_1, h(I_1) = I_0\}.$$

Proof:

It is sufficient to show that if I_p, I_{p+1} are two sets in ϑ_0 respectively with p and $p+1$ elements, there exists a set $h(I) \in \vartheta_0$ with $p+1$ or more elements such that $I_p \subseteq h(I) \subseteq I_p \cup I_{p+1}$. Let $M_2 = (E, \vartheta_2)$ be a partition matroid where

$$\vartheta_2 = \{I_2 \subseteq E: \ |I_2 \cap h^{-1}(e)| \leq 1, \text{ for all } e \in E_0\}.$$

Let I'_p, I'_{p+1} be sets in ϑ_1, respectively with p and $p+1$ elements, such that $h(I'_p) = I_p$ and $h(I'_{p+1}) = I_{p+1}$. The sets I'_p, I'_{p+1} are independent in M_2 as well as M_1, and we can apply Theorem 1.2. Thus there is a set $I \in \vartheta_1 \cap \vartheta_2$ such that

$$sp^1(I) \supseteq sp^1(I'_{p+1}),$$

hence $|I| \geq p+1$, and

$$sp^2(I) \supseteq sp^2(I'_p),$$

from which it follows that

$$h(I) \supseteq h(I'_p) = I_p.$$

Also h is one-one on I and $I \subseteq I'_p \cup I'_{p+1}$, which implies that $|h(I)| \geq p+1$ and $h(I) \subseteq I_p \cup I_{p+1}$. Thus $\mathcal{I}_0$ defines the independent sets of a matroid.

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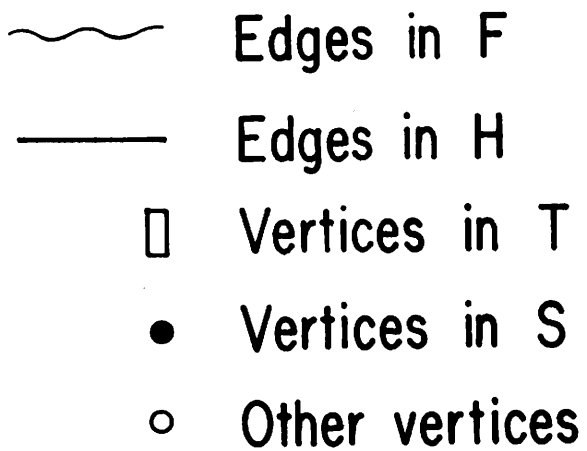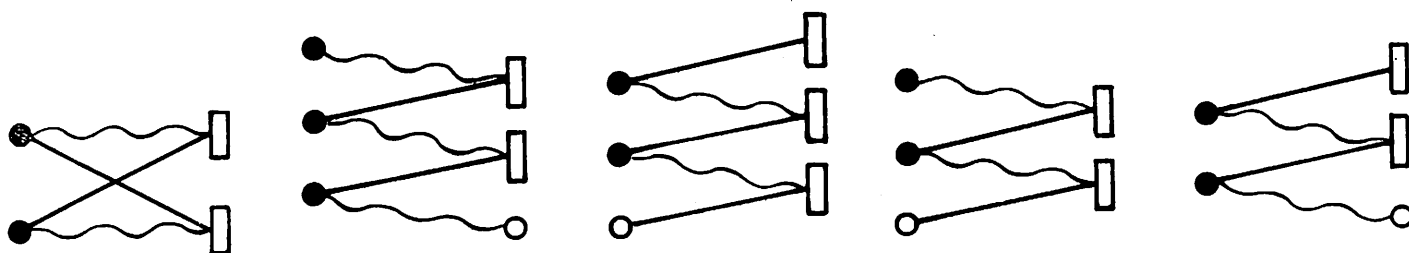
Edges in F
Edges in H
Vertices in T
Vertices in S
Other vertices

Fig. 1. Components of F Δ H