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TRAJECTORIES OF NONLINEAR RLC NETWORKS: A GEOMETRIC APPROACH

by

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Memorandum No. ERL-M324

28 February 1972

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ABSTRACT

We consider the response of a nonlinear, time-varying, coupled RLC network starting from a given operating point. We view the response as motion occurring in a differentiable manifold Σ in $\mathbb{R}^{2b} \times \mathbb{R}_+$, where b is the number of branches. We impose two basic manifold conditions on the network. First, the resistor characteristics are required to be a manifold Λ. Second, the resistor characteristics and their connections are such that the set of branch-voltages and branch-currents satisfying both the Kirchhoff laws and the resistor characteristics is a manifold Σ. We then show that under the conditions imposed on the RLC elements and the topology of the network, the network has a unique response specified by a flow on Σ if and only if the capacitor voltages, inductor currents, and time constitute a parametrization for Σ. Finally we show that our conditions include as special cases the determinateness conditions previously obtained by several authors.

Research sponsored by the National Science Foundation, Grant GK-10656X1.

I. INTRODUCTION

The formulation of the nonlinear network problem requires answers to two questions: (I) is there an operating point and is it unique? (II) given an operating point what conditions are required in order to have a unique well-defined trajectory? The first question concerns with the existence and uniqueness of solutions of resistive networks and is considered, for example, in [20]-[21], [12] for certain nonlinear networks. In this paper we consider exclusively the second question.

The conventional formulation of the dynamic equations of nonlinear network views the motion as occurring in the linear vector space $\mathbb{R}^n$ where n is the dimension of the state-vector. Following the geometric viewpoint first explicitly stated by Smale [1], we consider the motion as occurring in a differentiable manifold(1) Σ in a bigger space $\mathbb{R}^{2b} \times \mathbb{R}_+$, where b is the number of branches. We allow elements to be nonlinear, time-varying, and coupled (among similar elements).

Our basic assumptions are in the form of two "manifold conditions" (MCI) and (MCII) and some natural positive definiteness conditions on the L's and C's. (MCI) requires that the characteristics of the ρ nonlinear time-varying coupled resistors constitute a manifold in $\mathbb{R}^{2\rho} \times \mathbb{R}_+$. (MCI) is a straightforward generalization of the basic assumption of Smale. Physically, the operating point $(v,i,t) \in \mathbb{R}^{2b} \times \mathbb{R}_+$ must

(1) Roughly speaking, a manifold in $\mathbb{R}^m$ can be thought of as a smooth "surface" such that at every point of the surface there is a local parametric representation of the surface (e.g. longitude and latitude for a sphere in $\mathbb{R}^3$). We will explain this in detail in Sec. II. For an introduction to differentiable manifolds, see [2]-[5].

satisfy the Kirchhoff laws and lie on the resistor characteristics; thus (MCII) requires that the set of points which satisfy these two conditions constitute a manifold Σ in $\mathbb{R}^{2b} \times \mathbb{R}_+$. In Theorem 1, we show that, under the conditions imposed on the elements and the topology of the network, the physical laws specify a unique flow on the manifold Σ (hence the network has a unique well-defined trajectory starting from any given operating point) if and only if the capacitor voltages, inductor currents, and time constitute a parametrization for Σ. An immediate consequence of the manifold condition is that, at every operating point, the small-signal equivalent network must satisfy the determinateness conditions that Purslow [6] discovered for linear time-invariant networks. We then show that conditions that several authors [9],[10],[13]-[17] needed in order to write the network dynamic equations in normal form are in fact sufficient conditions for our manifold conditions.

II. FORMULATION

II.1 Network $\mathcal{N}$

We consider a nonlinear, time-varying RLC network $\mathcal{N}$ which, for simplicity, is assumed to have a connected graph with b branches and $n_t = n + 1$ nodes. Electrical coupling among branches of the same kind is allowed; thus dependent sources are viewed as coupled resistors. Each set of network variables (v,i), where v is the set of branch voltages and i is the set of branch currents, is a vector in $\mathbb{R}^{2b}$ partitioned as $(v_c, v_r, v_\ell, i_c, i_r, i_\ell)$, where subscript c (r,$\ell$ resp.)

denotes the variable pertaining to capacitors (resistors, inductors, resp.). Thus $\mathbb{R}^{2b}$ is a direct sum

$$\mathbb{R}^{2b} = \mathcal{C}^v \oplus \mathcal{R}^v \oplus \mathcal{L}^v \oplus \mathcal{C}^i \oplus \mathcal{R}^i \oplus \mathcal{L}^i \tag{1}$$

II.2 Independent sources

Without loss of generality, we can assume that for any RLC netowrk, given any tree, the independent sources are distributed in such a way that each independent voltage source is connected in series with a link and each independent current source is connected in parallel with a tree-branch. This can always be brought about by source transformation [7, pp. 709-414].

Thus a typical link and a typical tree-branch are of the form in Fig. 1, where the rectangular box represents an element which is not a source.

Let V_k (resp. I_k) denote the voltage across (resp. current through) the composite branch, then we have

$$V_k = \begin{cases} v_k - e_k(t) & \text{if branch } k \text{ is a link} \\ v_k & \text{if branch } k \text{ is a tree-branch} \end{cases}$$

$$I_k = \begin{cases} i_k & \text{if branch } k \text{ is a link} \\ i_k - j_k(t) & \text{if branch } k \text{ is a tree-branch.} \end{cases} \tag{2}$$

We assume, for simplicity, that the independent sources are C^2-functions of time. As will be clear later the whole formulation is still valid if the independent sources are continuous functions of time, in which case, however, the only modification needs to be made is to invoke, instead, a continuous version of implicit function theorem [5, Th. 9.3, p. 230; 8, Th. 5.2.4., p. 128].

II.3 Kirchhoff laws

Given any tree, KVL is expressed by $B(v-e(t)) = 0$, and KCL is expressed by $Q(i-j(t)) = 0$, where B and Q are the corresponding fundamental loop and cutset matrices, respectively; and as a result of source transformation $e(t)$ is a b-vector whose k-th component is equal to $e_k(t)$, the voltage source in series with branch k, if branch k is a link and 0 if branch k is a tree-branch; $j(t)$ is a b-vector whose k-th component is equal to $j_k(t)$, the current source in parallel with branch k, if branch k is a tree-branch and 0 if branch k is a link. Let $K \subset \mathbb{R}^{2b} \times \mathbb{R}_+$ be the set of all (v,i,t) such that (v,i) is a set of branch-voltages and branch-currents that satisfy Kirchhoff laws at time t, i.e.,

$$K = \{(v,i,t) \in \mathbb{R}^b \times \mathbb{R}^b \times \mathbb{R}_+ \mid B(v-e(t)) = 0, \quad Q(i-j(t)) = 0\} \tag{3}$$

Tellegen theorem states that $(v-e(t))$ and $(i-j(t))$ lie in the complementary orthogonal subspaces of each other [9; also 7, p. 422]. Since for any linear map A: $\mathbb{R}^k \to \mathbb{R}^m$, $\mathcal{R}(A^T) \overset{\perp}{\oplus} \mathcal{N}(A) = \mathbb{R}^k$,

$$K = \{(v,i,t) \in \mathbb{R}^{2b} \times \mathbb{R}_+ \mid v = Q^T v_\tau + e(t), \; i = B^T i_\lambda + j(t)\} \tag{4}$$

Note that v_τ and i_λ are the set of tree-branch voltages and the set of link currents, respectively. (See equation (2)). The map ψ_K: $(v_\tau, i_\lambda, t) \mapsto (v,i,t)$ is a C^2-diffeomorphism of $\mathbb{R}^b \times \mathbb{R}_+$ onto K. Let $K_{t'}$ denote the intersection of K and $t - t' = 0$. The map $\phi_K(\cdot,\cdot,t')$: $(v_\tau, i_\lambda) \rightarrow (v,i)$ is a C^∞-diffeomorphism of $\mathbb{R}^b$ onto $K_{t'}$. Thus K is a (b+1)-dimensional C^2 manifold of $\mathbb{R}^{2b} \times \mathbb{R}_+$ and ψ_K constitutes a global parametrization for K. Moreover, for each time t', $K_{t'}$ is a b-dimensional C^∞ manifold (in fact, an affine subspace) of $\mathbb{R}^{2b}$ and $\phi_K(\cdot,\cdot,t')$ is a global parametrization for $K_{t'}$.

II.4 Resistor characteristics

Let ρ be the number of (possibly electrically coupled) resistive branches, which, as indicated before, may include dependent sources. Let Λ be the set of all $(v_r, i_r, t) \in \mathcal{R}^v \times \mathcal{R}^i \times \mathbb{R}_+$, which satisfy the resistor characteristics. Let $\Lambda_{t'}$ denote the intersection of Λ and $t - t' = 0$. We impose the following basic requirement on the resistor characteristics:

(MCI): Λ is a C^2 submanifold of $\mathbb{R}^{2\rho} \times \mathbb{R}_+$, of dimension $(\rho+1)$; furthermore, for each t', $\Lambda_{t'}$ is a C^2 submanifold of $\mathbb{R}^{2\rho}$, of dimension ρ.

MCI means, by definition, that for each $(p,t') \in \Lambda$, there exist an open neighborhood $U_p \subset \mathbb{R}^{2\rho}$ of p, an open neighborhood $T \subset \mathbb{R}_+$ of t', an open set V_p of $\mathbb{R}^\rho$, and a C^2-function ψ_Λ: $(u,t) \mapsto (\hat{v}_{rV}(u,t), \hat{i}_{rV}(u,t), t)$ with ψ_Λ being a C^2-diffeomorphism of $V_p \times T$ onto $\Lambda \cap (U_p \times T)$.[2] ψ_Λ is a (local) parametrization of Λ. The local inverse

[2] Subscript V is used to emphasize that these functions depend on V.

of ψ_Λ is called a (local) coordinate system of Λ about (p,t'). Furthermore, for fixed $t \in T$, the map $\phi_\Lambda(\cdot,t)$: $u \mapsto (\hat{v}_{rV}(u,t), \hat{i}_{rV}(u,t))$ is a local parametrization of Λ_t. Hence the $(2\rho) \times \rho$ Jacobian matrix $D_1\, \phi_\Lambda(u,t)$ is of rank ρ for all $u \in V_p$ and $t \in T$.(3) In short, $(v_r, i_r, t) \in \Lambda \cap (U_p \times T)$ if and only if

$$\begin{aligned} v_r &= \hat{v}_{rV}(u,t) \\ i_r &= \hat{i}_{rV}(u,t) \end{aligned} \tag{5}$$

and the $2\rho \times \rho$ matrix $\begin{bmatrix} D_1\hat{v}_{rV}(u,t) \\ D_2\hat{i}_{rV}(u,t) \end{bmatrix}$ is of full rank for all (6)

$$u = (u_1, \ldots, u\ ) \in V_p \subset \mathbb{R}^\rho \text{ and } t \in T \subset \mathbb{R}_+.$$

An equivalent formulation of MCI is often useful [3, pp. 122; 4, pp. 71-72]: for each $(p,t') \in \Lambda$ there exist an open neighborhood $U_p \subset \mathbb{R}^{2\rho}$ of p. An open neighborhood $T \subset \mathbb{R}_+$ of t' and a C^2-function g: $U_p \times T \to \mathbb{R}^\rho$ such that

$$z \in \Lambda_t \cap U_p \subset \mathbb{R}^{2\rho} \text{ if and only if } g(z,t) = 0$$

and (7)

$$\text{rank}[D_1 g(z,t)] = \rho \text{ for all } z \in U_p,\ t \in T.$$

Remarks: (1) If all resistors are time-invariant. Λ_t is the same for all t. Therefore, (MCI) can be simplified as

Λ_t is a C^2-submanifold of $\mathbb{R}^{2\rho}$, of dimension ρ.

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(3) $D_1\, \phi_\Lambda(u,t)$ is the derivative map of the function $\phi_\Lambda(\cdot,t)$ at u.

(2) In some cases, U_p can be chosen to be $\mathbb{R}^{2\rho}$ itself and V_p to be $\mathbb{R}^{\rho}$, then $\phi(\cdot,t) = (\hat{v}_r(\cdot,t), \hat{i}_r(\cdot,t))$ is called a global parametrization for Λ_t. For example, if all resistors are voltage-controlled, then we can choose $u = v_r$ and consequently $\hat{v}_r(\cdot,t)$ is the identity map on $\mathbb{R}^{\rho}$.

(3) If Λ can be expressed as

$$\Lambda = \{(z,t) | g(z,t) = 0,\ g \in C^2 \text{ and } D_1 g(z,t) \text{ is of full rank}\}, \qquad (7)$$

then MCI is satisfied.

(4) Note that MCI is more restrictive than characterizing a resistive n-port element by $g(v_r,i_r,t) = 0$ where $g\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$. For example, let $n = 1$ and $g(v_r,i_r,t) = \sin^2 \pi v_r^2 + \sin^2 \pi i_r^2 = 0$ (which consists of all points in $\mathbb{R}^2$ with integer coordinates) does not qualify to be a resistor characteristic according to MCI, even though $g(\cdot,\cdot,\cdot)$ is C^∞.

(5) If a two-terminal resistor characteristic is parametrized by arc length via an injective C^2-map defined over a compact interval, then it satisfies MCI. It is a special case of the unicursal resistor of Chua and Rohrer [10]. Note that the unicursal resistor is allowed to have a characteristic that crosses itself.

II.5 Configuration space

Let Σ be the set of all $(v,i,t) \in \mathbb{R}^{2b} \times \mathbb{R}_+$ such that (i) $(v,i,t) \in K$ and (ii) $(v_r,i_r,t) \in \Lambda$. In Lagrangian mechanics, Σ would be the configuration space; any $(v,i,t) \in \Sigma$ satisfies both the Kirchhoff constraints and the constraints imposed by the resistor characteristics. Let $\Sigma_{t'}$ denote the intersection of Σ and $t - t' = 0$. We impose the additional basic assumption:

(MCII): Σ is a C^2 submanifold of $\mathbb{R}^{2b} \times \mathbb{R}_+$, of dimension $(b-\rho+1)$; furthermore, for each $t' \geq 0$, $\Sigma_{t'}$ is a C^2 submanifold of $\mathbb{R}^{2b}$, of dimension $(b-\rho)$.

(MCII) means that for any $(m,t') \in \Sigma$, there exist an open neighborhood $W_m \subset \mathbb{R}^{2b}$ of m, an open neighborhood T of t', an open set N_m in $\mathbb{R}^{b-\rho}$, and a C^2 -function ϕ_Σ such that

$$(v,i,t) \in \Sigma \cap (W_m \times T) \quad \text{if and only if}$$

$$(v,i,t) = (\phi_\Sigma(\alpha,t),\ t) \quad \forall\ (\alpha,t) \in N_m \times T$$

where $\phi_\Sigma(\cdot,t)$: $N_m \to \mathbb{R}^{2b}$ is a C^2-diffeomorphism of N_m onto $\Sigma_t \cap W_m$.

Let us bring in: (i) the parametrization (4) of K;

$$\begin{aligned} v &= Q^T v_\tau + e(t) \\ i &= B^T i_\lambda + j(t) \end{aligned} \tag{8}$$

and (ii) the parametrization (5) of Λ: locally, $(v_r,i_r,t) \in \Lambda$ if and only if

$$\begin{aligned} v_r &= \hat{v}_{rV}(u,t) \\ i_r &= \hat{i}_{rV}(u,t) \end{aligned} \tag{9}$$

and $\begin{bmatrix} D_1\hat{v}_{rV}(u,t) \\ D_1\hat{i}_{rV}(u,t) \end{bmatrix}$ is of full rank for u in some open set V of $\mathbb{R}^\rho$ and t in an open neighborhood $T \subset \mathbb{R}_+$ of t'. Classifying the branches as in (1), we partition B and Q as

$$[B_c \vdots B_r \vdots B_\ell] \qquad [Q_c \vdots Q_r \vdots Q_\ell] \tag{10}$$

Thus by combining (8) and (9), we conclude that, locally, $(v,i,t) \in \Sigma$ if and only if

$$\begin{aligned} &\hat{v}_{rV}(u,t) - Q_r^T v_\tau - e_r(t) = 0 \\ &\hat{i}_{rV}(u,t) - B_r^T i_\lambda - j_r(t) = 0 \end{aligned} \tag{11}$$

and $\begin{bmatrix} D_1\hat{v}_{rV}(u,t) \\ D_1\hat{i}_{rV}(u,t) \end{bmatrix}$ is of full rank for $u \in V$ and $t \in T$. Note that the left hand side of (11) defines a map $g(u,v_\tau,i_\lambda,t)$ from a subset $V \times \mathbb{R}^b \times T$ of $\mathbb{R}^{\rho+b+1}$ into $\mathbb{R}^{2\rho}$. MCII is equivalent to requiring that the Jacobian matrix $[D_1 g \vdots D_2 g \vdots D_3 g]$ be of full rank, i.e.,

$$\text{rank } J(u,t) = 2\rho \qquad \forall\, u \in V \quad \forall\, t \in T \tag{12}$$

where

$$J(u,t) = \begin{bmatrix} D_1\hat{v}_{rV}(u,t) & Q_r^T & 0 \\ D_1\hat{i}_{rV}(u,t) & 0 & B_r^T \end{bmatrix} \tag{13}$$

Remarks: (1) Equation (8) shows that the configuration space Σ depends on the waveforms of the independent sources.

(2) Loosely speaking, if (MCI) is satisfied, (MCII) has the additional requirement that the Kirchhoff constraints and the constraints imposed by the resistor characteristics be "independent" in the sense that at no point will the resistor characteristics duplicate any one of the Kirchhoff constraints. Let us illustrate the point by the following example (Fig. 2). In this example, $\mathbb{R}^{2\rho} = \{(v_1,v_2,i_1,i_2)\}$ and $\Lambda_t = \{(v_1,v_2,i_1,i_2) \mid v_1 = Ri_1,\ v_2 = ki_1\}$, which is a two-dimensional C^∞ submanifold of $\mathbb{R}^4$ for all k. However when k = R, the resistor constraints $v_1 = ri_1$, $v_2 = ki_1$ duplicate the Kirchhoff law $v_1 - v_2 = 0$, MCII is not satisfied.

II.6 Capacitor and inductor characteristics

As indicated above, the capacitors and inductors are not-necessarily-linear, not-necessarily-time-invariant, and possibly coupled. We make the assumption:

(LC): (a) All capacitors are voltage-controlled and all inductors are current-controlled, i.e., $q_c = \hat{q}_c(v_c,t)$, $\phi_\ell = \hat{\phi}_\ell(i_\ell,t)$, with $\hat{q}_c$ and $\hat{\phi}_\ell$ in C^2, hence (17)

$$C(v_c,t)\,\frac{dv_c}{dt} = i_c - \frac{\partial}{\partial t}\,\hat{q}_c(v_c,t)$$

$$L(i_\ell,t)\,\frac{di_\ell}{dt} = v_\ell - \frac{\partial}{\partial t}\,\hat{\phi}_\ell(i_\ell,t) \tag{18}$$

where $C(v_c,t) = D_1\hat{q}\,(v_c,t)$ and $L(i_\ell,t) = D_1\hat{\phi}_\ell(i_\ell,t)$. (19)

(b) The matrices $C(v_c,t)$ and $L(i_\ell,t)$ are positive definite[4] for all (v_c,i_ℓ,t).

Remarks: (1) It can be shown [8,pp. 142-143] that the positive definiteness assumption (b) requires that at every time t, the mappings $\hat{q}_c(\cdot,t)$ and $\hat{\phi}_\ell(\cdot,t)$ be strictly monotone.[5]

(2) In the linear time-invariant case, passivity of the inductors and capacitors is equivalent to the positive semidefiniteness of the L and C matrices [19, pp. 127-148]. Our assumption of positive definiteness at every point is stronger and it rules out degenerate networks of "perfectly coupled" L's and C's [7, pp. 568-570].

II.7 Flows on Σ

Let $x = (m,t)$ be a point on the manifold Σ. A (differentiable) curve on Σ through x is, by definition, a differentiable map α: $I \to \Sigma$, where $I \subset \mathbb{R}$ is an interval containing the origin, such that $\alpha(0) = x$. Let $w = \frac{d}{dt}\alpha(t)\Big|_{t=0}$, then w is called the tangent vector to the curve $\alpha(\cdot)$ at x. The set of all tangent vectors to Σ at x is called the tangent space to Σ at x and is denoted by $T_x(\Sigma)$. Indeed, if ψ is a parametrization of Σ in the neighborhood of x with $\psi(0) = x$, then $T_x(\Sigma)$ is the image of the linear map $D\psi(0)$: $\mathbb{R}^{b-\rho+1} \to \mathbb{R}^{2b+1}$, hence it is a linear subspace of $\mathbb{R}^{2b+1}$, of dimension $(b-\rho+1)$ [4, p. 74]. The

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(4) A not-necessarily symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive definite (semidefinite) iff $x^TAx > 0$ (≥ 0) $\forall x \neq 0$.

(5) A map F: $D \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone (strictly monotone) on D if $\langle x - y | Fx - Fy \rangle \geq 0$ (> 0, for $x \neq y$, resp.), $\forall x,y \in D$.

tangent space $T_x(\Sigma)$ can be viewed as the best "local approximating linear space" to the differentiable manifold Σ at the point x. The network interpretation of the tangent space is as follows; if x is the operating point, $T_x(\Sigma)$ would be the configuration space of the small-signal equivalent network obtained by replacing all nonlinear resistors by linear resistors whose resistances are equal to the small-signal incremental resistances at the operating point $x = (m,t)$ [7, pp. 720-724].

A <u>flow</u> on Σ is, by definition, a differentiable map $s: U \subset \mathbb{R} \times \Sigma \to \Sigma$, where U is an open set containing $\{0\} \times \Sigma$, such that (i) $s(0,x) = x$; and (ii) $s(t'+t,x) = s(t, s(t',x))$, whenever both sides of the equation are defined. Thus, for each $x \in \Sigma$, $s(\cdot,x)$ is a curve on Σ through x. Under certain regularity conditions, Σ can be considered as the state space of the dynamical system describing $\mathcal{N}$ and s is then the state transition function [11, pp. 46-49]. On the other hand, a <u>vector field</u> on Σ is, by definition, a map which assigns to each $x \in \Sigma$ a tangent vector in $T_x(\Sigma)$. If s is a flow on Σ, then for each $x \in \Sigma$ the tangent vector to the curve $s(\cdot,x)$ is defined. In this manner, the flow s gives rise to a vector field X on Σ, indeed $\frac{d}{dt} s(t,x) = X(s(t,x)) \;\; \forall\, (t,x) \in U$. This can be visualized as analogous to the velocity field of a moving fluid. Conversely, it can be shown that every C^1 vector field on Σ defines a flow [5, p. 381].

The network variables are constrained by Kirchhoff laws and the branch characteristics. Kirchhoff laws and resistor characteristics force the network variables to lie on Σ. Therefore the network solution, i.e., the function $t \mapsto (v(t), i(t), t)$, for a given set of initial conditions will be a curve on Σ. Now at each point $x \in \Sigma$ the capacitor and inductor equations (18) specify certain components, namely $(\dot{v}_c, \dot{i}_\ell)$, of the tangent vector $(\dot{v}, \dot{i}, \dot{t}) \in T_x(\Sigma)$ to the solution curve. Note that the last component of the tangent vector, $\dot{t}$, is always equal to 1. If equation (18) can specify a unique smooth vector field on Σ the network will have a unique response; in which case the corresponding flow of the vector field is the solution of the network analysis problem.

Let π denote the natural projection of $\mathbb{R}^{2b} \times \mathbb{R}_+$ into $\mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+$, i.e., π: $(v_c, v_r, v_\ell, i_c, i_r, i_\ell, t) \to (v_c, i_\ell, t)$. To simplify notation, we let π restricted to Σ be denoted by $\sigma \triangleq \pi|\Sigma$: $\Sigma \to \mathcal{C}^r \times \mathcal{L}^i \times \mathbb{R}_+$.

Theorem 1. Let $\mathcal{N}$ be a network as described in Sec. II.1 and satisfy (MCI), (MCII) and (LC). Under these conditions, the capacitor and inductor characteristics specify a unique C^1 vector field on Σ, hence $\mathcal{N}$ has a unique response starting from any operating point, if and only if given any point $x \in \Sigma$, σ: $\Sigma \to \mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+$ is a coordinate system for Σ about x (or, equivalently, σ^{-1} is a parametrization of Σ).

Remarks: (1) In a network analysis problem, the first step is to determine a set of variables which completely specify all the network variables, i.e., to determine a coordinate system for Σ. Theorem 1 shows that under the stated conditions, this set must be (up to local diffeomorphism) capacitor voltages, inductor currents, and time. This justifies the normal tree approach.

(2) If a C^1 vector field is defined on Σ, the network, starting from a given operating point, will have a unique response specified by the

corresponding flow. The flow is determined from the vector field by solving the differential equations.

(3) Chua and Rohrer [10] have considered nonlinear time-invariant networks with unicursal elements. They have given conditions for the existence of normal form equations. Note that if the nonlinear mapping F that they defined has an inverse, then the mappings $(v_c, i_\ell, t) \mapsto (-F_{RC}v_C + e_R(t), F_{LG}^T i_L + j_G(t)) \mapsto u$ exist, where the last mapping is F^{-1}. Hence (v_c, i_ℓ, t) is a parametrization for Σ.

Proof. First note that Σ is a C^2 submanifold, hence σ is a C^2-map.

$\Leftarrow$ By assumption σ is a local diffeomorphism. Hence $D\sigma(x)$ is a linear homeomorphism of $T_x(\Sigma)$ onto $T_{\sigma(x)}(\mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+)$. Therefore any $(\dot{v}_c, \dot{i}_\ell, 1)$ specifies, via $(D\sigma(x))^{-1}$, a unique tangent vector in $T_x(\Sigma)$. It remains to show that the vector field thus defined on Σ is C^1. Equation (18) defines, as a consequence of assumption (LC), a C^1 map $(v,i,t) \mapsto (\dot{v}_c, \dot{i}_\ell, 1)$ and we have a C^1-map $(D\sigma(x))^{-1}$: $(\dot{v}_c, \dot{i}_\ell, 1) \mapsto (\dot{v}, \dot{i}, 1) \in T_x(\Sigma)$. Therefore, the vector field is C^1.

$\Rightarrow$ We know that σ is C^2. Now we claim that the derivative map $D\sigma(x)$: $T_x(\Sigma) \to T_{\sigma(x)}(\mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+)$ is nonsingular. Hence σ will be a local diffeomorphism. Suppose $D\sigma(x)$ were singular, then some point $(\dot{v}_c, \dot{i}_\ell, 1) \in T_{\sigma(x)}(\mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+)$ would have no preimage and even if it had, it would not be unique. This would mean that equation (18) could not specify a unique vector field on Σ. We reach the desired contradiction. $\boxtimes$

III. CIRCUIT-THEORETIC INTERPRETATION

If at every point x on the manifold Σ, σ: $\Sigma \to \mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+$ is a coordinate system about x, then the tangent space of Σ at x, $T_x(\Sigma)$, is isomorphic to $\mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}$. With the interpretation that the tangent space as being the configuration space of the small-signal equivalent network at $x = (v,i,t)$, the fact that σ is a coordinate system about x means that (v_c, i_ℓ, t) determines a unique set of voltages and currents in the small-signal equivalent network. Let us justify this from the rank condition.

Let us for the time being assume that $\mathcal{N}$ has no capacitor-only loop and no inductor-only cutset. Networks having capacitor-only loops and inductor-only cutsets will be discussed later. Let v be partitioned into (v_R, v_L, v_C, v_G), where the subscripts R, L, C, and G denote resistive links, inductive links, capacitive tree-branches, and resistive tree branches, respectively. Similarly for i. Hence, the fundamental loop matrix is expressed as [7, pp. 516-521]

$$B = \begin{bmatrix} I & 0 & F_{RC} & F_{RG} \\ 0 & I & F_{LC} & F_{LG} \end{bmatrix}$$

Equation (11) becomes

$$\hat{v}_{RV}(u,t) + F_{RC}v_C + F_{RG}v_G - e_R(t) = 0$$

$$\hat{v}_{GV}(u,t) - v_G = 0$$

$$\hat{i}_{RV}(u,t) - i_R = 0$$

$$\hat{i}_{GV}(u,t) - F_{RG}^T i_R - F_{LG}^T i_L - j_G(t) = 0 \tag{20}$$

and $\begin{bmatrix} D_1\hat{v}_{RV}(u,t) \\ D_1\hat{v}_{GV}(u,t) \\ D_1\hat{i}_{RV}(u,t) \\ D_1\hat{i}_{GV}(u,t) \end{bmatrix}$ is of full rank for all $u \in V$ and $t \in T$. (21)

MCII further requires (equations (12) and (13)) that

$$\text{rank } J(u,t) = 2\rho \qquad \forall\ u \in V, \qquad \forall\ t \in T. \tag{22}$$

where $J(u,t)$ is the $2\rho \times (\rho+b)$ matrix defined by

$$J(u,t) = \begin{bmatrix} D_1\hat{v}_{RV}(u,t) & -F_{RC} & -F_{RG} & 0 & 0 \\ D_1\hat{v}_{GV}(u,t) & 0 & I & 0 & 0 \\ D_1\hat{i}_{RV}(u,t) & 0 & 0 & I & 0 \\ D_1\hat{i}_{GV}(u,t) & 0 & 0 & F_{RG}^T & F_{LG}^T \end{bmatrix} \tag{23}$$

Let us consider Theorem 1 and think in terms of the implicit function theorem. The projection σ: $\Sigma \to \mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+$ is a coordinate system for Σ (i.e., Σ is parametrizable by capacitor voltages, inductor currents, and time) if and only if the $2\rho \times 2\rho$ matrix

$$K(u,t) \triangleq \begin{bmatrix} D_1\hat{v}_{RV}(u,t) & -F_{RG} & 0 \\ D_1\hat{v}_{GV}(u,t) & I & 0 \\ D_1\hat{i}_{RV}(u,t) & 0 & I \\ D_1\hat{i}_{GV}(u,t) & 0 & F_{RG}^T \end{bmatrix} \tag{24}$$

is nonsingular $\forall\ u \in V,\ \ \forall\ t \in T$.

By elementary row operations, it can easily be shown that $K(u,t)$ is nonsingular if and only if

$$\begin{bmatrix} D_1\hat{v}_{RV}(u,t) & + F_{RG}\ D_1\hat{v}_{GV}(u,t) \\ -F_{RG}^T\ D_1\hat{i}_{RV}(u,t) & + D_1\hat{i}_{GV}(u,t) \end{bmatrix} \tag{25}$$

is nonsingular, i.e., if and only if

$$\begin{aligned} [I \quad F_{RG}] \begin{bmatrix} D_1\hat{v}_{RV}(u,t) \\ D_1\hat{v}_{GV}(u,t) \end{bmatrix} \Delta u &= 0 \\ [-F_{RG}^T \quad I] \begin{bmatrix} D_1\hat{i}_{RV}(u,t) \\ D_1\hat{i}_{GV}(u,t) \end{bmatrix} \Delta u &= 0 \end{aligned} \tag{26}$$

implies $\Delta u = 0$. Since the $2\rho \times \rho$ matrix $\begin{bmatrix} D_1\hat{v}_{RV}(u,t) \\ D_1\hat{v}_{GV}(u,t) \\ D_1\hat{i}_{RV}(u,t) \\ D_1\hat{i}_{GV}(u,t) \end{bmatrix}$ is assumed to be of full rank, $\Delta u = 0$ if and only if

$$\begin{bmatrix} D_1\hat{v}_{RV}(u,t) \\ D_1\hat{v}_{GV}(u,t) \\ D_1\hat{i}_{RV}(u,t) \\ D_1\hat{i}_{GV}(u,t) \end{bmatrix} \Delta u = 0. \tag{27}$$

Therefore, $K(u,t)$ is nonsingular if and only if (26) implies (27).

Now let us derive a small-signal equivalent network $\mathcal{N}_R$ at (u,t) from $\mathcal{N}$. First, remove all inductors, replace all capacitors by short-circuits, and set all independent sources to zero. Second, replace resistors by linear resistors which are specified by the following parametric representation $\Delta v_r = [D_1\hat{v}_r(u,t)]\Delta u$ and $\Delta i_r = [D_1\hat{i}_r(u,t)]\Delta u$. Thus equations (26) are exactly the KVL and KCL for $\mathcal{N}_R$. Hence (26) will imply (27) if and only if

$$\mathcal{N}_R \text{ has only the trivial solution } \Delta v = 0,\ \Delta i = 0. \tag{28}$$

Remarks. (1) Purslow [6] has shown that for a linear time-invariant network, if there exists a cutset (resp. loop) of dependent and independent current sources and none of the branch voltages (resp. currents) in the cutset (resp. loop) controls any dependent sources, then the network does not have a unique solution. This is in fact true for nonlinear networks. We are going to show that if such a cutset (resp. loop) exists, then (28) can not be satisfied. Consider the small-signal equivalent linear network $\mathcal{N}_R$ at (u,t). Let all the branch-currents and branch-voltages other than the branch-voltages (resp. currents) in the cutset (resp. loop) be zero; and let each branch in the cutset (resp. loop) have a nonzero branch-voltage (resp. branch-current) δ. Clearly this is a nontrivial solution for $\mathcal{N}_R$, i.e., it satisfies KVL, KCL, and branch-relations for $\mathcal{N}_R$.

(2) For a monotone network, i.e., a network whose branches are two-terminal elements having monotone increasing characteristics [12], (28) is satisfied at all points if after removing all inductors and

replacing all capacitors by short-circuits, the resulting resistive network satisfies the following conditions:

(U_ℓ) every loop made of c.c. resistors contains at least one strictly increasing resistor;

(U_c) every cutset made of v.c. resistors contains at least one strictly increasing resistor.

IV. SUFFICIENT CONDITIONS

Let $\mathcal{N}$ be a network as described in Sec. II.1 and satisfy (MCI), (MCII), and (LC). We shall give some sufficient conditions under which equation (24) is satisfied at an operating point, i.e., the projection σ: $\Sigma \to \mathcal{C}^v \times \mathcal{L}^i \times \mathbb{R}_+$ is a coordinate system for Σ about the operating point. If at every point $x \in \Sigma$ any one of the sufficient conditions given below is satisfied, then as a consequence of Theorem 1, the capacitor and inductor characteristics (18) define a unique C^1 vector field on Σ, hence the network will have a unique response specified by the corresponding flow.

In this section, conditions are examined for fixed t, we will henceforth suppress the variable t as if we were in the time-invariant case. Also the subscript V will be dropped, however we must keep in mind that all the conditions in this section are local conditions.

Case I. All resistors are voltage-controlled; in this case, $u = (v_R, v_G)$.

(SI): If $\begin{bmatrix} D_1\hat{i}_R(v_R,v_G) & D_2\hat{i}_R(v_R,v_G) \\ D_1\hat{i}_G(v_R,v_G) & D_2\hat{i}_G(v_R,v_G) \end{bmatrix}$ is positive definite (30)

then condition (24) is satisfied.

Proof: Since $K(v_R,v_G) = \begin{bmatrix} I & 0 & -F_{RG} & 0 \\ 0 & I & I & 0 \\ D_1\hat{i}_R & D_2\hat{i}_R & 0 & I \\ D_1\hat{i}_G & D_2\hat{i}_G & 0 & F_{RG}^T \end{bmatrix}$ (31)

is nonsingular if and only if

$$[-F_{RG}^T \quad I] \begin{bmatrix} D_1\hat{i}_R & D_2\hat{i}_R \\ D_1\hat{i}_G & D_2\hat{i}_G \end{bmatrix} \begin{bmatrix} -F_{RG} \\ I \end{bmatrix} \text{ is nonsingular.} \tag{32}$$

Hence (SI) follows. ☒

Remarks: (1) Condition (SI) requires the resistor characteristics $(v_R,v_G) \mapsto (\hat{i}_R(v_R,v_G), \hat{i}_G(v_R,v_G))$ be strictly monotone [8, pp. 142-143].

(2) Condition (SI) does not require the matrix be symmetric, hence nonreciprocal elements, for example, dependent sources, are allowed.

(3) Quasilinear resistors [13, p. 35] constitute a special case of condition (SI). Indeed, it is required for quasilinear resistors that the matrix in equation (30) be uniformly positive definite, as well as symmetric.

Case II. All resistors are current-controlled. (Dual to Case I.)

Case III. Some resistors are voltage-controlled, and some are current-controlled.

Assumption T: There exists a normal tree such that all resistors whose voltages are controlling variables can be put in the links and all resistors whose currents are controlling variables can be put in the tree, i.e., $u = (i_G, v_R)$.

(SIIIA): If every resistor whose voltage (resp. current) is a controlling variable can form a loop (resp. cutset) with only capacitors (resp. inductors), then condition (24) is satisfied.

Proof: Since $K(i_G, v_R) = \begin{bmatrix} 0 & I & -F_{RG} & 0 \\ D_1\hat{v}_G & D_2\hat{v}_G & I & 0 \\ D_1\hat{i}_R & D_2\hat{i}_R & 0 & I \\ I & 0 & 0 & F_{RG}^T \end{bmatrix}$ (33)

is nonsingular if and only if

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -F_{RG}^T \\ F_{RG} & 0 \end{bmatrix} \begin{bmatrix} D_1\hat{v}_G & D_2\hat{v}_G \\ D_1\hat{i}_R & D_2\hat{i}_R \end{bmatrix} \tag{34}$$

is nonsingular.

Note that the assumption in (SIIIA) implies $F_{RG} = 0$. Hence (SIIIA) follows. $\boxtimes$

Remarks: (1) Condition (SIIIA) imposes only topological constraint on the network. Indeed, electric coupling between links and tree-branches is allowed. The resistor characteristics $\begin{bmatrix} D_1\hat{v}_G & D_2\hat{v}_G \\ D_1\hat{i}_R & D_1\hat{i}_R \end{bmatrix}$ is not necessarily symmetric, hence nonreciprocal elements are allowed.

(2) The class of complete networks [9] for which $F_{RG} = 0$ satisfies (SIIIA). However, Brayton and Moser considered only uncoupled resistors.

(SIIIB): If $\begin{bmatrix} D_1\hat{v}_G & D_2\hat{v}_G \\ D_1\hat{i}_R & D_2\hat{i}_R \end{bmatrix}$ is positive semidefinite and symmetric, (resp. is positive definite); then condition (24) is satisfied.

Proof: It follows from (34) and Fact 1 (resp. Fact 2) in the Appendix. $\boxtimes$

Remark: It can be shown [8, pp. 142-143] that $\begin{bmatrix} D_1 v_G & D_2 v_G \\ D_1 i_R & D_2 i_R \end{bmatrix}$ is positive semidefinite if and only if the resistor characteristics $(i_G, v_R) \mapsto (\hat{v}_G(i_G, v_R),\ \hat{i}_R(i_G, v_R))$ is monotone (see Footnote (5)). If the matrix is positive definite, then the resistor characteristics are strictly monotone.

(SIIIC): Suppose there is no electric coupling between resistive links and resistive tree-branches. If $D\hat{v}_G(i_G)$ and $D\hat{i}_R(v_R)$ are positive semidefinite, and either one of them is symmetric, (resp. either one of them is positive definite); under these conditions, then (24) is satisfied.

Proof: Equation (34) becomes

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -F_{RG}^T \\ F_{RG} & 0 \end{bmatrix} \begin{bmatrix} D\hat{v}_G & 0 \\ 0 & D\hat{i}_R \end{bmatrix} \tag{35}$$

which is nonsingular if and only if

$$[I + F_{RG}^{T}(D\hat{i}_R)F_{RG}(D\hat{v}_G)] \text{ is nonsingular, or} \tag{36}$$

$$[I + F_{RG}(D\hat{v}_G)F_{RG}^{T}(D\hat{i}_R)] \text{ is nonsingular.} \tag{37}$$

Hence (SIIIC) follows from Fact 1 (resp. Fact 2) in the Appendix. $\boxtimes$

Remarks: (1) For the monotone networks considered by Desoer and Katzenelson [14], if we further require that the resistor characteristics be differentiable, then $D\hat{v}_G$ and $D\hat{i}_R$ are both positive semidefinite and diagonal, hence satisfy (SIIIC).

(2) Varaiya and Liu [15] have considered the following classes of networks; $D\hat{v}_G$ and $D\hat{i}_R$ are positive semidefinite and either one of them is positive definite and symmetric (resp. either one of them is positive semidefinite and diagonal). These classes of networks satisfy (SIIIC).

(3) Ohtsuki and Watanabe [16] have considered the class of netowrks for which elements characteristics are uniformly positive definite, hence they satisfy (SIIIB).

(4) Fujisawa and Kuh [17] have considered the following classes of networks; (i) $(D\hat{v}_G)F_{RG}^{T}(D\hat{i}_R)F_{RG} \in P_0$, (ii) $D\hat{v}_G$ is positive semidefinite and diagonal, and $F_{RG}^{T}(D\hat{i}_R)F_{RG} \in P_0$. They have shown that in these cases (36) is nonsingular. Hence these classes of networks satisfy condition (24).

V. CAPACITOR LOOPS AND INDUCTOR CUTSETS

Suppose that $\mathcal{N}$ has γ capacitor-only loops and λ inductor-only cutsets. Because of the constraints imposed by the Kirchhoff laws, not all the capacitor voltages and inductor currents are independent. One would expect from Theorem 1 a posteriori that solution curves are constrained to a submanifold of Σ. We are going to justify this fact.

Pick a normal tree. Let v be partitioned as $(v_S, v_R, v_L, v_C, v_G, v_\Gamma)$; similarly for i, where the subscripts S, R, L (C,G,Γ) denote link (tree-branch) capacitors, resistors, and inductors, respectively. The corresponding fundamental loop matrix is then given by [8]:

$$B = \begin{bmatrix} I & 0 & 0 & F_{SC} & 0 & 0 \\ 0 & I & 0 & F_{RC} & F_{RG} & 0 \\ 0 & 0 & I & F_{LC} & F_{LG} & F_{L\Gamma} \end{bmatrix} \tag{38}$$

The capacitor and inductor equations (18) written in terms of the above partition, become

$$\begin{aligned} \begin{bmatrix} i_S \\ i_C \end{bmatrix} &= C(v_S, v_C, t) \frac{d}{dt} \begin{bmatrix} v_S \\ v_C \end{bmatrix} + \frac{\partial}{\partial t} \begin{bmatrix} \hat{q}_S(v_S, v_C, t) \\ \hat{q}_C(v_S, v_C, t) \end{bmatrix} \\ \begin{bmatrix} v_L \\ v_\Gamma \end{bmatrix} &= L(i_L, i_\Gamma, t) \frac{d}{dt} \begin{bmatrix} i_L \\ i_\Gamma \end{bmatrix} + \frac{\partial}{\partial t} \begin{bmatrix} \hat{\phi}_L(i_L, i_\Gamma, t) \\ \hat{\phi}_P(i_L, i_\Gamma, t) \end{bmatrix} \end{aligned} \tag{39}$$

Note that the Kirchhoff laws require

$$[I \quad F_{SC}]\begin{bmatrix} v_S \\ v_C \end{bmatrix} = e_S(t) \tag{40}$$

$$[-F_{L\Gamma}^T \quad I]\begin{bmatrix} i_L \\ i_\Gamma \end{bmatrix} = j_\Gamma(t)$$

Thus,

$$\begin{gathered}
[I \quad F_{SC}]\; C^{-1}(v_S,v_C,t)\begin{bmatrix} i_S - \frac{\partial}{\partial t}\,\hat{q}_S(v_S,v_C,t) \\ i_C - \frac{\partial}{\partial t}\,\hat{q}_C(v_S,v_C,t) \end{bmatrix} = \frac{d}{dt}\,e_S(t) \\
[-F_{L\Gamma}^T \quad I]\; L^{-1}(i_L,i_\Gamma,t)\begin{bmatrix} v_L - \frac{\partial}{\partial t}\,\hat{\phi}_L(i_L,i_\Gamma,t) \\ v_\Gamma - \frac{\partial}{\partial t}\,\hat{\phi}_\Gamma(i_L,i_\Gamma,t) \end{bmatrix} = \frac{d}{dt}\,j_\Gamma(t)
\end{gathered} \tag{41}$$

Hence the Kirchhoff laws (40) and the capacitor and inductor characteristics impose on the network variables $(\gamma+\lambda)$ additional algebraic constraints, namely (41). Let K' be the set of all (v,i,t) satisfying both Kirchhoff laws (3) and (41). Note that $K' \subset K$. Our assumptions imply that K' defines a $(b-\gamma-\lambda+1)$-dimensional C^2-submanifold in $\mathbb{R}^{2b} \times \mathbb{R}_+$. Indeed, first note that (3) and (41) are in the form of $(b+\gamma+\lambda)$ equations $g(v,i,t) = 0$ (see Eq. (7)), second, the Jacobian matrix of g is of full rank because

$$[I \quad F_{SC}]C^{-1}(v_S,v_C,t)\begin{bmatrix} I \\ F_{SC} \end{bmatrix} \text{ and } [-F_{L\Gamma}^T \quad I]L^{-1}(i_L,i_\Gamma,t)\begin{bmatrix} -F_{L\Gamma} \\ I \end{bmatrix}$$

are both nonsingular by (LC).

Let Σ' be the set of all (v,i,t) such that (a) $(v,i,t) \in K'$ and (b) $(v_R, v_G, i_R, i_G, t) \in \Lambda$. Therefore, for the present case, (MCII) must be modified to read:

(MCII'): Σ' is a C^2-submanifold of $\mathbb{R}^{2b} \times \mathbb{R}_+$ of dimension $(b-\rho-\gamma-\lambda+1)$; furthermore, for each $t' \geq 0$, $\Sigma'_{t'}$ is a C^2-submanifold of $\mathbb{R}^{2b}$, of dimension $(b-\rho-\gamma-\lambda)$.

It turns out that (MCII') holds if and only if equations (20) hold and the rank condition on $J(u,t)$ (defined by (23)) holds.

Let σ' be the restriction to Σ' of the projection map $(v,i,t) \mapsto (v_C, i_L, t)$, then for the present case Theorem 1 becomes:

Theorem 1'. Let $\mathcal{N}$ be a network as described in Sec. II.1 and having γ capacitor-only loops and λ inductor-only cutsets. Suppose that $\mathcal{N}$ satisfies (MCI), (MCII'), and (LC). Under these conditions, the capacitor and inductor characteristics specify a unique C^1 vector field on Σ' if and only if given any point $x \in \Sigma'$, σ': $\Sigma' \to \mathcal{C}^{v_C} \times \mathcal{L}^{i_L} \times \mathbb{R}_+$ is a coordinate system for Σ' about x.

With the implicit function theorem in mind, it can be shown that σ' is a coordinate system for Σ' if and only if the $2\rho \times 2\rho$ matrix $K(u,t)$ (which turns out to have precisely the same form as (24)) is nonsingular, for all $u \in V$ and $t \in T$.

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Appendix

Fact 1. Let A and B be two nxn matrices with real elements. If A and B are positive semidefinite, and either one of them is symmetric, then (I+AB) is nonsingular.

Proof: Suppose B is symmetric. First we claim that $x^T Bx = 0$ implies $Bx = 0$. Since B is symmetric, we can write $x = \sum_{i=1}^{n} \alpha_i e_i$ where e_i's are orthonormal eigenvectors of B. $Bx = \sum_{i=1}^{n} \lambda_i \alpha_i e_i$ where λ_i's are eigenvalues of B. Hence $x^T Bx = \sum_{i=1}^{n} \lambda_i |\alpha_i|^2$, since $e_i^T e_j = \delta_{ij}$. Note that $\lambda_i \geq 0$ because B is positive semidefinite and symmetric. Therefore, $x^T Bx = 0$ implies $\lambda_i \alpha_i = 0$ for all i, i.e., $Bx = 0$.

Now we are going to show (I+AB) is nonsingular by contradiction. Suppose there were an $x \neq 0$ such that $(I+AB)x = 0$, i.e., $x = -ABx$. Therefore, $x^T B^T x = - x^T B^T ABx$; but A and B are both positive semidefinite, hence $x^T Bx = 0$. This implies $Bx = 0$. But this would lead to $x = 0$ because $x = -ABx$, hence we reach a contradiction. If A is symmetric, consider $(I+AB)^T = I + B^T A^T$. ☒

Fact 2. Let R and S be two nxn matrices with real elements. If one of them is positive definite and the other is positive semidefinite, then (I+RS) is nonsingular.

Proof: Suppose R is p.d. and S is p.s.d. For the other case one needs only to take the transpose. By contradiction. Suppose there were an $x \neq 0$ for which $x + RSx = 0$. Then $x^T S^T x = - x^T S^T RSx$, hence $x^T S^T RSx = 0$.

Now note that $Sx \neq 0$, because if it were, then $x = 0$. So we have $(Sx)^T R(Sx) = 0$ and $Sx \neq 0$, which contradicts to the fact that R is p.d. $\boxed{X}$

FOOTNOTES

(1) Roughly speaking, a manifold in $\mathbb{R}^m$ can be thought of as a smooth "surface" such that at every point of the surface there is a local parametric representation of the surface (e.g. longitude and latitude for a sphere in $\mathbb{R}^3$). We will explain this in detail in Sec. II. For an introduction to differentiable manifolds, see [2]-[5].

(2) Subscript V is used to emphasize that these functions depend on V.

(3) $D_1 \phi_\Lambda(u,t)$ is the derivative map of the function $\phi_\Lambda(\cdot,t)$ at u.

(4) A not-necessarily symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive definite (semidefinite) iff $x^T Ax > 0$ (≥ 0) $\forall\, x \neq 0$.

(5) A map $F: D \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone (strictly monotone) on D if $\langle x - y | Fx - Fy \rangle \geq 0$ (> 0, for $x \neq y$, resp.), $\forall\, x,y \in D$.

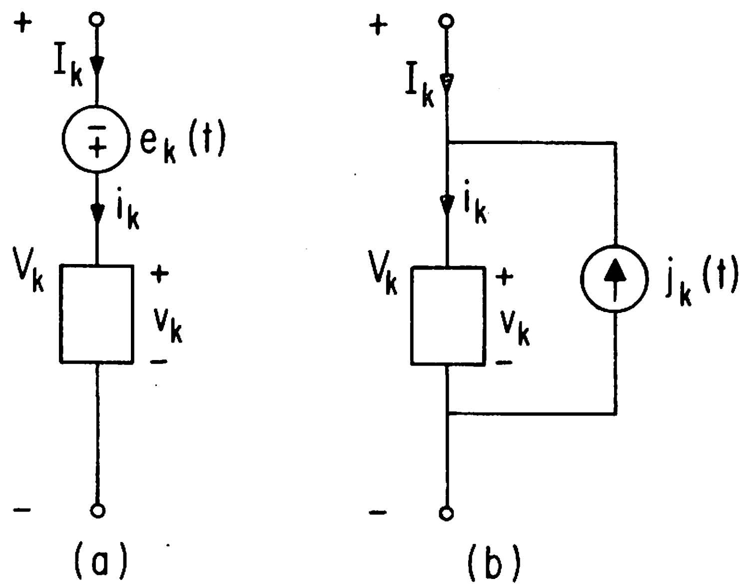

Fig. 1. Network element (a) a typical link,

(b) a typical tree-branch.

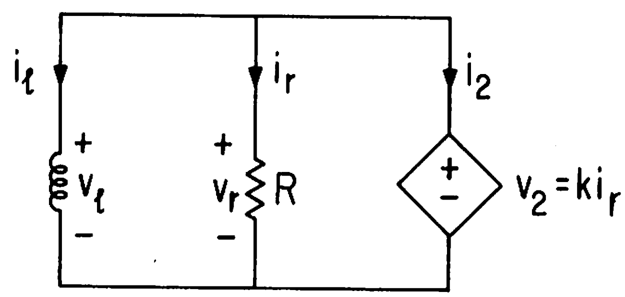

Fig. 2. A network satisfying (MCI) but not (MCII) when k = R.