OPTIMUM CITY SIZE:
A MINIMUM CONGESTION COST APPROACH

by

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ABSTRACT

The optimal allocation of urban land between the generation of traffic and the carrying of traffic which minimizes congestion costs is derived for a circular city. Unlike previous work which dealt only with the suburbs, this paper deals with both the suburbs and the central business district. This wider view of the problem leads to different conclusions, the most important of which are the fact that at no point in the city is all the land used for transportation and secondly that there is a maximum size for a city of a given working population which is independent of relative land and congestion costs.

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1. Introduction

Two recent papers have analyzed the optimal allocation of urban land between the generation of traffic and the carrying of traffic. Mills and de Ferranti [3] worked with a circular city, and Solow and Vickery [4] with a long narrow city; both used a measure of congestion costs as the principal criterion by which the optimal allocation of land was determined. The Solow-Vickery model was a homogeneous business district whereas the Mills-de Ferranti problem dealt with the suburbs outside the central business district. This paper adopts the circular city approach of Mills and de Ferranti but considers the allocation of land within both the C.B.D. (central business district) and the suburbs. Whereas Mills and de Ferranti found that at certain points of the city all available land is used for transportation, it is shown here that this never occurs when land in both the CBD and the suburbs is optimally allocated. It is also shown that for a city of a given population there exists a maximum and a minimum bound on the radius of the city.

We begin by presenting the basic assumptions about congestion costs, traffic flows and land values. The Mills-de Ferranti problem is then presented so that it can be contrasted with the the CBD problem which is its complement. A graphical solution is presented for the Mills-de Ferranti problem which clarifies several points and the case of all available land being used for transportation purposes is fully analyzed. A solution for the complete city follows straightforwardly from the results obtained for the Mills-de Ferranti and CBD problems.

The Maximum principle is used in the analysis of the allocation problem since this allows us to deal easily with the boundary conditions that arise at the edge of the CBD and also with the possible existence of constrained solutions (i.e. where all available land is used for transportation.)

2. Traffic Patterns, Land Use Densities and Congestion Costs

The question which this paper attempts to answer is; given that N people work in the CBD of a circular city and live in the suburbs which lie beyond the CBD, how should the land available be allocated between transportation use and business use in the CBD and between transportation use and residential use in the suburbs so as to minimize the social costs arising from congestion when these N people travel to and from work? Since the land used for the city has social value how large should the city be?

Although the city is assumed to be circular the analysis is generalized slightly by assuming that only $\theta (\leq 2\pi)$ radians of land is available. Each point in the city is identified by its distance from the city center, u. The number of travellers at a point is denoted by $T(u)$; the amount of land used for transportation by $L_2(u)$; and $L_1(u)$ denotes land used for business in the CBD and for residence in the suburbs. All available land at any radius, θu, is allocated for one purpose or another; hence

$$L_1(u) + L_2(u) = \theta u. \qquad (2.1)$$

Also no more than all the available land can be allocated for a single use, i.e.

$$L_1(u) \geq 0 \quad \text{and} \quad L_2(u) \geq 0 \quad . \tag{2.2}$$

The boundary of the CBD with the suburbs occurs at a radius ε from the city center and the boundary of the city at a radius $\bar{u}$. Since all the N workers in the CBD live in the suburbs it follows that

$$T(0) = 0 = T(\bar{u}) \quad \text{and} \quad T(\varepsilon) = N. \tag{2.3}$$

The density of workers in the CBD is assumed to be a constant a_c and the residential density in the suburbs, a_s, is also assumed constant. The residential density term can be adjusted to cover the case where each worker has a family which stays at home and doesn't travel to work in the CBD. Hence within the CBD the number of people working at a radius u from the city center, $N_c(u)$, is given by

$$N_c(u) = a_c L_1(u) = a_c(\theta u - L_2(u)), \tag{2.4}$$

and similarly the number of workers residing in the suburbs at a radius u from the cith center, $N_s(u)$, by

$$N_s(u) = a_s L_1(u) = a_s(\theta u - L_2(u)). \tag{2.5}$$

The land on which the city is to be built is assumed to have a rental value of R_A. This valuation can either be thought of as the market rent for an alternative nonurban use or as the social valuation of open space and parks. The congestion costs per traveller, p(u), are given by the function

$$p(u) = \bar{p} + \rho_1 \left[\frac{T(u)}{L_2(u)} \right]^{\rho_2} \qquad (2.6)$$

which is the one used by both the Solow-Vickery and the Mills-de Ferranti papers. In order to simplify the analysis $\bar{p}$ is taken to be zero. Hence total social costs are given by the integral

$$\int [\rho_1 \, C(u)^{\rho_2} + R_A \theta u] du, \qquad (2.7)$$

where $C(u) = T(u)/L_2(u)$. The limits of the integral depend upon the part of the city being studied; i.e. 0 to ε for the CBD, ε to $\bar{u}$ for the suburbs and of course 0 to $\bar{u}$ for the whole city.

3. The Mills-de Ferranti Problem

Before going on to extend the basic problem which Mills and de Ferranti studied it is useful to analyze their problem so that parallels between it and work in later sections of this paper are obvious. Indeed some of the conclusions reached about the Mills-de Ferranti model go beyond the statements in their paper and are fundamental to the solution of the extended problem.

The city with which Mills and de Ferranti worked had a CBD of given radius, ε, all the residents of the city worked in the CBD and lived in the suburbs. The route to work is assumed to be along the radius of the city passing through the place of residence. Social costs arise from these journeys to work and from the use of land for the city. Given that N people work in the CBD how big should the suburbs be and what is the optimal allocation of land between residential and transportation uses which minimizes social costs?

Given the assumption of the route to work above, it follows that the number of travellers at a point u, within the suburbs, is given by

$$T(u) = \int_u^{\bar{u}} N_s(v)dv. \tag{3.1}$$

Equation (3.1) can be differentiated, using the definition of $N_s(v)$ given by (2.5), yielding the differential equation

$$T'(u) = a_s L_2(u) - a_s\theta_u. \tag{3.2}$$

The Mills-de Ferranti problem is to minimize the costs given by (2.7) subject to the differential equation (3.2) and the inequalities (2.2), which can be rewritten in terms of $L_2(u)$ alone.

$$L_2(u) \geq 0, \qquad L_2(u) \leq \theta u \tag{3.3}$$

Since C(u) becomes infinite when $L_2(u) = 0$ and $T_2(u) \neq 0$ clearly no part of the optimally planned city can have zero land allocated for transportation as it would imply infinite social costs.

The Hamiltonian, $\mathcal{H}^s$, for the Mills-de Ferranti problem is

$$\mathcal{H}^s = \rho_1 C(u)^{\rho_2} + R_A\theta u + [-a_s\lambda(u) + \mu(u)][\theta u - L_2(u)], \tag{3.4}$$

where $\mu(u)$ is the multiplier applying to the constraint $L_2(u) \leq \theta u$ and $\lambda(u)$ is the langrangian multiplier for the differential equation (3.2). $\mu(u) \geq 0$ if $L_2(u) = \theta u$ and $\mu(u) = 0$ if $L_2(u) < \theta u$. The usual first order conditions follow immediately from (3.4);

$$\frac{\partial \mathcal{H}^s}{\partial T} = -\frac{d}{du}\lambda(u) = \rho_1(\rho_2 + 1)C^{\rho_2} \tag{3.5}$$

and

$$\frac{\partial \mathcal{H}^s}{\partial L_2} = -\rho_1\rho_2\, C^{\rho_2+1} + a_s\lambda - \mu = 0. \tag{3.6}$$

At the edge of the city, $u = \bar{u}$, we have a transversality condition, i.e. $\mathcal{H}^s(\bar{u}) = 0$, which gives the following condition

$$R_A\theta\bar{u} + [-a_s\lambda(\bar{u}) + \mu(\bar{u})][\theta\bar{u} - L_2(\bar{u})] = 0, \tag{3.7}$$

since $T(\bar{u}) = 0$. If the problem is to have a meaningful solution then $\bar{u} > \varepsilon > 0$ so (3.7) implies that $L_2(\bar{u}) \neq \bar{u}$. If we take $L_2(\bar{u}) = 0$ then (3.6) and (3.7) become[1]

$$\lambda(\bar{u}) = \frac{\rho_1\rho_2}{a_s} C(\bar{u})^{\rho_2+1} = \frac{R_A}{a_s} . \tag{3.8}$$

Since we have shown that for $u \simeq \bar{u}$, $\mu(u) = 0$, equations (3.5) and (3.6) can be solved to yield

$$C(u) - C(\bar{u}) = \frac{a_s}{\rho_2}[\bar{u} - u], \tag{3.9}$$

i.e. congestion is a linear function of u. Equation (3.9) may be rewritten as

$$C(u) = \frac{a_s}{\rho_2} Q(u) \tag{3.10}$$

where $Q(u) = \bar{u} - u + H$ and $H = \frac{\rho_2\, C(\bar{u})}{a_s} = \frac{\rho_2}{a_s}\left(\frac{R_A}{\rho_1\rho_2}\right)^{1/\rho_2+1}$. Thus in Eq. (3.10) we have a description of the optimal solution to the

Mills-de Ferranti problem for those parts of the suburbs beyond the last point at which all land is used for transportation. As [3] showed (3.10) together with (3.2) can be combined to give the following function for T(u):

$$T(u) = \frac{a_s \theta}{\rho_2+1} Q(u)^{-\rho_2} \left[uQ(u)^{\rho_2+1} - \bar{u} H^{\rho_2+1} + \frac{Q(u)^{\rho_2+2} - H^{\rho_2+2}}{\rho_2+2} \right] \quad (3.11)$$

A more useful approach is to use the same two equations to derive a relation describing the optimal allocation of land:

$$L_2'(u) = \frac{d}{du} L_2(u) = \frac{\rho_2+1}{Q(u)} \left[L_2(u) - \frac{\rho_2 \theta u}{\rho_2+1} \right]. \quad (3.12)$$

From (3.12) it follows that

$$L_2''(u) = \frac{d^2}{du^2} L_2(u) = \frac{(\rho_2+1)(\rho_2+2)}{Q(u)^2} \left[L_2(u) - \frac{\rho_2 \theta u}{\rho_2+2} - \frac{-\rho_2 \theta(\bar{u}+H)}{(\rho_2+1)(\rho_2+2)} \right]. \quad (3.13)$$

Using (3.12) and (3.13) we can sketch the form of the optimal solution for $L_2(u)$, see Fig. 1. For each pair (N,R_A) there is a unique $\bar{u}$ solution and Fig. 1 shows the optimal allocation of land for transportation associated with a particular $\bar{u}$ and varying values of N and R_A.

Thus for some values of N and R_A the allocation of land for transportation, $L_2(u)$, increases monotonically as one moves in from the city boundary, for lower values of (N,R_A) $L_2(u)$ reaches a maximum value and then falls for the rest of the distance to the CBD boundary. The rental value of land R_A may be sufficiently low that at some point $L_2(u)$

reaches its constrained value of θ; from this point inwards Eq. (3.11) is no longer valid and perhaps nearer the city center $L_2(u)$ again becomes less than θu. This is what we consider next.

When $L_2(u) = \theta u$, $\mu \neq 0$, and we can differentiate (3.6) and use the value of $\lambda'(u)$ given by (3.5) to give the following differential equation in $\mu(u)$:

$$\frac{d}{du}\mu(u) = \rho_1(\rho_2+1)C^{\rho_2}\left[a_s + \rho_2\frac{dC}{du}\right] \tag{3.14}$$

Along the constraint $L_2(u) = \theta u$, $T(u)$ remains constant at some value, $\bar{N}$ say; thus $C(u) = \frac{\bar{N}}{\theta u}$ whenever (3.14) is valid. Since

$$\frac{d}{du}C(u) = -\frac{\bar{N}}{\theta u^2} = -\frac{C}{u}$$

along the constraint, from (3.14) it follows that $\frac{d\mu}{du} < 0$ for all u such that $\frac{\rho_2 C(u)}{a_s} = Q(u) > u$.

Equations (3.10) and (3.11) give an expression for $L_2(u)$, which is valid from $\bar{u}$ inwards until the constraint $L_2(u) = \theta u$ is reached, it may be written as,

$$L_2(u) = \frac{\rho_2\,\theta u}{\rho_2+2} + \frac{\rho_2\,\theta(H+\bar{u})}{(\rho_2+1)(\rho_2+2)} - \frac{\rho_2\,\theta}{(\rho_2+1)(\rho_2+2)}\left(\frac{H}{Q}\right)^{\rho_2+1}(H+(\rho_2+2)\bar{u}). \tag{3.15}$$

Noting that the third term on the right-hand side of (3.15) is always positive and referring to (3.13) we see that $L_2(u)$ always lies below the $L_2''(u) = 0$ line. If u^* is the point at which $L_2(u)$ reaches the constraint then

$$\theta u^* < \frac{\rho_2 \ \theta u^*}{\rho_2+2} + \frac{\rho_2 \ \theta(H+\bar{u})}{(\rho_2+1)(\rho_2+2)} ,$$

or

$$u^* < \frac{\rho_2}{\rho_2+2} [H + \bar{u} - u^*] = \frac{\rho_2 \ Q(u^*)}{\rho_2+2} < Q(u^*). \tag{3.16}$$

Condition (3.16) proves that $\mu(u)$ is a monotonic function and hence that once $L_2(u)$ reaches its constrained maximum value, θu, it remains constrained from that point inwards to the CBD boundary.

Figure 1 therefore illustrates all possible forms of the solution to the Mills-de Ferranti problem. The most alarming aspect of the results is that near the CBD all available land may well be used for roads. One is tempted to suggest that perhaps the size of the CBD should be expanded under these circumstances. As we shall show when we plan for both the CBD and the suburbs at no point is all available land used for transportation.

4. The CBD Problem

We now consider a problem which is complementary to the Mills-de Ferranti problem, just as they ignored social costs arising within the CBD so we ignore social costs arising in the suburbs. Thus the city is considered to be made up entirely of a CBD with all the workers living outside the city limits. Land has to be allocated between transportation and business so that for a given number of workers, N, the social costs given by (2.7) are minimized yielding an optimum radius for the city, ε. All the workers are assumed to travel to the city boundary along the radius passing through their place of work, thus the number of travellers is given by

$$T(u) = \int_0^u N_c(v)dv. \tag{4.1}$$

$N_c(v)$ is defined in (2.4) and we get the differential equation for the number of travellers

$$T'(u) = a_c \theta u - a_c L_2(u) \tag{4.2}$$

As in the previous section we can rule out the possibility of their being no land allocated for transportation, i.e. $L_2(u) = 0$, except when $T(u) = 0$.

The Hamiltonian, $\mathcal{H}^C$, for the CBD problem is

$$\mathcal{H}^C = \rho_1 C(u)^{\rho_2} + R_A \theta u + [a_c \lambda(u) + \mu(u)][\theta u - L_2(u)], \tag{4.3}$$

where $\mu(u)$ is, as before, the multiplier applying to the constraint $L_2(u) \leq \theta u$. The first order conditions are

$$\frac{\partial \mathcal{H}^C}{\partial T} = -\lambda'(u) = \rho_1(\rho_2+1)C^{\rho_2} \tag{4.4}$$

and

$$\frac{\partial \mathcal{H}^C}{\partial L_2} = -\rho_1\rho_2 C^{\rho_2+1} - a_c\lambda - \mu = 0 \tag{4.5}$$

At the boundary of the city, $\mu = \varepsilon$, we know that $T(\varepsilon) = N$, and there is the traversality condition that $\mathcal{H}^C(\varepsilon) = 0$. Equations (4.3) and (4.5) yield the expression for the Hamiltonian along the optimal path,

$$\mathcal{H}^C = R_A\theta u + \rho_1\, C(u)^{\rho_2}\, T(u) - \rho_1\rho_2\, C(u)^{\rho_2+1}\,[\theta u - L_2(u)]. \qquad (4.6)$$

Hence the transversality condition is

$$R_A\theta\varepsilon = \rho_1\rho_2\, C(\varepsilon)^{\rho_2+1}\left[\theta\varepsilon - \frac{(\rho_2+1)L_2(\varepsilon)}{\rho_2}\right] \qquad (4.7)$$

A priori we assume $R_A > 0$ and congestion at the boundary $C(\varepsilon)$ must also be positive which implies the inequality

$$L_2(\varepsilon) < \frac{\rho_2\,\theta\varepsilon}{\rho_2+1}\quad . \qquad (4.8)$$

Thus without a doubt the $L_2(u)$ trajectory close to the city boundary is unconstrained and is governed by the congestion equation

$$C(u) = \frac{a_c}{\rho_2}\,[u - \varepsilon + G] \quad \text{where } G = \frac{\rho_2\, C(\varepsilon)}{a_c}\quad . \qquad (4.9)$$

This equation is derived from the adjoint equation (4.4) and (4.5), as was Eq. (3.10) in the previous section. Define $P(u) = u - \varepsilon + G$ and (4.2) and (4.9) may be integrated to yield

$$P(u)^{\rho_2}T(u) = G^{\rho_2}N + \frac{a_c\,\theta}{\rho_2+1}\left[uP(u)^{\rho_2+1} - \varepsilon G^{\rho_2+1} - \frac{P(u)^{\rho_2+2} - G^{\rho_2+2}}{\rho_2+2}\right] \qquad (4.10)$$

If Eq. (4.10) holds, i.e. $L_2(u) \neq \theta u$, over the range $0 \leq u \leq \varepsilon$, then we can use the condition T(0) = 0 and get

$$G^{\rho_2} N = \frac{a_c \theta}{\rho_2+1}\left[\varepsilon G^{\rho_2+1} + \frac{(G-\varepsilon)^{\rho_2+2} - G^{\rho_2+2}}{\rho_2+2}\right]. \tag{4.11}$$

Equations (4.7) and (4.11) can theoretically be solved to yield the optimal ε as a function of N and R_A since $L_2(\varepsilon)$ in (4.7) can be replaced by $\frac{\rho_2 N}{a_c \varepsilon}$.

The equation determining the amount of land allocated for transportation in the city can easily be obtained from (4.10) and (4.11) and the identity, $T(u) = \frac{a_c P(u) L_2(u)}{\rho_2}$. Hence:

$$L_2(u) = \frac{\rho_2 \theta}{\rho_2+1} \frac{1}{P(u)^{\rho_2+1}}\left[uP(u)^{\rho_2+1} + \frac{(G-\varepsilon)^{\rho_2+2} - P(u)^{\rho_2+2}}{\rho_2+2}\right] \tag{4.12}$$

The validity of this relationship depends upon the assumption that $L_2(u) \neq \theta u$ in the range $0 \leq u \leq \varepsilon$. It can be rewritten in the form:

$$L_2(u) = \frac{\rho_2 \theta u}{\rho_2+2} - \frac{\rho_2 \theta(G-\varepsilon)}{(\rho_2+1)(\rho_2+2)}\left[1 - \left(\frac{G-\varepsilon}{G-\varepsilon+u}\right)^{\rho_2+1}\right]. \tag{4.13}$$

As u becomes large so the $L_2(u)$ curve asymptotically approaches the line,

$$L_2(u) = \frac{\rho_2 \theta u}{\rho_2+2} - \frac{\rho_2 \theta(G-\varepsilon)}{(\rho_2+1)(\rho_2+2)} . \tag{4.14}$$

It is easily shown that

$$L_2'(u) = \frac{\rho_2+1}{P(u)}\left[\frac{\rho_2\ \theta u}{\rho_2+1} - L_2(u)\right] \tag{4.15}$$

and

$$L_2''(u) = \frac{(\rho_2+1)(\rho_2+2)}{P(u)^2}\left[L_2(u) - \frac{\rho_2\ \theta u}{\rho_2+2} + \frac{\rho_2\ \theta(G-\varepsilon)}{(\rho_2+1)(\rho_2+2)}\right]. \tag{4.16}$$

The $L_2'(u) = 0$ line is the same as the Mills-de Ferranti case and the $L_2''(u) = 0$ line is Eq. (4.14). Thus we are able to sketch the $L_2(u)$ curve on the basis of the above information; and noticing also from (4.13) that $L_2(u) \leq \frac{\rho_2\ \theta u}{\rho_2+2}$ for all $u > 0$.

Let us now consider possible alternatives to the solution shown in Fig. 2. Suppose the number of travellers is zero within a radius r of the city center, then the land allocated to business within that zone will be zero and all the land will be used for transportation even though no one is travelling. The cost of this land, $1/2\ R_A\ \theta r^2$, in no way reduces overall congestion costs so we may rule it out as a possible optimal solution. Having previously shown that $L_2(u) > 0$ for all $u \in (0,\varepsilon]$ and $L_2(\varepsilon) \neq \theta\varepsilon$, the only possible exception to Fig. 2 is $L_2(u) = \theta u$ for some $u \in (0,\varepsilon)$.

Along the arc $L_2(u) = \theta u$ it follows from differentiating (4.5) with respect to u and substituting for $\lambda'(u)$ from (4.4) that:

$$\lambda'(u) = \rho_1(\rho_2+1)C^{\rho_2}\left[\frac{dC}{du}\rho_2 - a_c\right]. \tag{4.17}$$

But since $C(u) = \frac{\bar{N}}{\theta u}$ along the arc, where $\bar{N}$ is a constant, then

$$\mu'(u) = -\rho_1(\rho_2+1)C^2\left[\frac{\rho_2 C}{u} + a_c\right], \tag{4.18}$$

showing that $\mu(u)$ is a monotonically decreasing function of u for positive u. Thus once the allocation of land for transportation has reached its maximum, it remains constrained at that value for all points closer to the city center. $\bar{N}$ in (4.17) is therefore equal to 0.

Let $\hat{u}$ be the distance from the city center at which the amount of land allocated for transportation becomes equal to θu. We have already shown that $T(\hat{u}) = 0$, hence from (4.10),

$$G^{\rho_2} N = \frac{a_c \theta}{\rho_2+1}\left[\varepsilon G^{\rho_2+1} - \hat{u}P(\hat{u})^{\rho_2+1} + \frac{P(\hat{u})^{\rho_2+2} - G^{\rho_2+2}}{\rho_2+2}\right] \tag{4.19}$$

but $L_2(\hat{u})a_c P(\hat{u}) = \rho_2 T(\hat{u}) = 0$ which implies $P(\hat{u}) = 0$ when $L_2(\hat{u}) = \theta\hat{u} \neq 0$. Thus (4.19) becomes

$$G^{\rho_2} N = \frac{a_c \theta}{\rho_2+1}\left[\varepsilon G^{\rho_2+1} - \frac{G^{\rho_2+2}}{\rho_2+2}\right], \tag{4.20}$$

and (4.10) can be simplified to

$$T(u) = \frac{a_c \theta}{\rho_2+1}\left[u - \frac{P(u)}{\rho_2+2}\right]P(u). \tag{4.21}$$

From (4.21) we can also obtain an expression for the allocation of land for transportation and it proves to be linear in u, the distance from the city center..

$$L_2(u) = \frac{\rho_2 \theta}{\rho_2+1}\left[u - \frac{P(u)}{\rho_2+2}\right] \tag{4.22}$$

However, when we solve (4.22) for the distance $\hat{u}$ at which all the land is allocated for transportation, we discover that this implies that $P(\hat{u}) = (\rho_2+2)\hat{u}$. $P(\hat{u})$ has already been shown to be zero and hence the only place in the city where all the land is allocated for transportation is at the city center. We have thus shown that at no point in the CBD city is all the land allocated for transportation. The case where $P(u)$ becomes zero at the city center proves to be a limiting case as we shall see.

Figure 2 shows the set of possible solutions to the CBD problem for a given number of people working in the city and varying values of land rent, R_A. If R_A is very high the city will be small and congestion costs high since not much land is allocated to transportation. For increasingly high R_A we approach but never reach the limiting case where no land is used for transportation, i.e. $L_1(u) = \theta u$, implying that

$$\frac{a_C \; \theta\varepsilon_{min}^2}{2} = N. \tag{4.23}$$

As R_A decreases so the relative costs arising from congestion can be decreased by expanding the city's size. Since in (4.9) we showed that congestion increases linearly with the distance from the city center (as the size of the city expands), it will be the city center which has zero congestion costs before any another point. Once we have attained zero congestion costs at the city center ($P(0) = 0$) there is no point in expanding the city further since taking a person from the center where he creates zero congestion costs to the edge of the city is clearly non-optimal. Thus as we have previously shown mathematically no solutions lie above the $L_2(u) = \frac{\rho_2 \theta u}{\rho_2+2}$ line.

If we examine this upper bound on the solution of $L_2(u)$, we find, since as was shown above it implies $G = \varepsilon$, that

$$a_c \theta \varepsilon_{max}^2 = (\rho_2+2)N. \tag{4.24}$$

Thus ε_{min} and ε_{max} are the lower and upper limits on the radius of the CBD city irrespective of the value of R_A for a given working population and at no point is all the land used for transportation purposes.

5. The Unified City Problem

One of the conclusions of the Mills-de Ferranti work was that close to the boundary of the CBD with the rest of the city all available land is under certain circumstances used for transportation. This is hardly a satisfactory "optimal" result and once this state of saturation has been reached the obvious answer is to extend the boundary of the CBD. The problem tackled here is to accommodate N workers within the CBD who all live outside the CBD and travel to work along the radius of the city which passes both through their place of work and their homes. Again the city should be designed to minimize social costs defined by (2.8), this time however the limits of integration are from $u = 0$, the city center, to $u = \bar{u}$, the boundary of the city which is unknown, as is the radius of the CBD, ε.

The solution can be tackled in two parts which are in essence the CBD problem of the previous section and the Mills-de Ferranti problem. Now that we are considering both the CBD and the suburbs the assumption made above concerning the direction of travel to work and the directional location of a workers residence are somewhat more unrealistic than in the previous sections. The assumption does have the advantage of keeping Eqs. (3.2) and (4.2) as the differential equations relating to travellers within the suburbs and the CBD respectively.

Again we may rule out the possibility of $L_2(u) = 0$ except where $\dot{T}(u) = 0$ and so the Hamiltonian for the unified problem is equal to $\mathcal{H}^c$, (4.3), within the CBD and equal to $\mathcal{H}^s$, (3.4), within the suburbs. The transversality condition at the edge of the city, (3.7), also applies but at ε, the junction of the CBD and the suburbs, the transversality condition (4.7) is replaced by a set of boundary conditions[2] relating variables on either side of ε. These are,

$$\begin{aligned} T(\varepsilon^-) &= T(\varepsilon^+) = N \\ \mathcal{H}^c(\varepsilon^-) &= \mathcal{H}^s(\varepsilon^+) \\ \lambda(\varepsilon^-) &= \lambda(\varepsilon^+) + v, \\ \varepsilon^- &= \varepsilon^+ = \varepsilon \end{aligned} \tag{5.1}$$

where ε^- denotes "on the CBD side of the boundary" and ε^+ "on the suburban side of the boundary." From (4.6) we know that

$$\mathcal{H}^c(\varepsilon^-) = R_A\, \theta\varepsilon - \rho_1\, C(\varepsilon^-)^{\rho_2+1} [\rho_2\, \theta\varepsilon - (\rho_2+1)L_2(\varepsilon^-)] \tag{5.2}$$

and similarly

$$\mathcal{H}^s(\varepsilon^+) = R_A\ \theta\varepsilon + \rho_1\ C(\varepsilon^+)^{\rho_2+1}\ [\rho_2\ \theta\varepsilon - (\rho_2+1)L_2(\varepsilon^+)]. \tag{5.3}$$

Equating $\mathcal{H}^c(\varepsilon^-)$ and $\mathcal{H}^s(\varepsilon^+)$ gives

$$\left[\frac{L_2(\varepsilon^-)}{L_2(\varepsilon^+)}\right]^{\rho_2+1} - \frac{(\rho_2+1)L_2(\varepsilon^-) - \rho_2\ \theta\varepsilon}{(\rho_2+1)L_2(\varepsilon^+) - \rho_2\ \theta\varepsilon} = 0, \tag{5.4}$$

since

$$\frac{C(\varepsilon^+)}{C(\varepsilon^-)} = \frac{T(\varepsilon^+)}{L_2(\varepsilon^+)} \cdot \frac{L_2(\varepsilon^-)}{T(\varepsilon^-)} = \frac{L_2(\varepsilon^-)}{L_2(\varepsilon^+)} .$$

One solution to (5.4) is $L_2(\varepsilon^+) = L_2(\varepsilon^-)$, i.e. there is continuity in the allocation of land across the CBD boundary. Differentiating (5.4) with respect to $L_2(\varepsilon^-)$ we obtain

$$\frac{\rho_2(\rho_2+1)(L_2(\varepsilon^-)-\theta\varepsilon)}{((\rho_2+1)L_2(\varepsilon^+)-\rho_2\theta\varepsilon)L_2(\varepsilon^-)} \tag{5.5}$$

which shows (5.4) to be a monotonic function of $L_2(\varepsilon^-)$ for positive values of ε. Hence $L_2(\varepsilon^+) = L_2(\varepsilon^-)$ is the only feasible solution for the boundary condtions. Note that (5.2) and (5.3) apply even if at one or both sides of the boundary $L_2(\varepsilon)$ has a constrained valued.

Let us consider initially the case where $L_2(\varepsilon) < \theta\varepsilon$, then

$$\lambda(\varepsilon^-) = \frac{-\rho_1\rho_2}{a_c}\ C(\varepsilon)^{\rho_2+1} = \frac{-\lambda(\varepsilon^+)\ a_s}{a_c} = \lambda(\varepsilon^+) + v.$$

However

$$G = \frac{C(\varepsilon)\rho_2}{a_c} = \frac{\rho_2}{a_c} \cdot \frac{a_s}{\rho_2} [H + \bar{u} - \varepsilon] = \frac{a_s}{a_c} Q(\varepsilon) \tag{5.6}$$

where

$$G^{\rho_2} N = \frac{a_c \theta}{\rho_2+1} \left[\varepsilon G^{\rho_2+1} + \frac{(G-\varepsilon)^{\rho_2+2} - G^{\rho_2+2}}{\rho_2+2} \right] \tag{5.7}$$

and

$$Q(\varepsilon)^{\rho_2} N = \frac{a_s \theta}{\rho_2+1} \left[\varepsilon Q(\varepsilon)^{\rho_2+1} - \bar{u} H^{\rho_2+1} + \frac{Q(\varepsilon)^{\rho_2+2} - H^{\rho_2+2}}{\rho_2+2} \right] \tag{5.8}$$

three equations in three unknown variables, G, ε and $\bar{u}$. Equation (5.7) is (4.11) and (5.8) comes directly from (3.1) by setting $u = \varepsilon$ and $T(\varepsilon) = N$.

From our analysis of the CBD problem we know that $L_2(\varepsilon) \leq \frac{\rho_2 \theta\varepsilon}{\rho_2+2}$ so in this section we are not troubled by the possibility of $L_2(u)$ ever equalling θu since we have shown continuity of $L_2(\varepsilon)$ across the boundary. Also we know from our analysis of the Mills-de Ferranti problem that $L_2(u)$ has the maximum value of $L_2(\varepsilon)$ when $L_2(\varepsilon) \leq \frac{\rho_2 \theta\varepsilon}{\rho_2+1}$. The solution to the unified problem for a given working population for the city can easily be presented as in Fig. 3 by combining the solutions to the CBD and Mills-de Ferranti problems.

We may summarize the findings by stating that the optimal allocation of land for transportation as we move out from the city center is a monotonically increasing concave function until the boundary of the CBD is reached when it becomes a monotonically decreasing convex function until the boundary of the city is reached when it is zero.

It only remains to find the maximum and minimum bounds on the city's size for a given working population, N. ε_{min} and ε_{max} are as in (4.23) and (4.24) respectively. From (5.6) given that we know $\varepsilon_{max} = G$,

$$\varepsilon_{max} = \frac{a_s}{a_c} [H + \bar{u}_{max} - \varepsilon_{max}], \tag{5.9}$$

and from (3.8) and (4.7),

$$H = \left(\frac{R_A}{\rho_1 \rho_2}\right)^{1/\rho_2+1} = \frac{a_c}{\rho_2} \varepsilon_{max} \left(\frac{1}{\rho_2+2}\right)^{1/\rho_2+2} . \tag{5.10}$$

These two equations provide the relationship

$$\frac{\bar{u}_{max}}{\varepsilon_{max}} = 1 + \frac{a_c}{a_s}\left[1 - \left(\frac{1}{\rho_2+2}\right)^{1/\rho_2+1}\right] \tag{5.11}$$

where $\varepsilon^2_{max} = \frac{(\rho_2+2)N}{a_c \theta}$.

It can easily be shown that

$$\frac{\bar{u}_{min}}{\varepsilon_{min}} = 1 + \frac{a_c}{a_s} , \qquad \varepsilon^2_{min} = \frac{2N}{a_c \theta} ; \tag{5.12}$$

from which it follows that

$$\frac{\bar{u}_{max}}{\bar{u}_{min}} = \sqrt{\frac{\rho_2+2}{2}} \left[1 - \frac{a_c}{a_s + a_c} \left(\frac{1}{\rho_2+2} \right)^{1/\rho_2+1} \right] . \qquad (5.13)$$

6. Conclusions

In this paper we have shown that by extending the Mills-de Ferranti problem to deal with the optimal allocation of land for transportation both in the suburbs and the CBD the conclusions of Mills that at some point in the city all land is used for transportation is no longer valid. Indeed we have proved the existence of an upper limit on the amount of land used for transportation and of a maximum city size for a given working population regardless of the alternative rental value of land. It is comforting to find that the Mills-de Ferranti vision of a city consisting entirely of roads at some points is false when we adopt a wider view of the problem.

The assumptions made in analyzing this problem are quite severe and in particular no one can seriously pretend that the major item of social costs arising in a city is congestion. Artle [1] in discussing the original Mills-de Ferranti paper raises many valid points the only one of which we have attempted to answer here is the neglect of social costs within the CBD.

<ins>Footnotes</ins>

1. Suppose $L_2(\bar{u}) \neq 0$, then $\lambda(\bar{u}) = \dfrac{R_A \; \theta\bar{u}}{a_s[\theta\bar{u} - L_2(u)]} = \dfrac{\rho_1\rho_2}{a_s}\left(\dfrac{T(\bar{u})}{L_2(\bar{u})}\right)^{\rho_2+1}$

   but $T(\bar{u}) = 0$ by assumption and hence implies that $\bar{u} = 0$ if $L_2(\bar{u}) \neq 0$. $\bar{u} > 0$ by assumption and hence $L_2(\bar{u}) = 0$.

2. See Bryson and Ho [2] pp. 101-106. v is the multiplier associated with the constraint $[T(u) - N]_{u=\varepsilon} = 0$.

References

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[2] A. E. Bryson and Y. C. Ho, "Applied Optimal Control," Blaisdell, Waltham, Mass., 1969.

[3] E. S. Mills and D. M. de Ferranti, "Market Choices and Optimum City Size," Amer. Econ. Rev. Proc., 61 (1971), pp. 340-345.

[4] R. M. Solow and W. S. Vickery, "Land Use in a Long Narrow City," Journal of Economic Theory, 3 (1971), pp. 430-447.

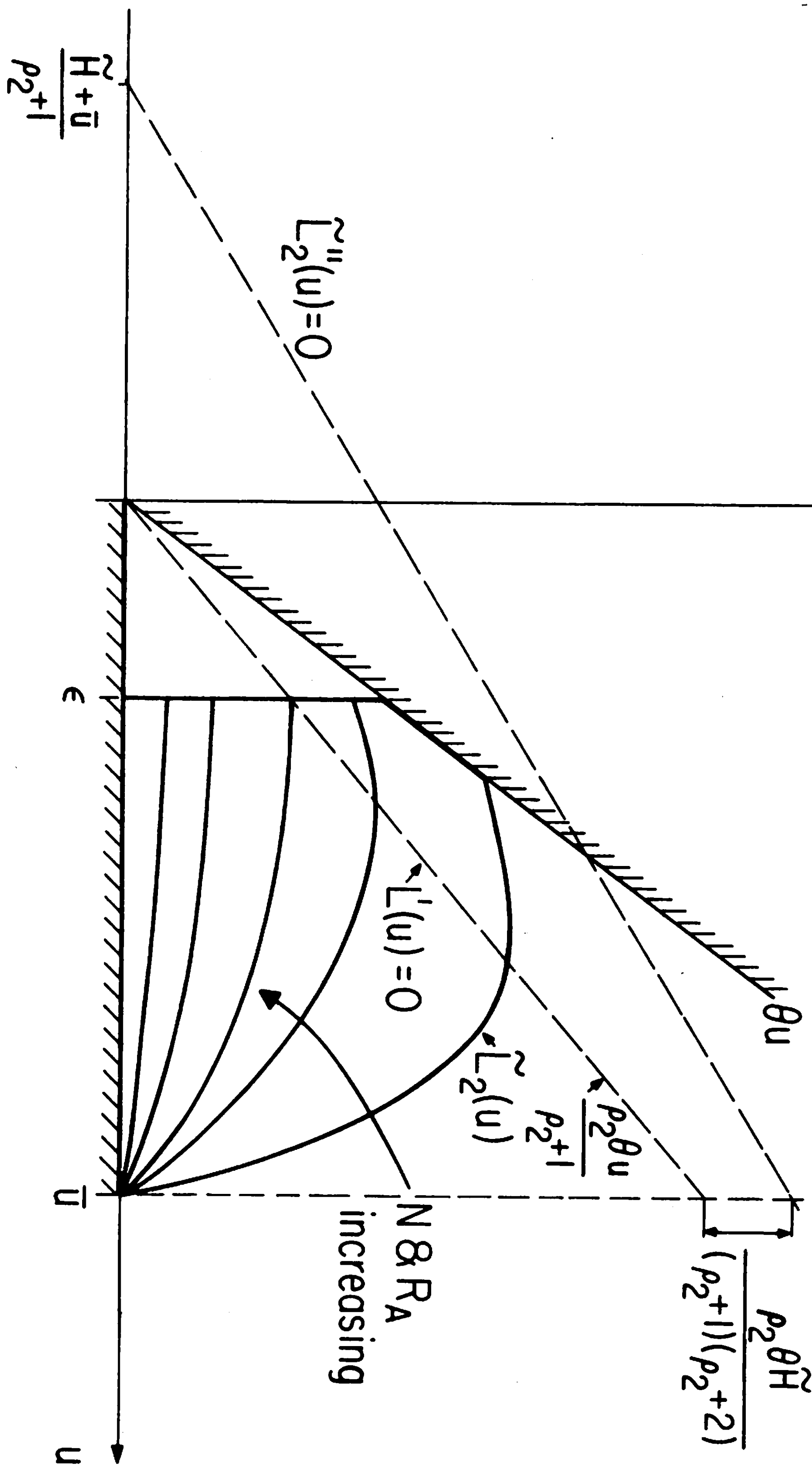

Fig. 1. Optimal allocation of land for transportation in the suburbs.

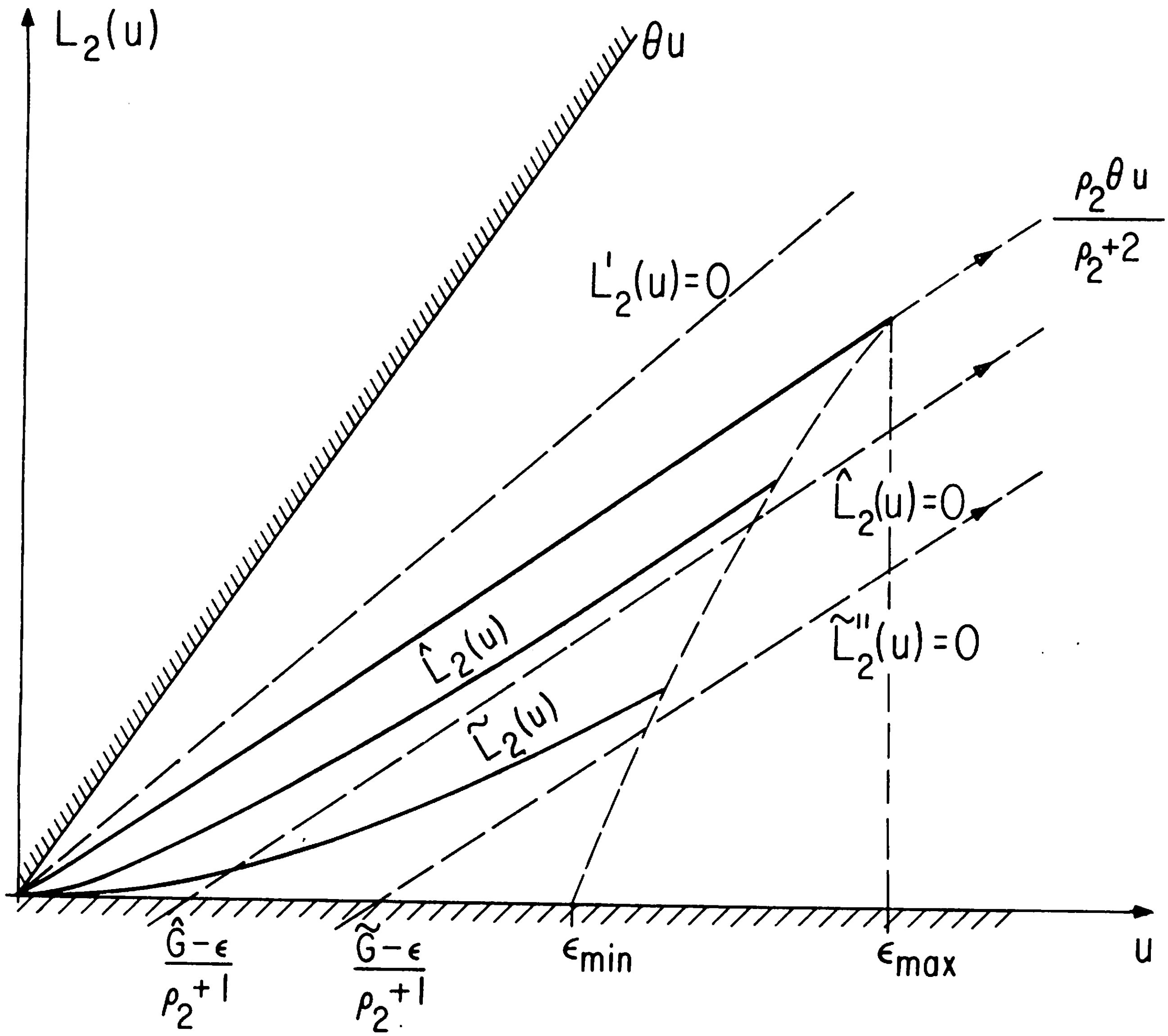

Fig. 2. Optimal allocation of land for transportation within the CBD for a fixed working population.

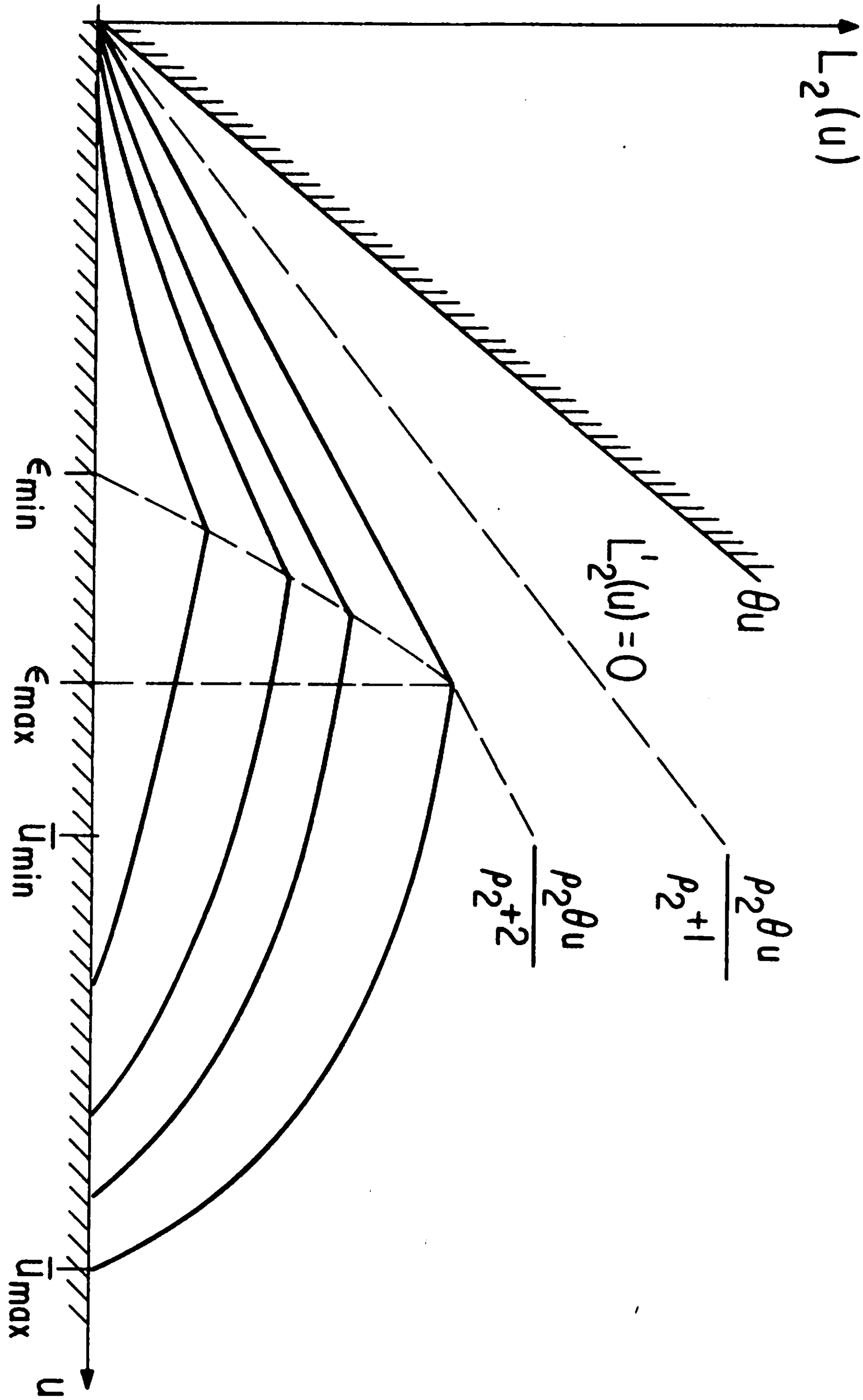

Fig. 3. Optimal allocation of land for transportation in both the CBD and the suburbs for a fixed working population.