EFFECTS OF CONGESTION ON THE SHAPE OF A CITY

by

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Memorandum No. ERL-M330

15 March 1972

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Abstract. This paper developes a model for the allocation of urban land among the residential and transportation sectors and the Central Business District. The model is used to derive the intensity of land use, that is the capital per unit of land, in the residential and transportation sectors. Two different institutional arrangements are discussed: a central planning agency, and a competitive market. It is shown that the externalities imposed by traffic congestion results in the market city being larger (for the same population), and with flatter density profiles.

Research sponsored by the National Science Foundation, Grant GK-10656X1.
Also supported by Conselho Nacional de Pesquisas, Brasil. (Ripper)
Also supported by a Fellowship from the Guggenheim Foundation. (Varaiya)

1. Introduction and Summary.

Consider a circular city with rotational symmetry. There is a dominant pole of attraction - the Central Business District (CBD) - at its center. The land outside the CBD can be put to one of two uses: it can be used for residences or for providing transportation to the CBD. Households have an inelastic demand for living space, but by erecting structures, that is, by substituting capital for land, larger number of households can be accomodated per acre of (physical) land. Similarly the effective surface available for transportation can be increased by investment of capital. Thus there are five variables: size of the CBD, land and capital devoted to housing, land and capital devoted to transportation.

Every household has a member who works in the CBD. Consequently each household incurs a private cost of transportation to the CBD. This cost increases with the traffic density so that each household imposes cost on the other households. The sum of three costs - interest on capital, transportation cost, and an opportunity cost for land is the total social cost. We are interested in determining the allocation of land and capital and the resulting city shape and social costs under two institutional arrangements.

In the first setting there is a central authority which determines the allocation that minimizes total social cost. This allocation will be called the optimal solution. In the second alternative we suppose that the allocative power is decentralized. On the one hand landlords, taking the rent profile as given, determine the amount of capital devoted to housing which maximizes their profit. On the other hand the city government, taking the rent as given, determines the amount of land and capital devoted to transportation which maximizes net benefit. We will call the resulting allocation the market solution.

Section 2 displays the mathematical model. The optimal solution is derived in Section 3, and in Section 4 it is shown that this solution cannot be sustained by a market institution of the kind described above. However any one of a number of appropriate taxation schemes in combination with a market will sustain the optimum solution. In Section 5 we derive the market solution. The two solutions are compared in Section 6. In particular it is shown that the market city is more spread-out than the optimal city, it has flatter density profiles. The market city devotes too much resources to transportation at the center and too much resources to housing at the periphery. We also obtain the ratio of the two city sizes and the two CBD sizes.

Three recent papers [1-3] have explored issues similar to those considered here. However none of these allows for possibilities of substitution between capital and land. Furthermore they mainly discuss the optimal solution, and do not develop the market solution. Of course, many people [4-6] have explored the possibilities of substitution between land and capital, in the absence of externalities. The contribution of this paper lies in combining both of these aspects.

2. The Model

Every point of the city is at some distance u from the center. This distance may be the usual Euclidean distance but it does not need to be. The CBD consists of all points within a fixed distance $\varepsilon > 0$ from the center. ε is given exogenously, but as we shall see later ε will have to increase as the size of the population increases. The area of (physical) land which lies at a distance between u and u+du from the center is $\theta(u)du$ where $\theta(u) > 0$ is given exogenously. For example, if $\frac{\theta(u)}{u} \equiv 2\pi$ we have a circular city, if $\frac{\theta(u)}{u} \equiv \theta < 2\pi$ we have the pie-shape city of [1], if $\frac{\theta(u)}{u} \equiv 16\pi$ we have the eight-layer city of [7]. The total area $\theta(u)$, at distance $u \geq \varepsilon$, is divided into two parts. $\ell_T(u)$ is devoted to transportation, and the remainder $\ell_H(u) = \theta(u) - \ell_T(u)$ is devoted to housing. Both ℓ_T, ℓ_H are determined within the model.

Each household demands one unit area of living space. If the density of households at u is $m_H(u)$ per unit area of land, giving a total of $m_H(u)\ell_H(u)$ households at u, then the capital costs necessary to support this density is $\alpha_H(m_H(u))^{\beta_H}$, where $\alpha_H > 0$ and $\beta_H > 1$ are constants. Similarly the effective surface available for transportation at u is $m_T(u)\ell_T(u)$, at a capital cost of $\alpha_T(m_T(u))^{\beta_T}$ where $\alpha_T > 0$ and $\beta_T > 1$ are constants. Let $n(u)$ be the number of households living at a distance at least u from the center. Then

$$\dot{n}(u) = \frac{dn(u)}{du} = - m_H(u)\ell_H(u), \tag{1}$$

and if the total number of households in the city is n, given exogenously, then

$$n(\varepsilon) = n, \qquad n(\bar{u}) = 0, \tag{2}$$

where $\bar{u}$ is the edge of the city. $\bar{u}$ is determined endogenously

We suppose that everyone travels to the CBD at the same time. Then the peak density of traffic at u is $d(u) = \frac{n(u)}{m_T(u)\ell_T(u)}$. Following [1-3] we assume that the transport cost per person per annum over a distance u to u+du is $\rho_1(d(u))^{\rho_2}$ where $\rho_1 > 0$ and $\rho_2 > 0$ are constants. Hence the total transport cost per annum is

$$\int_{\varepsilon}^{\bar{u}} \rho_1(d(u))^{\rho_2} n(u)du. \tag{3}$$

The interest on capital is r per annum, so that the total annual cost on capital is

$$r \int_{\varepsilon}^{\bar{u}} \{\alpha_H(m_H(u))^{\beta_H} \ \ell_H(u) + \alpha_T(m_T(u))^{\beta_T} \ \ell_T(u)\}du. \tag{4}$$

Finally if we take an opportunity cost for land (arising say from agriculture) we get a total of

$$R_a \int_{\varepsilon}^{\bar{u}} \theta(u)du. \tag{5}$$

The social cost to the city is the sum of items (3), (4), and (5).

3. The Optimal Solution

We want to determine the land use, $\ell_T(u)$, $\ell_H(u) = \theta(u) - \ell_T(u)$, the investment program $m_T(u) \geq 0$, $m_H(u) \geq 0$, and the city boundary $\bar{u}$, which accomodates n households i.e., satisfies

$$\dot{n}(u) = - m_H(u) \ (\theta(u) - \ell_T(u)), \qquad \varepsilon \le u \le \bar{u}, \tag{6}$$

$$n(\bar{u}) = 0, \qquad n(\varepsilon) = n, \tag{7}$$

and at the same time minimize the total social cost

$$\int_{\varepsilon}^{\bar{u}} \left\{ \rho_1 \left(\frac{n(u)}{m_T(u)\ell_T(u)}\right)^{\rho_2} n(u) + r\alpha_H (m_H(u))^{\beta_H} (\theta(u) - \ell_T(u)) + r\alpha_T (m_T(u))^{\beta_T} \ell_T(u) + R_a \theta(u) \right\} du.$$

This is a relatively straightforward optimal control problem [8]. Let p denote the adjoint or co-state variable, and define the Hamiltonian

$$H(u,n,p,m_H,m_T,\ell_T)$$
$$= \rho_1 \left(\frac{n}{m_T \ell_T}\right)^{\rho_2} n + r\alpha_H \ m_H^{\beta_H} (\theta(u) - \ell_T) + r\alpha_T \ m_T^{\beta_T} \ \ell_T + R_a \theta(u) - p m_H (\theta(u) - \ell_T) \tag{8}$$

Then at the optimal solution, the adjoint variable must satisfy the adjoint equation

$$\dot{p}(u) = - \frac{\partial H}{\partial n} (u) = - \rho_1(\rho_2+1)(d(u))^{\rho_2}, \qquad \varepsilon \le u \le \bar{u}. \tag{9}$$

Furthermore the Hamiltonian must be minimized over the set $m_H \ge 0$,

$m_T \geq 0$, $\theta(u) \geq \ell_T \geq 0$, which implies that m_H and m_T are given respectively by

$$r\alpha_H\beta_H(m_H(u))^{\beta_H - 1} = p(u), \tag{10}$$

$$r\alpha_T\beta_T(m_T(u))^{\beta_T - 1} = \rho_1\rho_2(d(u))^{\rho_2 + 1}, \tag{11}$$

whereas ℓ_T is given by

$$\frac{\partial H}{\partial \ell_T} = -\rho_1\rho_2 m_T(u)\ (d(u))^{\rho_2+1} + r\alpha_T(m_T(u))^{\beta_T} - r\alpha_H(m_H(u))^{\beta_H}$$

$$+ m_H(u)p(u) = 0, \tag{12}$$

provided that (12) yields $\ell_T \leq \theta(u)$, otherwise

$$\left.\frac{\partial H}{\partial \ell_T}\right|_{\ell_T = \theta(u)} < 0, \text{ and then}$$

$$\ell_T(u) = \theta(u). \tag{13}$$

The set of u for which ℓ_T is given by (12) will be called the <u>unsaturated</u> region, whereas the set of u for which $\ell_T(u) = \theta(u)$ will be called the <u>saturated</u> region.

Finally, the optimal value of $\bar{u}$ must be such that the Hamiltonian vanishes at $u = \bar{u}$, which implies that

$$p(\bar{u}) = \frac{R_a + r\alpha_H(m_H(\bar{u}))^{\beta_H}}{m_H'(\bar{u})}\ . \tag{14}$$

From (10) and (14) we can see that the optimal value of the adjoint variable at $\bar{u}$, $p(\bar{u})$, is independent of $\bar{u}$ and hence of n. Let this value be $\bar{p}$.

We can now solve (6)-(14) and obtain the optimal solution. We start with a trial choice of $\bar{u}$, and integrate the system (6) and (9) backwards using the known boundary conditions $n(\bar{u}) = 0$, $p(\bar{u}) = \bar{p}$. The system is completely deterministic. Since the values of m_H, m_T, and ℓ_T can be calculated in terms of $p(u)$ from (10)-(13). We check to see if the boundary condition $n(\varepsilon) = n$ is satisfied. If it is, then we have the optimal solution, otherwise we have to try a new value for $\bar{u}$. We see that the optimal solution is parametrized by $\bar{u}$, each value of $\bar{u}$ corresponding to a particular value of $n = n(\bar{u})$. We will sometimes index the variables by $\bar{u}$ to denote different optimal solutions: $p(u,\bar{u})$, $\ell_T(u,\bar{u})$, $d(u,\bar{u})$, etc.

It will prove convenient to change the independent variable from u to $t = \bar{u}-u$, the distance from the city's edge. Also define,

$$\pi(t) = \pi(t,\bar{u}) = p(\bar{u}-t,\bar{u}),$$

$$\delta(t) = \delta(t,\bar{u}) = d(\bar{u}-t,\bar{u}),$$

$$\lambda_T(t) = \lambda_T(t,\bar{u}) = \ell_T(\bar{u}-t,\bar{u}),$$

$$\nu(t) = \nu(t,\bar{u}) = n(\bar{u}-t,\bar{u}),$$

$$\mu_H(t) = \mu_H(t,\bar{u}) = m_H(\bar{u}-t,\bar{u}),$$

$$\mu_T(t) = \mu_T(t,\bar{u}) = m_T(\bar{u}-t,\bar{u}).$$

In the unsaturated region, ℓ_T is given by (12). From (10)-(12) we obtain

$$\pi(t) = A^{\frac{\beta_H-1}{\beta_H}} (\delta(t))^{(\rho_2+1)\left(\frac{\beta_T}{\beta_T-1}\right)\left(\frac{\beta_H-1}{\beta_H}\right)}, \tag{15}$$

where

$$A = (\rho_1\rho_2)^{\frac{\beta_T}{\beta_T-1}} \left(\frac{\beta_T-1}{\beta_T}\right)\left(\frac{\beta_H}{\beta_H-1}\right) \frac{(r\alpha_H\beta_H)^{\frac{1}{\beta_H-1}}}{(r\alpha_T\beta_T)^{\frac{1}{\beta_T-1}}} > 0.$$

Differentiating (15), and substituting from (9), we obtain

$$\dot{\delta}(t) = B(\delta(t))^{\gamma}, \tag{16}$$

where

$$B = \rho_1 \left(\frac{\beta_T-1}{\beta_T}\right)\left(\frac{\beta_H}{\beta_H-1}\right) A^{-\left(\frac{\beta_H-1}{\beta_H}\right)} > 0,$$

and

$$\gamma = (1+\rho_2)\left[1 - \left(\frac{\beta_T}{\beta_T-1}\right)\left(\frac{\beta_H-1}{\beta_H}\right)\right] . \tag{17}$$

Thus $-\infty < \gamma < 1+\rho_2$. Similarly,

$$\dot{\lambda}_T(t) = -(1+\rho_2)\left(\frac{\beta_T}{\beta_T-1}\right) B(\delta(t))^{\gamma-1} \lambda_T(t) + \rho_2 B(\delta(t))^{\gamma-1} \theta(\bar{u}-t), \tag{18}$$

and

$$\dot{\nu}(t) = -\rho_2 \, B(\delta(t))^{\gamma-1}\nu(t) + \left(\frac{1}{r\alpha_H\beta_H}\right)^{\frac{1}{\beta_H-1}} A^{\frac{1}{\beta_H}} (\delta(t))^{\left(\frac{1+\rho_2}{\beta_H}\right)\left(\frac{\beta_T}{\beta_T-1}\right)} \theta(\bar{u}-t). \tag{19}$$

The analysis is considerably simpler in the saturated region. We must have $\lambda_T(t) \equiv \theta(\bar{u}-t)$. Hence $\dot{\nu}(t) \equiv 0$, so that $\nu(t)$ is constant over every connected interval in the saturated region. The traffic density is

$$\delta(t) \equiv \frac{\nu(t)}{\theta(\bar{u}-t)\mu_T(t)}, \tag{20}$$

where μ_T can be expressed in terms of $\delta(t)$ via (11). Finally, $\pi(t)$ still continues to be governed by (9).

It is clear that $t = 0$ $(u = \bar{u})$ is in the unsaturated region. Furthermore, $\pi(0) = p(\bar{u}) = \bar{p} > 0$, and hence from (15), $\delta(0) = d(\bar{u}) = \bar{d} > 0$, independent of $\bar{u}$. Since $\nu(0) = 0$, we must have $\lambda_T(0) = 0$.

In the rest of this paper we will make some of the following assumptions.

<u>A1</u> $\quad \theta(u_2) \geq \theta(u_1)$ for $u_2 \geq u_1$.

<u>A2</u> $\quad \dfrac{\dot{\theta}(u_2)}{\theta(u_2)} \leq \dfrac{\dot{\theta}(u_1)}{\theta(u_1)}$ for $u_2 \geq u_1$.

<u>A3</u> $\quad \dfrac{\theta(u)}{u} \equiv$ constant.

<u>Proposition 1</u>. Suppose A1 holds. Then $\delta(t_2,\bar{u}) \geq \delta(t_1,\bar{u})$ for $t_2 \geq t_1$.

Proof. In the unsaturated region $\dot{\delta}(t) \geq 0$ by (16). In the saturated region the monotonicity of $\delta(t)$ follows from A1, (20), and (11).

Corollary 1. $\pi(t,\bar{u})$, $\mu_H(t,\bar{u})$ are non-decreasing in t, and if A1 holds, then $\mu_T(t,\bar{u})$ is also non-decreasing.

Proof. Since $\delta(t) > 0$, the monotonicity of π follows from (9), and then the assertion for μ_H follows from (10). Since $\delta(t)$ is non-decreasing, we can deduce the monotonicity of μ_T from (11).

As far as the behavior of $\lambda_T(t)$ is concerned, three cases may arise as shown in Fig. 2. Case 1 is the normal one where the entire city outside the CBD, $\varepsilon \leq u \leq \bar{u}$, is in the unsaturated region. In case 2 there is a ring, $\varepsilon \leq u \leq \bar{\varepsilon}$, immediately surrounding the CBD which is entirely devoted to transportation, whereas beyond $\bar{\varepsilon}$ the city is unsaturated. Case 3 is quite bizarre with alternating rings of saturated and unsaturated regions. We now give two sufficient conditions which rule out Case 3. Define $\tilde{\lambda}_T$ to be the fraction of land devoted to transportation,

$$\tilde{\lambda}_T(t) = \tilde{\lambda}_T(t,\bar{u}) = \frac{\lambda_T(t,\bar{u})}{\theta(\bar{u}-t)} .$$

Proposition 2. Suppose A_1 and A_2 hold. Suppose further that $\gamma \leq 1$ (see (17)). Then,

$$\tilde{\lambda}_T(t_2,\bar{u}) \geq \tilde{\lambda}_T(t_1,\bar{u}) \text{ when } t_2 \geq t_1.$$

Proof. From (18) we can obtain the differential equation satisfied by

$\tilde{\lambda}_T$ in the unsaturated region,

$$\dot{\tilde{\lambda}}_T(t) = \left\{-(1+\rho_2)\, \frac{\beta_T}{\beta_T - 1}\, B(\delta(t))^{\gamma-1} + \frac{\dot{\theta}(\bar{u}-t)}{\theta(\bar{u}-t)}\right\} \tilde{\lambda}_T(t) + \rho_2\, B(\delta(t))^{\gamma-1}. \tag{21}$$

Also $\tilde{\lambda}_T(0) = 0$. Let us suppose that $(0,\bar{t})$ is in the unsaturated region but $\bar{t}$ is saturated i.e., $\tilde{\lambda}_T(t) < 1$ for $0 \leq t < \bar{t}$, and $\tilde{\lambda}_T(\bar{t}) = 1$. Suppose that $\tilde{\lambda}_T$ is not non-decreasing over $(0,\bar{t})$. Then there must exist $0 \leq t_1 < t_2 \leq \bar{t}$ such that (see Fig. 3)

$$\tilde{\lambda}_T(t_1) = \tilde{\lambda}_T(t_2), \tag{22}$$

and

$$\dot{\tilde{\lambda}}_T(t_1) > 0 > \dot{\tilde{\lambda}}_T(t_2). \tag{23}$$

Now since δ is non-decreasing and $\gamma \leq 1$ we must have

$$(\delta(t_2))^{\gamma-1} \leq (\delta(t_1))^{\gamma-1}, \tag{24}$$

and by hypothesis

$$\frac{\dot{\theta}(\bar{u}-t_1)}{\theta(\bar{u}-t_1)} \leq \frac{\dot{\theta}(\bar{u}-t_2)}{\theta(\bar{u}-t_2)} \quad . \tag{25}$$

From (21), (22), (24), and (25) it is easy to get a contradiction of (23). Thus $\tilde{\lambda}_T$ is non-decreasing over $(0,\bar{t})$. Next we show that if $\tilde{\lambda}_T(\bar{t}) = 1$ then $\tilde{\lambda}_T(t) \equiv 1$ for $t \geq \bar{t}$. Indeed, if this is not the case, there must exist $\bar{\bar{t}} > \bar{t}$ such that (see Fig. 3)

$$\tilde{\lambda}_T(t) \equiv 1, \qquad \bar{t} \leq t \leq \bar{\bar{t}} ,$$

and

$$\dot{\tilde{\lambda}}(\bar{t}) > 0 > \dot{\tilde{\lambda}}(\bar{\bar{t}}). \tag{26}$$

Now, in the saturated region $\delta(t)$ is given by (20) and (11). It is easy to check that $\delta(t)$ is again non-decreasing so that $\delta(\bar{\bar{t}}) \geq \delta(\bar{t})$ and hence

$$(\delta(\bar{\bar{t}}))^{\gamma-1} \leq (\delta(\bar{t}))^{\gamma-1}.$$

But then the same argument as before will result in a contradiction of (26).

Corollary 2. Under the hypothesis of Proposition 2, Case 3 of Fig. 2 cannot occur in an optimal solution.

Note that Proposition 2 gives conditions which guarantee that the fraction of land devoted to transportation increases as we approach the CBD. If we are interested only in avoiding Case 3, a much weaker condition is possible. The proof of the next proposition is omitted since the argument is quite similar to the one given above.

Proposition 3. Suppose A1 holds. Suppose further that

$$\dot{\theta}(u)[\theta(u)]^{\frac{(1+\rho_2)}{(\beta_T+\rho_2)}(\gamma-1)-1}$$

is a non-increasing function of u.[1] Then, in the optimal solution for $\tilde{\lambda}_T(t,\bar{u})$ either

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[1] For example, $\frac{\theta(u)}{u}$ constant and $\frac{(1+\rho_2)}{(\beta_T+\rho_2)}(\gamma-1) - 1 \leq 0$, or $\theta(u) \equiv$ constant.

$$\tilde{\lambda}_T(t,\bar{u}) < 1 \quad \text{for all } t \le \bar{u},$$

or there exists $\bar{t} < \bar{u}$ such that

$$\tilde{\lambda}_T(t,\bar{u}) < 1 \quad \text{for} \quad t < \bar{t},$$

$$\tilde{\lambda}_T(t,\bar{u}) \equiv 1 \quad \text{for} \quad t \ge \bar{t}.$$

We will now derive some qualitative properties of the optimal solution. More detailed properties will be derived in Section 6 for a special case.

Definition. Let $\bar{t} = \bar{t}(\bar{u})$ be the smallest value[2] of t for which $\tilde{\lambda}_T(t,\bar{u}) = 1$, where $\tilde{\lambda}_T(t,\bar{u})$ is the solution of (21) with the initial condition $\tilde{\lambda}(0,\bar{u}) = 0$. Let $\bar{\varepsilon} = \bar{\varepsilon}(\bar{u}) = \bar{u} - \bar{t}(\bar{u})$. We call $\bar{t}$ the size of the residential ring, and $\bar{\varepsilon}$ the size of the CBD. Let $\bar{\nu} = \bar{\nu}(\bar{u}) = \nu(\bar{t}(\bar{u}),\bar{u})$ where $\nu(t,\bar{u})$ is the solution of (19) with initial condition $\nu(0,\bar{u}) = 0$. $\bar{\nu}$ is called the population of the city.

Proposition 4. Suppose A1 and A2 hold. Then $\tilde{\lambda}(t,\bar{u}_1) \ge \tilde{\lambda}(t,\bar{u}_2)$ for $0 \le t \le \bar{u}_1 \le \bar{u}_2$.

Proof. The differential equation (21) is of the form

$$\dot{\tilde{\lambda}}_T(t,\bar{u}) = f(t,\bar{u},\tilde{\lambda}_T(t,\bar{u})).$$

By A2 $\frac{\partial f}{\partial \bar{u}}(t,\bar{u},\tilde{\lambda}) \le 0$ for $\tilde{\lambda} \ge 0$. The assertion follows.

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[2] If $\tilde{\lambda}_T(t,\bar{u}) < 1$ for all $t \le \bar{u}$, let $\bar{t}(\bar{u}) = \bar{u}$.

Corollary 3. Suppose A1 and A2 hold. Then $\bar{t}(\bar{u})$ is a non-decreasing function of $\bar{u}$.

Corollary 4. Suppose A1 and A2 hold. Then $\nu(t,\bar{u})$ and $\bar{\nu}(\bar{u})$ are non-decreasing functions of $\bar{u}$.

Proof. Follows from (19), and Corollary 3.

Thus both the size of the residential ring and the population increase as the size of the city increases. Surprisingly, it is not in general true that the size of the CBD increases with the city size. However the following special case does hold. The proof is straightforward, but laborious, and it is omitted.

Proposition 5. Suppose that A3 holds. Then $\bar{\varepsilon}(\bar{u})$ is non-decreasing in $\bar{u}$.

4. Benefit-Cost Analysis.

Let Q(n,u) denote the minimum social cost of accomodating n households in the city beyond distance u including transportation cost up to u. The adjoint variable $p(u,\bar{u})$ satisfies the following relation,

$$p(u,\bar{u}) = \frac{\partial Q}{\partial n}(n(u,\bar{u}),\bar{u}).$$

Since there are no congestion externalities beyond $\bar{u}$, $p(\bar{u},\bar{u}) = \bar{p}$ is just the marginal cost of accomodating one household when the opportunity cost of land is R_a, and the cost of residential density of m_H is $\alpha_H(m_H)^{\beta_H}$. It follows that the 'optimal' rent profile per household at a distance u should be $p(u,\bar{u})$.

But the rent profile $R(u,\bar{u})$, which would arise from a competitive market, in the absence of any taxes, will be such that $R(\bar{u},\bar{u}) = \bar{p}$, and $R(u,\bar{u})$ + (Average transport cost from u to CBD) = constant. Hence

$$\dot{R}(u) = -\rho_1(d(u))^{\rho_2}, \quad R(\bar{u}) = \bar{p}. \tag{27}$$

Comparison of (27) with (9) shows that $p(u) - R(u)$ increases as u decreases. The distorting effects on allocation which would result if R instead of p were the rent per household are evident. In the first place, if residential construction were to be undertaken by profit-maximizing landlords, then m_H would be determined by (10) with p replaced by R, which would result in a lower residential density. On the other hand, suppose the city government takes the opportunity cost of land to be $m_H(u)R(u)$ instead of $m_H(u)p(u)$. The net marginal benefit resulting from devoting land at u to transportation (with the same intensity $m_T(u)$) is

$$\begin{aligned} MB(u) &= -\frac{\partial}{\partial \ell_T}\left[\rho_1\left(\frac{n(u)}{m_T(u)\ell_T(u)}\right)^{\rho_2} n(u)\right] - r\alpha_T(m_T(u))^{\beta_T} - m_H(u)R(u) \\ &= \rho_1\rho_2\, m_T(u)\,(d(u))^{\rho_2+1} - r\alpha_T(m_T(u))^{\beta_T} - m_H(u)R(u). \end{aligned}$$

Substituting from (10) and (12) we get

$$MB(u) = \{(1 - \frac{1}{\beta_H})p(u) - R(u)\}m_H(u).$$

It then follows that $MB(u) > 0$ for $\bar{\varepsilon} \leq u < \tilde{u}$ and $MB(u) < 0$ for $\tilde{u} < u \leq \bar{u}$, where $\tilde{u}$ is given by $MB(\tilde{u}) = 0$. Thus there will be a tendency to devote too much land to transportation near the CBD and too little land at the fringe. This argument is not completely correct, since what we must do is to compare the optimal solution with the allocation that results from a market equilibrium. This will be done in Section 6. For the moment, all we can assert is that in the absence of taxes or tolls the optimal solution cannot be sustained by a competitive rent mechanism. Of course various taxation and toll schemes are possible which would equate the private cost faced by a household at u with p(u).

5. The Market Solution

We use corresponding capital letters to denote variables. Thus P replaces p, M_H and M_T replace m_H, m_T respectively etc. Similarly Λ_T replaces λ_T, Δ replaces δ, Π replaces π, etc., however V replaces ν. To further aid comparison, we use the same equation numbers superscripted with a prime.

Let P(u) be the competitive rent paid by a household located at u. Since the only locational advantage in the model arises from differences in transport cost, household equilibrium requires

$$\dot{P}(u) = -\rho_1(D(u))^{\rho_2}, \qquad E \leq u \leq \bar{U}, \qquad (9)'$$

$$\dot{N}(u) = -M_H(u)\ (\theta(u)-L_T(u)), \qquad E \leq u \leq \bar{U}, \qquad (6)'$$

$$N(\bar{u}) = 0, \qquad N(E) = N. \qquad (7)'$$

In $(9)'$, the traffic density $D(u)$ is given by

$$D(u) = \frac{N(u)}{M_T(u)L_T(u)} \quad .$$

We assume that all land is owned by landlords, who take the rent as given, and adjust M_H so as to maximize profits. This leads to

$$r\alpha_H\beta_H(M_H(u))^{\beta_H-1} = P(u). \tag{10$'$}$$

At $\bar{U}$, the edge of the city, returns from household per unit of land must equal the opportunity cost of land. Thus

$$M_H(\bar{U})P(\bar{U}) - r\alpha_H(M_H(\bar{U}))^{\beta_H} = R_a, \tag{14$'$}$$

which gives $P(\bar{U}) = p(\bar{u}) = \bar{p}$, independent of $\bar{U}$.

The value of a unit area of land at distance u is

$$M_H(u)P(u) - r\alpha_H(M_H(u))^{\beta_H}.$$

We suppose that the city government buys land, $L_T(u)$, and invests in transportation, $M_T(u)$, so as to minimize {Transportation cost + capital costs + Value of land}. This leads to

$$r\alpha_T\beta_T(M_T)^{\beta_T-1} = \rho_1\rho_2(D(u))^{\rho_2+1}, \tag{11$'$}$$

and

$$- \rho_1\rho_2\ M_T(u)\ (D(u))^{\rho_2+1} + r\alpha_T(M_T(u))^{\beta_T} - r\alpha_H(M_H(u))^{\beta_H} + M_H(u)P(u) = 0, \tag{12$'$}$$

provided that $(12)'$ yields $L_T(u) \le \theta(u)$. Otherwise the city government converts all land to transportation,

$$L_T(u) = \theta(u). \tag{13$'$}$$

In the unsaturated region, we obtain from $(10)'$-$(12)'$,

$$\Pi(t) = A^{\frac{\beta_H - 1}{\beta_H}} (\Delta(t))^{(\rho_2+1)\left(\frac{\beta_T}{\beta_T-1}\right)\left(\frac{\beta_H-1}{\beta_H}\right)}, \tag{15$'$}$$

where $\Pi(t,\bar{U}) = P(\bar{U}-t,\bar{U})$, $\Delta(t,\bar{U}) = D(\bar{U}-t,\bar{U})$. In particular, $\Delta(0) = \delta(0) = \bar{d}$, independent of $\bar{U}$. From $(15)'$ and $(9)'$ we get

$$\dot{\Delta}(t) = \frac{B}{1+\rho_2} (\Delta(t))^{\gamma}. \tag{16$'$}$$

Similarly, in the unsaturated region, we have

$$\dot{\Lambda}_T(t) = -\left(\rho_2 + \frac{1}{1+\rho_2} + \frac{1}{\beta_T - 1}\right) B(\Delta(t))^{\gamma-1} \Lambda_T(t) + \rho_2 B(\Delta(t))^{\gamma-1}\theta(\bar{U}-t), \tag{18$'$}$$

$$\dot{V}(t) = -\rho_2 B(\Delta(t))^{\gamma-1}V(t) + \left(\frac{1}{r\alpha_H\beta_H}\right)^{\frac{1}{\beta_H-1}} A^{\frac{1}{\beta_H}} (\Delta(t))^{\left(\frac{1+\rho_2}{\beta_H}\right)\left(\frac{\beta_T}{\beta_H-1}\right)} \theta(\bar{U}-t), \tag{19$'$}$$

and

$$\dot{\tilde{\Lambda}}_T(t) = -\left\{\left(\rho_2 + \frac{1}{1+\rho_2} + \frac{1}{\beta_T-1}\right) B(\Delta(t))^{\gamma-1} + \frac{\dot{\theta}(\bar{u}-t)}{\theta(\bar{u}-t)}\right\} \tilde{\Lambda}_T(t)$$
$$+ \rho_2 B(\Delta(t))^{\gamma-1}. \tag{21$'$}$$

It follows that all the qualitative results derived in Section 3, continue to hold for the market solution.

6. Comparison of Optimal and Market Solutions

We first compare the two solutions when the city sizes are the same i.e., $\bar{u} = \bar{U}$. Next we consider the case where the two populations are the same i.e., $n = N$. Finally we derive asymptotic relations for the special case $\frac{\theta(u)}{u} \equiv$ constant.

The differential equations (16), (16)' yield respectively[3]

$$\delta(t) = [B(1-\gamma)(\bar{\delta}+t)]^{\frac{1}{1-\gamma}} \quad , \tag{28}$$

$$\Delta(t) = [B(1-\gamma)(\bar{\delta}+\frac{t}{1+\rho_2})]^{\frac{1}{1-\gamma}} \tag{28'}$$

where

$$\bar{\delta} = \frac{(\delta(0))^{1-\gamma}}{B(1-\gamma)} = \frac{\bar{d}^{1-\gamma}}{B(1-\gamma)} \quad .$$

Note that (28), (28)' are valid only in the unsaturated region. Also $\bar{\delta} < 0$ if $\gamma > 1$, which leads to the following surprising result.

<u>Definition</u>. For $\gamma > 1$, define $\bar{t}_\infty$ and $\bar{T}_\infty$ by the conditions $\bar{\delta} + \bar{t}_\infty = 0$, $\bar{\delta} + \bar{T}_\infty = 0$.

Note $\bar{T}_\infty = (1+\rho_2)\bar{t}_\infty$.

[3] If $\gamma = 1$, (28) becomes $\delta(t) = \bar{d} \exp(Bt)$. This can be obtained directly from (16) or by taking limits in (28) as γ approaches 1.

Proposition 6. Suppose $\gamma > 1$. Then the size of the residential rings $\bar{t}(\bar{u}) < \bar{t}_\infty$, $\bar{T}(\bar{u}) < \bar{T}_\infty$.

Proof. Follows from the fact that $\lim_{t \to \bar{t}_\infty} \delta(t,\bar{u}) = \infty$, $\lim_{t \to \bar{T}_\infty} \Delta(t,\bar{u}) = \infty$, independent of $\bar{u}$.

Returning to (28), (28)' we note that $\delta(t) > \Delta(t)$ for $t > 0$ since $\rho_2 > 0$. It follows from (11), (11)' that $m_T(u) > M_T(u)$. Similarly from (15), (15)' and (10), (10)' we can conclude that $p(u) > P(u)$ and $m_H(u) > M_H(u)$. This can be summarized as follows.

Proposition 7. Suppose $\bar{u} = \bar{U}$. Then, the intensity of land use in both the transportation and housing sectors is greater in the optimal solution than the market solution.

The relation between λ_T and Λ_T is not unequivocal, and depends upon the value of γ. The differential equations (18), (18)' are of the form

$$\dot{\lambda}_T(t,\bar{u}) = (\delta(t))^{\gamma-1} \phi(t,\bar{u},\lambda_T(t,\bar{u})), \tag{29}$$

$$\dot{\Lambda}_T(t,\bar{u}) = (\Delta(t))^{\gamma-1} \Phi(t,\bar{u},\Lambda_T(t,\bar{u})) \tag*{(29)'}$$

with

$$\phi(t,\bar{u},\lambda) < \Phi(t,\bar{u},\lambda) \quad \text{for} \quad 0 \le t \le \bar{u},\ 0 < \lambda. \tag*{(30).}$$

Proposition 8. Suppose $\gamma \le 1$. Then $\Lambda_T(t,\bar{u}) > \lambda_T(t,\bar{u})$ for $t > 0$.

Proof. Since $\delta(t) \ge \Delta(t)$ and $\gamma \le 1$, therefore $(\delta(t))^{\gamma-1} \le (\Delta(t))^{\gamma-1}$. The assertion follows from (29), (29)', and (30).

<u>Corollary 5</u>. Suppose $\gamma \leq 1$. Then $\bar{t}(\bar{u}) > \bar{T}(\bar{u})$ i.e., the size of the optimal residential ring is greater than the market residential ring. Also $n(\bar{u}) > N(\bar{u})$ i.e., the population of the optimal city exceeds that of the market city.

<u>Proof</u>. $\bar{t}(\bar{u})$, $\bar{T}(\bar{u})$ are respectively given by $\lambda_T(\bar{t}(\bar{u}),\bar{u}) = \theta(\bar{u}-t(\bar{u}))$, $\Lambda_T(\bar{T}(\bar{u}),\bar{u}) = \theta(\bar{u}-\bar{T}(\bar{u}))$. Since $\lambda_T(t,\bar{u}) < \Lambda_T(t,\bar{u})$, it follows that $\bar{T}(\bar{u}) < \bar{t}(\bar{u})$. Next,

$$n(\bar{u}) = \int_0^{\bar{t}(\bar{u})} m_H(\bar{u}-t,\bar{u})[\theta(\bar{u}-t) - \lambda_T(t,\bar{u})],$$

$$N(\bar{u}) = \int_0^{\bar{T}(\bar{u})} M_H(\bar{u}-t,\bar{u})[\theta(\bar{u}-t) - \Lambda_T(t,\bar{u})]dt.$$

By Proposition 7 $m_H(\bar{u}-t,\bar{u}) \geq M_H(\bar{u}-t,\bar{u})$, and since $\lambda_T < \Lambda_T$, $\bar{t}(\bar{u}) > \bar{T}(\bar{u})$, it follows that $N(\bar{u}) < n(\bar{u})$.

If $\gamma > 1$, a comparison of (18), (18)' shows that allocation of land between transportation and residences for the two institutions can take one of the two forms shown in Fig. 4. Similarly, if we compare (19) with (19)' we can see that the population of the market city may be greater or smaller than the population of the optimal city. However, since the case $\gamma > 1$ does not appear to be the normal situation, we do not pursue it any further.

We now turn to a comparison of the two situations when the populations are the same i.e., $n = N$. We denote by $\bar{u}(n)$, $\bar{\varepsilon}(n)$, $\bar{t}(n)$, respectively, the size of the city, the size of the CBD, and the size of the residential ring for the optimal solution when the total population is n. Similarly, we define $\bar{U}(n)$, $\bar{E}(n)$, and $\bar{T}(n)$.

Proposition 9. Suppose A1 holds. Then

$$\bar{V}((1+\rho_2)\bar{u}) \geq (1+\rho_2)\bar{\nu}(\bar{u}) \text{ for all } \bar{u} \geq 0.$$

Proof. Define $W(t) = W(t,\bar{u}) = \dfrac{V((1+\rho_2)t,\ (1+\rho_2)\bar{u})}{(1+\rho_2)}$. Then from (19)' we see that

$$\dot{W}(t) = -\rho_2 B(\delta(t))^{\gamma-1} W(t)$$

$$+\left(\frac{1}{r\alpha_H\beta_H}\right)^{\frac{1}{\beta_H-1}} A^{\frac{1}{\beta_H}} (\delta(t))^{\left(\frac{1+\rho_2}{\beta_H}\right)\left(\frac{\beta_T}{\beta_T-1}\right)} \theta((1+\rho_2)(\bar{u}-t)).$$

If we compare this with (19), and note that $\theta((1+\rho_2)(\bar{u}-t)) \geq \theta(\bar{u}-t)$ by A1, we can immediately conclude that

$$W(t,\bar{u}) = \frac{V((1+\rho_2)t,\ (1+\rho_2)\bar{u})}{1+\rho_2} \geq \nu(t,\bar{u}) \quad \text{for } 0 \leq t \leq \bar{t}(\bar{u}).$$

Hence $\dfrac{\bar{V}((1+\rho_2)\bar{u})}{1+\rho_2} \geq \bar{\nu}(\bar{u})$.

Corollary 6. Suppose A1 and A2 hold, and suppose $\gamma \leq 1$. Then

$$\bar{u}(n) < \bar{U}(n) \leq (1+\rho_2)\bar{u}(n) \quad \text{for all} \quad n \geq 0.$$

Proof. Since A1 and A2 hold, it follows from Corollary 3 that $\bar{u}(n)$

and $\bar{U}(n)$ are non-decreasing functions of n, and so the first inequality follows from Corollary 5. The second inequality is an immediate consequence of Proposition 9.

The result above establishes bound between the size of the optimal and market cities with the same population. Proposition 9 implies also that the sizes of the two residential rings satisfy the inequality $\bar{T}(n) \leq (1+\rho_2)\bar{t}(n)$. By laborious manipulation one can refine this to obtain the following result. The proof is omitted.

Proposition 10. Suppose A3 holds, and $\gamma \leq 1$. Then,

$$\bar{\varepsilon}(n) \leq \bar{E}(n) \leq (1+\rho_2)\bar{\varepsilon}(n),$$

$$\bar{t}(n) \leq \bar{T}(n) \leq (1+\rho_2)\bar{t}(n).$$

Now suppose A3 holds. Then we can explicitly solve for the differential equation (21), (21)$'$ and obtain $\bar{t}(\bar{u})$, $\bar{T}(\bar{U})$ by setting $\tilde{\lambda}_T(\bar{t}(\bar{u}),\bar{u}) = 1$, $\tilde{\Lambda}_T(\bar{T}(\bar{U}),\bar{U}) = 1$. The asymptotic behavior of these functions is given below. The derivation is omitted.

Proposition 11. Suppose A3 holds, and suppose $\gamma \leq 1$. Then

$$\lim_{\bar{u} \to \infty} \frac{\bar{t}(\bar{u})}{\bar{u}} = w, \quad \lim_{\bar{U} \to \infty} \frac{\bar{T}(\bar{U})}{\bar{U}} = W,$$

where the constants w, W are given by

$$w = \frac{\left(1 + \frac{1-\gamma}{w_1}\right)}{\left(1 + \frac{1-\gamma}{w_1 - w_2}\right)}, \qquad W = \frac{\left(1 + \frac{1-\gamma}{(1+\rho_2)w_1}\right)}{\left(1 + \frac{1-\gamma}{w_1 - w_2}\right)},$$

with

$$w_1 = (1+\rho_2)\left(\frac{\beta_T}{\beta_T-1}\right), \qquad w_2 = \rho_2$$

In particular, $0 < W < w < 1$.

We can also solve (19), (19)$'$ and use the preceding result to obtain the asymptotic behavior of $\bar{\nu}(\bar{u})$, $\bar{V}(\bar{U})$.

Proposition 12. Suppose A3 holds, and $\gamma < 1$. Let $\frac{\dot{\theta}(u)}{u} \equiv \theta > 0$. Then

$$\lim_{\bar{u} \to \infty} \frac{\bar{\nu}(\bar{u})}{\bar{u}^{-e+2}} = \theta_0 \, s,$$

$$\lim_{\bar{U} \to \infty} \frac{\bar{V}(\bar{U})}{\bar{U}^{-e+2}} = \theta_0 \, S$$

where s, S are constants with $0 < S < s$, and $e = \frac{1}{\beta_H - 1}\left(1 + \frac{\rho_2}{1-\gamma}\right)$.

From this result we can conclude that the average gross density $\frac{\bar{\nu}(\bar{u})}{\theta_0 \, \bar{u}^2}$, $\frac{\bar{V}(\bar{U})}{\theta_0 \bar{U}^2}$ becomes unbounded as the population grows. Furthermore the ratios of the city sizes $\frac{\bar{u}(n)}{\bar{U}(n)}$ approaches a constant less than 1 as the population grows.

7. A Critical Comment

The model presented here has two serious defects. First of all, taking access to the CBD as the sole determinant of rent is a very crude assumption. More serious is the fact that the market solution is an equilibrium solution in a very weak sense. That is to say, no realistic adjustment process can be conceived which can bring about this solution. The reason for this is that capital in residences and transportation is 'sunk' capital, with little or no <u>ex-post</u> substitutability. Hence it is impossible that the allocation of capital in city structures is in static equilibrium.

Acknowledgement. The authors wish to express gratitude to D. Livesey, R. Artle, B. von Rabineau, O. Fisch, and P. Macri for valuable comments.

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Footnotes

1 For example, $\frac{\theta(u)}{u} \equiv$ constant and $\frac{(1+\rho_2)}{(\beta_T+\rho_2)}(\gamma-1) - 1 \leq 0$, or

$\theta(u) \equiv$ constant.

2 If $\tilde{\lambda}_T(t,\bar{u}) < 1$ for all $t \leq \bar{u}$, let $\bar{t}(\bar{u}) = \bar{u}$.

3 If $\gamma = 1$, (28) becomes $\delta(t) = \bar{d}\exp(Bt)$. This can be obtained directly from (16) or by taking limits in (28) as γ approaches 1.

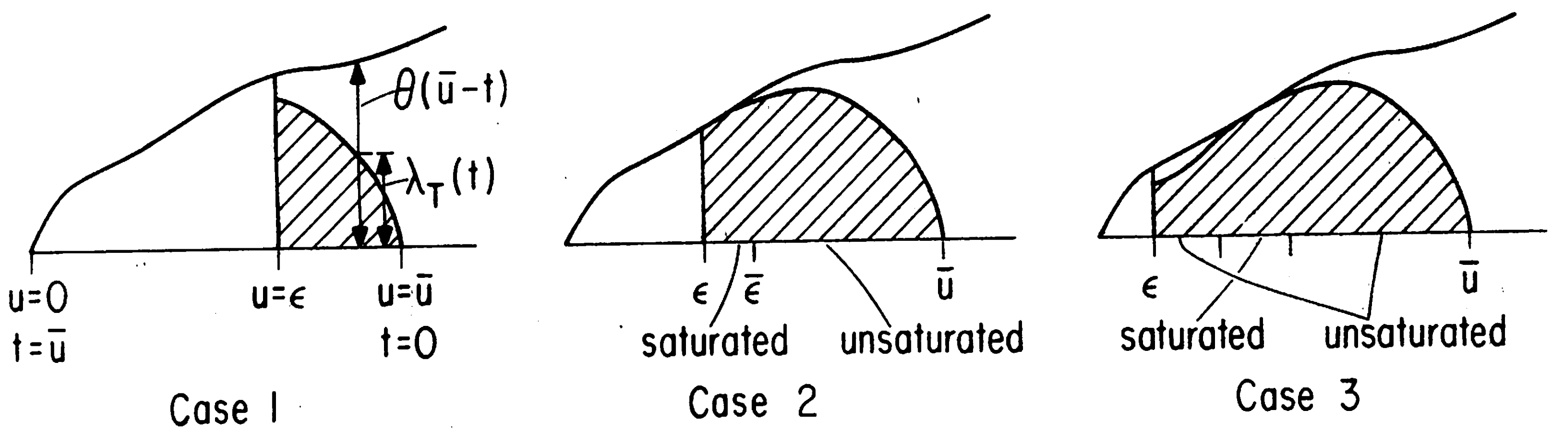

Fig. 2. Possible behaviors of $\lambda_T(t)$.

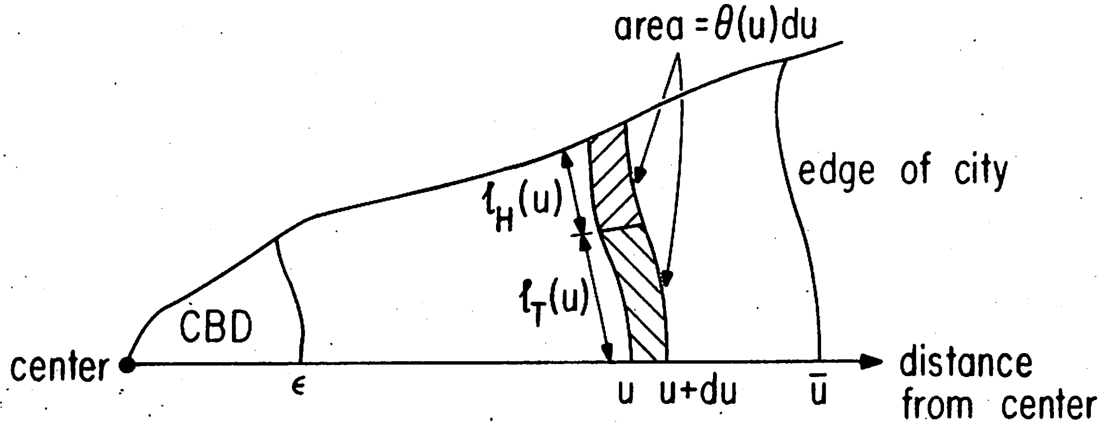

Fig. 1. Land use in the city.

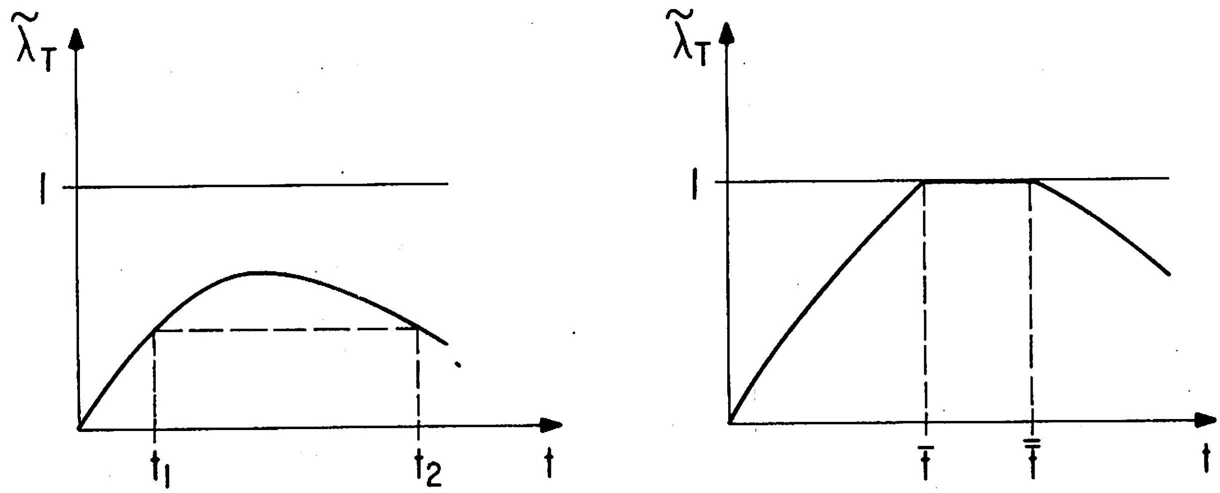

Fig. 3. Illustration for Proposition 2.

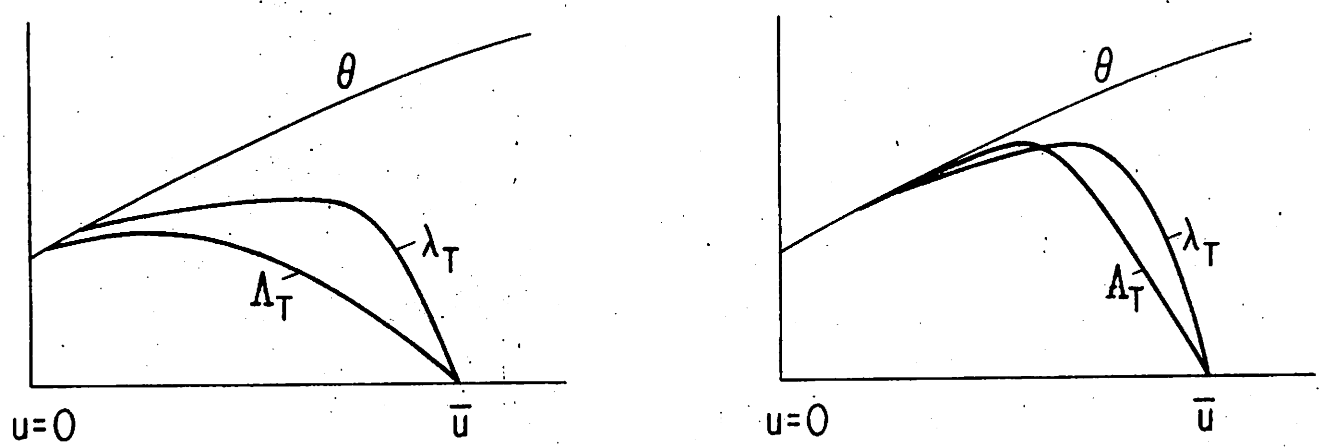

Fig. 4. Land allocation when $\gamma > 1$.