ON THE INPUT-OUTPUT PROPERTIES OF CONVOLUTION FEEDBACK SYSTEMS

by

Frank M. Callier

Memorandum No. ERL-M338

16 June 1972

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Research sponsored by the National Aeronautics ans Space Administration Grant NGL-05-003-016 and is with the Belgian National Fund for Scientific Research.

ON THE INPUT-OUTPUT PROPERTIES OF CONVOLUTION FEEDBACK SYSTEMS

by

Frank M. Callier

ABSTRACT

This dissertation considers n-input n-output convolution feedback systems characterized by $y = G*e$ and $e = u-y$. The continuous-time case as well as the discrete-time case are considered in the framework of the convolution algebras $\mathcal{A}$ and ℓ^1 respectively.

A graphical test is developed for checking the condition $\inf_{\text{Re } s \geq 0} |1 + \hat{g}(s)| > 0$ where $\hat{g}$ is the sum of a finite number of poles and a term in $\hat{\mathcal{A}}$ (i.e. the Laplace transform of an integrable function plus a series of delayed impulses). This is a significant extension of the Nyquist plot test because, in our case, as $|\omega| \to \infty$ the function $\omega \mapsto \hat{g}(j\omega)$ is asymptotically almost periodic rather than tending to zero as in the classical case. The discrete-time counterpart of this test as well as its extension to the n-input n-output case are also given.

The relation between the open-loop operator $\underset{\sim}{G}$ and the closed-loop operator $\underset{\sim}{H}$ is discussed. Thereby the importance of systems considered by Vidyasagar is demonstrated i.e. of systems with open-loop transfer function $\hat{G}(s) = \hat{P}(s)[\hat{Q}(s)]^{-1}$ where $\hat{P}, \hat{Q} \in \hat{\mathcal{A}}^{n\times n}$ or $\tilde{G}(z) = \tilde{P}(z)[\tilde{Q}(z)]^{-1}$ where $\tilde{P}, \tilde{Q} \in \tilde{\ell}^1_{n\times n}$. It is shown that if the

closed-loop impulse response H is stable in the sense that $H \in \mathcal{A}^{n\times n}$ or $H \in \ell^1_{n\times n}$ then the open-loop transfer function is of the above form. Moreover necessary and sufficient conditions for stability are given using this open-loop transfer function representation. Finally necessary and sufficient conditions for stability are given when the open-loop transfer function is of the above form with a finite number of poles in the open right half-plane Re $s > 0$ or in the open annulus $|z| > 1$.

The dissertation concludes by giving (a) necessary and sufficient conditions for stability when constant singular feedback is present in a simple case and (b) an application of the above theory to the stability analysis of nonlinear feedback systems.

ACKNOWLEDGEMENT

It is the explicit wish of the author to thank Professor C. A. Desoer for his uncountable suggestions and constructive criticisms that made possible this dissertation. The author wishes hereby also to thank Professors A. R. Bergen and H. Lewy who read the manuscript as well as Mrs. D. Simpson who typed it.

TABLE OF CONTENTS

1. INTRODUCTION

This dissertation considers linear time-invariant feedback systems with n inputs and n outputs. As it will become apparent there is no loss of generality in taking the feedback to be unity. We shall consider both continuous-time and discrete-time convolution feedback systems. Therefore a description and preliminaries are given for both cases.

1.1.1 System Description in the Continuous-time Case.

For the feedback system under consideration, the input u, output y and error e are functions from $\mathbb{R}_+$, (defined as $[0,\infty)$), to $\mathbb{R}^n$ or corresponding distributions on $\mathbb{R}_+$. The open-loop system is of the convolution type so that we have:

(1.1) $$y = G*e$$

(1.2) $$e = u-y$$

where G is an n×n matrix whose elements are distributions on $\mathbb{R}_+$. Let H denote the closed-loop impulse response of the feedback system, i.e.:

(1.3) $$y = H*u$$

and let $\underset{\sim}{G}$, $\underset{\sim}{H}$ denote the maps $\underset{\sim}{G}: e \longmapsto G*e$, $\underset{\sim}{H}: u \longmapsto H*u$ respectively.

We shall repeatedly use the convolution algebra $\mathcal{A}$ [1,2]: f is said to be in $\mathcal{A}$ iff

(1.4)
$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ f_a(t) + \sum_{i=0}^{\infty} f_i\, \delta(t-t_i) & \text{for } t \geq 0 \end{cases}$$

where $f_a(t) \in L^1[0,\infty)$ (i.e. $\int_0^\infty |f_a(t)|dt < \infty$), $f_i \in \mathbb{R}$ for all i, $\delta(\cdot)$ is the Dirac δ-function, $\{f_i\}_{i=0}^{\infty} \in \ell^1$ (i.e. $\sum_{i=0}^{\infty} |f_i| < \infty$) and $0 = t_0 < t_1 < t_2 < \ldots < t_i < \ldots$. Thus f is a distribution of order 0 with support on $\mathbb{R}_+$. An n-vector v(n×n matrix A) is said to be in $\mathcal{A}^n$ ($\mathcal{A}^{n\times n}$) iff all its elements are in $\mathcal{A}$. Let $\hat{f}$ denote the Laplace transform of $\hat{f}$:f belongs to the convolution algebra $\mathcal{A}$ if and only if $\hat{f}$ belongs to the algebra $\hat{\mathcal{A}}$ with pointwise product. Similarly $\hat{v} \in \hat{\mathcal{A}}^n$, $\hat{A} \in \hat{\mathcal{A}}^{n\times n}$. We shall also use the Banach spaces $L_n^q[0,\infty)$ for some $q \in [1,\infty]$. A function v mapping $\mathbb{R}_+$ into $\mathbb{R}^n$ belongs to $L_n^q[0,\infty)$ for some $q \in [1,\infty]$ iff the function $t \longmapsto |v(t)|$, where $|\cdot|$ denotes any vector norm in $\mathbb{R}^n$, belongs to $L^q[0,\infty)$ for the same $q \in [1,\infty]$ (i.e. $\int_0^\infty |v(t)|^q dt < \infty$ when $q \in [1,\infty)$ or $|v(\cdot)|$ essentially bounded when $q = \infty$).

1.1.2 System Description in the Discrete-time Case.

For the feedback system under consideration, the input u, output y and error e are functions from $\mathbb{Z}_+$ (the set of non-negative integers) into $\mathbb{R}^n$. The open-loop system is of the convolution type so that we have

(1.1') $y = G*e$

(1.2') $e = u-y$

where G is specified by a sequence of real n×n matrices $\{G_i\}_{i=0}^{\infty}$; thus (1.1') is equivalent to $y_m = \sum_{i=0}^{m} G_{m-i}\, e_i$ for $m = 0, 1, 2, \ldots$. Let H denote the closed-loop impulse response of the feedback system i.e.:

(1.3') $y = H*u$

and let G, H denote the maps $G: e \longmapsto G*e$, $H: u \longmapsto H*u$ respectively.

We shall repeatedly use the convolution algebra ℓ^1 : f is said to be in ℓ^1 iff

(1.4') $f = (f_0, f_1, f_2, \ldots)$

where $f_i \in \mathbb{R}$ for all i and $\sum_{i=0}^{\infty} |f_i| < \infty$. The product of two elements $f, g \in \ell^1$ is given by their convolution: $(f*g)_m = \sum_{i=0}^{m} f_{m-i} g_i$ for $m = 0, 1, 2, \ldots$ and it is easy to show that $f*g \in \ell^1$[3]. An n vector v (n×n matrix A) is said to be in $\ell_n^1(\ell_{n\times n}^1)$ iff all its elements are in ℓ^1. Let $\tilde{f}$ denote the z-transform of f, i.e. $\tilde{f}(z) = \sum_{i=0}^{\infty} f_i z^{-i}$: f belongs to the convolution algebra ℓ^1 iff $\tilde{f}$ belongs to the algebra $\tilde{\ell}^1$ with pointwise product. Similarly $\tilde{v} \in \tilde{\ell}^1_n$, $\tilde{A} \in \tilde{\ell}^1_{n\times n}$. We shall also use the Banach spaces ℓ^q_n for some $q \in [1,\infty]$. A function v mapping $\mathbb{Z}_+$ into $\mathbb{R}^n$ belongs to ℓ^q_n for some $q \in [1,\infty]$ iff the function $i \longmapsto |v_i|$, where $|\cdot|$ denotes any vector norm in $\mathbb{R}^n$, belongs to ℓ^q for the same $q \in [1,\infty]$ (i.e. $\sum_{i=0}^{\infty} |v_i|^q < \infty$ when $q \in [1,\infty)$ or $\{|v_i|\}_{i=0}^{\infty}$ essentially bounded when $q = \infty$).

1.2 General Remarks.

Notation

Unless specified otherwise explicitly, following notation is used throughout the dissertation:

$\mathbb{R}$ ($\mathbb{C}$)	field of real (complex) numbers
$\mathbb{Z}$	ring of integers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{Z}_+$	set of nonnegative integers
$\mathbb{R}^n$	set of real n-vectors
$\mathbb{R}^{n\times n}$	set of real n×n matrices
j	the imaginary unit
s	complex variable of the Laplace transform
z	complex variable of the z-transform
Re s	real part of the complex quantity s
I	unity matrix

lower case letters are used to indicate scalar-valued or vector-valued quantities

Capital letters are used to indicate matrix-valued quantities.

$\hat{f}$	Laplace-transform of f
$\tilde{f}$	z-transform of f
g (G)	scalar-valued, (respectively, matrix-valued) open-loop impulse response
h (H)	scalar-valued, (respectively, matrix-valued) closed-loop impulse response
(σ,ω)	cartesian coordinates of the complex variable s
(ρ,γ)	polar coordinates of the complex variable z

Note to the Reader

For reasons of clarity all statements, formulas, remarks, conventions and facts are indexed by a reference number. Theorems and corollaries have their own indexing system. Index numbers terminated by a prime refer to discrete-time convolution feedback systems.

Except for section 1, i.e. the introduction, an attempt has been made to group as much as possible in self-contained blocks results on the discrete-time case and results on the continuous-time case. In each block reference numbers and the index numbers of theorems and corrolaries are monotonically increasing.

For ease of reference on top of each page appears the first new reference number used and the theorems or corrolary stated on this page, as soon as they occur.

Special terminology used in the dissertation

minor	determinant of a square submatrix

principal part of $\hat{G}$ ($\tilde{G}$)   sum of the local principal parts of $\hat{G}$ ($\tilde{G}$) in Re $s \geq 0$ ($|z| \geq 1$)

1.3 Historical Background

Convolution feedback systems have been studied for a long time in control and circuit theory. In fact it may be said that they gave the starting pulse for the set-up of linear control and systems theory as we know them today. The elegance of these systems lies in the fact a) that through the superposition principle one needs only to study the impulse response of such systems and b) that through transform-techniques such as the Laplace- and z-transform the convolution of two operators is replaced by a pointwise product of them in the transform-space, where the powerfull results of analytic function theory and linear algebra are available. Therefore very sharp results concerning the input-output properties of convolution feedback systems can be obtained.

Initially authors were mainly concerned with single-input single-output lumped convolution feedback systems [8,9] and the extension of the ideas in [8] to n-input n-output lumped convolution feedback systems [e.g., 10]. Of central importance for lumped systems is Nyquist's graphical test in [8] which in the single-input single-output continuous-time case is necessary and sufficient to insure that the input and output of the system are related by an ordinary differential equation with stable modes.

In the sequel authors tried to extend Nyquist's graphical test to distributed convolution feedback systems. A first step is Desoer's paper [11], who works in the Banach space $L^1[0,\infty)$ and uses the Paley-Wiener Theorem [12]. A second step was the definition by Desoer and Wu [1] of the convolution algebra $\mathcal{A}$ which was possible by a result of Hille and Phillips [13]. The central fact here is that it is possible to handle distributions in $\mathcal{A}$ whereas in $L^1[0,\infty)$ this is not so, moreover $\mathcal{A}$ is an algebra isomorphic to the algebra of its Laplace transforms $\hat{\mathcal{A}}$. Based on these ideas results on the input-output properties of convolution feedback systems were further presented in J. C. Willems [14,15], Baker and Vakharia [16], Desoer and Wu [2,3], Desoer and Vidyasagar [17], Desoer and Lam [18,19,20], Vidyasagar [21], Nasburg and Baker [22] and Desoer and Callier [4,5,6,7].

1.4 Contributions of this Dissertation.

This dissertation is partly a reorganization of recent contributions of Callier and Desoer [4,5,6,7], and is therefore subdivided in following sections below:

Sec. 2. A graphical test for checking the stability of a single-input, single-output convolution feedback system.

Sec. 3. Continuous-time n-input n-output convolution feedback systems.

Sec. 4. Discrete-time n-input n-output convolution feedback systems.

Sec. 5. Conclusion

A In section 2 a graphical test is developed for checking the condition

$$(1.5) \qquad \inf_{\mathrm{Re}\, s \geq 0} |1 + \hat{g}(s)| > 0$$

where $\hat{g}$ consists of a term in $\hat{\mathcal{A}}$ and a finite number of poles in Re $s \geq 0$. As a consequence $1 + \hat{g}$ is, in Re $s \geq 0$, asymptotic to an almost periodic function say $\hat{f}$ as $|s| \longmapsto \infty$.

<u>Theorem 2.1</u> gives a necessary and sufficient condition involving the curve $\{\hat{f}(j\omega) | \omega \in \mathbb{R}\}$ to insure that $\inf_{\mathrm{Re}\, s \geq 0} |\hat{f}(s)| > 0$; <u>corollary 2.1</u> gives a corresponding graphical test.

<u>Theorem 2.2 and Corollary 2.2</u> give necessary and sufficient conditions involving the curve $\{1 + g(j\omega) | \omega \in \mathbb{R}\}$ and a graphical test, to insure (1.5) a condition essential for the L^q - stability of the continuous-time feedback system under study. Theorem 2.2 and Corollary 2.2 constitute a two-way generalization of Willems result [14,15], first we do not assume that the impulses of g are equally spaced and second we allow $\hat{g}$ to contain a finite number of poles in Re $s \geq 0$.

Next by decomposition-lemma A.2 we show that the graphical test as given by Corollary 2.2 can be used for checking the condition

$$(1.6) \qquad \inf_{\mathrm{Re}\, s \geq 0} |\det [I + \hat{G}(s)]| > 0$$

where $\hat{G}(s)$ is a matrix-valued transfer function consisting of a term in $\hat{\mathcal{A}}^{n\times n}$ and a principal part corresponding to a finite number of poles in Re $s \geq 0$.

Finally a graphical test is developed for checking the discrete-time counterpart of condition (1.5) in Theorem 2.2' and Corollary 2.2'. We show thereby the simplification that occurs due to the fact that, in this case, the open-loop transfer function $\tilde{g}$ becomes asymptotically constant as $|z| \to \infty$. We show also that Corollary 2.2' has implications for the discrete-time counterpart of (1.6) as well. Thus section 2 shows the applicability of a graphical test to check conditons (1.5), (1.6) and their discrete-time counterparts. It is interesting to note that recently J. H. Davis [34] has obtained similar results using different techniques.

B Sections 3 and 4 present input-output properties of continuous-time respectively discrete-time n-input n-output convolution feedback systems. Their structures are analoguous

First the overwhelming importance of systems in Vidyasagar's setting [21] is shown, i.e. of systems described by either (1.1) - (1.2) and

(1.7) $$\hat{G}(s) = \frac{\hat{P}(s)}{\hat{Q}(s)}$$

where $\hat{P}(\cdot)$, $\hat{Q}(\cdot)$ belong to $\hat{\mathcal{A}}^{n\times n}$

or (1.1') - (1.2') and

(1.7') $$\tilde{G}(z) = \frac{\tilde{P}(z)}{\tilde{Q}(z)}$$

where $\tilde{P}(\cdot)$ and $\tilde{Q}(\cdot)$ belong to $\tilde{\ell}^1_{n\times n}$.

Indeed in Theorems 3.1 and 4.1' we show that, under mild assumptions, $\hat{G}(s)$ (and $\tilde{G}(z)$) are of the structure (1.7) (respectively, (1.7')) if the closed-loop impulse response H is supposed to be in $\mathcal{A}^{n\times n}$ (respectively, $\ell^1_{n\times n}$). These theorems extend results of Nasburg and Baker [22] to the n-input n-output case and greatly relax the conditions imposed by previous authors. Moreover in Theorems 3.2 and 4.2' necessary and sufficient conditions involving (1.7) (respectively, (1.7')) are presented for H to be an element of $\mathcal{A}^{n\times n}$ (respectively, $\ell^1_{n\times n}$) if the open-loop impulse response G is supposed to be Laplace - (respectively, z-transformable). These theorems again constitute an extension of a result of Nasburg and Baker [22] to the n-input, n-output case. As a second part of sections 3 and 4 necessary and sufficient conditions for H to belong to $\mathcal{A}^{n\times n}$ (respectively, $\ell^1_{n\times n}$) are presented when the open-loop transfer function $\hat{G}$ (respectively, $\tilde{G}$) consists of a term in $\hat{\mathcal{A}}^{n\times n}$ (respectively, $\tilde{\ell}^1_{n\times n}$) and a principal part due to a finite number of poles in Re $s \geq 0$ (respectively, $|z| \geq 1$).

Theorems 3.3 and 4.3' culminates a series of investigations by Desoer, Wu and Lam [1,2,3,18,19,20], and handles the case where $\hat{G}$ (respectively, $\tilde{G}$) consists of a term in $\hat{\mathcal{A}}^{n\times n}$ (respectively, $\tilde{\ell}^1_{n\times n}$) and a principal part due to a higher order pole in Re $s \geq 0$ (respectively, $|z| \geq 1$). Theorem 3.4 makes an interpretation of the conditons of Theorem 3.3 possible and establishes a link with C. T. Chen's result [10]. Theorems 3.5 and 4.4' state the reinterpreted conditions for H to belong to $\mathcal{A}^{n\times n}$ (respectively, $\ell^1_{n\times n}$) when $\hat{G}$ (respectively, $\tilde{G}$) consists of a term in $\hat{\mathcal{A}}^{n\times n}$ (respectively, $\ell^1_{n\times n}$) and a principal part due to a finite number of poles in Re $s \geq 0$ respectively $|z| \geq 1$. These conditions are new and have not yet appeared in the literature.

C In section 5 after discussing the previous results, we state in Theorem 5.1 necessary and sufficient conditions for H to belong to $\mathcal{A}^{n\times n}$ for a simple convolution feedback system where a singular nonunity constant feedback is present. Finally in Theorem 5.2 we state sufficient conditions for the input-output stability of a nonlinear time-varying 2n-input 2n-output feedback system, where we use a result of the dissertation namely Theorem 3.5. We show thereby that the results of this dissertation have immediate implications for the stability analysis of nonlinear feedback systems through the application of the small gain theorem, passivity theorem, and loop transformation theorem [15,20].

1.5.1 Preliminary Results for the Continuous-time Case

We state now some well-known results concerning the algebra $\mathcal{A}$. (See among others [1],[2],[13] p. 150)

$$(1.13)\quad \left\{\begin{array}{l}\text{If } g \text{ belong to } \mathcal{A}, \text{ then } \hat{g} \text{ is analytic in Re } s > 0, \text{ bounded} \\ \text{in Re } s \geq 0, \text{ and each function } \omega \longmapsto \hat{g}(\sigma+j\omega) \text{ where } s \triangleq \sigma + j\omega \\ \text{is uniformly continuous for all } \sigma \geq 0.\end{array}\right.$$

$$(1.14)\quad \left\{\begin{array}{l}\text{If } g \text{ belongs to } \mathcal{A} \text{ then } g \text{ is invertible in } \mathcal{A} \text{ if and only if} \\ \inf\limits_{\text{Re } s \geq 0} |\hat{g}(s)| > 0\end{array}\right.$$

(1.15) $\left\{\begin{array}{l}\text{If } G \text{ belongs to } \mathcal{A}^{n\times n} \text{ then } G \text{ is invertible in } \mathcal{A}^{n\times n} \\ \text{if and only if} \\ \inf\limits_{\mathrm{Re}\, s \geq 0} |\det \hat{G}(s)| > 0\end{array}\right.$

(1.16) $\left\{\begin{array}{l}\text{If } H \text{ belongs to } \mathcal{A}^{n\times n} \text{ and } y = H{*}u \text{ then} \\ u \in L_n^q[0,\infty) \Rightarrow y \in L_n^q[0,\infty) \quad \text{for all } q \in [1,\infty]\end{array}\right.$

(1.17) Remark. Concerning the system defined by (1.1) - (1.2) we see that $H \in \mathcal{A}^{n\times n}$ implies L_n^q - input - output stability for this system for any $q \in [1,\infty]$ and therefore the system (1.1) - (1.2) is said to be stable iff $H \in \mathcal{A}^{n\times n}$.

1.5.2 Preliminary Results for the Discrete-time Case.

We state next some well-known results concerning the algebra ℓ^1. (See among others [3]p.19)

(1.13') $\left\{\begin{array}{l}\text{If } g \text{ belongs to } \ell^1, \text{ then } \tilde{g} \text{ is analytic in } |z| > 1, \\ \text{bounded in } |z| \geq 1, \text{ each function } \gamma \longmapsto \tilde{g}(\rho e^{j\gamma}) \text{ is uniformly} \\ \text{continuous on } [0,2\pi] \text{ for all } \rho \geq 1 \text{ (after setting } z = \rho e^{j\gamma}) \\ \text{and } \lim\limits_{|z|\to\infty} \tilde{g}(z) = g_0 = \text{constant.}\end{array}\right.$

(1.14') $\left\{\begin{array}{l}\text{If } g \text{ belongs to } \ell^1, \text{ then } g \text{ is invertible in } \ell^1 \text{ if and} \\ \text{only if} \\ \inf\limits_{|z| \geq 1} |\tilde{g}(z)| > 0\end{array}\right.$

(1.15') $\left\{\begin{array}{l}\text{If } G \text{ belongs to } \ell^1_{n\times n} \text{ then } G \text{ is invertible in } \ell^1_{n\times n} \text{ if and} \\ \text{only if} \\ \inf\limits_{|z| \geq 1} |\det \tilde{G}(z)| > 0\end{array}\right.$

$$(1.16')\quad \begin{cases} \text{If } H \text{ belongs to } \ell^1_{n\times n} \text{ and } y = H*u \text{ then} \\ u \in \ell^q_n \Rightarrow y \in \ell^q_n \quad \text{for all } q \in [1,\infty] \end{cases}$$

(1.17') <u>Remark</u>. Concerning the system defined by (1.1') - (1.2') we see that $H \in \ell^1_{n\times n}$ implies ℓ^q_n - input - output stability for this system for any $q \in [1,\infty]$ and therefore we agree to say that the system (1.1') - (1.2') is stable iff $H \in \ell^1_{n\times n}$.

2. A GRAPHICAL TEST FOR CHECKING THE STABILITY OF A SINGLE-INPUT SINGLE-OUTPUT CONVOLUTION FEEDBACK SYSTEM

2.1 Introduction

It is well-known that the classical graphical test [8] for stability is extremely important for two reasons: (a) it is based on experimental data that are easy to obtain with great accuracy and (b) in case of instability it gives clear indications of the required design modifications. Recently J. C. Willems [14,15] developed a graphical test for a single-input single-output continuous-time convolution feedback system with constant feedback, where the open-loop impulse response g(t) belongs to the algebra $\mathcal{A}$ and contains equally spaced impulses. This section generalizes previous result in that (a) the open-loop transfer function $\hat{g}(s)$ is the sum of a term in $\hat{\mathcal{A}}$ and a finite number of poles in Re $s \geq 0$, and (b) it does not require that impulses of g(t) be equally spaced. As a consequence the funtion $s \longmapsto \hat{g}(s)$ is asymptotically almost periodic in Re $s \geq 0$ for $|s| \to \infty$, and the conformal mapping technique of Willems does not work. We have to rely heavily on the theory of almost periodic functions [23,24,25]. It should be stressed that this difficulty is not encountered in the case of an analog single-input single-output discrete-time system. Indeed if $z \longmapsto \hat{g}(z)$ consists of a term in $\hat{\ell}^1$ and a finite number of poles in $|z| \geq 1$ then $z \longmapsto \hat{g}(z)$ has constant asymptote in $|z| \geq 1$ for $|z| \to \infty$ and therefore a graphical test can be obtained by a simple technique. Furthermore an important observation will be that the problem handled here has implications for the n-input, n-output case as well. All this will be handled in the paragraphs below.

2.2 Graphical Test for the Continuous-time Case.

2.2.1 Description of the System and Assumptions.

We consider a continuous-time scalar linear time-invariant system with input u, error e and output y. The latter are functions mapping $\mathbb{R}_+$ into $\mathbb{R}$ and satisfy

(2.1) $$y = g*e$$

(2.2) $$e = u-y$$

where g is a real-valued distribution with support on $\mathbb{R}_+$. As will become apparent there is no loss of generality in taking a unity feedback. Let $\hat{g}$ denote the Laplace-transform of g. We assume that $\hat{g}$ has the following form

(2.3) $$\hat{g}(s) = \hat{g}_r(s) + \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} r_{km}(s-p_k)^{-m_k+m}$$

where

(2.4) $$g_r \in \mathcal{A} ;$$

$$\text{(2.5)} \quad \begin{cases} \text{the poles } p_k \text{ are either real with real coefficients } r_{km} \\ \text{or conjugate complex with conjugate complex coefficients } r_{km}; \end{cases}$$

$$\text{(2.6)} \qquad \operatorname{Re} p_k \geq 0 \quad \text{for} \quad k = 1, 2, \ldots, \ell .$$

Note that because of (1.13) and (2.4)

$$\text{(2.7)} \quad \begin{cases} \hat{g}_r(\cdot) \text{ is analytic in } \operatorname{Re} s > 0\text{, bounded in } \operatorname{Re} s \geq 0 \\ \text{and each function } \omega \longmapsto \hat{g}_r(\sigma + j\omega) \text{ with } s \triangleq \sigma + j\omega \\ \text{is uniformly continuous for all } \sigma \geq 0. \end{cases}$$

It follows therefore that

$$\text{(2.8)} \quad \begin{cases} g(\cdot) \text{ is meromorphic in } \operatorname{Re} s > 0\text{, well defined and} \\ \text{continuous almost everywhere in } \operatorname{Re} s \geq 0. \end{cases}$$

A necessary and sufficient condition that the closed-loop impulse response h of the system (2.1) - (2.6) is in $\mathcal{A}$ (and thus stable as defined by remark (1.17)) is:

$$\text{(2.9)} \qquad \inf_{\operatorname{Re} s \geq 0} |1 + \hat{g}(s)| > 0.$$

For a proof see the appendix, lemma A.1.

The problem is to develop a graphical test for (2.9) based on the curve $\{1 + \hat{g}(j\omega) | \omega \in \mathbb{R}\}$. Observe that because of (2.4) $\hat{g}_r$ has following structure:

$$\text{(2.10)} \qquad \hat{g}_r(s) = \hat{g}_a(s) + \sum_{i=0}^{\infty} g_i \, e^{-st_i}$$

where

(2.11) $g_a(\cdot)$ is a real-valued function belonging to $L^1[0,\infty)$;

(2.12) $g_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots$;

(2.13) $$\sum_{i=0}^{\infty} |g_i| < \infty \ ;$$

(2.14) $0 = t_0 < t_1 < \ldots < t_i < \ldots$.

Let

(2.15) $$\hat{f}(s) \triangleq 1 + \sum_{i=0}^{\infty} g_i \, e^{-st_i} \triangleq \sum_{i=0}^{\infty} f_i \, e^{-st_i}.$$

Then $\hat{f}(s)$ is a Dirichlet-series with Dirichlet-exponents $-t_i$ subject to (2.14) and Dirichlet-coefficients f_i such that

(2.16) $f_0 = 1 + g_o$; $f_i = g_i$ for $i = 1, 2, \ldots$

where the coefficients g_i satisfy (2.12) - (2.13). First we develop a condition expressed in terms of the curve $\{\hat{f}(j\omega) \,|\, \omega \in \mathbb{R}\}$ insuring that $\inf\limits_{\mathrm{Re}\, s \geq 0} |\hat{f}(s)| > 0$, and then we use this result to develop the condition involving $\{1 + \hat{g}(j\omega) \,|\, \omega \in \mathbb{R}\}$ that will insure (2.9).

Given $s = \sigma + j\omega$ we denote by V_σ the vertical line in $\mathbb{C}$ (i.e. the complex plane) with Re $s = \sigma$. Moreover by $\bar{\hat{f}}(s)$ we mean the complex conjugate of $\hat{f}(s)$. Finally let

(2.17) $$n_p \overset{\Delta}{=} \text{the number of poles of } \hat{g}(s) \text{ counting multiplicities with } \operatorname{Re} p_k > 0.$$

2.2.2 A Necessary and Sufficient Condition Involving the Almost Periodic Curve $\{\hat{f}(j\omega) \mid \omega \in \mathbb{R}\}$ to insure $\inf\limits_{\operatorname{Re} s \geq 0} |\hat{f}(s)| > 0$.

Note that $\hat{f}$ defined by (2.15) - (2.16), is in $\hat{\mathcal{A}}$ as a consequence of (2.12) - (2.14) and can be uniformly approximated in Re $s \geq 0$ by a finite number of terms of the series (2.15). Hence

(2.18) $$\begin{cases} \hat{f} \text{ is bounded and uniformly continuous in } \operatorname{Re} s \geq 0 \text{ and} \\ \hat{f} \text{ is analytic in } \operatorname{Re} s > 0 \,. \end{cases}$$

We state next some standard definitions [23,24] and facts which streamline the proof of Theroem 2.1.

Given a line L, a set $S \subset L$ is said to be ℓ-relatively dense on L iff any open interval of length ℓ on L contains at least one point of the set.

Given a complex-valued function $w : D \longmapsto \mathbb{C}$, an element τ of D is said to be an ε-translation-number of w on D iff

$$|w(x+\tau) - w(x)| \leq \varepsilon \quad \text{for all } x \in D.$$

A complex-valued function w of a real variable x is said to be almost periodic iff, given any $\varepsilon > 0$, there exists a real number $\ell = \ell(\varepsilon) > 0$ such that the set of ε-translation-numbers $\tau = \tau(\varepsilon)$ of w on $\mathbb{R}$ is ℓ-relatively dense on $\mathbb{R}$.

Let $-\infty \leq \alpha < \beta \leq \infty$. A complex-valued function w of a complex variable s, analytic in a (vertical) strip (α,β), is said to be almost periodic in a strip (α,β) $([\alpha,\beta])$, iff given any $\varepsilon > 0$, there exists a real number $\ell = \ell(\varepsilon) > 0$ such that the set of imaginary ε-translation numbers $j\tau = j\tau(\varepsilon)$ of w on the strip (α,β) $([\alpha,\beta])$ is ℓ-relatively dense on the imaginary axis.

Note that this last definition requires that the functions $\omega \longmapsto w(\sigma + j\omega)$ be almost periodic on any V_σ for $\sigma \in (\alpha,\beta)$ $([\alpha,\beta])$ with an almost periodicity that is independent of σ, for $\sigma \in (\alpha,\beta)$ $([\alpha,\beta])$.

(2.18a) Fact.

The function $\hat{f}$ defined by (2.15) - (2.16) is almost periodic in the strip $[0,\infty)$.

Proof:

i) $\hat{f}(j\omega)$ is almost periodic on V_0 because $f(j\omega)$ can be uniformly approximated by a trigonometric polynomial

$$\hat{f}_N(j\omega) = \sum_{i=0}^{N} f_i \cdot e^{-j\omega t_i} \quad ([25]\text{p. } 9).$$

ii) We claim that the set $T(\varepsilon) = \{j\tau(\varepsilon) \mid \tau(\varepsilon) = \varepsilon\text{-translation-number of } \hat{f}(j\omega) \text{ on } V_0\}$ is the set of ε-translation-numbers of $\hat{f}(s)$ on the strip $[0,\infty)$.

Indeed by (2.18)

$$\hat{\pi}_\tau(s) \triangleq \hat{f}(s + j\tau) - \hat{f}(s) = \sum_{i=1}^{\infty} f_i(e^{-j\tau t_i} - 1)e^{-st_i}$$

is a Dirichlet-series bounded and uniformly continuous in $\operatorname{Re} s \geq 0$, analytic in $\operatorname{Re} s > 0$ and by (2.14) all its Dirichlet-exponents $-t_i$, $i = 1, 2, \ldots$ are negative, which implies that

$$M_\tau(\sigma) \triangleq \sup_{\omega \in \mathbb{R}} |\hat{\pi}_\tau(\sigma + j\omega)|$$

is decreasing on $\sigma \geq 0$ for any τ ([25] p. 69-70). Let $\tau = \tau(\varepsilon)$ be any ε-translation-number of $\hat{f}(j\omega)$ on V_0, then

$$|\hat{f}(j\omega + j\tau) - \hat{f}(j\omega)| = |\hat{\pi}_\tau(j\omega)| \leq \varepsilon \quad \text{for all } \omega \in \mathbb{R};$$

in other words $M_\tau(0) \leq \varepsilon$. Therefore, for all s in the strip $[0,\infty)$,

$$|\hat{f}(s + j\tau) - f(s)| = \hat{\pi}_\tau(s) \leq M_\tau(\sigma) \leq M_\tau(0) \leq \varepsilon.$$

Thus $T(\varepsilon)$ is a set of ε-translation-numbers of $\hat{f}(s)$ on the strip $[0,\infty)$ and since, by definition, any ε-translation-number $j\tau = j\tau(\varepsilon)$ of $\hat{f}(s)$ on the strip $[0,\infty)$ must be an ε-translation-number of $\hat{f}(j\omega)$ on V_0, we obtain that $T(\varepsilon)$ is the complete set of ε-translation-numbers of $\hat{f}(s)$ on the strip $[0,\infty)$. Thus the claim is proved.

Finally i) and ii) imply that given any $\varepsilon > 0$, there exists a real number $\ell(\varepsilon) > 0$ such that the set of imaginary ℓ-translation-numbers of $\hat{f}(s)$ on the strip $[0,\infty)$ is ℓ-relatively dense on the imaginary axis. $\overline{\underline{X}}$

Let us now consider the distribution of zeros of $\hat{f}(s)$ in the strip $(0,\infty)$

(2.18b) Fact.

If $\hat{f}(s)$, defined by (2.15) - (2.16), has a zero $s_0 = \sigma_0 + j\omega_0$ in the strip $(0,\infty)$, then $\hat{f}(s)$ has infinitely many zeros in a strip (α,β) (with $0 < \alpha < \beta < \infty$) containing s_0, and their imaginary parts are relatively dense on the imaginary axis (i.e. there exists a number $\ell > 0$, such that the imaginary parts are ℓ - relatively dense on the imaginary axis).

Proof:

Without loss of generality we assume that $\hat{f}(s)$ is not identically zero in Re $s \geq 0$. Since $\hat{f}(s)$ is analytic in Re $s > 0$, its zeros are not dense in Re $s > 0$, therefore we can choose $0 < r < \sigma_0$ such that $|\hat{f}(s)| \geq m > 0$ on $|s - s_0| = r$. By fact (2.18a) for any $0 < \varepsilon < m$ there exists a set of ε-translation-numbers $j\tau = j\tau(\varepsilon)$ of $\hat{f}$ on the strip $[0,\infty)$ that is relatively dense on the imaginary axis. Hence by $\hat{f}(s + j\tau) = \hat{f}(s + j\tau) - \hat{f}(s) + \hat{f}(s)$ it follows by (2.18) and Rouche's theorem (see [26], theorem 9.2.3, p. 254) that $\hat{f}(s)$ has a zero in any disc $|s - (s_0 + j\tau)| < r$, which proves the fact. $\bar{\underline{X}}$

Definition of the Argument $\phi(s)$ of $\hat{f}(s)$

By definition

$$\text{(2.19)} \qquad \phi(s) = \arg \hat{f}(s) = \operatorname{Im} \log \hat{f}(s) \quad \text{in Re } s \geq 0$$

with two additional conventions.

(2.19a) Convention. Let L denote a straight oriented line in Re $s \geq 0$. By convention we take $\phi(s)$, $s \in L$, as the right argument of $\hat{f}(s)$ on L, i.e. $\phi(s)$, $s \in L$, is an arbitrary branch of the argument which is continuous except at the zeros of $\hat{f}(s)$ on L, while it is discontinuous with a jump of $+m\pi$, when s goes through a zero of $\hat{f}(s)$ of order m in the positive direction of L. At any discontinuity point we assign to ϕ the mean value of its one-sided limits. The function $\phi(s)$, $s \in L$, is then well defined (mod. 2π) because of (2.18).

(2.19b) <u>Convention</u>. Because $\hat{f}(j0)$ is real and because it will later be assumed to be nonzero we pick for $\omega \longmapsto \phi(j\omega)$, $\omega \in \mathbb{R}$, that branch of the argument such that $\phi(j0) = 0$ (or π) according as $\hat{f}(j0)$ is positive (or negative, respectively).

<u>Remarks</u>.

(2.19c) <u>Remark</u>. It is important to observe that by convention (2.19a) and (2.18) the principle of the argument may be applied to $\hat{f}(s)$ on any rectangle in Re $s \geq 0$ which is oriented in clockwise sense and which has no zeros of $\hat{f}(s)$ on its corners.

(2.19d) <u>Remark</u>. Because of (2.18) and (2.19), for any strip (α,β) in Re $s \geq 0$ such that $0 \leq \alpha < \beta \leq \infty$ and $\inf\limits_{\beta \geq \mathrm{Re}\, s \geq \alpha} |\hat{f}(s)| > 0$, $\phi(s)$ is well defined (mod. 2π) and uniformly continuous in the strip $[\alpha,\beta]$ and analytic in the strip (α,β).

Since by Fact (2.18.a), $\omega \longmapsto \hat{f}(j\omega)$ is almost periodic we have:

(2.19e) <u>Fact [24]</u>.

Let $\hat{f}$ be defined by (2.15) - (2.16). If

(2.20) $$\inf_{\omega \in \mathbb{R}} |\hat{f}(j\omega)| \triangleq K > 0$$

then

(a) $\omega \longmapsto \phi(j\omega)$ is well defined on $\mathbb{R}$ and is of the form

$$\phi(j\omega) = \lambda\omega + w(j\omega) \tag{2.21}$$

where λ is a constant and $\omega \longmapsto w(j\omega)$ is almost periodic; the constant λ will be called "the mean angular velocity of $\hat{f}(j\omega)$" (In the literature the term "mean motion of $\hat{f}(j\omega)$" is used, however this is borrowed from celestial mechanics),

(b) if N is the least [redacted] integer such that

$$\sum_{N+1}^{\infty} |f_i| \leq K \sin(\delta/2) \quad \text{for some } 0 < \delta < \pi \tag{2.22}$$

then the mean angular velocity λ of $\hat{f}(j\omega)$ may be written in both the forms

$$\lambda = - h_0 t_0 - h_1 t_1 - \dots - h_N t_N \tag{2.23}$$

where the coefficients $h_0, h_1, \dots, h_N$ are integers with sum 1 and

$$\lambda = - r_0 t_0 - r_1 t_1 - \dots - r_N t_N \tag{2.24}$$

where the coefficients $r_0, r_1, \dots, r_N$ are nonnegative rationals with sum 1,

(c) with

$$\varepsilon \leq K \sin(\delta/2) \qquad \delta < \pi \tag{2.25}$$

any ε-translation-number $\tau(\varepsilon)$ of $\omega \longmapsto \hat{f}(j\omega)$ satisfies

(2.26) $|\phi(j\omega + j\tau) - \phi(j\omega) - c_\tau 2\pi| \leq \delta$ for all $\omega \in \mathbb{R}$

(2.27) $|\lambda\tau - c_\tau 2\pi| \leq \delta$

where c_τ is an integer depending on τ,

(d) the function $\omega \longmapsto \phi(j\omega)$ is almost periodic if and only if the mean angular velocity λ of $\hat{f}(j\omega)$ is zero or equivalently if and only if there exists an increasing sequence $\{\omega_n\}_{n=-\infty}^{+\infty}$ satisfying

(2.28) $\ldots < \omega_{-n} < \ldots < \omega_{-1} < \omega_0 = 0 < \omega_1 < \ldots < \omega_n < \ldots$

(2.29) $-\omega_{-n} = \omega_n$ for $n = 0, 1, 2, \ldots$

(2.30) $\lim_{n\to\infty} \omega_n = \infty$

such that

(2.31) $\phi(j\omega_n) = \phi(j0)$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$.

Proof

Part (a) is a straightforward transcription of [24] p. 167 Theorem 1. Part (b): the mean angular velocity of $\hat{f}(j\omega)$ is the same as the mean angular velocity of an exponential polynomial $\hat{f}_N(j\omega) = \sum_{i=0}^{N} f_i e^{-j\omega t_i}$ where N satisfies (2.22) (see [24] p. 170-176). Part (c) follows from [24] p. 168 - 170. The first statement of part (d) is obvious from part (a). The second statement is established as follows:

$\Rightarrow$ Since

(2.32) $\hat{f}(-j\omega) = \overline{\hat{f}}(j\omega)$ for all $\omega \in \mathbb{R}$

(2.33) $\phi(j\omega) - \phi(j0) = \phi(j0) - \phi(-j\omega)$ for all $\omega \in \mathbb{R}$

So unless $\phi(j\omega) \equiv \phi(j0)$, then for some $\omega' > 0$ either

$$\phi(j\omega') > \phi(j0) > \phi(-j\omega')$$

or

$$\phi(j\omega') < \phi(j0) < \phi(-j\omega')$$

Then (2.28) - (2.31) follows by the continuity and almost periodicity of $\phi(j\omega)$ on $\mathbb{R}$ and by (2.33).

$\Leftarrow$ The existence of the sequence $\{\omega_n\}_{n=-\infty}^{\infty}$ implies that $\phi(j\omega)$ is bounded on $\mathbb{R}$, hence the mean angular velocity of $\hat{f}(j\omega)$ is zero and thus by (2.21) $\omega \longmapsto \phi(j\omega)$ is almost periodic $\bar{\underline{X}}$

Before we give Theorem 2.1 we give a last interesting result.

(2.33a) Fact.

Let $\hat{f}$ be defined by (2.15) - (2.16). If

(2.20) $$\inf_{\omega \in \mathbb{R}} |\hat{f}(j\omega)| \triangleq K > 0$$

Then:

given any $\sigma > 0$, there exists a positive real number C_σ depending on σ such that

$|\phi(\sigma + j\omega) - \phi(j\omega)| < C_\sigma$ uniformly in ω.

Proof:

Because of (2.20) and (2.18) there exists a $\sigma^* > 0$, $\sigma^* < \sigma$ such that $\inf_{\sigma^* \geq \text{Re } s \geq 0} |\hat{f}(s)| > 0$, so by remark (2.19d) and convention (2.19b), $\phi(s)$ is well-defined and uniformly continuous in the strip $[0,\sigma^*]$ such that there exists a positive constant C_{σ^*} depending on σ^* for which

$$|\phi(\sigma^* + j\omega) - \phi(j\omega)| < C_{\sigma^*} \text{ uniformly in } \omega.$$

Observing that $[\sigma^*,\sigma]$ is a closed substrip of the strip $[0,\infty)$, in which $\hat{f}(s)$ is almost periodic by fact (2.18a) it follows that there exists a positive constant $C_{\sigma-\sigma^*}$ depending on $\sigma - \sigma^*$ such that

$$|\phi(\sigma + j\omega) - \phi(\sigma^* + j\omega)| < C_{\sigma-\sigma^*} \text{ uniformly in } \omega,$$

([24], p. 178-179, Theorem 3(iv)). Combining the two results we obtain that with $C_\sigma = C_{\sigma^*} + C_{\sigma-\sigma^*}$ the fact is true.

Theorem 2.1

Let $\hat{f}(s)$ be the Dirichlet-series defined by (2.15) - (2.16). Under these conditions

$$\inf_{\text{Re } s \geq 0} |\hat{f}(s)| > 0 \tag{2.34}$$

if and only if

i)

$$f_0 = 1 + g_0 \neq 0 \tag{2.35}$$

ii)

(2.20) $$\inf_{\omega \in \mathbb{R}} |\hat{f}(j\omega)| = K > 0$$

iii)

(2.36) the mean angular velocity λ of $\hat{f}(j\omega)$ is zero.

<u>Proof</u>:

$\Leftarrow$ a) Observe that because of (2.15) - (2.16), (2.18) and (2.35)

(2.37) $$\lim_{\sigma\to\infty} \hat{f}(\sigma + j\omega) = f_0 \neq 0 \text{ uniformly in } \omega$$

Thus there exists a $\sigma^* > 0$ such that

(2.38) $$\inf_{\text{Re } s \geq \sigma^*} |\hat{f}(s)| > 0.$$

So by Remark (2.19d) $\phi(s)$ is well defined (mod. 2π) and uniformly continuous in the strip $[\sigma^*,\infty)$. Hence (2.37) implies for any branch of $\phi(s)$

(2.39) $$\lim_{\sigma\to\infty} \phi(\sigma + j\omega) = \phi_\infty \quad \text{uniformly in}$$

where $\phi_\infty = \arg f_0$ (mod. 2π). Moreover we can pick a $\sigma_1 > 0$ so large that

(2.40) $$\sigma_1 \geq \sigma^*$$

and

(2.41) $$|\phi(\sigma_1 + j\omega) - \phi_\infty| < 1, \quad \text{uniformly in } \omega.$$

In view of (2.20), (2.40) and (2.38), we will have established (2.34) if we show that $\inf_{\sigma_1 > \text{Re } s > 0} |\hat{f}(s)| > 0$.

b) As a first step let us establish that

$$f(s) \neq 0 \text{ in } \sigma_1 > \text{Re } s > 0 \Rightarrow \inf_{\sigma_1 > \text{Re } s > 0} |\hat{f}(s)| > 0. \tag{2.42}$$

Indeed assume $\inf_{\sigma_1 > \text{Re } s > 0} |\hat{f}(s)| = 0$, then there exists a sequence $\{s_k\}_{k=2}^{\infty} \subset \{s | \sigma_1 > \text{Re } s > 0\}$, $s_k \triangleq \sigma_k + j\omega_k$ such that $\lim_{k\to\infty} |\hat{f}(s_k)| = 0$. Because by assumption $\hat{f}(s) \neq 0$ in $\sigma_1 > \text{Re } s > 0$ and because of (2.20) and (2.38) it follows then for this sequence that $\lim_{k\to\infty} |\omega_k| = \infty$ and $\lim_{k\to\infty} \inf \sigma_k > 0$, $\lim_{k\to\infty} \sup \sigma_k < \sigma_1$. The sequence $\{\sigma_k\}_{k=2}^{\infty}$ is a bounded infinite set of real numbers and so by Bolzano-Weierstrass' Theorem it contains an accumulation point. Thus without loss of generality we may assume that $\{\sigma_k\}_{k=2}^{\infty}$ is convergent say to σ_0, thus $\lim_{k\to\infty} \sigma_k \triangleq \sigma_0$, $0 < \sigma_0 < \sigma_1$. Finally by the uniform continuity of $\hat{f}(s)$ in $\text{Re } s \geq 0$ we conclude:

$$\left\{ \begin{array}{l} \text{there exists a real number } \sigma_0 \text{ and a sequence} \\ \{\omega_k\}_{k=2}^{\infty} \text{ with } 0 < \sigma_0 < \sigma_1 \text{ and } \lim_{k\to\infty} |\omega_k| = \infty, \text{ such that} \\ \lim_{k\to\infty} |\hat{f}(\sigma_0 + j\omega_k)| = 0. \end{array} \right. \tag{2.43}$$

Observe that f(s) is nonzero and almost periodic on V_{σ_0} so:

$$(2.44)\quad \begin{cases} \text{there exist positive real numbers } \ell \text{ and d such that} \\ \text{any interval of length } \ell \text{ contains a point } \sigma_0 + j\omega \text{ for} \\ \text{which } |\hat{f}(\sigma_0 + j\omega)| > d. \end{cases}$$

Therefore by a result of [25] (Theorem 3.6, p. 71), (2.18) and (2.43) - (2.44) imply that $\hat{f}(s)$ vanishes in the strip $(\sigma_0 - \delta, \sigma_0 + \delta)$ for any $\delta > 0$. Clearly this contradicts our assumption that $\hat{f}(s) \neq 0$ in $\sigma_1 > \text{Re } s > 0$. So our claim is true.

c) Having established claim (2.42) we will have proved (2.34) if we show that $\hat{f}(s) \neq 0$ in the strip $(0,\sigma_1)$. By contraposition of fact (2.18b) this will be true if we show that the number of zeros N of $\hat{f}(s)$ in the strip $(0,\sigma_1)$ is bounded. Consider therefore a sequence of rectangles $\{R_n\}_{n=1}^{\infty}$ defined by

$R_n \triangleq [0,\sigma_1] \times [-n,n]$ for n = 0, 1, 2, ..., and let the corresponding number of zeros of $\hat{f}(s)$ inside R_n be N_n for n = 1, 2, ... Observe that because of (2.20), (2.38) and Remark (2.19c) the principle of the argument may be applied to each of these rectangles oriented in the clockwise sense. We show now that the sequence $\{\Delta\phi_n\}_{n=1}^{\infty}$ (where $\Delta\phi_n$ is the net increase in argument around the rectangle R_n) is bounded. This follows easily if one observes

i) that by (2.20) and (2.36) and Fact (2.19e)(d) $\omega \longmapsto \phi(j\omega)$ is almost periodic and hence bounded

ii) that by (2.20) and Fact (2.33a)

$$|\phi(\sigma_1 + j\omega) - \phi(j\omega)| < C_{\sigma_1} \text{ uniformly in } \omega.$$

iii) that by (2.41) any branch of $\omega \longmapsto \phi(\sigma_1 + j\omega)$ is bounded. In view of this it follows then that there exists a positive constant C such that

$$N \stackrel{\Delta}{=} \lim_{n\to\infty} N_n = \lim_{n\to\infty} |\Delta\phi_n| < C.$$

$\Rightarrow$ First observe that the first equality of (2.37) still holds, so (2.34) implies (2.35). Next (2.34) implies (2.20), hence by fact (2.19e)(a) $\omega \longmapsto \phi(j\omega)$ is well-defined and satisfies (2.21). Furthermore by (2.34) and (2.18) and convention (2.19b), $s \longmapsto \phi(s)$ is well-defined and uniformly continuous in Re $s \geq 0$. Hence again (2.37) implies (2.39) and again we can pick a $\sigma_1 > 0$ such that (2.41) is true.

We claim now that $\omega \longmapsto \phi(j\omega)$ is almost periodic. For this purpose, in view of (2.20), it is sufficient to show that $\phi(jn)$ for $n = 0, 1, 2, \ldots$ remains bounded as $n \to \infty$. Consider therefore the rectangles $\{R_n'\}_{n=1}^{\infty}$ defined by $R_n' = [0,\sigma_1] \times [0,n]$ for $n = 1, 2, \ldots$. By (2.34) and (2.18) it follows that the principle of the argument can be applied to each of these rectangles; hence the net change in ϕ around each R_n' is zero for $n = 1, 2, \ldots$. Now by the uniform continuity of $s \longmapsto \phi(s)$ in Re $s \geq 0$, there exists a constant C independent of n such that for any horizontal segment $H_n' \stackrel{\Delta}{=} \{s = \sigma + jn;\ 0 \leq \sigma \leq \sigma_1\}$ for $n = 0, 1, 2, \ldots$: $|\phi(\sigma_1 + jn) - \phi(jn)| < C$. This fact with (2.41) implies that the sequence $\{\phi(jn) - \phi(j0)\}_{n=0}^{\infty}$ bounded. Hence, because $\phi(j0)$ is 0 or π by (2.19b) the sequence $\{\phi(jn)\}_{n=0}^{\infty}$ is also bounded. So our claim is true and by fact (2.19e)(d) the mean angular velocity of $\hat{f}(j\omega)$ is zero which implies (2.36). $\bar{\underline{X}}$

It is interesting to observe that under the conditions of Theorem 2.1 sign $\hat{f}(j0)$ = sign f_0 = sign $(1 + g_0)$. Hence if $1 + g_0 > 0$ (respectively < 0) then $\phi(j0) = 0$ (or π).

We want now to develop a graphical test involving $\{\hat{f}(j\omega)\,|\,\omega \in \mathbb{R}\}$ to insure $\inf\limits_{\mathrm{Re}\, s \geq 0} |\hat{f}(s)| > 0$. Here again it will be the almost periodicity of $\omega \longmapsto \hat{f}(j\omega)$ that will save us. We start giving some definitions and two facts.

Let $\ell(\varepsilon)$ be the "density-length" for the ε-translation-numbers of $\omega \longmapsto \hat{f}(j\omega)$. Observe that ε-translation-number of $\hat{f}(j\omega)$ can be determined by diophantine analysis (see e.g. [25] p. 146-149). From their pattern a "density-length" can be determined.

Consider now the path

(2.45) $$\gamma(\varepsilon) \triangleq \{\hat{f}(j\omega)\,|\,\omega \in [0,\ell(\varepsilon)]\}$$

and its closed ε-neighborhood $N(\varepsilon)$ defined by

(2.46) $$N(\varepsilon) \triangleq \{x \in \mathbb{C} \,|\, |x - \hat{f}(j\omega)| \leq \;\; ; \omega \in [0,\ell(\varepsilon)]\}.$$

We prove now Fact (2.46a), which allows by the simply knowledge of the path $\gamma(\varepsilon) = \{\hat{f}(j\omega) | \omega \in [0,\ell(\varepsilon)]\}$ to locate the closure of the set $\{\hat{f}(j\omega) | \omega \in \mathbb{R}\}$, and Fact (2.46b) which informs us about the minimal value of λ if it is nonzero.

(2.46a) Fact (Fig. 2.1)

Let $\hat{f}(j\omega)$ be defined by (2.15) - (2.16) (setting $s = j\omega$). Consider the path $\gamma(\varepsilon)$ and neighborhood $N(\varepsilon)$ given by (43), respectively (44). Under these conditions

(a) for any $\varepsilon > 0$, $N(\varepsilon)$ contains the closure of the set $\{\hat{f}(j\omega) | \omega \in \mathbb{R}\}$.

(b) as $\varepsilon \downarrow 0$, then $N(\varepsilon)$ tends towards the closure of the set $\{\hat{f}(j\omega) | \omega \in \mathbb{R}\}$.

Proof:

Observe that for any $\varepsilon > 0$, the ε-translation-numbers $\tau = \tau(\varepsilon)$ of $\omega \longmapsto \hat{f}(j\omega)$ are ℓ-relatively dense on $\mathbb{R}$. So given any $\omega \in \mathbb{R}$, there exists an ε-translation-number $\tau = \tau(\varepsilon)$ belonging to $[-\omega,-\omega+\ell]$ and shifting ω into an element $\omega + \tau \in [0,\ell]$ such that $|\hat{f}(j\omega) - \hat{f}(j\omega + j\tau)| \leq \varepsilon$. Hence $\hat{f}(j\omega) \in N(\varepsilon)$ for any $\omega \in \mathbb{R}$. Hence the set $\{\hat{f}(j\omega) | \omega \in \mathbb{R}\}$ is contained in the closed set $N(\varepsilon)$ and so is its closure. This proves the first statement. The second statement is a direct consequence of the first one. $\bar{\underline{X}}$

(2.46b) Fact.

Let $\hat{f}(j\omega)$ be given by (2.15) - (2.16) (setting $s = j\omega$). Assume that

$$\text{(2.20)} \qquad \inf_{\omega \in \mathbb{R}} |\hat{f}(j\omega)| \triangleq K > 0.$$

Under these conditions: (a) it is possible to determine the set X given by:

$$\text{(2.47)} \qquad X \triangleq \{x = -\sum_{i=0}^{N} h_i t_i = -\sum_{i=0}^{N} r_i t_i;\ h_i = \text{integer}$$

$$\text{and } r_i = \text{nonnegative rational for all } i;$$

$$\sum_{i=0}^{N} h_i = 1; \sum_{i=0}^{N} r_i = 1; \text{ N is the least}$$

$$\text{integer, such that } \sum_{N+1}^{\infty} |f_i| \leq K \sin(\delta/2)$$

$$\text{some } 0 < \delta < \pi\},$$

where $-t_i$ and f_i are Dirichlet-exponents and -coefficients of $\hat{f}(s)$; (b) moreover there are only a finite number of elements in X and, if $N \geq 1$,† then $X \sim \{0\}$ is nonempty, such that

$$\text{(2.48)} \qquad \lambda_{min} \triangleq \min_{\substack{x \in X \\ x \neq 0}} |x|$$

is well defined, and (c) as soon as the mean angular velocity λ of $\hat{f}(j\omega)$ satisfies $|\lambda| < \lambda_{min}$ then $\lambda = 0$.

<u>Proof</u>:

(a) is an immediate consequence of (2.20). (b) is a consequence of the fact the set $\{-t_i\}_{i=0}^{N}$ admits a <u>finite integral base</u> i.e. a set of real numbers $\{\beta_j\}_{j=1}^{M}$ such that i) there exists

†If $N = 0$, then by Fact (2.19e)(b) $\lambda = 0$, therefore this case will be omitted in the sequel.

no integers h_j, $j = 1, \ldots, M$, not all zero, such that $\sum_{j=1}^{M} h_j \; \beta_j = 0$ and ii) each number $-t_i$ can be expressed in a unique manner in the form $-t_i = \sum_{j=1}^{M} h_j^{(i)} \; \beta_j$ for $i = 0, 1, \ldots, N$ where $h_j^{(i)}$ are integers (see [24] p. 146, [23] p. 82-83). Equivalently the N+1 numbers $-t_i$ can be represented by lattice points (i.e. with integer coordinates) in $\mathbb{R}^M$-space, indeed each point $-t_i$ may be represented by the M-vector $(h_1^{(i)}, h_2^{(i)}, \ldots, h_M^{(i)})$ with integer coordinates. Now (2.47) merely expresses the fact that the numbers x can be represented as a subset of lattice-points in $\mathbb{R}^M$ that are in the closure of the convex hull of the N+1 lattice-points $h^{(i)}$, $i = 0, 1, \ldots, N$ in $\mathbb{R}^M$. Hence the set X is finite. Moreover $-t_i$, for $i = 1, 2, \ldots, N$, belongs to $X \sim \{0\}$. Therefore (b) is true. Concerning (c) observe that because of Fact (2.19e) (b) the mean angular velocity λ of $\hat{f}(j\omega)$ belongs to X and that also 0 belongs to X, hence (c) is a direct consequence of (b). $\underline{\overline{X}}$

As a final remark, observe that because the ε-translation-numbers are relatively dense on $\mathbb{R}$ it follows that as soon as (2.20) is satisfied we can pick a translation-number $\tau(\varepsilon)$ such that

$$\frac{\pi}{\tau(\varepsilon)} \leq \lambda_{min}. \tag{2.49}$$

We are now ready for a graphical test insuring $\inf_{\mathrm{Re}\, s \geq 0} |\hat{f}(s)| > 0.$

Corollary 2.1

Let $\hat{f}(s)$ be the Dirichlet-series defined by (2.15) - (2.16) and (2.12) - (2.14). Let $\gamma(\varepsilon)$ and $N(\varepsilon)$ be given by (2.45) and (2.46). Under these conditions

(2.34) $$\inf_{\text{Re } s \geq 0} |\hat{f}(s)| > 0$$

if and only if

i)

(2.35) $$f_0 = 1 + g_0 \neq 0$$

ii) the origin 0 of the complex plane is positioned with respect to $\{\hat{f}(j\omega) | \omega \in \mathbb{R}$ such that

a)

(2.50) there exists an $\varepsilon > 0$ such that 0 does not belong to $N(\varepsilon)$

b)

(2.51) for an $\varepsilon > 0$, with $0 < \varepsilon < K \triangleq \inf_{\omega \in \mathbb{R}} |\hat{f}(j\omega)|$, for which the corresponding ε- translation-number $\tau(\varepsilon)$ satisfies (2.49), where λ_{min} is defined by (2.47) - (2.48), then $|\phi(j\tau) - \phi(j0)| < \pi$ must hold.

Proof:

Because of Theorem 2.1 we need only to show that (2.50) - (2.51) are equivalent to (2.20) and (2.36). Clearly by Fact 2.5 (2.50) $\Longleftrightarrow$ (2.20). So we are left to prove the equivalence of (2.51) and (2.36) under the assumption (2.20).

$\Rightarrow$ We assume that (2.51) is true. Then $\varepsilon < K$ implies $\varepsilon \leq K \sin(\delta/2)$ with some $\delta < \pi$. So immediately by Fact (2.19e)(c) $|\phi(j\tau) - \phi(j0) - c_\tau 2\pi| \leq \delta < \pi$. Thus by (2.51) $c_\tau = 0$ and hence by (2.27) $|\lambda\tau| \leq \delta$ or $|\lambda| \leq \frac{\delta}{\tau} < \frac{\pi}{\tau}$. So by (47) $|\lambda| < \lambda_{min}$ which by Fact (2.46b)(c) implies $\lambda = 0$. $\bar{\underline{X}}$

$\Leftarrow$ We assume that (2.36) is true. Let $0 < \varepsilon \leq K \sin(\delta/2)$, $\delta < \pi$ and let $\tau(\varepsilon)$ satisfy (2.49) then immediately from Fact (2.19e)(c) $|c_\tau 2\pi| \leq \delta < \pi$ i.e. $c_\tau = 0$. Hence from (2.26) $|\phi(j\tau) - \phi(j0)| \leq \delta < \pi$. $\bar{\underline{X}}$

<u>Remarks</u>

(2.51a) <u>Remark.</u> It is important to observe that the knowledge of the density-length $\ell(\varepsilon)$ allows us to locate the closure of the set $\{\hat{f}(j\omega)|\omega \in \mathbb{R}\}$ and that the knowledge of a translation-number $\tau(\varepsilon)$ allows us to replace the condition $\lambda = 0$ by a condition on the increase in argument.

(2.51b) <u>Remark.</u> If $\omega \longmapsto \hat{f}(j\omega)$ is periodic with period ω_0 two important simplifications occur i.e.:

a) $\{\hat{f}(j\omega)|\omega \in \mathbb{R}\} = \{\hat{f}(j\omega)|\omega \in [0,\omega_0]$

b) if (2.20) is satisfied then $\phi(j\omega) = \lambda\omega + w(j\omega)$ where $w(j\omega)$ is periodic with period ω_0.

Hence: $\lambda = 0 \Longleftrightarrow \phi(j\omega_0) = \phi(j0)$ and hence: for the case that $\omega \longmapsto \hat{f}(j\omega)$ is periodic with period ω_0 part ii) of Corollary 2.1 can be replaced by: the origin 0 of the complex plane is positioned with respect to $\{\hat{f}(j\omega)|\omega \in \mathbb{R}\}$ such that

a)

0 does not belong to $\{\hat{f}(j\omega)|\omega \in [0,\omega_0]\}$

b)

$\phi(j\omega_0) = \phi(j0)$.

2.2.3 A Necessary and Sufficient Condition involving $\{1 + \hat{g}(j\omega) \mid \omega \in \mathbb{R}\}$ to insure $\inf_{\mathrm{Re}\, s \geq 0} |1 + \hat{g}(s)| > 0$

Definition of the argument $\theta(s)$ of $1 + \hat{g}(s)$ subject to (2.3) - (2.6).

By definition

(2.52) $$\theta(s) = \arg[1 + \hat{g}(s)] = \mathrm{Im}\log[1 + \hat{g}(s)] \quad \text{for } \mathrm{Re}\, s \geq 0$$

with two additional conditions.

(2.52a) Convention. Let L denote a straight oriented line in Re $s \geq 0$ By convention we take $\theta(s)$, $s \in L$ as the right argument of $1 + \hat{g}(s)$ on L, i.e. $\theta(s)$, $s \in L$, is an arbitrary branch of the argument which is continuous except at the zeros and poles of $1 + \hat{g}(s)$ on L, while it is discontinuous with a jump of $+m\pi(-m_k\pi)$, when s passes, in the positive direction on L a zero (pole) of $1 + \hat{g}(s)$ of order $m(m_k)$. At a discontinuity-point we assign to θ the mean value of its one-sided limits. The funciton $\theta(s)$, $s \in L$, is then well defined (mod. 2π) because of (2.8)

(2.52b) Convention. Because $1 + \hat{g}(s)$ is real for $s = \sigma \geq 0$ and meromorphic in Re $s > 0$, there exists an interval $(0,\sigma^*)$ on which $1 + \hat{g}(\sigma)$ is real, finite and different from zero. We pick for $\omega \longmapsto \theta(j\omega)$ that branch of the argument such that

$\theta(j0) \triangleq 0$ (or π) according as $1 + \hat{g}(\sigma)$ is positive (or negative) on $(0,\sigma^*)$.

Theorem 2.2

Given $\hat{g}(s)$ defined by (2.3) - (2.6), let $\hat{f}(s)$ be the Dirichlet-series given by (2.15) - (2.16) and let n_p be given by (2.17). Under these conditions:

$$\inf_{\mathrm{Re}\, s \geq 0} |1 + \hat{g}(s)| > 0 \tag{2.9}$$

if and only if

i)

$$1 + g_0 \neq 0, \tag{2.35}$$

ii)

$$\inf_{\omega \in \mathbb{R}} |1 + \hat{g}(j\omega)| > 0, \tag{2.53}$$

iii)

(2.36) The mean angular velocity λ of $\omega \longmapsto \hat{f}(j\omega)$ is zero,

iv)

$$\lim_{\omega\to\infty} [\theta(j\omega) - \phi(j\omega)] = \theta(j0) - \phi(j0) + n_p\, \pi = b\, 2\pi \tag{2.54}$$

where b is an integer.

Proof:

(a) Let us first study the asymptotic behavior of $1 + \hat{g}(s)$. In view of (2.11), the Riemann-Lebesque lemma implies $g_a(s) \to 0$ as $|s| \to \infty$ in Re $s \geq 0$, hence by (2.3), (2.10) and (2.15) - (2.16)

(2.55) $\quad 1 + \hat{g}(s) \to \hat{f}(s)$ as $|s| \to \infty$ in Re $s \geq 0$.

An important conclusion is that because of (2.55) and Fact (2.18a) $\omega \longmapsto 1 + \hat{g}(j\omega)$ has an asymptotic almost periodic behavior on $\mathbb{R}$ and $s \longmapsto 1 + \hat{g}(s)$ has an asymptotic almost periodic behavior in Re $s \geq 0$ (for $|s| \to \infty$).

(b) $\Leftarrow$. We first show that

(2.34) $$\inf_{\text{Re } s \geq 0} |\hat{f}(s)| > 0.$$

Indeed (2.53), (2.55) imply $\liminf_{|\omega| \to \infty} |\hat{f}(j\omega)| > 0$. Hence, since $\omega \longmapsto \hat{f}(j\omega)$ is almost periodic on V_0 by Fact (2.18a),

(2.20) $$\inf_{\omega \in \mathbb{R}} |\hat{f}(j\omega)| \triangleq K > 0.$$

So by (2.35), (2.20) and (2.36) it follows that (2.34) is true by Theorem 2.1.

Observe that by Fact (2.19e)(d), (2.20) and (2.36) are equivalent to the existence of a sequence $\{\omega_n\}_{n=-\infty}^{\infty}$ satisfying (2.28) - (2.31). Now choose ω_n with positive index from this sequence and a $\sigma^* > 0$, both sufficiently large so that:

(a) The open rectangle ABCD $\triangleq (0,\sigma^*) \times (-\omega_n,\omega_n)$ (Fig. 2.2), i) has all poles of $1 + \hat{g}(s)$ with Re $p_k > 0$ in the interior of ABCD, ii) has all poles of $1 + \hat{g}(s)$ with Re $p_k = 0$ on AB, iii) neither A nor B are the location of a pole of $1 + \hat{g}(s)$;

(b) in the complement of this rectangle with respect to $\{s \mid \text{Re } s \geq 0\}$ except for AB: $1 + \hat{g}(s)$ is sufficiently close to $\hat{f}(s)$ such that $1 + \hat{g}(s)$ is bounded away from zero by (2.55) and

(2.34) in this complement, except for AB. The principle of the argument can be applied to ABCD. Denote by $\Delta\theta_{AB}$ the net change in argument on the oriented seqment AB. By the principle of the argument along with (2.17) it follows that:

$$\Delta\theta_{ABCDA} = (n_P - n_Z)2\pi \tag{2.56}$$

where n_Z is the number of zeros of $1 + \hat{g}(s)$ inside ABCD. Remember that $\hat{f}(s)$ is analytic inside ABCD, continuous on the boundary ABCDA and by (2.34) bounded away from zero in Re $s \geq 0$ hence again by the principle of the argument

$$\Delta\phi_{ABCDA} = 0. \tag{2.57}$$

Moreover since ω_n and $\tilde{\sigma}$ have been chosen sufficiently large it follows from (2.55) that

$$\Delta\phi_{BCDA} \simeq \Delta\theta_{BCDA} \tag{2.58}$$

where $\simeq$ indicates that equality is reached as $\omega_n \to \infty$ and $\sigma^* \to \infty$. From conditions (2.29), (2.31), (2.54), the fact that $\theta(j\omega) - \theta(j0) = \theta(j0) - \theta(-j\omega)$ because $1 + k\hat{g}(-j\omega) = \overline{1 + k\hat{g}(j\omega)}$ and (2.53),

$$\Delta\phi_{AB} = 0 \tag{2.59}$$

$$\Delta\theta_{AB} \simeq n_P\ 2\pi\ . \tag{2.60}$$

Hence (2.57) - (2.59) imply $\Delta\phi_{BCDA} \simeq \Delta\theta_{BCDA} \simeq 0$, which along with (2.60) and (2.56) implies $n_Z = 0$. Thus, for sufficiently large ω_n and σ^*, $1 + \hat{g}(s)$ has no zeros in ABCD. Furthermore by construction $1 + \hat{g}(s)$ is bounded away from zero in the complement of ABCD with respect to $\{s | \text{Re } s > 0\}$, and by (2.53) $1 + \hat{g}(s)$ is bounded away from zero in the complement of ABCD with respect to $\{s | \text{Re } s \geq 0\}$. Hence (2.9) follows.

(c) $\Rightarrow$. Immediately (2.9) implies (2.53). Thus because of (2.9) and (2.55), we can pick an $\omega = \omega^*$ and $\sigma = \sigma^*$ so large that the rectangle ABCD $= (0,\sigma^*) \times (-\omega^*,\omega^*)$ (see Fig. 2.2) is such that $\hat{f}(s)$ is bounded away from zero in the complement of ABCD with respect to $\{s | \text{Re } s \geq 0\}$ except on AB. Since by Fact (2.18a): $\omega \longmapsto \hat{f}(\sigma + j\omega)$ is almost periodic on any line V_σ, $\sigma \in [0,\infty)$, it follows then that $\hat{f}(s)$ is bounded away from zero on all these V_σ. Hence

$$(2.34) \qquad \inf_{\text{Re } s \geq 0} |\hat{f}(s)| > 0$$

which by Theorem 2.1 implies (2.35), (2.20), and (2.36). Hence because of Fact (2.19e)(d) there exists a sequence $\{\omega_n\}_{n=-\infty}^{\infty}$ such that (2.28) - (2.31) hold.

We show now that (2.54) holds. From now on, pick the parameters of ABCD, so that ω^* is an element of the above sequence with positive index and so that ω^* and σ^* are so large that (2.55) holds; finally all poles of $1 + \hat{g}(s)$ with $\text{Re } p_k > 0$ should be inside ABCD and all poles of $1 + \hat{g}(s)$ with $\text{Re } p_k = 0$ should be on AB but neither A nor B should be the location of a pole. Again the principle of the argument can be applied with ABCDA oriented in clockwise sense. Hence

$\Delta\theta_{ABCDA} = n_P\, 2\pi$. Similarly $\Delta\theta_{ABCDA} = 0$. By the construction of ABCD again $\Delta\theta_{AB} = 0$ implies $\Delta\theta_{BCDA} = 0$. Hence by (2.55) again $\Delta\theta_{BCDA} \simeq 0$ such that $\Delta\theta_{AB} \simeq n_P\, 2\pi$. Thus for ω_n, $n > 0$, sufficiently large, because $\theta(j\omega) - \theta(j0) = \theta(j0) = \theta(-j\omega)$ and (2.29): $\theta(j\omega_n) \simeq \theta(j0) + n_P\, \pi$. This implies by (2.30) and (2.31) $\lim_{\omega\to\infty} [\theta(j\omega) - \phi(j\omega)] = \theta(j0) - \theta(j0) + n_P\, \pi$ which because of (2.55) implies (2.54). $\bar{\underline{X}}$

In order to establish a graphical test it is interesting to observe that because of (2.52), the validity of condition (2.54) can be determined in principle by considering $1 + \hat{g}(j\omega)$ and $\hat{f}(j\omega)$ only over a <u>finite interval</u>. Moreover given the neighborhood $N(\varepsilon)$, defined by (2.46), it follows by the asymptotic and symmetry-properties that

(a) given any $\varepsilon > 0$, there exists $\Omega(\varepsilon)$ such that

(2.61) $$\omega > \Omega(\varepsilon) \Rightarrow |1 + \hat{g}(j\omega) - \hat{f}(j\omega)| \leq \varepsilon$$

$$\Rightarrow 1 + \hat{g}(j\omega) \in N(2\varepsilon), \text{ where}$$

(2.61a) $$N(2\varepsilon) \triangleq \{x \in \mathbb{C} \mid |x - \hat{f}(j\omega)| \leq 2\varepsilon;\ \omega \in [0,\ell(\varepsilon)]\}$$

(b) $$\inf_{\omega \in \mathbb{R}} |1 + \hat{g}(j\omega)| > 0$$

$\Leftrightarrow$ the origin 0 of the complex plane is positioned w.r.t. $\{1 + \hat{g}(j\omega) | \omega \in \mathbb{R}\}$ and $\{\hat{f}(j\omega) | \omega \in \mathbb{R}\}$ such that there exists an $\varepsilon > 0$ such that

i) 0 does not belong to $N(2\varepsilon)$

ii) 0 does not belong to $\{1 + \hat{g}(j\omega) | \omega \in [0,\Omega(\varepsilon)]\}$.

From this discussion, Theorem 2.2, Theorem 2.1 and Corollary 2.1 we conclude with the following graphical test:

Corollary 2.2 (Graphical test)

Given $\hat{g}(s)$ defined by (2.3) - (2.6). Let $\hat{f}(s)$ be the Dirichlet-series defined by (2.15) - (2.16). Let $\gamma(\varepsilon)$, $N(2\varepsilon)$, $\Omega(\varepsilon)$ be given by (2.34), (2.61a) and by (2.61). Under these conditions:

(2.9) $$\inf_{\mathrm{Re}\, s \geq 0} |1 + \hat{g}(s)| > 0$$

if and only if

i)

(2.35) $$f_0 = 1 + g_0 \neq 0$$

ii) the origin 0 of the complex plane is positioned with respect to $\{\hat{f}(j\omega) | \omega \in \mathbb{R}\}$ and $\{1 + \hat{g}(j\omega) | \omega \in \mathbb{R}\}$ such that

(a)

(2.62) $\left\{\begin{array}{l}\text{there exists an } \varepsilon > 0 \text{ such that 0 does not belong to } N(2\varepsilon) \\ \text{and } \{1 + \hat{g}(j\omega) | \omega \in [0,\Omega(\varepsilon)]\}\end{array}\right.$

(b)

(2.51) $\left\{\begin{array}{l}\text{for an } \varepsilon > 0 \text{ with } 0 < \varepsilon < K \triangleq \inf_{\omega \in \mathbb{R}} |\hat{f}(j\omega)|, \text{ for which} \\ \text{the corresponding } \varepsilon\text{-translation-number } \tau(\varepsilon) \text{ satisfies} \\ \text{(2.49), where } \lambda_{min} \text{ is defined by (2.47) - (2.48), then} \\ |\phi(j\tau) - \phi(j0)| < \pi \text{ must hold.}\end{array}\right.$

(c)

(2.54) $$\lim_{\omega \to \infty} [\theta(j\omega) - \phi(j\omega)] = \theta(j0) - \phi(j0) + n_p\, \pi = b\, 2\pi$$

where b is an integer. (Note that, by (2.17), n_p is the number of poles with positive real part).

Comment

a) It should be noted that the graphical test as given in Corollary 2.2 requires the knowledge a priori of the asymptotic part $\hat{f}(s)$ of $1 + \hat{g}(s)$. This however is the price we have to pay for admitting an almost periodic asymptote in the transfer function

b) J. C. Willems' conditions [14,15] are easily derived from Theorem 2.2, Corollary 2.2 and Remark (2.51b) if we take his assumptions i.e. $|g_0| < 1$; $\omega \longmapsto \hat{f}(j\omega)$ is periodic with period ω_0 and $\hat{g} \in \hat{\mathcal{A}}$. Then

$$\inf_{\mathrm{Re}\, s \geq 0} |1 + \hat{g}(s)| > 0$$

if and only if

i) $\inf_{\omega \in \mathbb{R}} |1 + \hat{g}(j\omega)| > 0$

ii) $\lim_{n\to\infty} \theta(jn\,\omega_0) = \theta(j0) = 0$ for $n = 0, 1, 2, \ldots$.

2.2.4 Implications for the n-input, n-output case

The aim of this paragraph is to show that the above theory allows us also to check the conditon

$$\inf_{\mathrm{Re}\, s \geq 0} |\det[I + \hat{G}(s)]| > 0 \tag{2.63}$$

when $\hat{G}(s)$ is the transfer function of a real n-input n-output convolution feedback system consisting of a term in $\hat{\mathcal{A}}^{n\times n}$ and a principal part due to a finite number of poles with <u>positive</u> real part, i.e.

$$\hat{G}(s) = \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} R_{km}(s - p_k)^{-m_k+m} + \hat{G}_r(s) \tag{2.64}$$

where

$$\begin{cases} G_r \text{ belongs to } \mathcal{A}^{n\times n} \\ \text{Re } p_k > 0 \quad \text{for } k = 1, 2, \ldots, \ell; \\ \text{the poles } p_k \text{ are real or pairwise conjugate complex;} \\ \text{the matrices } R_{km} \text{ are real or pairwise conjugate} \\ \text{complex } n\times n \text{ matrices according to the poles.} \end{cases}$$

Following the theory of the decomposition-Lemma A.2 in the Appendix and its subsequent corrolaries we finally get from Corollary A.2.4 after adding and subtracting 1 on the right hand side of (A.15) and regrouping (observe that $1 \in \hat{\mathcal{A}}$):

$$\det[I + \hat{G}(s)] = 1 + \hat{g}_r(s) + \sum_{k=1}^{\ell'} \sum_{m=0}^{m'_k-1} r_{km}(s - p_k)^{-m'_k+m} \tag{2.65}$$

where

$$\begin{cases} g_r \in \mathcal{A}; \\ \ell' \leq \ell; \\ m'_k \text{ is the order of the pole at } p_k \text{ of } \det[I + \hat{G}(s)], \\ \text{thus } r_{k0} \neq 0 \text{ for } k = 1, 2, \ldots, \ell; \\ \text{the coefficients } r_{km} \text{ are either real or conjugate} \\ \text{complex constants according to the corresponding poles.} \end{cases}$$

Observe that with $\hat{g}(s) \triangleq \hat{g}_r(s) + \sum_{k=1}^{\ell'} \sum_{m=0}^{m'_k-1} r_{km}(s-p_k)^{-m'_k+m}$

we get completely the same structure as in (2.3) - (2.6) and thus checking (2.63) is the same as checking (2.9) such that all the results of the previous paragraphs are applicable.

2.3 Graphical Test for the Discrete-time Case

2.3.1 Description of the Systems

We consider a discrete-time scalar linear time-invariant system with input u, error e and output y. The latter are sequences mapping $\mathbb{Z}_+$ into $\mathbb{R}$ and satisfy

(2.1') $\quad y = g*e$

(2.2') $\quad e = u-y$

where g is specified by a sequence of real numbers $\{g_i\}_{i=0}^{\infty}$ and (2.1') is equivalent to $y_i = \sum_{j=1}^{i} g_{i-j}\, e_j$ for $i = 0, 1, \ldots$.

As will become apparent there is no loss of generality in assuming a unity feedback. Let $\tilde{g}$ denote the z-transform of g. We assume that $\tilde{g}$ has following structure

(2.3') $$\tilde{g}(z) = \tilde{g}_r(z) + \sum_{k=1}^{\ell} \sum_{m=0}^{m_{k-1}} r_{km}(z - p_k)^{-m_k+m}$$

where

(2.4') $\quad g_r \in \ell^1;$

(2.5') $\begin{cases} \text{the poles } p_k \text{ are either real with real coefficients } r_{km} \\ \text{or conjugate complex with complex conjugate coefficients } r_{km}; \end{cases}$

(2.6') $|p_k| \geq 1$ for $k = 1, 2, \ldots, \ell$.

Note that because of (1.13') and (2.4')

(2.7') $\begin{cases} \tilde{g}_r(\cdot) \text{ is analytic in } |z| > 1, \text{ bounded on } |z| \geq 1, \\ \text{each function } \gamma \longmapsto \tilde{g}_r(\rho\, e^{j\gamma}) \text{ is uniformly continuous on} \\ [0,2\pi] \text{ (after setting } z = \rho\, e^{j\gamma}) \text{ and } \lim\limits_{|z|\to\infty} \tilde{g}(z) = g_0 = \text{constant} \end{cases}$

It follows therefore that

(2.8') $\begin{cases} \hat{g}(\cdot) \text{ is meromorphic in } |z| > 1, \text{ well defined and} \\ \text{continuous almost everywhere in } |z| \geq 1. \end{cases}$

A necessary and sufficient condition that the closed-loop impulse response h of the system (2.1') - (2.6') is in ℓ^1 (and thus stable as defined by remark (1.17')) is

(2.9') $$\inf_{|z| \geq 1} |1 + \tilde{g}(z)| > 0$$

For a proof see the appendix, lemma A.1'.

The problem is to develop a graphical test for (2.9') based on the closed path [27] $\{1 + \tilde{g}(z);\ z = e^{j\gamma};\ \gamma \in [0,2\pi]\}$. Observe that

(2.10') $$\lim_{|z|\to\infty} 1 + \tilde{g}(z) = 1 + g_0 = \text{constant.}$$

Thus here the asymptote of $1 + \tilde{g}(z)$ for $|z| \to \infty$ is <u>constant</u>

and therefore an investigation of the asymptotic case will not be necessary here.

As a last remark let

(2.11') $n_p \triangleq$ the number of poles of $\tilde{g}(z)$ counting multiplicities with $|p_k| > 1$.

<u>2.3.2 A Necessary and Sufficient Condition involving $\{1 + \tilde{g}(z);\ z = e^{j\gamma};\ \gamma \in [0,2\pi]\}$ to insure $\inf\limits_{|z| \geq 1} |1 + \tilde{g}(z)| > 0$</u>

<u>Definition of the argument $\theta(z)$ of $1 + \tilde{g}(z)$ subject to (2.3') - (2.6')</u>

By definition

(2.12') $$\theta(z) = \arg[1 + \tilde{g}(z)] = \operatorname{Im} \log[1 + \tilde{g}(z)] \quad \text{for } |z| \geq 1$$

with two additional conventions,

(2.12a') <u>Convention</u>. Let C denote a path [27] in $|z| \geq 1$. By convention we take $\theta(z)$, $z \in C$, as the right argument of $1 + \tilde{g}(z)$ on C, i.e. $\theta(z)$, $z \in C$, is an arbitrary branch of the argument, which is continuous except at the zeros and poles of $1 + \tilde{g}(z)$ on C, while it is discontinuous with a jump of $+m\pi$ $(-m_k\pi)$ on C, when s passes, in the positive direction on C, a zero (pole) of $1 + \tilde{g}(z)$ of order $m(m_k)$. At a discontinuity point we assign to θ the mean value of its one-sided limits. The function $\theta(z)$, $z \in C$ is then well defined (mod. 2π) because of (2.8').

(2.12b') <u>Convention</u>. Because $1 + \tilde{g}(z)$ is real for $z = \rho$, ρ real and $|\rho| \geq 1$, and meromorphic in $|z| > 1$, there exists an interval $(1,\rho^*)$ on $z = \rho$, ρ positive real and $|\rho| \geq 1$, on which $1 + \tilde{g}(\rho)$ is real, finite and different from zero. We pick for $\gamma \longmapsto \theta(e^{j\gamma})$, $\gamma \in [0,2\pi]$, that branch of the argument such that $\theta(e^{j0}) \triangleq 0$ (or π) according as $1 + \tilde{g}(\rho)$ is positive or negative on $(1,\rho^*)$.

Theorem 2.2'

Given $\tilde{g}(z)$ defined by (2.3') - (2.6') and let n_P be given by (2.11').

Under these conditions:

(2.9') $$\inf_{|z| \geq 1} |1 + \tilde{g}(z)| > 0$$

if and only if

i)

(2.13') $$\lim_{|z|\to\infty} 1 + \tilde{g}(z) = 1 + g_0 = \text{constant} \neq 0 \ ,$$

ii)

(2.14') $$\inf_{\theta \in [0,2\pi]} |1 + \tilde{g}(e^{j\theta})| > 0 \ ,$$

iii)

(2.15') $$\theta(e^{j2\pi}) - \theta(e^{j0}) = n_P \, 2\pi \ .$$

Proof

(a) $\Leftarrow$. Observe that because of (2.13') there exists a positive number $\rho^* > 1$ such that:

$1 + \tilde{g}(z)$ has no poles in $|z| \geq \rho^*$;

$1 + \tilde{g}(z)$ is uniformly continuous in $|z| \geq \rho^*$ and

(2.16') $$\inf_{|z| \geq \rho^*} |1 + \tilde{g}(z)| > 0.$$

It follows therefore that $\theta(z)$ as given by (2.12') is well defined (mod. 2π) and uniformly continuous in $|z| \geq \rho_1$ hence $\lim_{|z|\to\infty} \theta(z) = \theta_\infty$ = constant where

(2.17') $\qquad \theta_\infty \triangleq \arg(1 + g_0) \pmod{2\pi}$

and there exists a positive number ρ_1 such that

(2.18') $\qquad \rho_1 \geq \rho^*$

and

(2.19') $\qquad |\theta(\rho_1 e^{j\gamma}) - \theta_\infty| < 1 \quad$ for all $\gamma \in [0,2\pi]$.

Observe now that because of (2.18') and (2.16') we will have established (2.9') if we show that $\inf_{1 \leq |z| \leq \rho_1} |1 + g(z)| > 0$ or equivalently

(2.20') $\qquad 1 + \tilde{g}(z) \neq 0 \quad$ in $1 \leq |z| \leq \rho_1$

because the closed annulus $1 \leq |z| \leq \rho^*$ is compact. Observe further that because of (2.8') and convention (2.12a') the principle of the argument can be applied to the closed annulus $1 \leq |z| \leq \rho_1$ resulting in

(2.21') $\qquad \theta(e^{j2\pi}) - \theta(e^{j0}) - [\theta(\rho_1\, e^{j2\pi}) - \theta(\rho_1\, e^{j0})] = (n_p - n_z)2\pi$

where n_p is as given in (2.11') and n_z is the number of zeros of $1 + \tilde{g}(z)$ in the interior of $1 \leq |z| \leq \rho_1$). Note that on $|z| = 1$ ($|z| = \rho_1$) the positive direction is the counterclockwise (clockwise) sense and that the annulus $1 \leq |z| \leq \rho_1$ can be converted in a simply connected domain by making a cut along any radius $z = \rho\, e^{j\gamma}$, γ = constant and $\gamma \in [0,2\pi]$. Note also that because of convention (2.12b') $z \longmapsto \theta(z)$ is well defined on $|z| = 1$. Finally by (2.21') and (2.15') we obtain that $n_z\, 2\pi = \theta(\rho_1\, e^{j2\pi}) - \theta(\rho_1\, e^{j0})$, hence by (2.17') and (2.19') $|n_z\, 2\pi| < 2$ which because n_z is an integer implies $n_z = 0$. Hence $1 + \tilde{g}(z) \neq 0$ in the interior of $1 \leq z \leq \rho_1$. In addition because of (2.16'), (2.18') and (2.14') it follows that (2.20') is true and hence we are done. $\bar{\underline{X}}$

(b) $\Rightarrow$. Immediately (2.9') implies (2.14'). Next observe that (2.9') and (2.10') implies (2.13') such that by analog reasoning as in (a) there exist a positive number $\rho_1 > 1$ such that (2.19') with (2.17') is true.

Therefore on $|z| = \rho_1$

$$|\theta(\rho_1\, e^{j2\pi}) - \theta(\rho_1\, e^{j0})| < 2. \tag{2.22'}$$

Applying again the principle of the argument to the annulus $1 \leq |z| \leq \rho_1$ we obtain (2.21') where however $n_Z = 0$. Therefore along with (2.22')

$$\theta(e^{j2\pi}) - \theta(e^{j0}) \in (n_p\, 2\pi - 2,\ n_p\, 2\pi + 2). \tag{2.23'}$$

Observe now that $\theta(e^{j2\pi})$ and $\theta(e^{j0})$ can only differ by an integral multiple of 2π because they are arguments of the same complex number $1 + \tilde{g}(1)$. Therefore (2.23') implies (2.15') and we are done $\bar{\underline{X}}$

Comment

Observe the simplicity of the proof and note that this is caused precisely because of (2.10'). Indeed in the continuous-time case an analog result as (2.10') is valid for $1 + \hat{g}(s)$ when Re $s = \sigma$ tends to $+\infty$, however $1 + \hat{g}(s)$ does not alway tend towards a constant as $|s| \to \infty$ in Re $s \geq 0$ which causes precisely the difficulties encountered in paragraph 2.2.

We state now the graphical test for the discrete-time case which is now an easy translation of Theorem 2.2'.

Corollary 2.2'

Given $\tilde{g}(z)$ defined by (2.3') - (2.6') and let n_p be defined by (2.11').

Under these conditions:

(2.9') $$\inf_{|z| \geq 1} |1 + \tilde{g}(z)| > 0$$

if and only if

i)

(2.13') $$\lim_{|z| \to \infty} 1 + \tilde{g}(z) = 1 + g_0 = \text{constant} \neq 0 \ ,$$

ii) the origin 0 of the complex plane is positioned with respect to the closed path $\{1 + \tilde{g}(z);\ z = e^{j\gamma},\ \gamma \in [0,2\pi]\}$ such that

(a)

(2.23') 0 does not belong to the closed path $\{1 + \tilde{g}(z);\ z = e^{j\gamma};$ $\gamma \in [0,2\pi]\}$

(b)

(2.24') $\begin{cases} \text{the closed path } \{1 + \tilde{g}(z);\ z = e^{j\gamma},\ \gamma \in [0,2\pi]\} \\ \text{encircles 0 exactly } n_p \text{ times in counterclockwise} \\ \text{sense when } \gamma \text{ increases from 0 to } 2\pi. \end{cases}$

2.3.3 Implications for the n-input, n-output Case.

The aim of this paragraph is to show that the above theory allows us to check the condition

$$\inf_{|z| \geq 1} |\det[I + \tilde{G}(z)]| > 0 \tag{2.25'}$$

when $\tilde{G}(z)$ is the transfer function of a real n-input n-output convolution feedback system consisting of a term in $\tilde{\ell}^1_{n\times n}$ and a principal part due to a finite number of poles with absolute value larger than one, i.e.

$$\tilde{G}(z) = \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} R_{km}(z - p_k)^{-m_k+m} + \tilde{G}_r(z) \tag{2.26'}$$

where

$\begin{cases} G_r \text{ belongs to } \ell^1_{n\times n}; \\ |p_k| > 1 \text{ for } k = 1, 2, \ldots, \ ; \\ \text{the poles } p_k \text{ are real or pairwise conjugate complex;} \\ \text{the matrices } R_{km} \text{ are real or pairwise conjugate complex} \\ n\times n \text{ matrices according to the poles.} \end{cases}$

Following the theory of decomposition Lemma A.2' and Remark (A.12b') in the appendix we are able to rewrite $\det[I + \tilde{G}(z)]$

in the following form:

$$\det[I + \tilde{G}(z)] = 1 + \tilde{g}_r(z) + \sum_{k=1}^{\ell'} \sum_{m=0}^{m_k'-1} r_{km}(z - p_k)^{-m_k'+m} \tag{2.27'}$$

where

$$\begin{cases} g_r \in \ell^1; \\ \ell' \leq \ell\ ; \\ m_k' \text{ is the order of the pole at } p_k \text{ of } \det[I + \tilde{G}(z)], \\ \text{thus } r_{k0} \neq 0 \text{ for } k = 1, 2, \ldots, \ell; \\ \text{the coefficients } r_{km} \text{ are either real or complex} \\ \text{conjugate constants according to the corresponding poles.} \end{cases}$$

Observe that with $\tilde{g}(z) = \tilde{g}_r(z) + \sum\limits_{k=1}^{\ell} \sum\limits_{m=0}^{m_k'-1} r_{km}(z - p_k)^{-m_k'+m}$

we get completely the same structure as in (2.3') - (2.6') and thus checking (2.27') is the same as checking (2.9') such that all the previous results of paragraph 2.3.2 are applicable.

3. CONTINUOUS-TIME N-INPUT N-OUTPUT CONVOLUTION FEEDBACK SYSTEMS.

3.1 Introduction

This section considers continuous-time feedback systems with n inputs and n outputs as described in paragraph 1.1.1.

First the relation between the open-loop operator $\underset{\sim}{G}$ and closed-loop operator $\underset{\sim}{H}$ is discussed. (a) In Theorem 3.1 below we prove that, under very mild assumptions on the open-loop impulse response G and on the closed-loop system, if the closed-loop impulse response $H \in \mathcal{A}^{n\times n}$ then $\hat{G}$ is of the form

$$\hat{G}(s) = \hat{P}(s)[\hat{Q}(s)]^{-1} \tag{3.1}$$

where $\hat{P}$, $\hat{Q} \in \hat{\mathcal{A}}^{n\times n}$. Thus we show the importance of systems given by (1.1) - (1.2) and (3.1) which is the class of systems introduced by M. Vidyasagar [21]. Theorem 3.1 is also an extension of a result of Nasburg and Baker [22]: the extension is in two directions, first, the n-input n-output case is considered and, second, the requirements on G are greatly relaxed. (b) Theorem 3.2 is a straightforward extension of a result of [22]: it shows again the importance of systems introduced by M. M. Vidyasagar in that $\hat{H} \in \hat{\mathcal{A}}^{n\times n}$ if and only if $\hat{G}$ is of the form (3.1) and $\inf_{\text{Re } s \geq 0} |\det[\hat{P}(s) + \hat{Q}(s)]| > 0$.

Next necessary and sufficient conditions for stability are discussed when $\hat{G}$ is of the form (3.1) with a finite number of poles in Re $s > 0$. (a) Theorem 3.3 gives these conditions for a higher order pole in Re $s > 0$ and consecutive remarks take care of the multiple pole case. (b) Theorem 3.4 enables an interpretation of these condtions and a simpler formulation of the multiple pole case which is stated as Theorem 3.5. These theorems extend results of Desoer, Wu, Lam and Chen [1,2,10,19,20].

3.2 The Relation Between $\underset{\sim}{G}$ and $\underset{\sim}{H}$

Theorem 3.1

Let G be an $n\times n$ matrix whose elements are distributions with support on $\mathbb{R}_+$. Suppose that in a neighborhood of the origin, say $V \subset \mathbb{R}$, G includes at most δ-functions (i.e. on V, it is a distribution of at most order 0). For the system defined by (1.1) and (1.2), assume that the closed-loop impulse response H exists and is uniquely defined by

$$H + G*H = G. \tag{3.2}$$

Under these conditions, if $H \in \mathcal{A}^{n\times n}$, then

(a) G is Laplace-transformable and for some finite $\sigma \geq 0$ $e^{-\sigma t}\, G \in \mathcal{A}^{n\times n}$ (i.e. the product of each element of G with $e^{-\sigma t}$ belongs to $\mathcal{A}$) ;

(b) $\hat{G}$ is of the form

$$\hat{G}(s) = \hat{P}(s)[\hat{Q}(s)]^{-1} \quad \text{for Re } s > 0 \tag{3.1}$$

where $\hat{P}(\cdot)$ and $\hat{Q}(\cdot) \in \hat{\mathcal{A}}^{n\times n}$;

(c) $\hat{G}$ can at most have a countable number of poles in the vertical strip $0 < \text{Re } s \leq \sigma$, and has no poles in Re $s > \sigma$.

Comment. This theorem shows that under mild conditions on G regarding its behavior near $t = 0$, once the closed-loop system is well defined and stable, then $\hat{G}$ is necessarily of the form (3.1), can at most have poles in the strip $0 < \mathrm{Re}\, s \leq \sigma$ and is analytic for $\mathrm{Re}\, s > \sigma$.

Proof

(a) By assumption, $H \in \mathcal{A}^{n\times n}$, i.e.

$$H(t) \begin{cases} = H_a(t) + \sum_{i=0}^{\infty} H_i \delta(t-t_i) & \text{for } t \geq 0 \\ = 0 & \text{for } t < 0 \end{cases}$$

where $H_a(\cdot) \in L^1_{n\times n}[0,\infty)$, $H_i \in \mathbb{R}^{n\times n}$ for $i = 0, 1, 2, \ldots$ and $0 = t_0 < t_1 < t_2 < \ldots$. By assumption G can at most have an impulse at the origin. By the Abelian Theorem of the Laplace-transform [29] and the properties of distributions, if G has an impulse G_0 at $t = 0$, $\hat{G}(s) \to G_0$ as $\mathrm{Re}\, s \to \infty$. Clearly from (3.2), if G_0 is the zero matrix, then $H_0 = 0$. If $G_0 \neq 0$, then by balancing impulses at the origin in (3.2) we have $(I + G_0)H_0 = G_0$. By assumption H, hence H_0, is uniquely defined by (3.2) hence $\det(I + G_0) \neq 0$. Furthermore by direct calculation $(I + G_0)(I - H_0) = I$ so that $\det[I - H_0] \neq 0$.

The function $I - \hat{H}(s)$ is analytic and bounded for $\mathrm{Re}\, s > 0$, continuous on $\mathrm{Re}\, s = 0$, and tends to $I - H_0$ as $\mathrm{Re}\, s \to \infty$. Consequently, there exists a $\sigma \geq 0$ such that

$$\text{(3.3)} \qquad \inf_{\mathrm{Re}\, s \geq \sigma} \det[I - \hat{H}(s)]| > 0.$$

Next observe that $e^{-\sigma\cdot}\, H(\cdot)$ and $e^{-\sigma\cdot}\,[I\delta(\cdot) - H(\cdot)] \in \mathcal{A}^{n\times n}$ and that $\widehat{(e^{-\sigma\cdot}\, H(\cdot))}(s) = \hat{H}(s+\sigma)$ for Re $s \geq 0$ and

$$\widehat{\{e^{-\sigma\cdot}\,[I\delta(\cdot) - H(\cdot)]\}}\,(s) = I - \hat{H}(s+\sigma) \text{ for Re } s \geq 0.$$

Therefore by (3.3) and (1.15) $[I - \hat{H}(\cdot + \sigma)]^{-1}$, for Re $s \geq 0, \in \hat{\mathcal{A}}^{n\times n}$. Finally

$$\hat{G}^*(\cdot + \sigma) \triangleq \hat{H}(\cdot + \sigma)\,[I - \hat{H}(\cdot + \sigma)]^{-1}, \quad \text{for Re } s \geq 0, \in \mathcal{A}^{n\times n}. \tag{3.4}$$

Next from (3.2)

$$e^{-\sigma t}\, H + e^{-\sigma t}\, G * e^{-\sigma t}\, H = e^{-\sigma t}\, G \tag{3.5}$$

such that if G has a Laplace-transform

$$\hat{G}(\cdot + \sigma) = \hat{H}(\cdot + \sigma) + \hat{G}(\cdot + \sigma)\,\hat{H}(\cdot + \sigma). \tag{3.6}$$

Now observe that $\hat{G}^*(\cdot + \sigma)$ given by (3.4) satisfies (3.6). Because all terms are in $\hat{\mathcal{A}}^{n\times n}$ an inverse Laplace transform of Eq. (3.6) where $\hat{G}(\cdot + \sigma) = \hat{G}^*(\cdot + \sigma)$ may be performed. Therefore $e^{-\sigma t}\, G^*$ satisfies (3.5) and so by the uniqueness implied by the convolution algebra of distributions on $\mathbb{R}_+$

$$\hat{G}(\cdot + \sigma) = \hat{H}(\cdot + \sigma)\,[I - \hat{H}(\cdot + \sigma)]^{-1} \text{ for Re } s \geq 0$$

and $\hat{G}(\cdot + \sigma) \in \hat{\mathcal{A}}^{n\times n}$.

Thus $e^{-\sigma t}\, G \in \mathcal{A}^{n\times n}$ and $\hat{G}(\cdot) = \hat{H}(\cdot)\,[I - \hat{H}(\cdot)]^{-1}$ for Re $s \geq \sigma$. This proves (a).

b) Since by (1.13) $\hat{H}(\cdot)$ and $[I - \hat{H}(\cdot + \sigma)]^{-1}$ are analytic for Re $s > 0$, $[I - H(\cdot)]^{-1}$ has at most a countable number of poles in the strip $0 <$ Re $s \leq \sigma$ and by analytic continuation

$$\hat{G}(\cdot) = \hat{H}(\cdot)\,[I - \hat{H}(\cdot)]^{-1} \text{ for Re } s > 0. \tag{3.7}$$

Choose $\hat{P}(\cdot) = \hat{H}(\cdot)$, $\hat{Q} = [I - \hat{H}(\cdot)]$. Thus (b) and (c) have been established. $\underline{\bar{X}}$

Remarks

(3.7a) Remark. It is important to reflect on the fact that under the conditions of Theorem 3.1, we have

$$[I + \hat{G}(\cdot)][I - \hat{H}(\cdot)] = I \text{ for Re } s > 0$$

This expression emphasizes the symmetrical role played by $\underset{\sim}{H}$ and $\underset{\sim}{G}$: $\underset{\sim}{H}$ is obtained from $\underset{\sim}{G}$ by a negative feedback of I; $\underset{\sim}{G}$ is obtained from $\underset{\sim}{H}$ by a negative feedback of (-I) (to cancel the preceeding one!).

(3.7b) Remark. A little more can be said about the poles of $\hat{G}(\cdot)$:

$$\hat{G}(\cdot) = \hat{P}(\cdot)[\hat{Q}(\cdot)]^{-1} = \hat{P}(\cdot)\ \mathrm{Adj}[\hat{Q}(\cdot)]/\det \hat{Q}(\cdot).$$

The function $\phi : s \longmapsto \det \hat{Q}(s) \triangleq \det[I - \hat{H}(s)]$ is analytic and bounded in Re $s > 0$ and because of (3.3) $\inf_{\text{Re } s \geq \sigma} |\det Q(s)| > 0$.

Therefore ϕ has at most a countable number of zeros p_k for $k = 1,2,\ldots$ in the strip $0 <$ Re $s \leq \sigma$. Moreover by a theorem of [30, p. 457]

$\displaystyle\sum_{k=1}^{\infty} \frac{\text{Re } p_k}{1 + |p_k|^2} < \infty.$ Therefore $G(\cdot)$ either has a finite number of poles p_k in the strip $0 < \text{Re } s \leq \sigma$ or else it has an infinite sequence of them in the strip $0 \quad \text{Re } s \leq \sigma$ such that they accumulate on the imaginary axis {or and } at the point $|\text{Im } s| = \infty$ (i.e. they "shoot at infinity" along a vertical line in the strip $0 < \text{Re } s \leq \sigma$).

<u>Theorem 3.2</u>

Let G be an $n \times n$ matrix whose elements are Laplace-transformable distributions with support on $\mathbb{R}_+$. For the system defined by (1.1) - (1.2), assume that the closed-loop transfer function $\hat{H}$ is well defined for almost all s in the half plane of convergence of $\hat{G}(\cdot)$, i.e.

$$\hat{H}(s) = \hat{G}(s)[I + \hat{G}(s)]^{-1} \tag{3.8}$$

for almost all s in the half plane of convergence of $\hat{G}(\cdot)$. Under these conditions,

$$H \in \mathcal{A}^{n\times n} \tag{3.9}$$

if and only if there exists $\hat{P}, \hat{Q} \in \hat{\mathcal{A}}^{n\times n}$ such that

$$\hat{G}(s) = \hat{P}(s)\ [\hat{Q}(s)]^{-1} \tag{3.10}$$

and

$$\inf_{\text{Re } s \geq 0} |\det[\hat{P}(s) + \hat{Q}(s)]| > 0. \tag{3.11}$$

<u>Proof</u>

$\Rightarrow$. From (3.8) - (3.9) by algebra

$$\hat{G}(s) = \hat{H}(s)\ [I - \hat{H}(s)]^{-1} \text{ for Re } s \geq 0.$$

Choose $\hat{P} = \hat{H}$ and $\hat{Q} = I - \hat{H}$. Hence by (3.9) $\hat{P}$ and $\hat{Q} \in \hat{\mathcal{A}}^{n\times n}$ and (3.10) follows. Finally, since $\hat{P} + \hat{Q} = I$ (3.11) holds.

$\Leftarrow$. From (3.8) and (3.10)

$$\hat{H}(s) = \hat{P}(s)\ [\hat{P}(s) + \hat{Q}(s)]^{-1}.$$

In view of (1.15) and (3.11) $\hat{H} \in \hat{\mathcal{A}}^{n\times n}$ as the product of two elements of $\hat{\mathcal{A}}^{n\times n}$. $\underline{\bar{X}}$

Remarks

(3.11a) Remark. It is clear from (3.10) that a given $\hat{G}$ does not define the ordered pair $(\hat{P}, \hat{Q})$ uniquely; for example, they might have a right common factor. In order to be able to express the condition (3.11) in a form which depends on $\hat{G}$ only, we impose the Vidyasagar no-cancellation condition (N) [21]:

(N) $\left\{\right.$ the ordered pair (a,b) where $a, b: \mathbb{C} \longmapsto \mathbb{C}$ is said to satisfy the no-cancellation on a set $A \subset \mathbb{C}$ iff, for all sequences $\{s_k\}$ in A, $a(s_k) \to 0$ implies that $\liminf |b(s_k)| > 0$.

It is then easy to show that, [5], if $(\det \hat{Q}(s), \det[\hat{P}(s) + \hat{Q}(s)])$ satisfies (N) on Re $s \geq \sigma$, then (3.11) is equivalent to

$$\inf_{\text{Re } s \geq 0} |\det[I + \hat{G}(s)]| > 0.$$

(3.11b) Remark. Observe that (3.11) can always be tested graphically by the method described in paragraph 2.2 setting $\hat{g}(s) \triangleq \det[\hat{P}(s) + \hat{Q}(s)] - 1$ which is in $\hat{\mathcal{A}}$.

3.3 Necessary and Sufficient Conditions for Stability.

By stability we mean stability as defined in Remark (1.17)

In this paragraph we consider systems of the form (1.1) - (1.2) where the open-loop transfer function $\hat{G}$ is given by

$$(3.12) \qquad \hat{G}(s) = \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} R_{km}(s-p_k)^{-m_k+m} + \hat{G}_r(s)$$

where (a) the poles p_k and the corresponding matrices R_{km} are real or pairwise complex conjugate for $k = 1, \ldots, \ell$ and $m = 0, 1, \ldots, m_k-1$, (b) Re $p_k > 0$ for $k = 1, 2, \ldots \ell$ and (c) $\hat{G}_r \in \hat{\mathcal{A}}^{n\times n}$.

Observe that if

$$(3.13) \qquad \hat{G}(s) = \hat{P}_1(s)\left(\prod_{k=1}^{\ell} (s - p_k)^{m_k'} I\right)^{-1}$$

where $\hat{P}_1 \in \hat{\mathcal{A}}^{n\times n}$, I is the unit n×n matrix and m_k' are integers larger than or equal to m_k for $k = 1, 2, \ldots, \ell$, then (3.13) can be brought in the form (3.12). This follows from Corollary A.2.3 in the appendix. Furthermore observe that $\hat{G}(s)$ as given by (3.13) can be rewritten in the form.

$$(3.14) \qquad \hat{G}(s) = \left[\prod_{k=1}^{\ell} (s+1)^{-m_k'} \hat{P}_1(s)\right]\left[\prod_{k=1}^{\ell} \left(\frac{s - p_k}{s + 1}\right)^{m_k'} I\right]^{-1}$$

Hence, since $\left(\frac{1}{s+1}\right)$ and $\left(\frac{s - p_k}{s + 1}\right)$ for $k = 1, 2, \ldots, \ell \in \hat{\mathcal{A}}$ we obtain that the "numerator" and "denominator" on the R.H.S. of (3.14) are in $\hat{\mathcal{A}}^{n\times n}$, so $\hat{G}(s)$ as given by (3.14) is of the form (3.1). This establishes a link with previous paragraphs in that $\hat{G}$ as given by (3.12) can be derived from a form (3.1) where

$\hat{G}$ has a finite number of poles in Re s > 0.

We consider now first and in detail the case where $\hat{G}$ has a real pole of order m in Re s > 0. The extension to the case of a finite number of poles will be done in subsequent remarks. We consider thus the open-loop transfer function $\hat{G}$ defined by

$$\hat{G}(s) = \sum_{i=0}^{m-1} R_i(s - p)^{-m+i} + \hat{G}_r(s) \tag{3.15}$$

where $p \in \mathbb{R}$, $p > 0$, $\hat{G}_r \in \hat{\mathcal{A}}^{n\times n}$, $r_0 \triangleq$ rank of $R_0 \leq n$ and R_i ($i = 0, 1, \ldots, m-1$) are $n\times n$ matrices with real coefficients.

We start by pointing out some facts which will streamline the proof of Theorem 3.3.

(3.15a) Fact

Let

$$\hat{R}^*\left(\frac{1}{s+1}\right) \triangleq \left(\sum_{i=0}^{m-1} R_i(s-p)^{-m+i}\right)\left(\frac{s-p}{s+1}\right)^m \tag{3.16}$$

then $\hat{R}^*\left(\frac{1}{s+1}\right)$ is an $n\times n$ complex polynomial matrix in $\left(\frac{1}{s+1}\right)$ of degree m. This is obvious by considering the Laurent expression of $\hat{R}^*\left(\frac{1}{s+1}\right)$ about $s = -1$.

(3.16a) Fact. (Smith Canonical form [31]).

For the $n\times n$ polynomial matrix $R^*\left(\frac{1}{s+1}\right)$ there exist unimodular (i.e. with nonzero constant determinant) polynomial matrices in $\left(\frac{1}{s+1}\right)$ viz. $\hat{S}\left(\frac{1}{s+1}\right)$ and $\hat{T}\left(\frac{1}{s+1}\right)$, such that

$$\hat{T}\left(\frac{1}{s+1}\right)\hat{R}^*\left(\frac{1}{s+1}\right)\hat{S}\left(\frac{1}{s+1}\right) = \tag{3.17}$$

$$\mathrm{diag}\{\underbrace{a_1\left(\frac{1}{s+1}\right),\ldots,\hat{a}_j\left(\frac{1}{s+1}\right),\ldots,\hat{a}_{r^*}\left(\frac{1}{s+1}\right)}_{r^*},\underbrace{0,0,\ldots,0}_{n-r^*}\}$$

where i) r^* = rank of $\hat{R}^*\left(\frac{1}{s+1}\right)$ = order of the largest minor of $\hat{R}^*\left(\frac{1}{s+1}\right)$ which is not equal to the zero polynomial;

ii) the $\hat{a}_j\left(\frac{1}{s+1}\right)$, $j = 1,2,\ldots\ r^*$ are the invariant polynomials of $\hat{R}^*\left(\frac{1}{s+1}\right)$ and each polynomial $\hat{a}_j(\cdot)$ divides $a_{j+1}(\cdot)$, $j = 1,2,\ldots, r^*-1$;

iii) the diagonal matrix in the R.H.S. of (3.17) can be obtained by elementary operations.

(3.17a) Fact.

The polynomial matrices $\hat{S}\left(\frac{1}{s+1}\right)$ and $\hat{T}\left(\frac{1}{s+1}\right) \in \hat{\mathcal{Q}}^{n\times n}$ and their inverses are polynomial matrices in $\left(\frac{1}{s+1}\right)$ also in $\hat{\mathcal{Q}}^{n\times n}$.

(3.17b) Fact.

Let $\hat{a}_j(\cdot)$, $j = 1,2,\ldots, r^*$ be as in (3.17) and let r_0 be the rank of R_0, then

(a)

$$\begin{cases} \hat{a}_j(1/(p+1)) = 0 \text{ for } r_0 + 1 \le j \le r^* \\ \hat{a}_j(1/(p+1)) \ne 0 \text{ for } 1 \le j \le r_0 \end{cases} \quad \text{by definition of } r_0; \tag{3.18}$$

$$\hat{a}_j\left(\frac{1}{s+1}\right) = \hat{b}_j\left(\frac{1}{s+1}\right)\left(\frac{s-p}{s+1}\right)^{c_j} \quad \text{for } r_0 + 1 \leq j \leq r^* \tag{3.19}$$

where c_j is the order of the zero of $\hat{a}_j(\cdot)$ at $s = p$;

$b_j(\cdot)$ is a polynomial with

$$\hat{b}_j(1/(p+1)) \neq 0, \text{ (see [32]), and} \tag{3.20}$$

$$1 \leq c_{r_0+1} \leq c_{r_0+2} \leq \cdots \leq c_{r^*}.$$

Proof

Set $s = p$ in (3.17) and note that the L.H.S. becomes $\hat{T}(1/(p+1))\ R_0(p+1)^{-m}\ \hat{S}(1/(p+1))$. Since $\hat{S}(\cdot)$ and $\hat{T}(\cdot)$ are unimodular, exactly $(r-r_0)$ polynomial $\hat{a}_j(\cdot)$ are zero at $s=p$. By ii) of (3.17) $\hat{a}_j(1/(p+1)) = 0$ for $r_0 + 1 \leq j \leq r^*$. Hence (3.18) and (3.19) follow with the properties of the latter as a consequence of ii) of (3.17). $\bar{\underline{X}}$

Note that the exponents c_j in (3.19) may, for some j, be larger than m (in fact $c_{r^*} \leq r^*m$).

Therefore, since the c_j are monotonically increasing and since $c_j - m$ may be of any sign, partition the index set K = $\{r_0+1, r_0+2, \ldots, r^*\}$ into

$$K_- = \{r_0+1, r_0+2, \ldots, \alpha\} = \{j \mid 1 \leq c_j < m\} \tag{3.21}$$

$$K_0 = \{\alpha+1, \alpha+2, \ldots, \beta\} = \{j \mid c_j = m\} \tag{3.22}$$

$$K_+ = \{\beta+1, \beta+2, \ldots, r^*\} = \{j \mid c_j > m\}\ . \tag{3.23}$$

We are now ready for Theorem 3.3.

Theorem 3.3

Consider the system defined by (1.1), (1.2) and (3.15).

Let $\hat{S}\left(\frac{1}{s+1}\right)$ and $\hat{T}\left(\frac{1}{s+1}\right)$ be the polynomial matrices defined in (3.17). Suppose that the index sets K_-, K_0, K_+ as defined in (3.21) - (3.23), are not empty.

Consider the partitioning

$$(3.24) \qquad \hat{T}\left(\frac{1}{s+1}\right)[I + \hat{G}_r(s)]\,\hat{S}\left(\frac{1}{s+1}\right) = \begin{array}{r} \\ \alpha\{ \\ n-\alpha\{ \end{array}\begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad}^{\alpha} & \overbrace{\qquad\qquad}^{n-\alpha} \end{array} \\ \left[\begin{array}{c|c} \hat{L}_{11}(s) & \hat{L}_{12}(s) \\ \hline \hat{L}_{21}(s) & \hat{L}_{22}(s) \end{array}\right] \end{array}$$

and let $\hat{b}_j(\cdot)$ be the polynomials defined in (3.19). Finally let H be the closed-loop impulse response of the system considered. Under these conditions

$$(3.25) \qquad H \in \hat{\mathcal{A}}^{n\times n}$$

if and only if

$$(3.26) \qquad \inf_{\mathrm{Re}\, s \geq 0} |\det[I + \hat{G}(s)] > 0$$

and

$$(C) \qquad \det\{\hat{L}_{22}(p) + \mathrm{diag}[\hat{b}_{\alpha+1}(1/(p+1)),\ldots,\hat{b}_{\beta}(1/(p+1)),0,0,\ldots,0]\} \neq 0.$$

Proof

$\Leftarrow$. Since $I - \hat{H}(s) = [I + G(s)]^{-1}$, we need only show that

$$(3.27) \qquad [I + \hat{G}(\cdot)]^{-1} \in \hat{\mathcal{A}}^{n\times n} .$$

By fact (3.17a), (3.27) is equivalent to

$$\left\{\hat{T}\left(\frac{1}{\cdot+1}\right)[I+\hat{G}(\cdot)]\ \hat{S}\left(\frac{1}{\cdot+1}\right)\right\}^{-1} \in \hat{\mathcal{A}}^{n\times n}\ .$$

Introduce now the following multiplier:

(3.28) $$\hat{M}(s) \triangleq$$

$$\mathrm{diag}\{\underbrace{\hat{z}(s)^m,\hat{z}(s)^m,\ldots,\hat{z}(s)^m}_{r_0},\underbrace{\hat{z}(s)^{m-c_{r_0+1}},\hat{z}(s)^{m-c_{r_0+2}},\ldots,\hat{z}(s)^{m-c_\alpha}}_{\alpha-r_0},$$

$$\underbrace{1,1,\ldots,1}_{n-\alpha}\}$$

with

(3.29) $$\hat{z}(s) \triangleq \frac{s-p}{s+1} \in \hat{\mathcal{A}}.$$

By (3.21) and (3.29)

(3.30) $$\hat{M}(\cdot) \in \hat{\mathcal{A}}^{n\times n}\ .$$

Remark that

$$\left\{\hat{T}\left(\frac{1}{s+1}\right)[I+\hat{G}(s)]\ \hat{S}\left(\frac{1}{s+1}\right)\right\}^{-1} = \hat{M}(s)\ \hat{N}(s)^{-1} \quad \text{where}$$

(3.31) $$\hat{N}(s) \triangleq \left\{\hat{T}\left(\frac{1}{s+1}\right)[I+\hat{G}(s)]\ \hat{S}\left(\frac{1}{s+1}\right)\right\}\hat{M}(s).$$

Clearly by (3.30) we are done if we can show that

$$\hat{N}(\cdot)^{-1} \in \hat{\mathcal{A}}^{n\times n}$$

Therefore by (1.15) we prove that $\hat{N}(\cdot) \in \hat{\mathcal{A}}^{n\times n}$ and

$$\inf_{\text{Re } s \geq 0} |\det \hat{N}(s)| > 0.$$

Rewrite (3.28), therefore

(3.32) $$\hat{M}(s) = \hat{z}(s)^{m}\,\hat{\Delta}(s)$$

where

(3.33) $$\hat{\Delta}(s) \triangleq$$

$$\text{diag}\{\underbrace{,1,\ldots,1}_{r_0},\underbrace{\hat{z}(s)^{-c_{r_0+1}},\hat{z}(s)^{-c_{r_0+2}},\ldots,\hat{z}(s)^{c_\alpha}}_{\alpha-r_0},$$

$$\underbrace{\hat{z}(s)^{-m},\hat{z}(s)^{-m},\ldots,\hat{z}(s)^{-m}}_{n-\alpha}\}\;.$$

By (3.31), (3.15), (3.32), (3.33), (3.29), (3.16), (3.17), (3.19) and (3.20), we obtain

(3.34) $$\hat{N}(s) = \hat{N}_1(s) + \hat{N}_2(s) \text{ where}$$

(a)

(3.35) $$\hat{N}_1(s) = \hat{D}_1(s) \oplus \hat{D}_2(s) \text{ with}$$

(3.36) $$\hat{D}_1(s) =$$

$$\text{diag}\{\underbrace{\hat{a}_1\left(\frac{1}{s+1}\right),\hat{a}_2\left(\frac{1}{s+1}\right),\ldots,\hat{a}_{r_0}\left(\frac{1}{s+1}\right)}_{r_0},\underbrace{\hat{b}_{r_0+1}\left(\frac{1}{s+1}\right),\hat{b}_{r_0+2}\left(\frac{1}{s+1}\right),\ldots,\hat{b}_{\alpha}\left(\frac{1}{s+1}\right)}_{\alpha-r_0}\}$$

(3.37) $\hat{D}_2(s) =$

$$\operatorname{diag}\{\underbrace{\hat{b}_{\alpha+1}\left(\frac{1}{s+1}\right),\hat{b}_{\alpha+2}\left(\frac{1}{s+1}\right),\ldots,\hat{b}_{\beta}\left(\frac{1}{s+1}\right)}_{\beta-\alpha},\ \underbrace{\hat{b}_{\beta+1}\left(\frac{1}{s+1}\right)\hat{z}(s)^{c_{\beta+1}-m},}$$

$$\underbrace{\hat{b}_{\beta+2}\left(\frac{1}{s+1}\right)\hat{z}(s)^{c_{\beta+2}-m},\ldots,\hat{b}_{r*}\left(\frac{1}{s+1}\right)\hat{z}(s)^{c_{r*}-m}}_{r*-\beta},\underbrace{0,0,\ldots,0}_{n-r*}\}$$

and (b)

(3.38) $$\hat{N}_2(s) = \hat{T}\left(\frac{1}{s+1}\right)[I + \hat{G}_r(s)]\ \hat{S}\left(\frac{1}{s+1}\right)\ \hat{M}(s).$$

Immediately

(3.39) $$\hat{N}(\cdot) \in \hat{\mathcal{A}}^{n\times n}\ .$$

$\hat{N}_1(\cdot) \in \hat{\mathcal{A}}^{n\times n}$ because all its elements $\in \hat{\mathcal{A}}$ (indeed all its nonzero elements are polynomials in $\left(\frac{1}{s+1}\right)$ because there are no negative powers of $\hat{z}(\cdot)$ by (3.23)) and $\hat{N}_2(\cdot) \in \hat{\mathcal{A}}^{n\times n}$ by Fact (3.17a), (3.15) and (3.30).

Finally by (3.26) and since $\hat{S}\left(\frac{1}{s+1}\right)$ and $\hat{T}\left(\frac{1}{s+1}\right)$ are unimodular

$$\inf_{\mathrm{Re}\ s \geq 0} \left|\det \hat{T}\left(\frac{1}{s+1}\right)[I + \hat{G}(s)]\ \hat{S}\left(\frac{1}{s+1}\right)\right| > 0.$$

Hence, since by (3.28) - (3.29) det $\hat{M}(\cdot)$ has only one zero for Re $s \geq 0$ i.e. at p, we obtain with (3.31)

(3.40) $$\inf_{s\in U}\left|\det \hat{N}(s)\right| > 0$$

where U is the half plane Re s $\geq$ 0 with a small neighborhood of p deleted.

Consider now det $\hat{N}(p)$.

Observe that by (3.38), (3.24) and (3.28) - (3.29)

$$(3.41) \qquad N_2(s) = \begin{array}{c} \\ \alpha\{ \\ n-\alpha\{ \end{array} \begin{array}{c} \overbrace{\phantom{\hat{K}_{11}(s)}}^{\alpha} \quad \overbrace{\phantom{\hat{L}_{12}(s)}}^{n-\alpha} \\ \left[\begin{array}{c|c} \hat{K}_{11}(s) & \hat{L}_{12}(s) \\ \hline \hat{K}_{21}(s) & \hat{L}_{22}(s) \end{array}\right] \end{array}$$

with

$$(3.42) \qquad \hat{K}_{11}(p) = 0$$

$$(3.43) \qquad \hat{K}_{21}(p) = 0$$

Thus by (3.34), (3.35), (3.41) - (3.43)

det $\hat{N}(p)$ = det $\hat{D}_1(p)$ det$[\hat{L}_{22}(p) + \hat{D}_2(p)]$ with by (3.36), (3.18) and (3.20)

$$(3.44) \qquad \det \hat{D}_1(p) \neq 0$$

and by (3.37), (3.20), (3.29) and (3.23)

$$(3.45) \qquad \det[\hat{L}_{22}(p) + \hat{D}_2(p)] =$$

$$\det\{\hat{L}_{22}(p) + \mathrm{diag}[\hat{b}_{\alpha+1}(1/(p+1),\ldots,\hat{b}_\beta(1/(p+1)),0,\ldots,0]\}$$

which is nonzero by (C). Hence

$$(3.46) \qquad \det \hat{N}(p) \neq 0.$$

Since $\hat{N}(\cdot)$ is continuous in Re s $\geq$ 0, (3.39), (3.40) and (3.46) imply that $\hat{N}(\cdot)^{-1} \in \hat{\mathcal{A}}^{n\times n}$. $\underline{\bar{X}}$

$\Rightarrow$. Thus $\hat{H} \in \hat{\mathcal{A}}^{n\times n}$ by assumption.

(3.26) follows immediately by [17].

To establish (C) we use contradiction. So by (3.45) suppose that $\det[\hat{L}_{22}(p) + \hat{D}_2(p)] = 0$. We are going to show that, for some input $u \in L_n^2[0,\infty)$, the system defined by (1.1) - (1.2) has an error e not in $L_n^2[0,\infty)$. This is a contradiction because by (1.16) $u \in L_n^2[0,\infty)$ and $H \in \mathcal{A}^{n\times n}$ imply that $y = H*u \in L_n^2[0,\infty)$ and thus $e = u-y \in L_n^2[0,\infty)$.

The Laplace transforms of e and u are related by

(3.47) $$[I + \hat{G}(s)]\, \hat{e}(s) = \hat{u}(s).$$

Multiply (3.47) on the left by $\hat{T}\left(\frac{1}{s+1}\right)$ and define the n-vectors $\hat{e}^*(\cdot)$ and $\hat{u}^*(\cdot)$ by

(3.48) $$\hat{S}\left(\frac{1}{s+1}\right) \hat{M}(s)\, \hat{e}^*(s) = \hat{e}(s)$$

(3.49) $$\hat{T}\left(\frac{1}{s+1}\right) \hat{u}(s) = \hat{u}^*(s).$$

By (3.47) - (3.49) and (3.31) obtain

(3.50) $$\hat{N}(s)\, \hat{e}^*(s) = \hat{u}^*(s).$$

Because $\det[\hat{L}_{22}(p) + \hat{D}_2(p)] = 0$ we can pick a nonzero vector $\eta \in \mathbb{R}^{n-\alpha}$ in the null space of $[\hat{L}_{22}(p) + \hat{D}_2(p)]$, hence

(3.51) $$[L_{22}(p) + D_2(p)]\, \eta = 0.$$

Pick now the vector $\xi \in \mathbb{R}^{\alpha}$ such that

$$\xi \triangleq -[\hat{D}_1(p)]^{-1}\,\hat{L}_{12}(p)\,\eta \tag{3.52}$$

which is well defined because of (3.44) and the fact that all elements of $\hat{L}_{12}$ and $\hat{D}_1$ are in $\hat{\mathcal{A}}$

Hence with

$$\hat{e}^*(s) = \frac{1}{s-p}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{3.53}$$

and

$$\hat{u}^*(s) = \begin{pmatrix}\hat{u}_1^*(s)\\ \hat{u}_2^*(s)\end{pmatrix}\begin{matrix}\}\ \alpha\\ \}\ n-\alpha\end{matrix} \tag{3.54}$$

and (3.50), (3.34), (3.35), (3.41), we obtain

$$\hat{u}_1^*(s) = \{[\hat{D}_1(s) + \hat{K}_{11}(s)]\ \xi + \hat{L}_{12}(s)\eta\}/(s-p) \tag{3.55}$$

$$\hat{u}_2^*(s) = \{\hat{K}_{21}(s)\xi + [\hat{D}_2(s) + \hat{L}_{22}(s)]\eta\}/(s-p). \tag{3.56}$$

All the components of the numerators of (3.55) and (3.56) are in $\hat{\mathcal{A}}$; by virtue of (3.42) - (3.43) and (3.51) - (3.52) and $p > 0$ they have at least a first order zero at p. Therefore $\hat{u}_1^*(\cdot)$ and $\hat{u}_2^*(\cdot)$ are analytic and bounded at $s = p$.

Thus $\hat{u}^*(\cdot)$ is analytic for Re $s > 0$, bounded in Re $s \geq 0$ and, as $|\omega| \to \infty$, for each component $\hat{u}^*_{(i)}(\cdot)$ of $\hat{u}^*(\cdot)$ we obtain

$|\hat{u}^*_{(i)}(\text{Re } s + j\omega)|$ is at most $O\left(\frac{1}{|\omega|}\right)$ uniformly for any fixed Re $s \geq 0$.

It follows therefore that the components of $\hat{u}^*(\cdot)$ are the Laplace-transforms of elements of $L^2[0,\infty)$ (Wiener's Theorem [12] p. 8). From Fact (3.17a) and (3.49) we conclude that the same is true for the components of $\hat{u}(\cdot)$, hence

(3.57) $$u \in L_n^2[0,\infty).$$

Finally by (3.48), (3.53), (3.28) - (3.29 and since $\eta \neq 0$ and $\hat{S}\left(\frac{1}{s+1}\right)$ is unimodular, there exists at least one component of $\hat{e}(.)$ which has a nonzero residue at p. Thus

(3.58) $$e \notin L_n^2(0,\infty)$$

and by (3.57) and (3.58) we have established a contradiction. $\bar{X}$

<u>Remarks</u>

(3.58a) <u>Remark</u>. If in Theorem 3.3 $p = 0$ then (3.26) and (C) are still sufficient for stability; moreover (3.26) and (C) are also necessary if the magnitude of the components of the numerators on the R.H.S. of (3.55) and (3.56) are at least of order $O(|s - p|^{\delta})$ for some real number $\delta > 0$ at p.

The first statement is obvious from the sufficiency part of the proof of Theorem 3.3.

Concerning the second statement observe that everything carries over for $p = 0$ in the necessity part of the proof of Theorem 3.3 until (3.53).

Now with

(3.59) $$\hat{e}^*(s) = \frac{1}{(s-p)^{\gamma}}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \text{ where } \gamma = \frac{1+\varepsilon}{2} \quad \text{for some } \varepsilon \in (0,\delta],$$

and (3.54) we obtain

(3.60) $\hat{u}_1^*(s) = \{[\hat{D}_1(s) + \hat{K}_{11}(s)]\xi + \hat{L}_{12}(s)\eta\}/(s-p)^\gamma$

(3.61) $\hat{u}_2^*(s) = \{\hat{K}_{21}(s)\xi + [\hat{D}_2(s) + \hat{L}_{22}(s)]\eta\}/(s-p)^\gamma$.

Observe that all the components of the numerators on the R.H.S. of (3.60) - (3.61) are in $\hat{\mathcal{A}}$ and have a zero at p. By assumption their magnitudes are at least of order $O(|s - p|^\delta)$ at p, some $\delta > 0$.

Therefore because of (3.60) - (3.61)

(3.62) $\hat{u}^*(\cdot)$ is analytic in Re $s > 0$.

Next observe that the magnitude of each component $\hat{u}^*_{(i)}(\cdot)$ of $\hat{u}^*(\cdot)$ is at least of order $O(|s - p|^{\delta-\gamma})$ at p with, by (3.59), $\delta - \gamma > -\frac{1}{2}$. Therefore there exists positive numbers σ_0, ω_0 such that on all vertical lines V_σ, $\sigma \in [0,\sigma_0]$, the integrals

$$\int_{\text{Im } p-\omega_0}^{\text{Im } p+\omega_0} |\hat{u}^*_{(i)}(\sigma + j\omega)|^2 d\omega$$

are uniformly bounded for all $\sigma \in [0,\sigma_0]$. Furthermore note that $\hat{u}^*(\cdot)$ is bounded in $\{s|\text{Re } s \geq 0\} \sim [0,\sigma_0] \times [\text{Im } p-\omega_0, \text{Im } p+\omega_0]$. Finally as $|\omega| \to \infty$ for each i, for all fixed $\sigma \geq 0$, we obtain that

$$|\hat{u}^*_{(i)}(\sigma + j\omega)| \leq K_i|\omega|^{-\gamma}$$

where K_i is a positive constant independent of σ, and γ is larger than 1/2 by (3.59). Therefore for each i there exists a constant $M_i > 0$ such that

$$(3.63)\qquad \int_{-\infty}^{\infty} |\hat{u}^*_{(i)}(\sigma + j\omega)|^2 d\omega < M_i \quad \text{for all fixed } \sigma \geq 0.$$

Hence by (3.62) and (3.63) it follows that (Wiener's Theorem [12] p. 8) each component of $\hat{u}^*(\cdot)$ is the Laplace-transform of an element of $L^2[0,\infty)$. From Fact (3.17a) and (3.49) we conclude that the same is true for the components of $\hat{u}(\cdot)$, hence

$$(3.57)\qquad u \in L_2^2[0,\infty).$$

Finally by (3.48), (3.28) - (3.29) and since $\eta \neq 0$ and $\hat{S}\left(\frac{1}{s+1}\right)$ is unimodular, $\lim\limits_{s\to p} (s-p)^{\gamma}\hat{e}(s)$ is well defined and nonzero. Therefore at least one component of $\hat{e}(\cdot)$ is not locally absolutely square integrable on the vertical line V_0 and thus a fortiori not uniformly absolutely square integrable on all lines V_σ, $\sigma > 0$. Therefore by Wiener's Theorem ([12 p. 8) at least one of the components of $\hat{e}(\cdot)$ is not the Laplace transform of an element of $L^2[0,\infty)$, i.e.

$$(3.58)\qquad e \notin L_n^2[0,\infty).$$

So by (3.57) and (3.58) we arrive again at a contradiction. $\bar{X}$

(3.63a) Remark. Theorem 3.3 describes in detail what happens when the sets K_-, K_0, K_+ given by (3.21) - (3.23) are nonempty. When one or more of these sets are empty the required modifications of (C) and of the multiplier $\hat{M}(s)$ are straightforward.

(3.63b) <u>Remark</u>. If there are ℓ poles at $p_1, p_2, \ldots, p_\ell$ of orders $m_1, m_2, \ldots, m_\ell$ with positive real part, one proceeds similarly as in Theorem 3.3. The principal part of $\hat{G}$ is transformed into a polynomial matrix in $\frac{1}{s+1}$ and Facts (3.15a), (3.16a), (3.17a) and (3.17b) are repeated. In the proof of stability one uses a product of multipliers similar to $\hat{M}(s)$. Note that $\hat{M}(s)$ is diagonal. Observe that Condition (C) is used only to check that det $\hat{N}(s)$ does not vanish at $s = p$. Therefore for the more general case an appropriate condition (C) is required at each pole. This was checked by us.

In order to give an interpretation of the condition (C) we first give a result which will streamline the proof of Theorem 3.4.

(3.63c) <u>Fact</u>.

Given the system (3.15).

Let the principal part of $\hat{G}$ be denoted by

$$\hat{R}\left(\frac{1}{s-p}\right) \triangleq \sum_{i=0}^{m} R_i (s-p)^{-m+i}. \tag{3.64}$$

Let α be given by (3.21), let r^* be the rank of the polynomial matrix $\hat{R}^*(\cdot)$ defined by (3.16) and let $\{c_j\}_{j=r_0+1}^{r^*}$ be the set of exponents used in (3.19). Under these conditions the integer r defined by

$$r \triangleq m\alpha - \sum_{j=r_0+1}^{\alpha} c_j \tag{3.65}$$

is the maximal order of the pole at p of all minors of $\hat{R}\left(\frac{1}{\cdot - p}\right)$.

Proof

Immediately by (3.16)

$$\hat{R}\left(\frac{1}{s-p}\right) = \left(\frac{s+1}{s-p}\right)^m \hat{R}^*\left(\frac{1}{s+1}\right) . \tag{3.66}$$

Hence, denoting by $\hat{m}_{jk}\left(\frac{1}{s-p}\right)$ and $\hat{m}^*_{jk}\left(\frac{1}{s+1}\right)$ a minor of order j of $\hat{R}\left(\frac{1}{s-p}\right)$ respectively $\hat{R}^*\left(\frac{1}{s+1}\right)$ for $k = 1,2,\ldots,j^*$, where $j^* \triangleq \left(\frac{n!}{j!(n-j)!}\right)^2$, it follows that

$$\hat{m}_{jk}\left(\frac{1}{s-p}\right) = \left(\frac{1}{s-p}\right)^{jm} \hat{m}^*_{jk}\left(\frac{1}{s+1}\right) \text{ for } \begin{array}{l} j = 1,2,\ldots,n \\ k = 1,2,\ldots,j^* \end{array} . \tag{3.67}$$

Therefore since r^* is the rank of $\hat{R}^*(\cdot)$

$$\hat{m}_{jk}(\cdot) \equiv 0 \text{ for } r^* < j \leq n \text{ and } k = 1,2,\ldots,j^* . \tag{3.68}$$

Now let $\hat{q}^*_j(\cdot)$ be the greatest monic common divisor of all minors of order j of $\hat{R}^*(\cdot)$ then, by a result of Gantmacher [31, p. 141], the invariant polynomials $\hat{a}_j(\cdot)$ of $\hat{R}^*(\cdot)$ admit a representation

$$\hat{a}_j\left(\frac{1}{s+1}\right) = \hat{q}^*_j\left(\frac{1}{s+1}\right) / \hat{q}^*_{j-1}\left(\frac{1}{s+1}\right) \text{ for } j = 1,2,\ldots,r^*$$

where $\hat{q}^*_0(\cdot) \triangleq 1$. So immediately

$$\hat{q}^*_j\left(\frac{1}{s+1}\right) = \prod_{\ell=1}^{j} \hat{a}_\ell\left(\frac{1}{s+1}\right) \text{ for } j = 1,2,\ldots,r^* \tag{3.69}$$

Hence by setting

(3.70) $\quad c_j = 0 \quad \text{for } j = 1,2,\ldots,r_0$

we get by (3.69) - (3.70) and (3.18) - (3.20) that $\hat{q}_j^*\left(\frac{1}{\cdot+1}\right)$ admits a zero of order $\sum_{\ell=1}^{j} c_\ell$ at p for $j = 1,2,\ldots r^*$.

Observe now that the order of the zero at p of $\hat{q}_j^*\left(\frac{1}{\cdot+1}\right)$ is the minimal order of the zero at p of the minors of order j of $\hat{R}^*\left(\frac{1}{\cdot+1}\right)$ that is of the set $\{\hat{m}_{jk}^*\left(\frac{1}{\cdot+1}\right)\}_{k=1}^{j^*}$. Consider now the set of expressions $\{\left(\frac{s+1}{s-p}\right)^{jm} \hat{m}_{jk}^*\left(\frac{1}{s+1}\right)\}_{k=1}^{j^*}$ then it is immediately clear that the integer d_j

(3.71) $$d_j \triangleq mj - \sum_{\ell=1}^{j} c_\ell \quad \text{for } j = 1,2,\ldots,r^*$$

is the maximal order of the pole at p of the expressions this set. Therefore by (3.67) d_j is the maximal order of the pole at p of $\{\hat{m}_{jk}\left(\frac{1}{\cdot-p}\right)\}_{k=1}^{j^*}$ i.e. of all minors of order j of $\hat{R}\left(\frac{1}{\cdot-p}\right)$.

Consider now the map $j \longmapsto d_j$ for $j = 1,2,\ldots,r^*$. Observe that this map is concave because $j \mapsto c_j$ is nondecreasing. Furthermore by the definition of α, β, γ in (3.21) - (3.23) and by (3.70) it attains its maximum at $j = \alpha$. Therefore, along with (3.68), d_α is the maximal order of the pole at p of all minors of $\hat{R}\left(\frac{1}{\cdot-p}\right)$ and the same is true for the integer r since by (3.65) $d_\alpha = r$. $\underline{\bar{X}}$

We try now to investigate the nature of condition (C).

Theorem 3.4.

Given the system defined by (1.1), (1.2) and (3.15). Assume that the assumptions and preliminary operations of Theorem 3.3 are valid, respectively performed. Let $\hat{q}_{ij}(s)$ be the cofactor of the ij-th element of $I + \hat{G}(s)$ and let c_{ij} be the order of the pole at p of $\hat{q}_{ij}(\cdot)$ for i, j = 1,2,..., n. Moreover let

$$c \triangleq \max_{i,j} c_{ij}$$

i.e. let c be the maximal order of the pole at p of the cofactors of $I + \hat{G}(\cdot)$. Finally let d be the order of the pole at p of $\det[I + \hat{G}(\cdot)]$, and let r be the maximal order of the pole at p of all minors of the principal part of $\hat{G}$ given by (3.64).

Under these conditions

$$\inf_{\mathrm{Re}\, s \geq 0} |\det[I + \hat{G}(s)]| > 0 \tag{3.26}$$

and either

$$\det\{L_{22}(p) + \mathrm{diag}[\hat{b}_{\alpha+1}(1/(p+1)),\ldots,\hat{b}_{\beta}(1/(p+1)),0,0,\ldots,0]\} \neq 0 \tag{C}$$

or

$$d = r \tag{3.72}$$

or

$$c \leq d \tag{3.73}$$

are necessary and sufficient for

(3.25) $H \in \mathcal{A}^{n\times n}$

Proof

(a) Observe that (3.26) and (C) $\Longleftrightarrow$ (3.25) is precisely the statement of Theorem 3.3.

(b) In order to prove: (3.26) and (3.72) $\Longleftrightarrow$ (3.25) we show that (C) $\Longleftrightarrow$ (3.72).

First observe that because of decomposition-lemma A.2 and Corrolary A.2.4 of the appendix, $\det[I + \hat{G}(s)]$, where $\hat{G}(s)$ is given by (3.15), admits following representation:

$$\det[I + \hat{G}(s)] = \sum_{m=0}^{d-1} r_m (s-p)^{-d+m} + \hat{g}_r(s) \tag{3.74}$$

where $\hat{g}_r \in \hat{\mathcal{A}}$, the r_m are real coefficients for $m = 0,1,2,\ldots,$ $d-1$ and $r_0 \neq 0$.

Next we go back to the proof of Theorem 3.3 where

$$\hat{N}(s) \triangleq \hat{T}\left(\frac{1}{s+1}\right)[I + \hat{G}(s)]\,\hat{S}\left(\frac{1}{s+1}\right)\hat{M}(s). \tag{3.31}$$

Observe that $\hat{N} \in \hat{\mathcal{A}}^{n\times n}$ and that the expression of condition (C) is in $\hat{\mathcal{A}}$. Therefore from the proof of Theorem 3.3:

$$\begin{aligned} &\text{"(C)"} \Longleftrightarrow \det \hat{N}(p) \neq 0 \text{ and well defined;} \\ &\text{"not (C)"} \Longleftrightarrow \det \hat{N}(p) = 0. \end{aligned} \tag{3.75}$$

We rewrite now $\det \hat{N}(s)$.

Remember that $\hat{T}\left(\frac{1}{s+1}\right)$ and $\hat{S}\left(\frac{1}{s+1}\right)$ are unimodular polynomial matrices, therefore

(3.76) $$\det \hat{T}\left(\frac{1}{s+1}\right) = C_T = \text{a nonzero constant}$$

and

(3.77) $$\det \hat{S}\left(\frac{1}{s+1}\right) = C_S = \text{a nonzero constant.}$$

Moreover using (3.28) - (3.29) and Fact (3.63c) we obtain

(3.78) $$\det \hat{M}(s) = \left(\frac{s-p}{s+1}\right)^r .$$

Therefore using (3.31) and (3.74), (3.76) - (3.78)

$$\det \hat{N}(s) = C_T C_S \left(\frac{s-p}{s+1}\right)^r \left[\sum_{m=0}^{d-1} r_m (s-p)^{-d+m} + \hat{g}_r(s)\right] .$$

Finally by (3.75) and (3.74), (3.76) - (3.77):

(3.79) $$\begin{cases} \text{"(C)"} & \Leftrightarrow \; d = r \; ; \\ \text{"not (C)"} & \Leftrightarrow \; d < r \; . \end{cases}$$

So we are done . $\bar{\underline{X}}$

(c) We finally show that, (3.26) and (3.73) $\Leftrightarrow$ (3.25). First observe that, because of decomposition-lemma A.2 and its subsequent corrolaries in the appendix and because of (3.15), each cofactor $\hat{q}_{ij}(s)$ of $I + \hat{G}(s)$ admits a representation

(3.80) $$\hat{q}_{ij}(s) = \sum_{m=0}^{c_{ij}-1} r_{ijm}(s-p)^{-c_{ij}+m} + \hat{q}_{ijr}(s) \quad \text{for } i,j = 1,2,\ldots,n$$

where for i, j = 1,2,...,n: $\hat{q}_{ijr} \in \hat{\mathcal{A}}$, r_{ijm} are real coefficients for $m = 0,1,\ldots c_{ij} - 1$ and $r_{ij0} \neq 0$. Therefore by (3.74), (3.80) and Cramer's rule each element $\hat{f}_{ij}(\cdot)$ of $[I + \hat{G}(\cdot)]^{-1}$, where $\hat{G}(\cdot)$ is given by (3.64), admits a representation:

$$\text{(3.81)} \qquad \hat{f}_{ij}(s) = \frac{\hat{q}_{ji}(s)}{\det[I + \hat{G}(s)]} = \frac{\displaystyle\sum_{m=0}^{c_{ji}-1} r_{jim}(s-p)^{-c_{ji}+m} + \hat{q}_{jir}(s)}{\displaystyle\sum_{m=0}^{d-1} r_m(s-p)^{-d+m} + \hat{g}_r(s)}$$

where $\hat{q}_{jir}$ for $i, j = k,2,\ldots,n$ and $\hat{g}_r \in \hat{\mathcal{A}}$;

r_{jim} for $i, j = 1,2,\ldots,n$ and r_m are real constants for all m;

$r_{ji0} \neq$ for $i, j = 1,2,\ldots,n$ and $r_0 \neq 0$.

This holds in $\operatorname{Re} s \geq 0$ by analytic continuation if (3.26) is satisfied.

We prove now: (3.26) and (3.73 $\Longleftrightarrow$ (3.25).

$\Rightarrow$. Introduce a multiplier $\hat{\mu}(\cdot)$ given by $\hat{\mu}(s) \triangleq \left(\frac{s-p}{s+1}\right)^d \in \hat{\mathcal{A}}$. Observe that,because of (3.74) and (3.80), $\hat{q}_{ij}(\cdot)\ \hat{\mu}(\cdot)$ for $i, j = 1,2,\ldots,n$ and $\det[I + \hat{G}(\cdot)]\hat{\mu}(\cdot)$ belong to $\hat{\mathcal{A}}$. Moreover because of (3.26), (3.73) - (3.74) $\inf_{\operatorname{Re} s \geq 0} |\det[I + \hat{G}(s)])\hat{\mu}(s)| > 0.$

Therefore by (1.14) $\{(\det[I + \hat{G}(\cdot)]\hat{\mu}(\cdot)\}^{-1} \in \hat{\mathcal{A}}$ and hence,by (3.81), $\hat{f}_{ij}(\cdot) \in \hat{\mathcal{A}}$ for $i,j = 1,2,\ldots,n$ because $\hat{f}_{ij}$ can be written as a product of two elements in the algebra. Hence $[I + \hat{G}(\cdot)]^{-1} \in \hat{\mathcal{A}}^{n\times n}$ and, because $I - \hat{H}(\cdot) = [I + \hat{G}(\cdot)]^{-1}$, (3.25) follows. $\bar{\underline{X}}$

$\Leftarrow$. Immediately because of (3.25), $[I + \hat{G}(\cdot)]^{-1} \in \hat{\mathcal{A}}^{n\times n}$.

Therefore $\{\det[I + \hat{G}(\cdot)]\}^{-1} \in \hat{\mathcal{A}}$ which by (1.13) implies that $\{\det[I + \hat{G}(\cdot)]\}^{-1}$ is bounded in Re $s \geq 0$, thus (3.26) follows. Hence each element $\hat{f}_{ij}$ of $[I + \hat{G}(\cdot)]^{-1}$ admits a representation (3.81). Moreover for all $i, j = 1,2,\ldots,n$ $\hat{f}_{ij}(\cdot) \in \hat{\mathcal{A}}$, so $\hat{f}_{ij}(p)$ must be finite for $i, j = 1, 2,\ldots,n$, thus by (3.81) (3.73) follows. $\underline{\bar{X}}$

Observe that if (3.26) is satisfied but not (3.73), i.e. $c > d$, then at least one of the elements $\hat{f}_{ij}$ of $(I + \hat{G})^{-1}$ and, since $I - \hat{H} = [I + \hat{G}]^{-1}$, also at least one element of $\hat{H}$ has a pole at p.

<u>Remarks</u>

(3.81a) <u>Remark</u>. From the proof of Theorem 3.4 it follows that for the system of Theorem 3.3:

(i) "(C)" $\Longleftrightarrow d = r$

"not (C)" $\Longleftrightarrow d < r$

(ii) under the assumption of (3.26)

"(C)" $\Longleftrightarrow c \leq d$

"not (C)" $\Longleftrightarrow c > d$.

Hence following interpretation of condition (C) is possible: "(C)" is equivalent to require that the order of the pole at p of $\det[I + \hat{G}(\cdot)]$ equals the maximal order of the pole at p of all minors of the principal part of $\hat{G}$. Next, if

$\inf_{\text{Re } s \geq 0} |\det[I + \hat{G}(s)]| > 0$ but (C) is not satisfied, then at least one element of $[I + \hat{G}(\cdot)]^{-1}$ and thus also of $\hat{H}(\cdot)$ has a pole at p and instability results.

(3.81b) Remark. Theorem 3.4 describes what happens if the sets K_-, K_0, K_+ given by (3.21) - (3.23) are nonempty. When one or more of these sets are empty then, according to Remark (3.63b), the condition (C) must be modified. The interpretation as given in Remark (3.81a) however remains always valid.

(3.81c) Remark. If in (3.15) $\hat{G}_r$ is a matrix in $\hat{\mathcal{Q}}^{n\times n}$ whose elements are rational functions, then C. T. Chen's result [10, Theorem 9-10, p. 376] can be used and requires that, in addition to (3.26), "d = the maximal order of the pole at p of all minors of $I + \hat{G}(\cdot)$" is necessary and sufficient for stability. It should be stressed that this last requirement is equivalent to $c \leq d$. Indeed $I + \hat{G}$ is a matrix whose elements are rational functions in this case. Let $e_j \stackrel{\Delta}{=}$ the maximal order of the pole at p of the minors of order j of $I + \hat{G}(\cdot)$. Then, similarly as for the map $j \longmapsto d_j$ in the proof of Fact (3.63c) (see (3.71)), it can be shown that the map $j \longmapsto e_j$ is concave such that, since $e_{n-1} = c$ and $e_n = d$, $c \leq d \Leftrightarrow d = \max_j e_j$.

(3.81d) Remark. If there are ℓ poles at $p_1, p_2, \ldots, p_\ell$ of orders $m_1, m_2, \ldots, m_\ell$ with positive real part, which are either real or complex conjugate, then an analog interpretation of condition (C) at each pole can be obtained.

(3.81e) Remark. The maximal order of the pole at p of the principal part of $\hat{G}$ is the exponent of (s-p) in the least monic common denominator of all minors of this matrix whose elements are rational functions.

Finally to be more precise we state the extension of Theorem 3.3 to the multiple pole case.

Theorem 3.5

Consider the system defined by (1.1), (1.2) and (3.12). Let $\hat{d}(s)$ be the least monic common denominator of all minor of the matrix $\sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} R_{km}(s-p_k)^{-m_k+m}$, i.e. the principal part of G and let $\hat{d}(s)$ be given by

$$\hat{d}(s) = \prod_{k=1}^{\ell} (s-p_k)^{r_k}.$$

Let d_k be the order of the pole at p_k of $\det[I + \hat{G}(\cdot)]$ for $k = 1, 2, \ldots, \ell$. Under these conditions

$$H \in \mathcal{A}^{n\times n}$$

if and only if

$$\inf_{\mathrm{Re}\, s \geq 0} |\det[I + \hat{G}(s)]| > 0$$

and

$$d_k = r_k \quad \text{for } k = 1, 2, \ldots, \ell. \qquad \underline{\bar{X}}$$

4. DISCRETE-TIME N-INPUT N-OUTPUT CONVOLUTION FEEDBACK SYSTEMS.

4.1 Introduction

This section considers discrete-time convolution feedback systems with n inputs and n outputs as described in paragraph 1.1.2. An analog program is followed as in Section 3. In view of the simpler analytic nature of the problems discussed, more elementary tools can be used in the proofs such that the techniques used are clearer to the reader.

First the relation between the open-loop operator $\underset{\sim}{G}$ and the closed-loop operator $\underset{\sim}{H}$ is discussed. Thereby we show the importance of systems defined by (1.1') - (1.2') and

$$\tilde{G}(z) = \tilde{P}(z)[\tilde{Q}(z)]^{-1} \tag{4.1'}$$

where $\tilde{P}, \tilde{Q} \in \tilde{\ell}^{1}_{n\times n}$. Furthermore results of Baker and Nasburg [22] are extended.

Next necessary and sufficient conditions for stability are discussed when the open-loop transfer function $\tilde{G}$ is of the for (3.1') with a finite number of poles in $|z| > 1$. Thereby results of Desoer, Wu and Lam are extended [3,18].

Everywhere conpletely analog results as in section 3 are obtained.

4.2 The Relation Between $\underset{\sim}{G}$ and $\underset{\sim}{H}$

Theorem 4.1'

Let G be a sequence of real n×n matrices $\{G_i\}_{i=0}^{\infty}$. For the system defined by (1.1') and (1.2') assume that the closed-loop impulse response H exists and is uniquely defined by

$$H + G*H = G. \tag{4.2'}$$

Under these conditions, if $H \in \ell^1_{n\times n}$, then

(a) G is z-transformable and for some finite $\rho \geq 1$ the sequence of real n×n matrices $\{G_i \rho^{-i}\}_{i=0}^{\infty} \in \ell^1_{n\times n}$;

(b) $\tilde{G}$ is of the form

$$\tilde{G}(z) = \tilde{P}(z)[\tilde{Q}(z)]^{-1} \quad \text{for } |z| > 1 \tag{4.1'}$$

where $\tilde{P}(\cdot)$ and $\tilde{Q}(\cdot) \in \ell^1_{n\times n}$;

(c) $\tilde{G}$ can at most have a countable number of poles in the annulus $1 < |z| \leq \rho$, and has no poles in the annulus $|z| > \rho$.

Comment. This theorem shows that once the closed-loop system is well defined, then G is necessarily of the form (4.1'), can at most have a finite number of poles in any annulus of the form $1 + \varepsilon \leq |z| \leq \rho$ (ε small and positive) and is analytic for $|z| > \rho$.

Proof.

a) By assumption $H = \{H_i\}_{i=0}^{\infty}$ is well defined and belongs

$\ell^1_{n\times n}$. Therefore H_0 is a well defined real n×n matrix. From (4.2') $H_0 + G_0 H_0 = G_0$, so that, since H_0 is uniquely defined, $\det[I + G_0] \neq 0$. Hence since $(I + G_0)(I - H_0) = I$, $\det[I - H_0] \neq 0$.

The function $I - \tilde{H}(\cdot)$ is analytic and bounded for $|z| > 1$ and tends to $I - H_0$ as $|z| \to \infty$. Therefore there exists a $\rho > 1$ such that

$$\inf_{|z| > \rho} |\det[I - \tilde{H}(z)]| > 0. \tag{4.3'}$$

Next observe that $\{H_i \rho^{-i}\}_{i=0}^{\infty}$ and $\{[I\delta_{i0} - H_i]\rho^{-i}\}_{i=0}^{\infty}$ (where δ_{i0} = Kronecker delta = $\begin{cases} 1 \text{ for } i = 0 \\ 0 \text{ otherwise} \end{cases}$) $\in \ell^1_{n\times n}$ and that

$$\widetilde{\{\{H_i\rho^{-1}\}_{i=0}^{\infty}\}}(z) = \tilde{H}(\rho z) \quad \text{for } |z| \geq 1 \text{ and}$$

$$\widetilde{\{\{[I\delta_{i0} - H_i]\rho^{-i}\}_{i=0}^{\infty}\}}(z) = I - \tilde{H}(\rho z) \quad \text{for } |z| \geq 1.$$

Therefore, by (4.3') and (1.15'), $[I - \tilde{H}(\rho\cdot)]^{-1}$, for $|z| \geq 1$, $\in \tilde{\ell}^1_{n\times n}$. Finally

$$\tilde{G}^*(\rho\cdot) \triangleq \tilde{H}(\rho\cdot)[I - \tilde{H}(\rho\cdot)]^{-1}, \text{ for } |z| \geq 1, \in \tilde{\ell}^1_{n\times n}. \tag{4.4'}$$

Next from (4.2')

$$H_i\rho^{-i} + \sum_{j=0}^{i} (G_{i-j}\rho^{-(i-j)})(H_j\rho^{-j}) = G_i\rho^{-i} \quad \text{for } i = 0,1,2,\ldots, \tag{4.5'}$$

Observe that the second term on the L.H.S. of (4.5') is due to convolution of $\{G_i\rho^{-i}\}_{i=0}^{\infty}$ with $\{H_i\rho^{-i}\}_{i=0}^{\infty}$, such that if G has a z-transform

(4.6') $\tilde{G}(\rho\cdot) = \tilde{H}(\rho\cdot) + \tilde{G}(\rho\cdot)\,\tilde{H}(\rho\cdot)$.

Next note that $\tilde{G}^*(\rho\cdot)$ given by (4.4') satisfies (4.6'). Because all terms are in $\tilde{\ell}^1_{n\times n}$ an inverse z-transform of Eq. (4.6') where $\tilde{G}(\rho\cdot) = \tilde{G}^*(\rho\cdot)$ may be performed. Therefore $\{G_i^*\rho^{-i}\}_{i=0}^{\infty}$ satisfies (4.5') and so by the uniqueness implied by the convolution algebra of real sequences on $\mathbb{Z}_+$ (i.e. the nonnegative integers)

$$\tilde{G}(\rho\cdot) = \tilde{H}(\rho\cdot)[I - \tilde{H}(\rho\cdot)]^{-1} \quad \text{for } |z| \geq 1$$

and $\tilde{G}(\rho\cdot) \in \ell^1_{n\times n}$.

Thus $\{G_i\rho^{-i}\}_{i=0}^{\infty} \in \ell^1_{n\times n}$ and $\tilde{G}(\cdot) = \tilde{H}(\cdot)\,[I - \tilde{H}(\cdot)]^{-1}$ for $|z| \geq \rho$.

This proves (a).

b) Since by (1.13') $\tilde{H}(\cdot)$ and $[I - \tilde{H}(\rho\cdot)]^{-1}$ are analytic for $|z| > 1$, $[I - \tilde{H}(\cdot)]^{-1}$ has at most a finite number of poles in any annulus of the form $1 + \varepsilon \leq |z| \leq \rho$ (ε small and positive) and by analytic continuation in the annulus $1 < |z| \leq \rho$

$$\tilde{G}(\cdot) = \tilde{H}(\cdot)[I - \tilde{H}(\cdot)]^{-1} \quad \text{for } |z| > 1.$$

Choose $\tilde{P}(\cdot) = \tilde{H}(\cdot)$, $\tilde{Q}(\cdot) = [I - \tilde{H}(\cdot)]$. Thus (b) and (c) have been established. $\bar{\underline{X}}$

Remarks

(4.6a') Remark. Observe that under the conditions of Theorem 4.1' we have $[I + \tilde{G}(z)]\,[I - \tilde{H}(z)] = I$ for $|z| > 1$. Thus $\underset{\sim}{H}$ and $\underset{\sim}{G}$ play a symmetrical role:

$\underset{\sim}{H}$ is obtained from $\underset{\sim}{G}$ by negative feedback of I;

$\underset{\sim}{G}$ is obtained from $\underset{\sim}{H}$ by negative feedback of -I.

(4.6b') <u>Remark</u>. A little more can be said about the poles of $\tilde{G}(\cdot)$:

$$\tilde{G}(\cdot) = \tilde{P}(\cdot)[\tilde{Q}(\cdot)]^{-1} = \tilde{P}(\cdot)\ \mathrm{Adj}[\tilde{Q}(\cdot)]/\det\tilde{Q}(\cdot).$$

The function ϕ: $z \longmapsto \det \tilde{Q}(z) \triangleq \det[I - \tilde{H}(z)]$ is analytic and bounded in $|z| > 1$ and because of (4.3') $\inf_{|z| \geq \rho} |\det \tilde{Q}(z)| > 0$.

Therefore ϕ has at most a countable of zeros p_k, for $k = 1, 2, \ldots$ in the annulus $1 < |z| \leq \rho$. Moreover by a theorem of [33, pp. 63-64] $\sum_{k=1}^{\infty} (1 - |p_k|^{-1}) < \infty$. Therefore $\tilde{G}(\cdot)$ either has a finite number of poles in the annulus $1 < |z| \leq \rho$ or else it has an infinite sequence of them in the annulus $1 < |z| \leq \rho$ such that they accumulate on the unit circle.

<u>Theorem 4.2'</u>

Let G be a sequence of n×n real matrices which is z-transformable. For the system defined by (1.1') and (1.2') assume that the closed-loop transfer function $\tilde{H}$ is well defined for almost all z in the domain of convergence of $\tilde{G}$; more precisely,

$$\tilde{H}(z) = \tilde{G}(z)[I + \tilde{G}(z)]^{-1} \tag{4.8'}$$

for almost all z in the domain of convergence of $G(\cdot)$. Under these conditions, (4.9') $\qquad H \in \ell^1_{n\times n}$

if and only if there exists $\tilde{P}$, $\tilde{Q} \in \tilde{\ell}^1_{n\times n}$ such that

(4.10') $\qquad \tilde{G}(z) = \tilde{P}(z)\,[\tilde{Q}(z)]^{-1}$

and

(4.11') $\qquad \inf\limits_{|z| \geq 1} |\det[\tilde{P}(z) + \tilde{Q}(z)]| > 0.$

Proof

$\Rightarrow$ From (4.8') - (4.9') by algebra

$$\tilde{G}(z) = \tilde{H}(z)\,[I - \tilde{H}(z)]^{-1} \quad \text{for } |z| \geq 1$$

Choose $\tilde{P} = \tilde{H}$ and $\tilde{Q} = I - \tilde{H}$. Hence by (4.9') $\tilde{P}$ and $\tilde{Q} \in \tilde{\ell}^1_{n\times n}$ and (4.10') holds. Finally, since $\tilde{P} + \tilde{Q} = I$ (4.11') holds.

$\Leftarrow$ From (4.8')

$$\tilde{H}(z) = \tilde{P}(z)\,[\tilde{P}(z) + \tilde{Q}(z)]^{-1}.$$

In view of (1.15') and (4.11') $\tilde{H} \in \tilde{\ell}^1_{n\times n}$ as the product of two elements of the algebra $\tilde{\ell}^1_{n\times n}$. $\overline{\underline{X}}$

Remarks.

(4.11a') Remark. As in the continuous-time case (4.10') does not determine the ordered pair $(\tilde{P},\tilde{Q})$ uniquely. In order that condition (4.11') depend only on $\tilde{G}$ we may, as Vidyasagar, impose on the pair $(\tilde{P},\tilde{Q})$ a no-cancellation condition [21].

(4.11b') Remark. Observe that (4.11') can always be tested graphically by the method described in paragraph 2.3 by setting

$\tilde{g}(z) \triangleq \det[\tilde{P}(z) + \tilde{Q}(z)] - 1$ which is in $\tilde{\ell}_1$.

4.3 Necessary and Sufficient Conditions for Stability.

By stability we mean stability as defined in Remark (1.17').

In this paragraph we consider systems of the form (1.1') - (1.2') where the open-loop transfer function $\tilde{G}$ is given by

$$\tilde{G}(z) = \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} R_{km}(z - p_k)^{-m_k+m} + \tilde{G}_r(z) \tag{4.12'}$$

where (a) the poles p_k and the corresponding residue matrices R_{km} are real or pairwise complex conjugate for $k = 1, 2, \ldots, \ell$ and $m = 0, 1, \ldots, m_k - 1$, (b) $|p_k| > 1$ for $k = 1, 2, \ldots, \ell$ and (c) $\hat{G}_r \in \hat{\mathcal{A}}^{n\times n}$.

Observe that if

$$\tilde{G}(z) = \tilde{P}_1(z)\left(\prod_{k=1}^{\ell} (z - p_k)^{m'_k} I\right)^{-1} \tag{4.13'}$$

where $\tilde{P}_1 \in \tilde{\ell}^1_{n\times n}$, I is the unit n×n matrix and m'_k are integers larger than or equal to m_k for $k = 1, 2, \ldots, \ell$ then (4.13') can be brought in the form (4.12'). This can be derived from decomposition lemma A.2' and Remark (A.12b'). Furthermore observe that $\tilde{G}(z)$ as given by (4.13') can be rewritten in the form

$$\tilde{G}(z) = \left[\left(\prod_{k=1}^{\ell} z^{m'_k}\right)^{-1} \tilde{P}_1(z)\right] \left[\prod_{k=1}^{\ell}\left(\frac{z-p_k}{z}\right)^{m'_k} I\right]^{-1} \tag{4.14'}$$

Hence, since $\frac{1}{z}$ and $\left(\frac{z-p_k}{z}\right)$ for $k = 1, 2, \ldots, \ell \in \hat{\ell}_1$, we obtain that the "numerator" and the "denominator" on the R.H.S. of

(4.14') are in $\tilde{\ell}^1_{n\times n}$, so $\tilde{G}(z)$ is of the form (4.1'). This establishes a link with previous paragraphs in that $\tilde{G}$ as given by (4.12') can be derived from a form (4.1') where $\hat{G}$ has a finite number of poles in $|z| > 1$.

We consider now first and in detail the case where $\tilde{G}$ has a real pole of order m in $|z| > 1$. The extension to the case of a finite number of poles will be taken care of in subsequent remarks.

We consider thus the open-loop transfer function G defined by

$$\text{(4.15')} \qquad \tilde{G}(z) = \sum_{i=0}^{m-1} R_i (z-p)^{-m+1} + \tilde{G}_r(z)$$

where $p \in \mathbb{R}$, $|p| > 1$, $G_r \in \tilde{\ell}^1_{n\times n}$, r_0 rank of $R_0 \leq n$ and R_i $(i=0,1,2,\ldots,m-1)$ are n×n matrices with real coefficients.

We start by pointing out some facts which will streamline the proof of Theorem 4.3'. Because of the analogy with paragraph 3.3 proofs are omitted.

(4.15a') <u>Fact</u>.

Let

$$\text{(4.16')} \qquad \tilde{R}^*(1/z) \triangleq \left(\sum_{i=0}^{m-1} R_i (z-p)^{-m+i}\right) \left((z-p)/z\right)^m$$

then $\tilde{R}^*(1/z)$ is a polynomial matrix in $(1/z)$ of degree m.

(4.16a') <u>Fact. (Smith Canonical form [31])</u>.

For the n×n polynomial matrix $\tilde{R}^*(1/z)$ there exist unimodular (i.e. with nonzero <u>constant</u> determinant) <u>polynomial</u> matrices in (1/z), viz. $\tilde{T}(1/z)$ and $\tilde{S}(1/z)$, such that

$$(4.17') \qquad \tilde{T}(1/z)\tilde{R}^*(1/z)\tilde{S}(1/z) =$$

$$\mathrm{diag}\{\underbrace{\tilde{a}_1(1/z),\ldots,\hat{a}_j(1/z),\ldots,\tilde{a}_{r^*}(1/z)}_{r^*},\underbrace{0,0,\ldots,0}_{n-r^*}\}$$

where i) r^* = rank of $\tilde{R}^*(1/z)$ = order of the largest minor of $\tilde{R}^*(1/z)$ which is not equal to the zero polynomial; ii) $\tilde{a}_j(1/z)$, $j = 1,2,\ldots,r^*$ are the invariant polynomials of $R^*(1/z)$ and each polynomial $\tilde{a}_j(\cdot)$ divides $\tilde{a}_{j+1}(\cdot)$, $j = 1,2,\ldots, r^*-1$; iii) the diagonal matrix on the R.H.S. of (4.17') can be obtained by elementary operations.

(4.17a') <u>Fact</u>.

The polynomial matrices $\tilde{S}(1/z)$ and $\tilde{T}(1/z) \in \tilde{\ell}^1_{n\times n}$ and their inverses are polynomial matrices also in $\tilde{\ell}^1_{n\times n}$.

(4.17b') <u>Fact</u>.

Let $\tilde{a}_j(\cdot)$ for $j = 1, 2, \ldots, r^*$ be as in (4.17') and let r_0 be the rank of R_0, then

(a)

$$(4.18') \qquad \begin{cases} \tilde{a}_j(1/p) \neq 0 & \text{for } 1 \leq j \leq r_0 ; \\ \tilde{a}_j(1/p) = 0 & \text{for } r_0 + 1 \leq j \leq r^* ; \end{cases}$$

(b) by the factorization of the last $r^* - r_0$ polynomials

$$(4.19') \qquad \tilde{a}_j(1/z) = \tilde{b}_j(1/z)((z-p)/z)^{c_j} \text{ for } r_0 + 1 \leq j \leq r^*, \text{ where}$$

c_j is the order of the zero of $\tilde{a}_j(\cdot)$ at $z = p$, $\tilde{b}_j(\cdot)$ is a polynomial with

(4.20') $\tilde{b}_j(1/p) \neq 0$, and

$$1 \leq c_{r_0+1} \leq c_{r_0+2} \leq \cdots \leq c_{r^*} .$$

Note. The c_j may be larger than m (in fact $c_{r^*} \leq r^*m$ and are monotonically increasing. Thus the (c_j-m)'s take on any sign. Therefore partition the index set $K \triangleq \{r_0+1, r_0+2, \ldots, r^*\}$ into:

(4.21') $K_- = \{r_0+1, r_0+2, \ldots, \alpha\} = \{j \mid 1 \leq c_j < m\}$

(4.22') $K_0 = \{\alpha+1, \alpha+2, \ldots, \beta\} = \{j \mid c_j = m\}$

(4.23') $K_+ = \{\alpha+1, \alpha+2, \ldots, r^*\} = \{j \mid c_j > m\}$

We are now ready for Theorem 4.3'

Theorem 4.3'

Consider the system given by (1.1') - (1.2') and (4.15'). Let $\tilde{S}(1/z)$ and $\tilde{T}(1/z)$ be the polynomial matrices defined in (4.17'). Suppose that the index-sets K_-, K_0, K_+, as defined in (4.21') - (4.23') are not empty.

Consider the partitioning

(4.24')
$$\tilde{T}(1/z)\,[I + \tilde{G}_r(z)]\tilde{S}(1/z) = \begin{array}{c c} & \begin{array}{cc} \overbrace{\qquad\quad}^{\alpha} & \overbrace{\qquad\quad}^{n-\alpha} \end{array} \\ \begin{array}{c} \alpha\{ \\ n-\alpha\{ \end{array} & \left[\begin{array}{c|c} \tilde{L}_{11}(z) & \tilde{L}_{12}(z) \\ \hline \tilde{L}_{21}(z) & \tilde{L}_{22}(z) \end{array}\right] \end{array}$$

and let $\tilde{b}_j(\cdot)$ be the polynomials defined in (4.19'). Under these conditions,

(4.25') $H \in \ell^1_{n\times n}$

if and only if

(4.26') $$\inf_{|z| \geq 1} |\det[I + \tilde{G}(z)]| > 0$$

and at the pole p the following condition holds

(C') $$\det\{\tilde{L}_{22}(p) + \mathrm{diag}[\tilde{b}_{\alpha+1}(1/p), \ldots, \tilde{b}_{\beta}(1/p), 0, 0, \ldots, 0]\} \neq 0$$

Proof

$\Leftarrow$. Note that (4.25') is equivalent to $[I + \tilde{G}(z)]^{-1} \in \tilde{\ell}^1_{n\times n}$, which by Fact (4.17a') is again equivalent to $\{\tilde{T}(1/z)[I + \hat{G}(z)]$ $\tilde{S}(k/z)\}^{-1} \in \tilde{\ell}^1_{n\times n}$. Take now as multiplier

(4.27') $$\tilde{M}(z) \triangleq \mathrm{diag}\{\underbrace{\tilde{s}(z)^m, \tilde{s}(z)^m, \ldots, \tilde{s}(z)^m}_{r_0}, \underbrace{\tilde{s}(z)^{m-c_{r_0+1}}, \tilde{s}(z)^{m-c_{r_0+2}}, \ldots, s(z)^{m-c_\alpha}}_{\alpha-r_0}, \underbrace{1, 1, \ldots, 1}_{n-\alpha}\}$$

with

(4.28') $$\tilde{s}(z) \triangleq ((z-p)/z) \in \tilde{\ell}^1.$$

As in the continuous-time case, write

$$\{\tilde{T}(1/z)\ [I + \tilde{G}(z)]\tilde{S}(1/z)\}^{-1} = \tilde{M}(z)\ \tilde{N}(z)^{-1}.$$

Then by using the facts above, (1.15') and (4.26') - (4.28'), a detailed calculation shows that $\tilde{N} \in \tilde{\ell}^1_{n\times n}$ and as a consequence of (C') $\inf_{|z| \geq 1} |\det N(z)| > 0$, so $\tilde{N}^{-1} \in \tilde{\ell}^1_{n\times n}$. Since $\tilde{M}$ is also in this algebra, the claim follows. $\bar{\underline{X}}$

$\Rightarrow$ Thus by assumption $\tilde{H} \in \tilde{\ell}^1_{n\times n}$.

(4.26') follows immediately by [17].

To establish (C') we use contradiction. We show that if the L.H.S. of (C') is zero, then there exists an input $u \in \ell^2_n$ which results in an error e not in ℓ^2_n. This is a contradiction, since $u \in \ell^2_n$ and $H \in \ell^1_{n\times n}$ imply $y \in \ell^2_n$ by (1.16'), so $e = u - y$ should be in ℓ^2_n.

The z-transforms of e and u are related by:

(4.29') $$[I + \tilde{G}(z)]\, \tilde{e}(z) = \tilde{u}(z).$$

Multiplying (4.29') on the left by $\tilde{T}(1/z)$ and setting

(4.30') $$\tilde{N}(z) = \tilde{T}(1/z)\ [I + \tilde{G}(z)]\tilde{S}(1/z)\ \tilde{M}(z)$$

(4.31') $$\tilde{u}^*(z) = \tilde{T}(1/z)\tilde{u}(z)$$

(4.32') $$\tilde{e}(z) = \tilde{S}(1/z)\tilde{M}(z)\tilde{e}^*(z) \quad ,$$

we obtain

(4.33') $$\tilde{N}(z)\tilde{e}^*(z) = \tilde{u}^*(z).$$

Observe that

(4.34') $$\tilde{M}(z) = \tilde{s}(z)^m\tilde{\Delta}(z)$$

where

$$\tilde{\Delta}(z) \triangleq \mathrm{diag}\{\underbrace{1,1,\ldots,1}_{r_0},\underbrace{\tilde{s}(z)^{-c_{r_0+1}},\tilde{s}(z)^{-c_{r_0+2}},\ldots,\tilde{s}(z)^{-c_\alpha}}_{\alpha-r_0}, \tag{4.35'}$$

$$\underbrace{\tilde{s}(z)^{-m},\tilde{s}(z)^{-m},\ldots,s(z)^{-m}}_{n-\alpha}\}.$$

With

$$\tilde{N}_1(z) = \tilde{T}(1/z)\left(\sum_{i=0}^{m-1} R_i(z-p)^{-m+i}\right)\tilde{S}(1/z)\tilde{M}(z) \tag{4.36'}$$

$$\tilde{N}_2(z) = \tilde{T}(1/z)\ [I + \tilde{G}_r(z)]\tilde{S}(1/z)\tilde{M}(z) \tag{4.37'}$$

(4.30') and (4.15') imply

$$\tilde{N}(z) = \tilde{N}_1(z) + \tilde{N}_2(z)\ . \tag{4.38'}$$

By (4.36'), (4.34'), (4.35'), (4.28'), (4.16'), (4.17'), (4.19') and (4.22')

$$\tilde{N}_1(z) = \tilde{D}_1(z) \oplus \tilde{D}_2(z) \tag{4.39'}$$

where

$$\tilde{D}_1(z) = \mathrm{diag}\{\underbrace{\tilde{a}_1(1/z),\tilde{a}_2(1/z),\ldots,\tilde{a}_{r_0}(1/z)}_{r_0}, \tag{4.40'}$$

$$\underbrace{\tilde{b}_{r_0+1}(1/z),\tilde{b}_{r_0+2}(1/z),\ldots,\tilde{b}_\alpha(1/z)}_{\alpha-r_0}\}$$

$$(4.41')\qquad D_2(z) = \mathrm{diag}\{\underbrace{\tilde{b}_{\alpha+1}(1/z),\tilde{b}_{\alpha+2}(1/z),\ldots,\tilde{b}_{\beta}(1/z)}_{\beta-\alpha},$$

$$\underbrace{\tilde{b}_{\beta+1}(1/z)\tilde{s}(z)^{c_{\beta+1}-m},\tilde{b}_{\beta+2}(1/z)\tilde{s}(z)^{c_{\beta+2}-m},\ldots,\tilde{b}_{r*}(1/z)\tilde{s}(z)^{c_{r*}-m}}_{r*-\beta},$$

$$\underbrace{0,0,\ldots,}_{n-r*}\}.$$

By (4.37'), (4.24'), (4.27') and (4.28')

$$(4.42')\qquad \tilde{N}_2(z) = \begin{array}{c} \\ \alpha\{ \\ n-\alpha\{ \end{array}\begin{array}{c} \overbrace{\phantom{\tilde{K}_{11}(z)}}^{\alpha}\quad\overbrace{\phantom{\tilde{L}_{12}(z)}}^{n-\alpha} \\ \left[\begin{array}{c|c} \tilde{K}_{11}(z) & \tilde{L}_{12}(z) \\ \hline \tilde{K}_{21}(z) & \tilde{L}_{22}(z) \end{array}\right] \end{array}$$

where

$$(4.43')\qquad \tilde{K}_{11}(p) = 0$$

$$(4.44')\qquad \tilde{K}_{21}(p) = 0\ .$$

Furthermore by (4.40'), (4.18') and (4.20')

$$(4.45')\qquad \det \tilde{D}_1(p) \neq 0$$

and by (4.41'), (4.20'), (4.28') and (4.23')

(4.46') "(C') not true" is equivalent to $\det[\tilde{D}_2(p)+\tilde{L}_{22}(p)] = 0$.

In order to establish the contradiction, using (4.46') we can pick a <u>nonzero</u> vector $\eta \in \mathbb{R}^{n-\alpha}$ in the null-space of $\tilde{D}_2(p) + \tilde{L}_{22}(p)$, thus

$$(4.47') \qquad [\tilde{D}_2(p) + \tilde{L}_{22}(p)]\eta = 0.$$

Pick now the vector $\xi \in \mathbb{R}^{\alpha}$ such that

$$(4.48') \qquad \xi = - [\tilde{D}_1(p)]^{-1} \tilde{L}_{12}(p)\ \eta$$

which is well defined because of (4.45') and because all elements of $\tilde{D}_1(.)$ and $\tilde{L}_{12}(\cdot)$ are in $\tilde{\ell}^1$. Hence, setting

$$(4.49') \qquad \tilde{e}^*(z) = \frac{z}{z-p} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\tilde{u}^*(z) = \begin{pmatrix} \tilde{u}_1^*(z) \\ \\ \tilde{u}_2^*(z) \end{pmatrix} \begin{matrix} \}\alpha \\ \\ \}\ n-\alpha \end{matrix}$$

(4.33'), (4.38'), (4.39') and (4.42') imply

$$(4.50') \qquad \tilde{u}_1^*(z) = \{[\tilde{D}_1(z) + \tilde{K}_{11}(z)]\xi + \tilde{L}_{22}(z)\eta\}(z/(z-p))$$

$$(4.51') \qquad \tilde{u}_2^*(z) = \{\tilde{K}_{21}(z)\xi + [\tilde{D}_2(z) + \tilde{L}_{22}(z)]\eta\}(z/(z-p)).$$

Observe that because of (4.33') - (4.44') and (4.47') - (4.48') the expressions between the braces in the R.H.S. of (4.50') - (4.51') have a zero at p. All components of the expressions between the braces on the R.H.S. of (4.50') - (4.51') are in $\tilde{\ell}_1$, hence are analytic in $|z| > 1$, such that their zero at p is at least of first order ($|p| > 1$!). Therefore $\tilde{u}^*(z)$ is well behaved and bounded at p these remarks and the properties of the components of $\tilde{u}_1^*(\cdot)$ and $\tilde{u}_2^*(\cdot)$ imply that $\tilde{u}^*(\cdot)$ is analytic in $|z| > 1$, bounded in $|z| \geq 1$, continuous on $|z| = 1$ and, as $|z| \to \infty, \tilde{u}^*(z) \to$ a constant. Therefore the conditions of lemma A.3', i.e. the discrete-time counterpart of Wiener's Theorem, are satisfied and so $\tilde{u}^*(\cdot) \in \tilde{\ell}_n^2$. Furthermore by (1.16'), (4.31') and Fact (4.17a') $\tilde{u}(\cdot) \in \tilde{\ell}_n^2$ or

(4.52') $$u \in \ell_n^2 .$$

Finally by (4.49'), (4.32'), (4.27') - (4.28') and since $S(1/z)$ is unimodular and $\eta \neq 0$, $e(z)/z$ has a pole at p with nonzero residue, thus

(4.53') $$e \notin \ell_n^2$$

and by (4.52') - (4.53') we have shown the contradiction we were after. $\bar{\underline{X}}$

Remarks

(4.53a') Remark. If $|p| = 1$ then (4.26') and (C') are still sufficient for stability; moreover (4.26') and (C') are also necessary if the magnitudes of the components of the expressions between the braces on the R.H.S. of (4.50') - (4.51') are at least of order $O(|z-p|^{\delta})$ for some real $\delta > 0$ at p.

The first statement is obvious from the sufficiency part of Theorem 4.3'.

The second statement can be shown by (a) using an analog reasoning as in Remark (3.58a), (b) replacing (4.49') in the necessity part of the proof of Theorem 4.3' by

$\tilde{e}*(z) = (z/(z-p))^{\gamma} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ where $\gamma = \frac{1+\varepsilon}{2}$, $\varepsilon \in (0,\delta]$, and (c) by using lemma A.3'.

(4.53b') Remark. Theorem 4.3' describes in detail what happens when the sets K_-, K_0, K_+ given by (4.21') - (4.23') are nonempty. When one or more of these sets are empty the required modifications of (C') and the multiplier $\tilde{M}(z)$ are straightforward.

(4.53c') Remark. In case there are ℓ poles at $p_1, p_2, \ldots, p_\ell$ of order $m_1, m_2, \ldots, m_\ell$ with magnitude larger than one, which are either real or complex conjugate, one proceeds similarly as in Theorem 4.3'. First the principal part of $\tilde{G}$ is transformed into a polynomial matrix in $\frac{1}{z}$ and Facts (4.15a'), (4.16a'), (4.17a') and (4.17b') are repeated. Next in the proof of stability one uses a product of multipliers similar to $\tilde{M}(z)$. Observe that condition (C') is used only to check that det $\tilde{N}(z)$ does not vanish at $z = p$. Therefore for the more general case an appropriate condition (C) is required at each pole. This was checked by us.

(4.53d') Remark. Using completely similar techniques as in the continuous-time case a completely similar interpretation can be obtained for the condition (C').

Finally to be more precise we state the extension of Theorem 4.3' to the multiple pole case where we use the least monic common denominator of all minors of the principal part of the open-loop transfer function $\tilde{G}$.

Theorem 4.4'.

Consider the system defined by (1.1'), (1.2') and (3.12'). Let $\tilde{d}(z)$ be the least monic common denominator of all minors of the matrix $\sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} R_{km}(s-p_k)^{-m_k+m}$, i.e. the principal part of $\tilde{G}$, and let $\tilde{d}(z)$ be given by

$$\tilde{d}(z) = \prod_{k=1}^{\ell} (z-p_k)^{r_k}. \tag{4.59'}$$

Let d_k be the order of the pole at p_k of $\det[I + \tilde{G}(\cdot)]$ for $k = 1,2,\ldots\ell$. Under these conditions

$$H \in \ell^1_{n\times n} \tag{4.25'}$$

if and only if

$$\inf_{|z| \geq 1} |\det[I + \tilde{G}(z)]| > 0 \tag{4.26'}$$

and

$$d_k = r_k \quad \text{for } k = 1, 2, \ldots, \ell . \tag{C,k'}$$

$\bar{\underline{X}}$

CONCLUSION

5.1 Discussion of the Main Results of the Dissertation.

In this dissertation we have presented a series of results related to the input-output properties of both continuous-time and discrete-time convolution feedback systems. First a graphical test was developed to check $\inf_{\mathrm{Re}\, s \geq 0} |1 + \hat{g}(s)| > 0$, where $\hat{g}$ is the sum of a term in $\hat{\mathcal{A}}$ and a finite number of poles in Re s $\geq$ 0. Implications for the n-input n-output case and the counterpart of this test for analog discrete-time systems were discussed in the sequel. Next in analog treatments, (a) the representation of the open-loop transfer function given a stable system and (b) necessary and sufficient conditions for stability for both continuous-time and discrete-time convolution feedback systems were discussed.

In all cases treated unity feedback was assumed. It is now proper to make the following remark. The extension of the graphical test and the representation of the open-loop transfer function, given a stable system, to the case of a constant non-unity feedback is straightforward. The same can be said about the necessary and sufficient conditions for stability as given by Theorems 3.2 and 4.2'. Special care however should be used concerning the necessary and sufficient conditions for stability as given by Theorems 3.3, 3.4, 3.5, 4.3' and 4.4'. Indeed these latter theorems can only be extended without difficulty to a constant non-unity nonsingular feedback. For example for the case of Theorem 3.3 one still has the following relation between open-loop and closed-loop transfer functions $\hat{G}$, respectively $\hat{H}$, when a constant non-unity feedback matrix K is present, i.e.

$$I - K\hat{H} = (I + K\hat{G})^{-1}.$$

A straightforward adaptation of Theorem 3.3 can only determine necessary and sufficient conditions such that $(I + K\hat{G})^{-1} \in \hat{\mathcal{A}}^{n\times n}$ i.e. $K\hat{H} \in \hat{\mathcal{A}}^{n\times n}$. Therefore if K is a singular matrix in $\mathbb{R}^{n\times n}$ then only some components of $\hat{H}$ are guaranteed to be in $\hat{\mathcal{A}}$, not all of them! Hence additional research is needed if one allows a singular constant non-unity feedback. As an introduction to this, a necessary and sufficient condition for stability is stated in Theorem 5.1 for the case of a simple n-input n-output continuous-time convolution feedback system with a singular constant feedback matrix.

Finally in order to show that the results of this dissertation have implications in the investigations of nonlinear stability-analysis a sufficient condition for the $L^q_{2n}[0,\infty)$ input-output stability [any $q \in [1,\infty]$] of a nonlinear time-varying 2n-input 2n-output feedback system is stated in Theorem 5.2.

5.2 Necessary and Sufficient Conditions for Stability of a Simple n-input n-output Continuous-time Convolution Feedback System With Singular Constant Feedback.

System Description

We consider an n-input, n-output continuous-time convolution feedback system. The input u, output y and error e are functions from $\mathbb{R}_+$ to $\mathbb{R}^n$ or corresponding distributions on $\mathbb{R}_+$. They are related by (see Fig. 5.1)

$$u = G*e \tag{5.1}$$

$$e = u - Ky \tag{5.2}$$

where G is an n×n matrix whose elements are distributions on $\mathbb{R}_+$ and K is a singular real n×n constant matrix with rank $\rho < n$. The Laplace transform $\hat{G}$ of G is given by

$$\hat{G}(s) = R(s-p)^{-1} + \hat{G}_r(s) \tag{5.3}$$

where $\begin{cases} p \in \mathbb{R} \text{ and } p > 0 \,; \\ R \text{ is a real } n{\times}n \text{ constant matrix} \\ \hat{G}_r \in \hat{\mathcal{A}}^{n\times n}. \end{cases}$

Theorem 5.1

Given the system defined by (5.1) - (5.3).

Let d be the order of the pole p of $\det[I + K\hat{G}(\cdot)]$.

Let r be the rank of the constant matrix KR.

Let P, Q be two nonsingular matrices, elements of $\mathbb{R}^{n\times n}$ such that

$$QKP = \begin{array}{c} \\ \rho\{ \\ n-\rho\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\quad\quad}^{\rho} & \overbrace{\quad}^{n-\rho} \end{array} \\ \left[\begin{array}{c|c} I_{\rho\times\rho} & 0 \\ \hline 0 & 0 \end{array}\right] \end{array} \tag{5.4}$$

where $I_{\rho\times\rho}$ is the $\rho \times \rho$ unit matrix, and let

$$P^{-1} R \triangleq R^* \tag{5.5}$$

with R^* partitioned into

$$R^* = \begin{array}{r} \\ \rho\{ \\ n-\rho\{ \end{array} \overbrace{\left[\begin{array}{c} R_2^* \\ \hline R_2^* \end{array}\right]}^{n} . \tag{5.6}$$

Finally let H be the closed-loop impulse response of the system defined by (5.1) - (5.3).

Under these conditions,

$$H \in \mathcal{A}^{n\times n} \tag{5.7}$$

if and only if,

$$\inf_{\mathrm{Re}\, s \geq 0} \left|\det[I + K\hat{G}(s)]\right| > 0 \tag{5.8}$$

$$d = r \tag{5.9}$$

<u>and</u>

$$R_2^*[I + K\hat{G}(p)] = 0_{(n-\rho)\times n} \tag{5.10}$$

where $0_{(n-\rho)\times n}$ is the $(n-\rho)\times n$ zero matrix.

<u>Sketch of the Proof</u>

Observe that under the conditions of the theorem

$$\hat{H} = \hat{G}[I + K\hat{G}]^{-1} \tag{5.11}$$

and

$$(5.12)\qquad I - K\hat{H} = [I + K\hat{G}]^{-1} .$$

Moreover r is the maximal order of the pole at p of all minors of the principal part of $K\hat{G}$. Therefore by an anolog reasoning as in Theorems 3.3 and 3.4 and by (5.12)

$$(5.8) \text{ and } (5.9) \Longleftrightarrow K\hat{H} \in \hat{\mathcal{A}}^{n\times n}$$

Define now

$$\hat{H}^* \triangleq P^{-1}\hat{H} \text{ with } \hat{H}^* = \begin{array}{r} \\ \rho\,\{ \\ n-\rho\,\{ \end{array}\!\!\begin{array}{c} \overbrace{\qquad\quad}^{n} \\ \left[\begin{array}{c} H_1^* \\ \hline H_2^* \end{array}\right] \end{array} .$$

Observe that because P is nonsingular

$$\hat{H} \in \hat{\mathcal{A}}^{n\times n} \Longleftrightarrow \hat{H}^* \in \hat{\mathcal{A}}^{n\times n} .$$

It can then be shown that

$$K\hat{H} \in \hat{\mathcal{A}}^{n\times n} \Longleftrightarrow \text{all elements of } \hat{H}_1^* \text{ are in } \hat{\mathcal{A}}$$

and given (5.8) and (5.9)

$$(5.10) \Longleftrightarrow \text{all elements of } \hat{H}_2^* \text{ are in } \hat{\mathcal{A}} .$$

$\underline{\bar{X}}$

<u>Remark</u>. Observe that when K is nonsingular then $\hat{H}_1^* = \hat{H}$ such that no additional condition (5.10) is needed. Moreover if K is nonsingular then the maximal order of the pole at p of the principal parts of $\hat{G}$ and $K\hat{G}$ are the same [Both principal parts of $\hat{G}$ and $K\hat{G}$ are equivalent polynomial matrices in $\frac{1}{s-p}$]. Next if K is

nonsingular then r is equal to maximal order order of the pole at p of all minors of the principal part of $\hat{G}$: in our case the rank of the constant matrix R.

5.3 A Sufficient Condition for the input-output Stability of a Given 2n-input 2n-output Nonlinear Time-varying Feedback System.

System Description (Fig. 5.2)

In this paragraph we consider a 2n-input 2n-output nonlinear time-varying feedback system S as shown in Fig. 5.2. The inputs u_1, u_2, errors e_1, e_2, outputs y_1, y_2 are functions of time mapping $\mathbb{R}_+$ into $\mathbb{R}^n$.
The block labeled Φ_t is a memoryless, time-varying nonlinearity whose input-output relation is defined in terms of a nonlinear function ϕ: $\mathbb{R}^n \times \mathbb{R}_+ \longmapsto \mathbb{R}^n$ by

(5.13) $$y_1(t) = \phi[e_1(t),t]$$

The block labeled G is a linear time-invariant subsystem whose input-output relation is defined in terms of its impulse response matrix G by convolution, i.e.

(5.14) $$y_2(t) = (G*e_2)(t) \quad .$$

The Laplace transform $\hat{G}$ of G satisfies (3.12).
The system equations are (5.13), (5.14), (3.12) and

(5.15) $$e_1 = u_1 - y_2$$

(5.16) $$e_2 = u_2 + y_1 \ .$$

Definition

The system S (Fig. 5.2) defined by (5.13) - (5.16) and (3.12) is said to be input-output stable if, given any $q \in [1,\infty]$, to any input (u_1,u_2) belonging to $L^q_{2n}[0,\infty)$ corresponds an output (y_1,y_2) belonging to $L^q_{2n}[0,\infty)$.

Note.

Let F be an element of $\mathcal{A}^{n\times n}$, i.e.

$$(5.17) \qquad F(t) = \begin{cases} F_a(t) \sum_{i=0}^{\infty} F_i\, \delta(t-t_i) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where $F_a(\cdot) \in L^1_n[0,\infty)$, $F_i \in \mathbb{R}^{n\times n}$ for all i = 0, 1, 2, ..., $\sum_{i=0}^{\infty} |F_i| < \infty$ (where $|\cdot|$ is any matrix norm in $\mathbb{R}^{n\times n}$) and $0 = t_0 < t_1 < t_2 < \dots$.

Define a norm on $\mathcal{A}^{n\times n}$ by

$$(5.18) \qquad \|F\|_a = \int_0^\infty |F_a(t)|dt + \sum_{i=0}^{\infty} |F_i|$$

where $|\cdot|$ is any matrix norm in $\mathbb{R}^{n\times n}$.

It is well known that the pair $(\mathcal{A}^{n\times n}, \|\cdot\|_a)$ is a Banach algebra [2,20].

Theorem 5.2

Consider the system S defined by (5.13) - (5.16) and (3.12) (see Fig. 5.2). Let $\hat{d}(s)$ be the greatest monic common divisor of all minors of the principal part of $\hat{G}$ and let $\hat{d}(s)$ be given by

$$(5.19) \qquad \hat{d}(s) = \prod_{k=1}^{\ell} (s-p_k)^{r_k}.$$

Let Φ_t be the time-varying nonlinearity whose characteristic $\phi(\cdot,\cdot)$ has the following properties:

N.1. $\phi(\cdot,\cdot)$: $\mathbb{R}^n \times \mathbb{R}_+ \longmapsto \mathbb{R}^n$ and ϕ is a continuous function with respect to its first argument and is a regulated function(1) with respect to its second argument.

N.2. There exist an $n \times n$ constant nonsingular matrix K and a positive real number μ such that

$$|\phi(x,t) - \phi(x',t) - K(x-x')| \leq \mu|x - x'|$$

for all $t \in \mathbb{R}_+$ and all $x, x' \in \mathbb{R}^n$;

moreover

$$\phi(0,t) = 0 \quad \text{for all } t \in \mathbb{R}_+ .$$

Let H be the closed-loop impulse response of the n-input n-output convolution feedback system with G as open-loop impulse response and K as constant feedback matrix, i.e.

$$\hat{H} \triangleq \hat{G}[I + K\hat{G}]^{-1}.$$

Let d_k be the order of the pole at p_k of $\det[I + K\hat{G}(\cdot)]$.

Under these conditions,

if

$$\inf_{\mathrm{Re}\, s \geq 0} |\det[I + K\hat{G}(s)]| > 0 ,$$

$$d_k = r_k \quad \text{for } k = 1, 2, \ldots, \ell,$$

---

(1) $\phi(x,t)$: $\mathbb{R}^n \times \mathbb{R}_+ \longmapsto \mathbb{R}^n$ is called regulated in t iff for all fixed $x \in \mathbb{R}^n$, $\phi(x,t)$ has finite one-sided limits at every $t \in \mathbb{R}_+$.

and

$$\|H\|_a \, \mu < 1,$$

<u>then</u> for any $q \in [1,\infty]$

the maps $(u_1,u_2) \longmapsto (e_1,e_2)$

$$(u_1,u_2) \longmapsto (y_1,y_2)$$

are well-defined sending $L^q_{2n}[0,\infty)$ into $L^q_{2n}[0,\infty)$; moreover they are bounded and uniformly continuous on $L^q_{2n}[0,\infty)$. (Note that the first statement implies that the given system is input-output stable).

<u>Sketch of the Proof</u>

This theorem follows essentially by loop-transformation [20], Theorem 3.5, (1.16), assumption N.2 and the incremental small gain Theorem [20].

APPENDIX

A.1 Results on Continuous-time Systems

Lemma A.1.

Given the system defined by (2.1) - (2.6). Let h be the closed-loop impulse response of the above system. Under these conditions

$$h \in \mathcal{A}$$

if and only if

(2.9) $$\inf_{\mathrm{Re}\, s \geq 0} |1 + \hat{g}(s)| > 0$$

Proof

$\Leftarrow$. Introduce a multiplier $\hat{m}(s) = \prod_{k=1}^{\ell} \left(\frac{s-p_k}{s+1}\right)^{m_k}$. Observe that $\hat{m}(s) \in \hat{\mathcal{A}}$ and that $\hat{m}(s)^{-1}$ is well defined except at $s = p_k$ for $k = 1, 2, \ldots, \ell$. Next observe that $1 - \hat{h} = (1 + \hat{g})^{-1}$. So we are done if $(1 + \hat{g})^{-1} \in \hat{\mathcal{A}}$. Observe that $(1 + \hat{g})^{-1} = \hat{m}[(1 + \hat{g})\hat{m}]^{-1}$, hence $h \in \mathcal{A}$ if $[(1 + \hat{g})\hat{m}]^{-1} \in \hat{\mathcal{A}}$. This follows by (1.14) observing that $(1 + \hat{g})\hat{m} \in \hat{\mathcal{A}}$ and by (2.9) and the structure of $\hat{m}$ that $\inf_{\mathrm{Re}\, s \geq 0} |(1 + \hat{g}(s)\hat{m}(s)| > 0$.

$\Rightarrow$. Immediately $\hat{h} \in \hat{\mathcal{A}}$ implies $(1 + \hat{g})^{-1} \in \hat{\mathcal{A}}$. Therefore by (1.13) $\sup_{\mathrm{Re}\, s \geq 0} \left| \frac{1}{1 + \hat{g}(s)} \right| < \infty$ which implies (2.9).

Decomposition-Lemma A.2

Let g be a complex-valued distribution of order 0 with support on $\mathbb{R}_+$, i.e.

$$g(t) \triangleq \begin{cases} 0 & \text{for } t < 0 \\ g_a(t) + \sum_{i=0}^{\infty} g_i \delta(t-t_i) & \text{for } t \leq 0 \end{cases} \tag{A.1}$$

where $g_a(\cdot)$ is a complex-valued function on $\mathbb{R}_-$; $g_i \in \mathbb{C}$ for all i and $0 = t_0 < t_1 \ldots < t_i < \ldots$.

Let the "magnitude" $|g|$ of g be defined by

$$|g(t)| \triangleq \begin{cases} 0 & \text{for } t < 0 \\ |g_a(t)| + \sum_{i=0}^{\infty} |g_i| \; \delta(t-t_i) & \text{for } t \geq 0 \end{cases} \tag{A.2}$$

and assume that g is magnitude integrable, i.e.

$$|g| \in \mathcal{A}. \tag{A.3}$$

[Observe that $\mathcal{A}$ considers only real-valued distributions]

Under these conditions if

$$\hat{v}(s) = \frac{\hat{g}(s)}{s-p} \text{ where } p \in \mathbb{C}, \text{ Re } p > 0 \tag{A.4}$$

then

$$\hat{v}(s) = \frac{\hat{g}(p)}{s-p} + \hat{v}_r(s) \text{ where } v_r \in L^1_{[0.\infty)}, \text{ thus } |v_r| \in \mathcal{A}. \tag{A.5}$$

Proof

We shall use the properties of convolution algebra LA

(LA_+) of real-valued distributions of order 0 with support on $\mathbb{R}$ ($\mathbb{R}_+$) ([14] § 8). f is said to be in LA (LA_+) iff on its support

$$f(t) = f_a(t)+\sum_{i=0}^{\infty} f_i \ \delta(t-t_i)$$

where $f_a(\cdot)$ is a real-valued function belonging to $L^1(-\infty,\infty)$ ($L^1[0,\infty)$), $f_i \in \mathbb{R}$ for $i = 0, 1, 2, \ldots, \sum_{i=0}^{\infty}|f_i| < \infty$ and $-\infty < t_0 < t_1 < \ldots < t_i < \ldots$ ($0 = t_0 < t_1 < \ldots < t_i < \ldots$).

Convolution in LA is defined by

$$(f_1*f_2)\ (t) \triangleq \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau \tag{A.6}$$

where integration is performed in the sense of distribution theory.

That is : if $f_1(t) = f_{1a}(t) + \sum_{i=0}^{\infty} f_{1i}\delta(t-t_i)$ and if $f_2(t) = f_{2a}(t) + \sum_{j=0}^{\infty} f_{2j}\delta(t-t_j)$ then

$$(f_1*f_2)(t) \triangleq \int_{-\infty}^{\infty} f_{1a}(\tau)\ f_{2a}(t-\tau)d\tau + \sum_{t_j\leq t} f_{2j}f_{1a}(t-t_j)$$
$$\sum_{t_i\leq t} f_{1i}f_{2a}(t-t_i) + \sum_{t_i+t_j\leq t} f_{1i}f_{2j}\delta(t-t_i-t_j)\ .$$

It is easy to check that f_1, $f_2 \in$ LA implies $f_1*f_2 \in$ LA.

Observe that $LA_+ \equiv \mathcal{A}$ and LA is an extension of $\mathcal{A}$ if the support of the distribution is extended to the whole real line. It is therefore an easy matter to check that if $f_1 \in LA$ and $f_2 \in L^q(-\infty,\infty)$, then $f_1 * f_2 \in L^q(-\infty,\infty)$ where $q \in [1,\infty]$. Finally observe that all the properties of LA are conserved if we consider complex-valued distributions instead of real-valued distributions. Hence without loss of generality the lemma will be proved if we proved if we restrict ourselves to the case where in (A.4)

$$g \in \mathcal{A},\ p \in \mathbb{R},\ p > 0. \tag{A.7}$$

Consider now the complex-valued function $\hat{v}_r$ defined

$$\hat{v}_r(s) \triangleq \frac{\hat{g}(s) - \hat{g}(p)}{s - p} \tag{A.8}$$

We are done if we show that $v_r \in L^1(0,\infty)$.

Now by (A.7) and (1.13) $\hat{g}(s)$ is analytic in Re $s > 0$, bounded and continuous in Re $s \geq 0$. Hence, because $p \in \mathbb{R}$ and $p > 0, \hat{v}_r(p) = \dot{\hat{g}}(p)$ which is well defined. Moreover $\dot{\hat{v}}_r(s)$ is well-defined in Re $s > 0$. Hence $\hat{v}_r(s)$ is analytic in Re $s > 0$. Furthermore $\hat{v}_r$ is bounded in Re $s \geq 0$ and continuous in Re $s \geq 0$ and, as $|\omega| \to \infty$,

$|\hat{v}_r(\sigma + j\omega)|$ is at most $O\left(\frac{1}{|\omega|}\right)$ uniformly for all fixed $\sigma \geq 0$.

Therefore by Wiener's Theorem ([12] p. 8)

$$\begin{cases} v_r \in L^2[o,\infty) \\ v_r(t) = \underset{N\to\infty}{\text{l.i.m.}} \dfrac{1}{2\pi} \displaystyle\int_{\omega=-N}^{N} \hat{v}_r(\sigma + j\omega) e^{(\sigma+j\omega)t} d\omega \end{cases} \tag{A.9}$$

where the integration may be performed on any vertical line $\{s = \sigma + j\omega \mid \sigma = \text{constant} \geq 0\}$.

Consider now a distribution h and functions e_+, e_- mapping $\mathbb{R}$ into $\mathbb{R}$ defined by:

$$h(t) \triangleq \begin{cases} 0 & \text{for } t < 0 \\ g(t) - \hat{g}(p)\delta(t) & \text{for } t \geq 0 \end{cases}$$

$$e_+(t) \triangleq \begin{cases} e^{pt} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$e_-(t) \triangleq \begin{cases} 0 & \text{for } t \geq 0 \\ -e^{pt} & \text{for } t < 0 \end{cases} .$$

Observe that $h \in \mathcal{A} \subset LA$ and that $e_- \in L^1(-\infty,\infty) \cap L^2(-\infty,\infty)$, therefore $h*e_- \in L^1(-\infty,\infty) \cap L^2(-\infty,\infty)$. Moreover using (A.6)

$$\text{(A.10)} \qquad (h*e_-)(t) = (h*e_+)(t) = \begin{cases} -\int_t^\infty g(\tau)e^{p(t-\tau)}d\tau & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

therefore

$$\text{(A.11)} \qquad (h*e_-) = (h*e_+) \in L^1[0,\infty) \cap L^2[0,\infty).$$

Note that because of (A.11), $\widehat{h*e_+}$ is well-defined in Re $s \geq 0$, analytic in Re $s > 0$ and bounded and continuous in Re $s \geq 0$. Moreover, [12],

$$\text{(A.12)} \qquad (h*e_+)(t) = \underset{N\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-N}^{+N} \widehat{(h*e_+)}(\sigma+j\omega)e^{(\sigma+j\omega)t}d\omega, \text{ any fixed } \sigma \geq 0.$$

We calculate now $\widehat{(h*e_+)}(s)$ for Re $(s-p) > 0$ using (A.10) and integration by parts. Observe that

$$\widehat{(h*e_+)}(s) = -\int_0^\infty \int_{\tau=t}^\infty g(\tau)e^{-p\tau}d\tau \; e^{-(s-p)t}dt.$$

Let us set

$$a(t) = -\int_{\tau=t}^\infty g(\tau)e^{-p\tau}d\tau$$

$$b(t) = -\frac{e^{-(s-p)t}}{(s-p)} \text{ where Re } (s-p) > 0.$$

Introducing generalized derivatives ([28] p. 38 example 2) and observing that integration by parts is allowed on $\mathbb{R}_+$, if Re $(s-p) > 0$, because all terms converge, we obtain:

$$\widehat{(h*e_+)}(s) = \int_0^\infty a(t)\dot{b}(t)dt = \left[a(t)b(t)\right]_0^\infty - \int_0^\infty \dot{a}(t)b(t)dt,$$

that is along with (A.8)

$$\widehat{h*e_+} = \hat{v}_r \quad \text{for Re } (s-p) > 0.$$

Therefore because of (A.9), (A.12) we obtain in L^2-sense and thus almost everywhere

$$\text{(A.13)} \qquad v_r(t) = (h*e_+)(t) = \begin{cases} 0 & \text{for } t < 0 \\ -\int_{\tau=t}^\infty g(\tau)e^{p(t-\tau)}d\tau & \text{for } t \geq 0 \end{cases}$$

which because of (A.11) implies $v_r \in L^1[0,\infty)$ Q.E.D. $\bar{\underline{X}}$

(A.13a) Remark. It is easy to check that formula (A.13) is also valid in the case where g is a complex-valued distribution, satisfying. (A.1) - (A.3), $p \in \mathbb{C}$ and Re $p > 0$.

(A.13b) Remark. If in (A.4) Re $p = 0$, then previous theory is not valid. In fact if v_r exists then v_r satisfies (A.13), such that for Re $p = 0$ the decomposition (A.5) is valid if and only if

$$\text{(A.14)} \qquad -\int_{\tau=t}^{\infty} g(\tau)e^{p(t-\tau)}d\tau \quad \text{belongs to } L^1[0,\infty).$$

this however is not automatically satisfied if $|g| \in \mathcal{A}$.

Consider following example:

$$g(t) = \begin{cases} 0 & t < 0 \\ 1 & t \in [0,1) \\ t^{-3/2} & T \in [1,\infty) \end{cases}$$

and let $p = 0$

Observe that $g \in \mathcal{A}$, however $-\int_t^{\infty} g(\tau)e^{p(t-\tau)}d\tau = -\int_t^{\infty} g(\tau)d\tau$

does not belong to $L^1[0,\infty)$, furthermore it does not even belong to $L^2[0,\infty)$. Therefore the decomposition of Lemma A.2 is not applicable if Re $p = 0$. However if (A.14) is satisfied then a decomposition (A.5) is applicable.

Some corrolaries from Lemma A.2 are now stated.

Corrolary A.2.1

Let

$\hat{v}(s) = \dfrac{\hat{g}(s)}{s-p}$ where $g \in \mathcal{A}$, $p \in \mathbb{R}$, $p > 0$.

Then

$$\hat{v}(s) = \frac{\hat{g}(p)}{s-p} + \hat{v}_r(s), \text{ where } v_r \in \mathcal{A}$$

[Observe that g and v_r are real-valued]

<u>Corollary A.2.2</u>

Let

$$\hat{v}(s) = \frac{\hat{g}(s)}{(s-p_1)(s-p_2)}$$

where $g \in \mathcal{A}$; $p_1, p_2 \in \mathbb{C}$; $p_1 = \bar{p}_2$ (i.e. they are complex conjugate) and Re $p_1 = R_e\, p_2 > 0$. Then

$$\hat{v}(s) = \frac{\hat{g}(p_1)}{p_1-p_2}\;\frac{1}{s-p_1} + \frac{\hat{g}(p_2)}{p_2-p_1}\cdot\frac{1}{s-p_2} + \hat{v}_r(s)$$

where $v_r \in \mathcal{A}$

[Observe that this result is obtained after applying two times Lemma A.2 and that g and v_r are real valued].

<u>Corollary A.2.3</u>

Let

$$\hat{v}(s) = \frac{\hat{g}(s)}{\prod_{k=1}^{\ell}(s-p_k)^{m_k}}$$

where $g \in \mathcal{A}$ and the poles p_k are such that (a) Re $p_k > 0$ for $k = 1, 2, \ldots, \ell$ and (b) they are either real or pairwise conjugate complex. Then

$$\hat{v}(s) = \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} r_{km}\left(s-p_k\right)^{-m_k+m} + \hat{v}_r(s)$$

where $v_r \in \mathcal{A}$ and the coefficients r_{km} are either real or conjugate complex according to the poles.

Corollary A.2.4

Let $\hat{G}(s)$ be the open-loop transfer function of a real n-input n-output convolution feedback system, given by

$$\hat{G}(s) = \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} R_{km}(s-1_k)^{-m_k+m} + \hat{G}_r(s) \tag{2.64}$$

where $G_r \in \mathcal{A}^{n\times n}$, Re $p_k > 0$ for $k = 1, 2, \ldots, \ell$,, the poles p_k are either real or complex conjugate and the matrices R_{km} are real or conjugate complex n×n matrices according to their corresponding poles.

Then

$$\det[I + \hat{G}(s)] = \sum_{k=1}^{\ell'} \sum_{m=1}^{m_k'-1} r_{km}(s-p_k)^{-m_k'+m} + \hat{g}_r(s) \tag{A.15}$$

where
$g_r \in \mathcal{A}$
$\ell' \leq \ell$
m_k' is the order of the pole at p_k of $\det[I + \hat{G}(s)]$, thus $r_{k0} \neq$ for $k = 1,2,\ldots,\ell$;
the coefficients r_{km} are either real or conjugate complex constants according to the corresponding poles.

[Observe that "crossterms" of the type $\frac{\hat{g}(s)}{\prod_{k=1}^{\ell''} (s-p_k)^{m_k''}}$ may give rise to "restterms" that are Laplace-transforms of complex-valued functions. However the sum of the "principal parts" and the sum of the "restterms" always are the Laplace-transforms of real-valued distributions or functions].

A.2 Results on Discrete-time Systems

Lemma A.1'

Given the system defined by (2.1') - (2.6'). Let h be the closed-loop impulse response of the above system. Under these conditions

$$h \in \ell^1$$

if and only if

$$\inf_{|z| \geq 1} |1 + \tilde{g}(z)| > 0 \tag{2.9'}$$

Proof

$\Leftarrow$. Introduce a multiplier $\tilde{m}(z) \triangleq \prod_{k=1}^{\ell} \left(\frac{z-p_k}{z}\right)^{m_k}$. Observe that $\tilde{m}(z) \in \tilde{\ell}^1$ and that $\tilde{m}(z)^{-1}$ is well defined except at $z = p_k$ for $k = 1,2,\ldots,\ell$. Next observe that $1 - \tilde{h} = (1 + \tilde{g})^{-1}$. So we are done if $(1 + \tilde{g})^{-1} \in \tilde{\ell}^1$. Observe that $(1 + \tilde{g})^{-1} = \tilde{m}[(1 + \tilde{g})\tilde{m}]^{-1}$, hence $h \in \ell^1$ if $[(1 + \tilde{g})\tilde{m}]^{-1} \in \tilde{\ell}^1$. This follows by (1.14') observing that $(1 + \tilde{g})\tilde{m} \in \tilde{\ell}^1$ and by (2.9') and the structure of (2.9') that

$$\inf_{|z| \geq 1} |(1 + \tilde{g}(z))\tilde{m}(z)| > 0.$$

$\Rightarrow$. Immediately $\tilde{h} \in \tilde{\ell}^1$ implies $(1 + \tilde{g})^{-1} \in \tilde{\ell}^1$. Therefore by (1.13') $\sup_{|z| \geq 1} \left|\frac{1}{1 + \tilde{g}(z)}\right| < \infty$ which implies (2.9').

Decomposition-Lemma A.2'

Let g be a complex-valued sequence, i.e.

(A.1') $g = \{g_0, g_1, g_2, \ldots, g_i, \ldots\}$

where $g_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots$, and assume that

(A.2') $|g| \triangleq \{|g_i|\}_{i=0}^{\infty} \in \ell^1$.

Under these conditions if

(A.3') $\tilde{v}(z) = \dfrac{\tilde{g}(z)}{z-p}$ where $p \in \mathbb{C}$, $|p| > 1$

then

(A.4') $\tilde{v}(z) = \dfrac{\tilde{g}(p)}{z-p} + \tilde{v}_r(z)$ where $|v_r| \triangleq \{|v_{ri}|\}_{i=0}^{\infty} \in \ell^1$

<u>Proof</u>

We shall use properties of the convolution algebra $\ell(\ell_+)$ of absolutely summable real-valued functions defined on $\mathbb{Z}$ (i.e. sequences) with support on $\mathbb{Z}$ ($\mathbb{Z}_+$): f is said to be in $\ell(\ell_+)$ iff on its support $f_i \in \mathbb{R}$ and $\sum_{i=-\infty}^{\infty} |f_i| < \infty$ $\left(\sum_{i=0}^{\infty} |f_i| < \infty\right)$. Convolution in ℓ is defined by

(A.5') $$(f_1 * f_2)_i \triangleq \sum_{j=-\infty}^{\infty} f_{1j} f_2(i-j) \quad \text{for } i = \ldots, -1, 0, 1, \ldots$$

It can be checked that $f_1, f_2 \in \ell$ implies $f_1 * f_2 \in \ell$. Observe that ℓ_+ can be identified with ℓ^1, by assigning the value zero to the elements of ℓ^1 on an additional subdomain $\mathbb{Z}_- - \{0\}$. Where necessary we assume that this has been done and denote that by $\ell_+ \equiv \ell^1$. Note further that all the properties of ℓ are conserved if we consider complex-valued sequences instead of real-valued sequences. Hence without loss of generality the lemma will be proved if we restrict ourselves to the case where in (A.3')

(A.6') $g \in \ell^1 \equiv \ell_+$, and $p \in \mathbb{R}$, $|p| > 1$.

Consider now the complex-valued function $\tilde{v}_r$ given by

(A.7') $\tilde{v}_r(z) \triangleq \dfrac{\tilde{g}(z)-\tilde{g}(p)}{z - p}$.

We are done if we show that $v_r \in \ell_+ \equiv \ell^1$.

Observe that because of (A.6') and (1.13') $\tilde{g}(z)$ is analytic in $|z| > 1$, bounded on $|z| \geq 1$ and continuous on $|z| \geq 1$ and, as $|z| \to \infty$, $g(z) \to g_0$ = constant. Hence, because $|p| > 1$, $\tilde{v}_r(p) = \dot{\tilde{g}}(p)$ which is well defined. Moreover $\dot{\tilde{v}}_r(z)$ is well defined in $|z| > 1$. Hence $\tilde{v}_r(z)$ is analytic in $|z| > 1$. Furthermore $\tilde{v}_r(z)$ is bounded and continuous in $|z| \geq 1$ and, as $|z| \to \infty$, $\tilde{v}_r(z) \to 0$. Therefore $\tilde{v}_r(z)$ admits a Laurent expansion ([27] pp. 239-240) about $z = 0$, i.e.

$$\tilde{v}_r(z) = \sum_{i=0}^{\infty} v_{ri} z^{-i} \quad \text{for } |z| > 1$$

such that $\tilde{v}_r(z)$ is the z-transform of a sequence $v_r = \{v_{ri}\}_{i=0}^{\infty}$ with $v_{r0} = 0$; from this, because of the uniform continuity of $\tilde{v}_r(z)$ in the compact annulus $1 \leq |z| \leq 2$, we obtain for the value v_{ri} of v_r

(A.8') $$v_{ri} = \frac{\rho^i}{2\pi} \int_0^{2\pi} \tilde{v}_r(\rho e^{j\gamma}) e^{j\gamma i} d\gamma \quad \text{for } i = 0,1,2,\ldots$$

with ρ any fixed positive number satisfying $\rho \geq 1$.

Consider now real-valued sequences h, e_+, e_- defined on $\mathbb{Z}$ and given by

$$h_i \triangleq \begin{cases} 0 & \text{for } i = -1, -2, \dots \\ g_0 - \tilde{g}(p) & \text{for } i = 0 \\ g_i & \text{for } i = 1, 2, \dots \end{cases}$$

$$e_{+i} \triangleq \begin{cases} 0 & \text{for } i = 0, -1, -2, \dots \\ p^{i-1} & \text{for } i = 1, 2, \dots \end{cases}$$

$$e_{-i} \triangleq \begin{cases} -p^{i-1} & \text{for } i = 0, -1, -2, \dots \\ 0 & \text{for } i = 1, 2, \dots \end{cases}$$

Observe that h and e_- belong to the convolution algebra ℓ, therefore $h*e_-$ is in ℓ.

Moreover using (A.5')

$$\text{(A.9')} \quad (h*e_+)_i = (h*e_-)_i = \begin{cases} 0 & \text{for } i = 0,-1,-2,\dots \\ -\sum_{j=1}^{\infty} g_j p^{(i-j)-1} & \text{for } i = 1,2,\dots \end{cases}$$

Therefore

$$\text{(A.10')} \qquad h*e_+ = h*e_- \in \ell_+ \equiv \ell^1$$

Note that because of (A.10'), $h*e_+$ is well-defined in $|z| \geq 1$, analytic in $|z| > 1$, continuous and bounded in $|z| \geq 1$. An analog reasoning as that used for deriving (A-8') leads to:

(A.11') $$(h{*}e_+)_i = \frac{\rho^i}{2\pi}\int_0^{2\pi} \widetilde{(h{*}e_+)}\,(\rho e^{j\gamma})e^{j\gamma i}d\gamma, \quad \text{any fixed } \rho \geq 1,$$

$$i = 0,1,2,\ldots .$$

Using (A.9') and summation by parts we obtain as in the continuous-time case

$$\widetilde{h{*}e_+} = \tilde{v}_r \quad \text{for } |z| > |p|.$$

Therefore because of (A.8'), (A.11') and (A.9')

(A.12') $$v_{ri} = (h{*}e_+)_i = \begin{cases} 0 & \text{for } i = 0 \\ -\sum\limits_{j=1}^{\infty} g_j p^{(i-j)-1} & \text{for } i = 1,2,\ldots \end{cases}$$

and hence by (A.10') $v_r \in \ell^1$ Q.E.D. $\bar{\underline{X}}$

Remarks

(A.12a') Remark. It is easy to check that formula (A.12') still holds when g is a complex-valued sequence satisfying (A.1') - (A.2') and $p \in \mathbb{C}$, $|p| > 1$.

Remark. If in (A.3') $|p| = 1$, then previous theory is not valid anymore. In fact if v_r exists then v_r satisfies (A.12') such that for $|p| = 1$ the decomposition (A.4') is valid if and only if

$$\left\{0, -\sum_{j=1}^{\infty} g_j p^{(1-j)-1}, \ldots, -\sum_{j=i}^{\infty} g_j p^{(i-j)-1}, \ldots\right\} \in \ell^1$$

This is not automatically satisfied.

Indeed consider $g_i \begin{cases} = 0 & \text{for } i = 0 \\ = \frac{1}{i^2} & \text{for } i = 1,2,\ldots \end{cases}$ and $p = 1$

then $g \in \ell^1$ however

$$\left\{0, \left\{-\sum_{j=i}^{\infty} \frac{1}{j^2}\right\}_{i=1}^{\infty}\right\} \notin \ell^1.$$

(A.12b') <u>Remark</u>. Using the same reasoning, completely analog counterparts of Corollaries A.2.1 - A.2.4 can be obtained.

<u>Lemma A.3' [Discrete-time counterpart of Wiener's Theorem [12,p.8]]</u>.

Let v be a real-valued sequence defined on $\mathbb{Z}_+$, i.e.

(A.13') $$v = \{v_i\}_{i=0}^{\infty}$$

where $v_i \in \mathbb{R}$ for all i. Let $\tilde{v}$ denote its z-transform. Under these conditions,

(A.14') $$v \in \ell^2$$

if and only if

(A.15') $$\begin{cases} \tilde{v} \text{ exists and is analytic for } |z| > 1, \\ \lim \tilde{v}(z) \to \text{a constant as } |z| \to \infty, \end{cases}$$

and

there exists a positive constant M such that

(A.16') $$\int_{-\pi}^{\pi} |\tilde{v}(e^{j\gamma})|^2 \, d\gamma < M \quad \text{for any } \rho > 1.$$

$$(A.22')\quad |\tilde{v}(\rho e^{j\gamma})|^2 = \sum_{i=0}^{\infty} v_i^2\rho^{-2i} + 2\sum_{i=1}^{\infty}\sum_{k=1}^{i-1} v_i v_k \rho^{-(i+k)}\cos((i-k)\gamma)$$

$$\leq \left(\sum_{i=0}^{\infty}|v_i|\rho^{-i}\right)^2 .$$

Therefore the integration of the second expression of (A.22') on $|z| = \rho$, for any $\rho > 1$, may be performed term by term as long as the integration interval is compact.

Next note that

$$\int_{-\pi}^{\pi} \cos(i-k)\gamma)\, d\gamma = 0 \quad \text{for all} \begin{cases} i = 1,2,\ldots\ . \\ k = 0,1,\ldots,\ i-1. \end{cases}$$

Therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |v(\rho e^{j\gamma})|^2\, d\theta = \sum_{i=0}^{\infty} v_i^2\rho^{-2i} < \sum_{i=0}^{\infty} v_i^2 < \infty, \text{ for any } \rho > 1.$$

Hence (A.16') follows with $M = \sum_{i=0}^{\infty} v_i^2 < \infty$. $\bar{\underline{X}}$

LIST OF REFERENCES

[1] C. A. Desoer and M. Y. Wu, "Stability of linear time-invariant systems," IEEE Transactions on Circuit Theory, CT-15, pp. 245-250, Set. 1968.

[2] C. A. Desoer and M. Y. Wu, "Stability of multiple-loop feedback linear time-invariant systems," Jour. Math. Anal. and Appl., 23, pp. 121-130, June 1968.

[3] C. A. Desoer and M. Y. Wu, "Input-output properties of multiple-input, multiple-output discrete systems, Part I," Jour. Franklin Institute, 290, pp. 11-24, July 1970 (pp. 12-13, pp. 19-20, esp.).

[4] C. A. Desoer and F. M. Callier, "Convolution feedback systems," to appear in SIAM Jour. on Control.

[5] F. M. Callier and C. A. Desoer, "Discrete-time convolution feedback systems," to appear in International Jour. of Control.

[6] C. A. Desoer and F. M. Callier, "Recent results in convolution feedback systems," 5th ASILOMAR Conference on Circuit and Systems pp. 252-255, Nov. 1971.

[7] F. M. Callier and C. A. Desoer, "A graphical test for checking the stability of a linear time-invariant feedback system," to appear in IEEE Trans. Automatic Control and to be presented at the 1972 JACC.

[8] H. Nyquist, "Regeneration Theory," Bell System Technical Journal, 2, pp. 126-147, Jan. 1932.

[9] E. I. Jury, "Sampled-data control systems," John Wiley, N. Y., 1958.

[10] C. T. Chen, "Introduction to linear system theory," Holt, Rinehart & Winston, N. Y., 1970, [pp. 376-378 esp.].

[11] C. A. Desoer, "A general formulation of the Nyquist criterion," IEEE Transations on Circuit Theory, CT-12, 2, pp. 230-234, June 1965.

[12] R.E.A.C. Paley and N. Wiener, "Fourier-transform in the complex-domain," N. Y., Am. Math. Soc., 1934, (pp. 60-61 and p. 8 esp.).

[13] E. Hille and R. S. Phillips, "Functional analysis and semi-groups," revised ed, Providence, R.I., Am. Math. Soc., 1957, (p. 150, p. 155 esp.)

[14] J. C. Willems, "Stability, instability, invertibility and causality," SIAM Jour. on Control, 7, 4, pp. 645-671, 1963. (section 8 esp.).

[15] J. C. Willems, "The analysis of feedback systems," Research Monograph No. 62, M.I.T. Press, Cambridge, Massachusetts, 1970.

[16] R. A. Baker and D. J. Vakharia, "Input-output stability of linear time-invariant systems," IEEE Transactions on automatic control, AC-15, pp. 316-319, June 1970.

[17] C. A. Desoer and M. Vidyasagar, "General necessary conditions for input-output stability," IEEE Proceedings, 59, 8, p. 1255-1256, Aug. 1971.

[18] C. A. Desoer and F. L. Lam, "Stability of linear time-invariant systems," IEEE Proceedings, 58, pp. 1841-1843, Nov. 1970.

[19] C. A. Desoer and F. L. Lam, "On the input-output properties of linear time-invariant systems," IEEE Transactions on Circuit theory, CT-15, 1, pp. 60-62, January 1972.

[20] C. A. Desoer, Notes on course EECS 290D, University of California, Berkeley, Department of Electrical Engineering and Computer Sciences, Berkeley, CA, April 1971, (pp. 75-79 esp.).

[21] M. Vidyasagar, "Input-output stability of a broad class of linear time-invariant systems," to appear in SIAM Jour. on Control.

[22] R. E. Nasburg and R. A. Baker, "Stability of linear time-invariant distributed parameter single-loop feedback systems," 5th ASILOMAR Conference on Circuit and Systems, pp. 110-113, Nov. 1971.

[23] H. Bohr, "Almost periodic functions," Chelsea, N.Y., 1951, (pp. 30-33, 82-83, 101 esp.).

[24] B. Jessen and H. Thornehave, "Mean motions and zeros of almost periodic functions," Acta Mathematica (Upsala), 77, pp. 137-279, 1945 (pp. 143-148, pp. 167-192 esp.).

[25] C. Corduneanu, "Almost periodic functions," Interscience, N.Y., 1961, (p. 9, pp. 69-72, pp. 146-149 esp.).

[26] E. Hille, "Analytic function theory," vol. I, Ginn, Boston, MA, 1959.

[27] J. Dieudonné, "Foundations of modern analysis," Academic Press, N.Y., 1969, (pp. 125-128, pp. 217-218, pp. 239-240, esp.)

[28] I. Stakgold, "Boundary value problems of mathematical physics," vol. 1, Mac Millan, N.Y., 1968 (p. 38 esp.).

[29] D. V. Widder, "The Laplace transform," vol. I, Chelsea, N.Y., 1952, (pp. 130-145, esp.)

[30] E. Hille, "Analytic function theory," vol. II, Ginn, Boston, MA, 1959, (pp. 457 esp.)

[31] F. R. Gantmacher, "Matrix theory," vol. I, Chelsea, N.Y., 1959 (pp. 130-145, esp.)

[32] K. Hoffman and R. Kunze, "Linear algebra," Prentice-Hall, Englewood Cliffs, N.J., 1961, (pp 108-128, esp.).

[33] K. Hoffman, "Banach spaces of analytic functions," Prentice-Hall, Englewood Cliffs, N.J., 1962, (pp. 63-64, esp.).

[34] J. H. Davis, "Encirclement conditions for stability and instability of feedback systems with delays," to appear in International Jour. of Control.

LIST OF FIGURE CAPTIONS

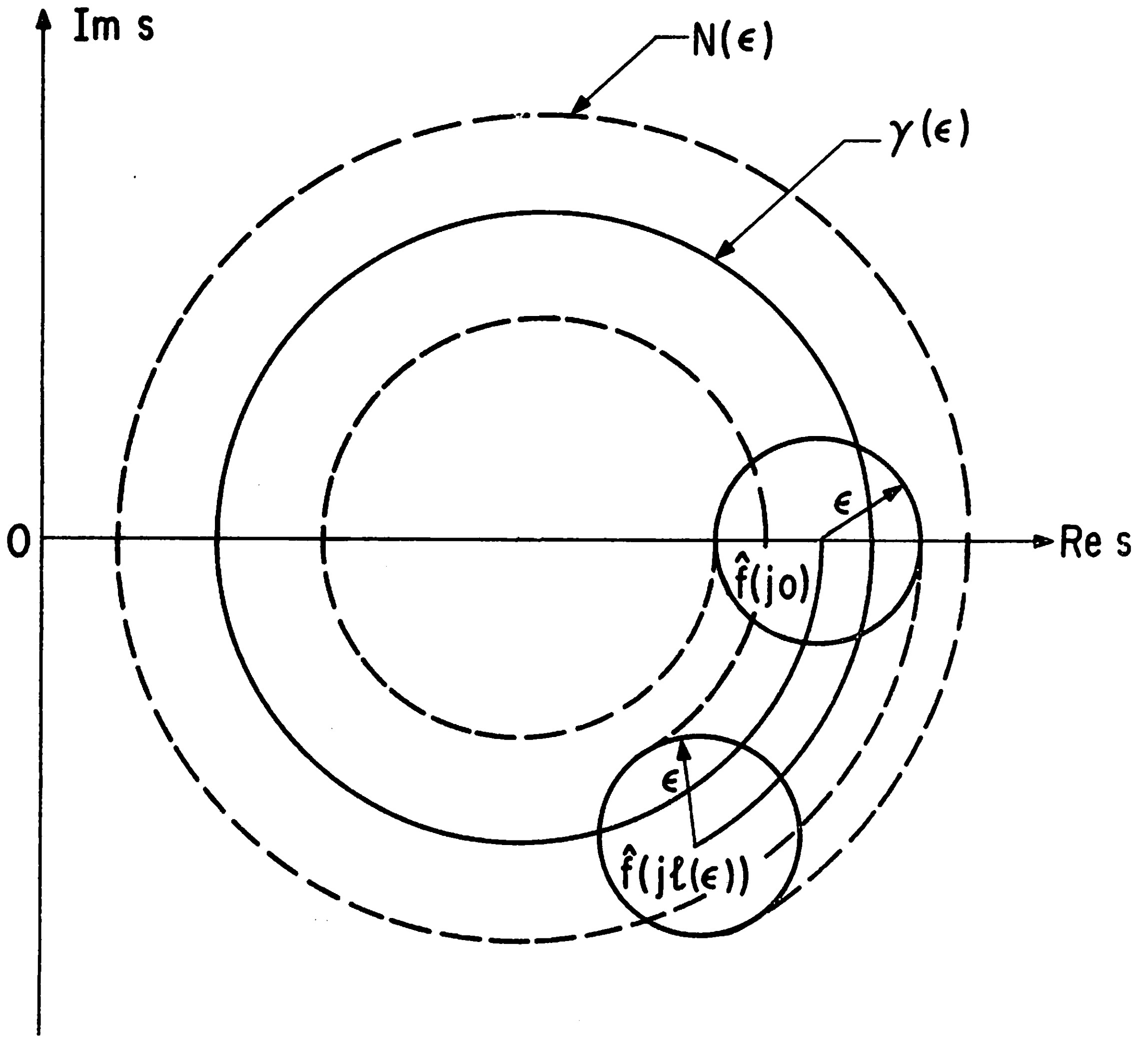

Fig. 2.1

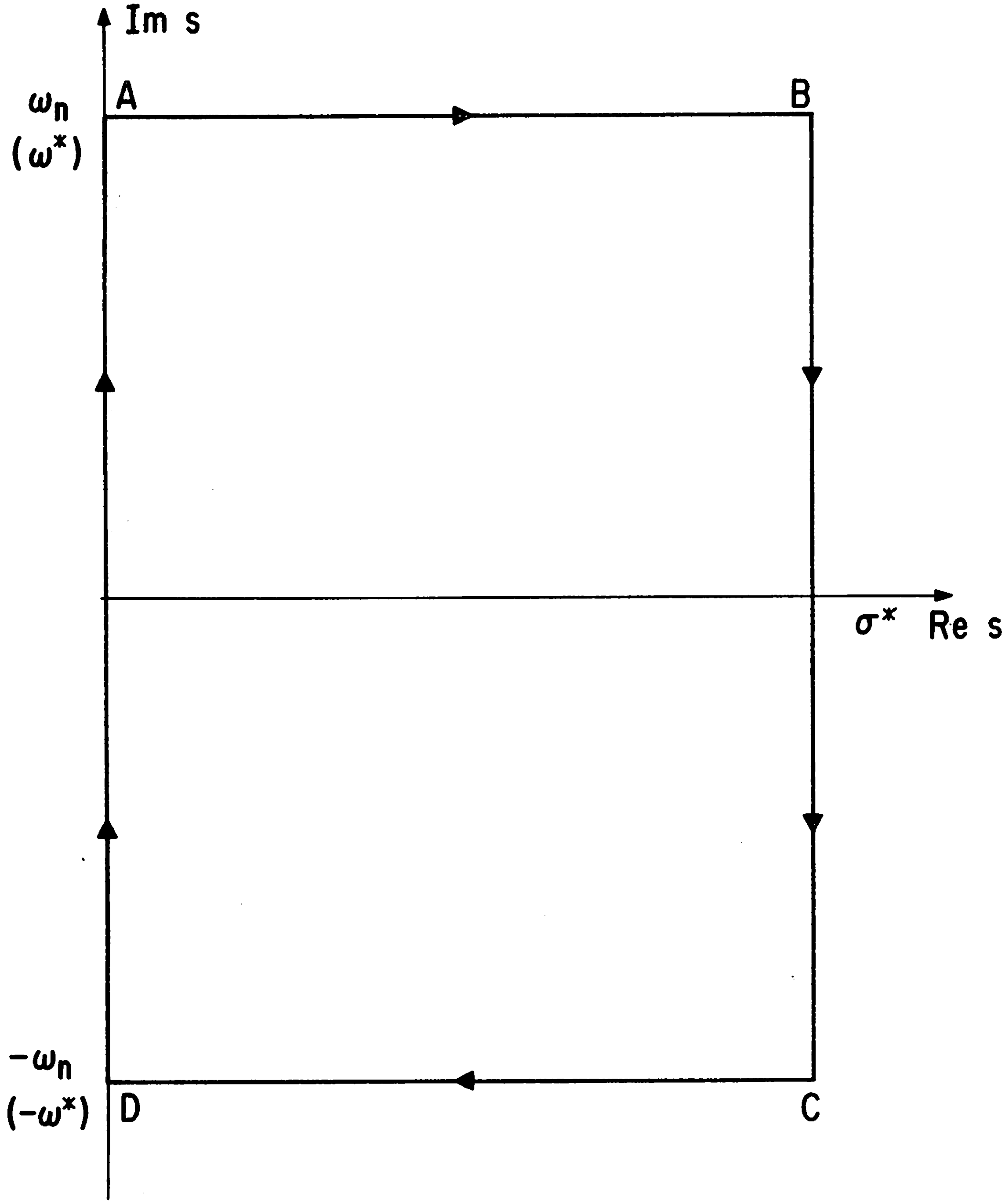

Fig. 2.2

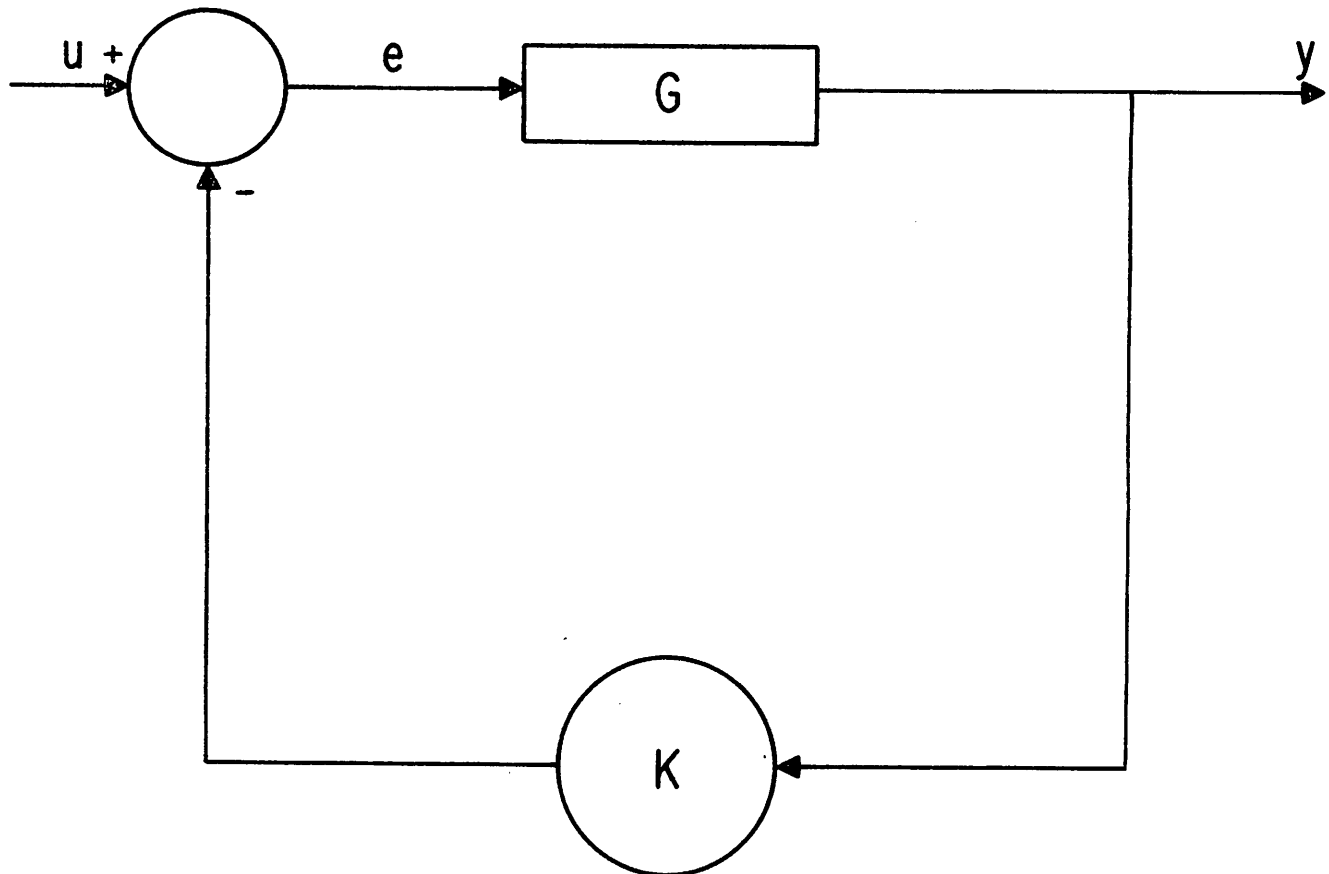

Fig. 5.1.

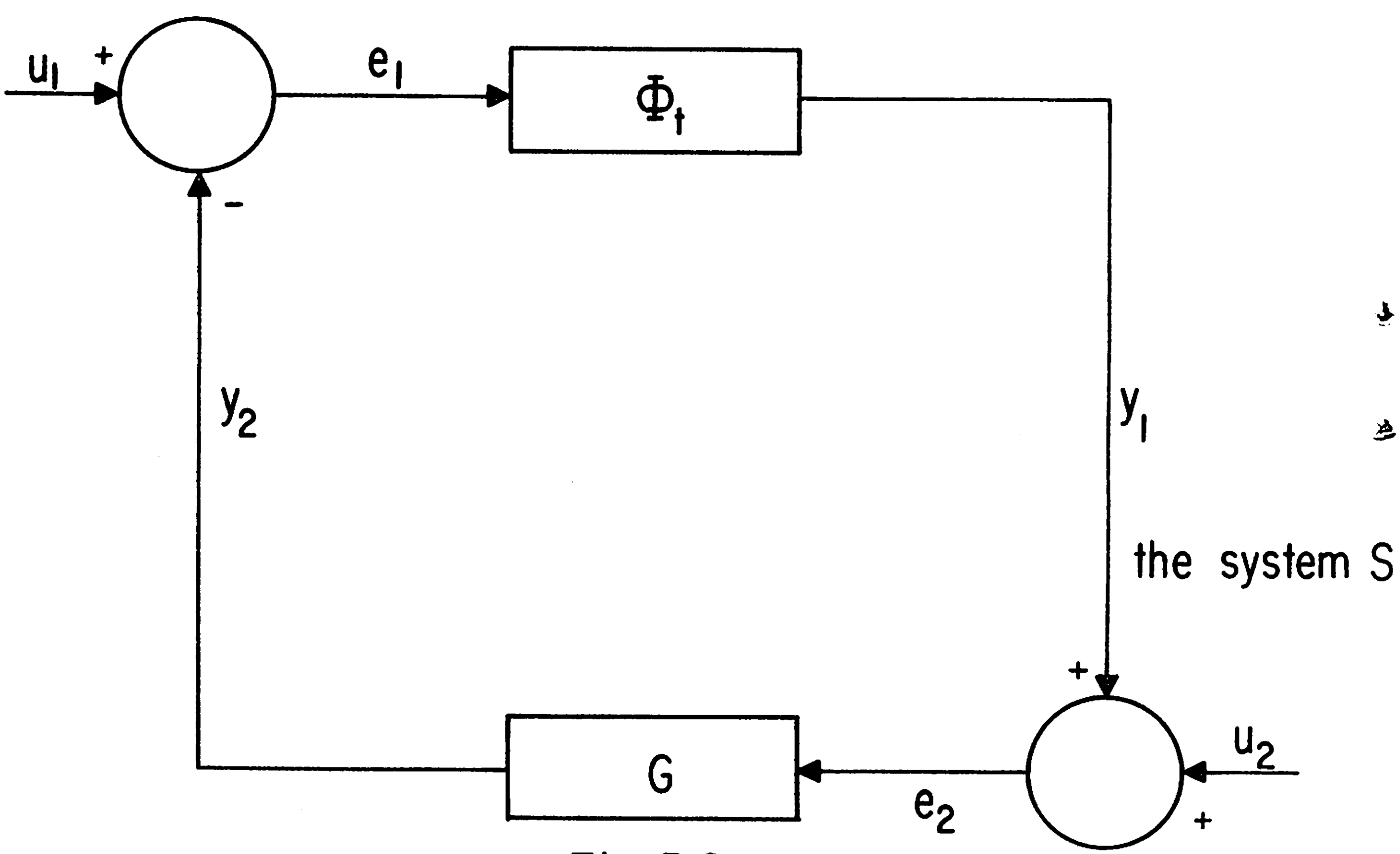

Fig. 5.2.