A THEORY OF ALGEBRAIC N-PORTS

by

L. O. Chua and Y-F Lam

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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L. O. Chua[†] and Y-F Lam[††]

ABSTRACT

This paper is concerned with the foundational aspects of an important subclass of nonlinear n-ports; namely, the class of algebraic n-ports which includes, among other things, the resistor, inductor, capacitor, and memristors as special cases. Sufficient conditions which guarantee an algebraic n-port to admit of all 2^n hybrid representations are given. Both global and local characterizations are considered in detail. In particular, certain global properties are shown to be invariants relative to the modes of hybrid representation. The concept of reciprocity is explored in depth and shown to play a surprising role in determining such global properties as losslessness and passivity.

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[†]Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California 94720.

[††]Department of Electrical Engineering, University of Waterloo, Waterloo, Ontario, Canada.

I. INTRODUCTION

An electrical n-port is a black box with n pairs of external terminals called "ports" such that the current entering a terminal of each port is equal to the current leaving the second terminal. The theory of n-ports is probably the most fundamental aspects of network theory since most network theoretic concepts such as reciprocity, passivity, losslessness, etc. are defined only for n-ports. In fact, with the help of the "connection n-port" recently introduced by Brayton [1], any network may be viewed as an interconnection of appropriate n-ports.[1] Although circuit theorists have succeeded in developing a unified theory of linear n-ports during the last two decades [3-6], very little has yet been done for nonlinear n-ports. The relatively slow progress in the theory of nonlinear n-ports is due not only to the difficulty in the mathematics involved, but also to the lack of a precise and logical characterization and classification of n-ports.

The class of nonlinear n-ports is very large. Indeed, it includes all n-ports! In order to obtain useful results, we will restrict ourselves in this paper to an important subclass; namely, the class of <u>algebraic</u> <u>n-ports</u> to be defined in Section II. This subclass includes not only the four basic n-ports--resistors, inductors, capacitors, and memristors [7], but many more. The various modes for representing algebraic n-ports are presented in Section II along with a theorem giving sufficient conditions for an n-port to admit of all 2^n hybrid representations [8-11]. In Section III, we present several <u>global</u> characterizations of algebraic n-ports which can be interpreted as generalizations of the monotone property of one-ports. Some of these characterizations are shown to be "invariant" in the sense that they are independent of the modes of representation. In Section IV, we introduce a new definition of reciprocity for algebraic n-ports which applies to a larger class of n-ports than that given by Brayton [1]. We prefer this definition not only because it is more general, but also because it bears the same familiar form as the well-known Lorentz reciprocity relation for linear n-ports [4], even though it is not as geometrically appealing as Brayton's definition. We then define two generalized potential functions for reciprocal algebraic n-ports which reduce to such well-known potential functions as content, co-content, energy, co-energy, etc. in special cases. The last section is concerned with the power and energy related properties of algebraic n-ports. Sufficient conditions are given for various algebraic n-ports to be passive or lossless. Contrary to the common belief that reciprocity, passivity, and losslessness are independent properties, we found reciprocity to play an important role on the passivity and losslessness of a nonlinear n-port. Among several surprising results, we prove that a linear non-reciprocal inductor cannot be lossless and a linear anti-reciprocal inductor cannot be passive.

Throughout this paper, we let R^k denote the Euclidean k-space and $\|\cdot\|$ the usual Euclidean norm. Vectors are denoted by lower case letters and matrices by upper case letters. A column vector will usually be denoted by $x = [x_1, x_2, \cdots, x_k]$. Since we will be dealing mostly with vector quantities, we will distinguish the scalar components of vectors by arabic subscripts. A literal subscript will normally denote sub-vectors. For example, we usually partition a vector $x = [x_1, x_2, \cdots, x_k, x_{k+1}, \cdots x_n] \in R^n$

into $x = [x_a, x_b]$, where $x_a = [x_1, x_2, \cdots, x_k]$ and $x_b = [x_{k+1}, x_{k+2}, \cdots, x_n]$. In addition, we let $\dot{x}$ denote the time-derivative of the vector x. Finally, the symbol $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

II. REPRESENTATION OF ALGEBRAIC N-PORTS

Let $\mathcal{N}$ be an n-port with the port voltage v_j and port current i_j defined with current entering the positive terminal. Let $v \triangleq [v_1, v_2, \cdots, v_n]$, $i \triangleq [i_1, i_2, \cdots, i_n]$, $\phi \triangleq [\phi_1, \phi_2, \cdots, \phi_n]$, and $q \triangleq [q_1, q_2, \cdots, q_n]$ denote the port voltage, current, flux-linkage, and charge vectors, respectively, where $\dot{\phi}_j = v_j$ and $\dot{q}_j = i_j$. A mixed vector is one whose components consist of a mixture of at least two different types of port variables. We say two mixed vectors $\xi \triangleq [\xi_1, \xi_2, \cdots, \xi_n]$ and $\eta \triangleq [\eta_1, \eta_2 \cdots, \eta_n]$ are dynamically independent if $\xi_j \neq \eta_j$, $\xi_j \neq \dot{\eta}_j$, and $\eta_j \neq \dot{\xi}_j$. For example, $\xi \triangleq [v_1, i_2, \phi_3]$ and $\eta \triangleq [i_1, v_2, q_3]$ are dynamically independent whereas $\xi \triangleq [v_1, i_2, \phi_3]$ and $\eta \triangleq [\phi_1, v_2, q_3]$ are not since $\xi_1 = v_1 = \dot{\phi}_1 = \dot{\eta}_1$. An n-port characterized by a constitutive relation

$$R(\xi, \eta) = 0 \qquad (1)$$

between two dynamically independent port vectors ξ and η is said to be an algebraic n-port. In the special case where $\{\xi, \eta\}$ takes on the 4 combinations $\{v,i\}$, $\{\phi,i\}$, $\{q,v\}$, and $\{q,\phi\}$, $\mathcal{N}$ is said to be an n-port resistor, inductor, capacitor, and memristor [7], respectively. However, the class of algebraic n-ports is much larger than these 4 basic types because there exist many more distinct combinations. For example, the 3rd degree traditor [12] defined by $v_1 = -Aq_2 i_3$, $v_2 = -Aq_1 i_3$, and $\phi_3 = Aq_1 q_2$ is an algebraic 3-port with $\xi \triangleq [v_1, v_2, \phi_3]$ and $\eta \triangleq [q_1, q_2, i_3]$.

Definition 1. The constitutive relation (1) is said to be C^k-parametrizable if it can be represented by a C^k-function ($k \geq 1$) $\mu \triangleq [\xi,\eta]$: $R^m \to R^{2n}$; i.e.,

$$\xi = \xi(\rho) \text{ and } \eta = \eta(\rho) \tag{2}$$

$\xi \in R^n$, $\eta \in R^n$, and $\rho \in R^m$, $0 \leq m \leq n$; such that the rank of the $2n \times m$ Jacobian matrix $\partial\mu(\rho)/\partial\rho$ is equal to m $\forall$ $\rho \in R^m$. In this case, the n-port is said to have <u>dimension</u> m and will be denoted by $\mathcal{N}(k,m)$.

The difference between the dimension "m" and the port number "n" is a measure of the pathological character of an n-port. For example, a <u>nullator</u> [6] is a 0-dimensional 1-port since it is characterized by μ: $R^0 \to R^2$, where $\mu(0) \triangleq [v(0),i(0)] = [0,0]$. A <u>norator</u> [6] on the other hand is a 2-dimensional 1-port since it is characterized by μ: $R^2 \to R^2$, where $\mu(\rho) \triangleq [v(\rho),i(\rho)] = [\rho_1,\rho_2]$, $\forall$ $\rho = [\rho_1,\rho_2] \in R^2$. Between these two pathological extremes lies the common class of n-ports having dimension $m = n$. In this case, if it is possible to choose $\eta(\rho) = \rho$ ($\xi(\rho) = \rho$), then (2) reduces to $\xi = \xi(\eta)$ ($\eta = \eta(\xi)$) and $\mathcal{N}$ is said to be <u>η-controlled</u> (<u>ξ-controlled</u>). For example, the 1-dimensional 1-port characterized by $\mu(\rho) \triangleq [v,i] = [\rho^3,\rho]$ is current-controlled since $i = \rho$ and hence $v = i^3$. Since any theorem formulated relative to the representation (2) automatically specializes to the η-controlled or the ξ-controlled case upon setting $\rho = \eta$ or $\rho = \xi$, it is usually more convenient to work with the parametric representation (2).

<u>Definition 2</u>. An algebraic n-port $\mathcal{N}(k,n)$ is said to admit a <u>C^ℓ-hybrid representation</u> ($\ell \leq k$) if it can be represented by $y = h(x)$ where $h(\cdot)$ is a C^ℓ-function on R^n, and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \triangleq \Sigma \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{3}$$

where A and B are <u>diagonal</u> $n \times n$ permutation matrices satisfying the property that either $A_{jj} = 1$, $B_{jj} = 0$ or $A_{jj} = 0$, $B_{jj} = 1$.

<u>Remarks 1</u>.

(i) It is obvious that $\mathcal{N}(k,n)$ admits a C^{ℓ}-hybrid representation if, and only if, there exists a pair of permutation matrices A and B such that $x = A\xi(\rho) + B\eta(\rho) \triangleq f(\rho)$ is a C^{ℓ}-diffeomorphism of R^n onto R^n. In this case, $y = B\,\xi(\rho) + A\eta(\rho) \triangleq g(\rho) = g \circ f^{-1}(x) \triangleq h(x)$.

(ii) It can be shown that the matrix Σ is orthogonal, symmetric, unimodular, and elementary. Hence, $\Sigma^{-1} = \Sigma$. Moreover, we have $A + B = I_n$, $AB = BA = 0_n$, $AA = A$, $BB = B$, where I_n and 0_n denote the identity and the zero matrix, respectively.

Although there are 2^n distinct hybrid representations, only four are commonly encountered in practice. They are the ξ-controlled, η-controlled, the hybrid I and the hybrid II representations as defined in Table 1 (rows 1 to 4). Closely related to the hybrid I and II representations are the four <u>conjugate hybrid representations</u> defined in Table 1 (rows 5 to 8). These conjugate representations are extremely convenient for the study of potential functions of reciprocal n-ports [8]. In addition, there are two other frequently encountered representations useful in studying the transformation properties of nonlinear n-ports. They are called the <u>transmission I and II representations</u> and are defined in Table 1 (rows 9 and 10).

Generally speaking, an algebraic n-port which admits of one hybrid representation may fail to admit another mode of hybrid representation. It is desirable therefore to derive conditions which prevent this from happening [9-10]. More recently, Desoer and Oster [11] have obtained conditions which guarantee a reciprocal n-port to admit all 2^n hybrid representations. We will consider the general case. Before we state the main theorem, we will rephrase the global implicit function theorem [13] as follow:

Lemma 1. Let an algebraic n-port $\mathcal{N}$ be characterized by a C^1-hybrid representation

$$y = \begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} h_a(x_a,x_b) \\ h_b(x_a,x_b) \end{bmatrix} \triangleq h(x)$$

where $[x,y] = \sum[\xi,\eta]$, $x_a \in X_a \triangleq R^s$, $x_b \in X_b \triangleq R^{n-s}$, $y_a \in Y_a \triangleq R^s$, $y_b \in Y_b \triangleq R^{n-s}$, and $0 \le s \le n$. If the following conditions are satisfied:

(1) $\det \partial h_a(x)/\partial x_a \neq 0 \quad \forall\, x \in R^n$

(2) $\lim\limits_{\|x_a\| \to \infty} \|h_a(x_a,x_b)\| = \infty \quad \forall\, x_b \in X_b$

then $\mathcal{N}$ admits of the following equivalent hybrid representation:

$$\begin{bmatrix} x_a \\ y_b \end{bmatrix} = \begin{bmatrix} g_a(y_a,x_b) \\ g_b(y_a,x_b) \end{bmatrix} \triangleq g(y_a,x_b)$$

where $g(\cdot,\cdot)$ is a C^1-function on $Y_a \times X_b = R^n$.

Theorem 1. Hybrid Representation Theorem.

If an algebraic n-port $\mathcal{N}(k,n)$ admits a C^1-hybrid representation $y = h(x)$ where $[x,y] = \Sigma[\xi,\eta]$ and $h: R^n \to R^n$ satisfies the following two conditions:

(1) $\partial h(x)/\partial x$ is a P-matrix[2], $\forall\, x \in R^n$ (4)

(2) $\lim\limits_{|x_j| \to \infty} |h_j(x)| = \infty, \quad \forall\, [x_1,x_2,\cdots,x_{j-1},x_{j+1},\cdots,x_n] \in R^{n-1},$

and $\forall\, j = 1,2,\cdots,n$ (5)

then $\mathcal{N}(k,n)$ admits of all 2^n distinct C^1-hybrid representations.

Proof. Let $[x^a,y^a] = \Sigma^a[\xi,\eta]$ where Σ^a is any one of the 2^n distinct permutations of $[\xi,\eta]$. Since $[x,y] = \Sigma[\xi,\eta]$ and $\Sigma^{-1} = \Sigma$, $[x^a,y^a] = \Sigma^a\Sigma[x,y] \triangleq \Sigma^b[x,y]$. Let A^b and B^b be the pair of permutation matrices associated with Σ^b. Then $x^a = A^b x + B^b y$ and $y^a = B^b x + A^b y$. If $B^b = 0_n$, then $y^a = h(x^a)$ since in this case, $A = I_n$, $x^a = x$, and $y^a = y$. If $A^b = 0_n$, then $B^b = I_n$ and hence $y^a = h^{-1}(x^a)$ since (4) and (5) imply that $h(\cdot)$ is a C^1-diffeomorphism of R^n onto R^n [15-16]. It suffices therefore to consider the case $A^b \neq 0_n$ and $B^b \neq 0_n$. Rearrange the n hybrid equations $y_j = h_j(x_1,x_2,\cdots,x_n)$, $j = 1,2,\cdots,n$ into the form:

$$y' \triangleq \begin{bmatrix} y_A \\ y_B \end{bmatrix} = \begin{bmatrix} h_A(x_A,x_B) \\ h_B(x_A,x_B) \end{bmatrix} = h'(x')$$

where the variables y_j and x_j associated with the non-zero columns of B^b are lumped together in y_B and x_B, respectively. The vectors y_A and x_A

are similarly defined. Clearly, $\partial h_B(x')/\partial x_B$ is a principal submatrix of $\partial h(x)/\partial x$ and has therefore a positive determinant in view of (4). Moreover, (5) implies that

$$\lim_{\|x_B\| \to \infty} \|h_B(x_A,x_B)\| = \infty \qquad \forall\ x_A \in R^{n-s}$$

where s is the total number of non-zero columns of B^b. Hence, it follows from Lemma 1 that $\mathcal{N}(k,n)$ admits of the equivalent representation:

$$x_B = g_B(x_A,y_B) \text{ and } y_A = g_A(x_A,y_B)$$

where both $g_A(\cdot,\cdot)$ and $g_B(\cdot,\cdot)$ are C^1-functions on R^n. It is clear that the components of $y^a \triangleq [y_1^a,y_2^a,\cdots,y_n^a]$ is a permutation of the components of $[y_A,x_B]$ and those of $x^a \triangleq [x_1^a,x_2^a,\cdots,x_n^a]$ is a permutation of the components of $[x_A,y_B]$. That is, we can write $y^a = E^y[y_A,x_B]$ and $x^a = E^x[x_A,y_B]$ where both E^x and E^y are non-singular, orthogonal, unimodular and orthogonal $n \times n$ matrices. Hence,

$$y^a = E^y \begin{bmatrix} y_A \\ x_B \end{bmatrix} = E^y \begin{bmatrix} g_A(x_A,y_B) \\ g_B(x_A,y_B) \end{bmatrix} = E^y \begin{bmatrix} g_A \circ (E^x x^a) \\ g_B \circ (E^x x^a) \end{bmatrix} \triangleq h^a(x^a)$$

where $h^a(\cdot)$: $R^n \to R^n$ is a C^1-function. Q.E.D.

III. GLOBAL CHARACTERIZATION OF N-PORTS

It is well known that the qualitative properties of nonlinear networks depend to a great extent on the global characteristics of the elements' nonlinearity. For networks made up of interconnection of 1-ports, various sufficient conditions have been obtained which ensure either the existence and uniqueness of solutions for resistive networks [17-21], or the global stability of dynamic networks [22-24]. Some of these conditions require the resistors to be characterized by strictly monotonically-increasing functions. Others require the resistors to be voltage controlled or current-controlled. Still others require the $v - i$ curves to be uniformly increasing, etc. A precise classification of n-ports in terms of their global characteristics is fundamental not only to the analysis of nonlinear networks but to synthesis as well [25]. In attempting to generalize the various global characterizations of 1-ports to n-ports, many subtleties and complications arise. For example, whereas a 1-port is strictly monotone if its constitutive relation is an injection from R^1 into R^1, there exist homeomorphic mappings from R^n onto R^n which are not monotone when $n \geq 2$ [26]. Moreover, as will be seen in the sequel, an n-port which is bijective with respect to one hybrid representation may fail to be bijective with respect to another representation if $n > 1$. On the other hand, there are characterizations of n-ports which are independent of all possible modes of hybrid representation and these characterizations are said to be hybrid invariants. These possibilities make it necessary for us to define many seemingly redundant but distinct global characterizations.

Definition 3. An algebraic n-port $\mathcal{N}(k,m)$ represented by (2) is said to be non-decreasing if

$$\alpha(\rho^a,\rho^b) \triangleq \langle \xi(\rho^a)-\xi(\rho^b),\ \eta(\rho^a)-\eta(\rho^b)\rangle \geq 0$$

$\forall\ \rho^a$ and $\rho^b \in R^m$. It can be shown that an n-port which admits of a hybrid representation is non-decreasing if, and only if, $\langle h(x^a)-h(x^b), x^a-x^b \rangle \geq 0$ $\forall\, x^a$ and $x^b \in R^n$ [26].

Definition 4. Let $\mathcal{N}(k,n)$ be an algebraic n-port which admits a hybrid representation $y = h(x)$ and let $\alpha(x^a,x^b) \triangleq \langle h(x^a)-h(x^b),\, x^a-x^b \rangle$. Then $\mathcal{N}(k,n)$ is said to be increasing if $\alpha(x^a,x^b) > 0$ $\forall\, x^a \neq x^b \in R^n$. It is said to be x-uniformly increasing if there exists a constant $c > 0$ such that $\alpha(x^a,x^b) \geq c\|x^a-x^b\|^2$ $\forall\, x^a$ and $x^b \in R^n$. If in addition, $h(\cdot) \in C^1$ and the Jacobian matrix $J_h(\cdot)$ is bounded on R^n, then $\mathcal{N}(k,n)$ is said to be strongly uniformly increasing. $\mathcal{N}(k,n)$ is said to be proper if $h(\cdot)$ is surjective on R^n.

Theorem 2. Hybrid Invariant Characterizations.

The definitions for an increasing, non-decreasing, strongly uniformly increasing, and proper n-port are independent of the mode of hybrid representation and are therefore hybrid invariant characterizations.

Proof. Let $y = h(x)$ and $y' = h'(x')$ denote any two distinct hybrid representations of $\mathcal{N}(k,n)$, where $[x,y] = \Sigma[\xi,\eta]$ and $[x',y'] = \Sigma'[\xi,\eta]$. It follows from $\Sigma^{-1} = \Sigma$ and $\Sigma'^{-1} = \Sigma'$ that $[x,y] = \Sigma\Sigma'[x',y']$ and $[x',y'] = \Sigma'\Sigma[x,y]$. Since permutation matrices form a group [27], there exist a permutation matrix $\Sigma^* = \Sigma\Sigma'$ with $\Sigma^{*-1} = \Sigma^*$, and a pair of associated permutation matrices A^* and B^* such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Sigma^* \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ B^* & A^* \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

and

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \Sigma^* \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ B^* & A^* \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now let $[\xi^a,\eta^a]$ and $[\xi^b,\eta^b]$ be any two points in R^{2k} and let $[x^a,y^a] = \Sigma[\xi^a,\eta^a]$, $[x'^a,y'^a] = \Sigma'[\xi^a,\eta^a]$, $[x^b,y^b] = \Sigma[\xi^b,\eta^b]$ and $[x'^b,y'^b] = \Sigma'[\xi^b,\eta^b]$. It follows from Remarks 1 that:

$$\begin{aligned} \langle y'^a-y'^b,x'^a-x'^b \rangle &= \langle B^*(x^a-x^b),A^*(x^a-x^b) \rangle + \langle B^*(x^a-x^b),B^*(y^a-y^b) \rangle \\ &\quad + \langle A^*(y^a-y^b),A^*(x^a-x^b) \rangle + \langle A^*(y^a-y^b),B^*(y^a-y^b) \rangle \\ &= \langle B^*(x^a-x^b),y^a-y^b \rangle + \langle A^*(x^a-x^b),y^a-y^b \rangle \\ &= \langle y^a-y^b,x^a-x^b \rangle \end{aligned}$$

$\therefore$

$$\langle h'(x'^a)-h'(x'^b),x'^a-x'^b \rangle = \langle h(x^a)-h(x^b),x^a-x^b \rangle .$$

This proves that the definitions for increasing and non-decreasing n-ports are invariants of the hybrid representations.

The definition for a strongly uniformly increasing n-port has been shown to be invariant in [9-10]. It remains to prove that the definition of proper n-ports is also an invariant representation. In view of Remarks 1, we can write $y = h(x) = g \circ f^{-1}(x)$ and $y' = h'(x) = g' \circ f'^{-1}(x)$, where $f(\cdot)$ and $f'(\cdot)$ are bijective maps. Now if $h(\cdot)$ is surjective, so is $g(\cdot)$. It follows from the fact that Σ^* is non-singular that $g'(\cdot)$ is also surjective. Hence, $h'(\cdot)$ is surjective since it is the composition of two surjective maps $g'(\cdot)$ and $f'^{-1}(\cdot)$ [28]. We have proved that $h(\cdot)$

is surjective $\Rightarrow h'(\cdot)$ is surjective. Hence the definition of proper n-ports is a hybrid invariant. Q.E.D.

Definition 5. An algebraic n-port $\mathcal{N}(k,n)$ which admits a hybrid representation $y = h(x)$ is said to be x-homeomorphic {x-bijective} if $h(\cdot)$ is an injection [bijection].

Remarks 2.

(i) The basis for defining an "x-homeomorphic" n-port in terms of an injective function is given by Brouwer's theorem on the invariance of domain [29-30]: "any injective continuous function $h: R^n \to R^n$ is homeomorphic."

(ii) It can be shown that any increasing n-port represented by $y = h(x)$ is x-homeomorphic [26]. However, the converse is false for $n \geq 2$. A case in point is as follows: Let $\mathcal{N}$ be represented by $i = g(v)$, where $i_1 = v_1+v_2$ and $i_2 = v_1-v_2$. Clearly, $\mathcal{N}$ is v-homeomorphic. However, $\mathcal{N}$ is not increasing since $\alpha(a,b) \triangleq \langle g(a)-g(b), a-b \rangle = -1$ when $a = [1,1]$ and $b = [1,0]$. Neither is $\mathcal{N}$ decreasing since $\alpha(a,b) = 1$ when $a = [1,1]$ and $b = [0,1]$.

(iii) The reason for attaching the prefix "x" to Definition 5 for homeomorphic and bijective n-ports is because these characterizations are not invariant relative to the different modes of hybrid representation. For example, let $\mathcal{N}$ be represented by $v_1(\rho) = \rho_1$, $v_2(\rho) = \rho_2$, $v_3(\rho) = \rho_3$; $i_1(\rho) = -\rho_1+3\rho_2$, $i_2(\rho) = \rho_1$ and $i_3(\rho) = -\rho_1+2\rho_2+\rho_3$ where $\rho = [\rho_1, \rho_2, {}_3] \in R^3$. Hence, $\mathcal{N}$ admits of the following two hybrid representations: $y = h(x)$ with $y = [i_1, i_2, i_3]$, $x = [v_1, v_2, v_3]$, and $y' =$

$h'(x')$ with $y' = [v_1,i_2,i_3]$, $x' = [i_1,v_2,v_3]$, where $h(\cdot)$ and $h'(\cdot)$ are defined respectively as follows:

$$h(x) \triangleq \begin{bmatrix} -1 & 3 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \triangleq Hx, \quad h'(x) \triangleq \begin{bmatrix} -1 & 3 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \\ v_3 \end{bmatrix} \triangleq H'x'$$

Since $\det H \neq 0$ and $\det H' = 0$, it follows that $\mathcal{N}$ is both x-homeomorphic and x-bijective. However, $\mathcal{N}$ is neither x'-homeomorphic nor x'-bijective.

IV. LOCAL CHARACTERIZATION OF N-PORTS

An algebraic n-port can be characterized locally according to whether it is reciprocal or not. As will be shown in Sec. V, this local property influences the global qualitative behaviors of n-ports in a significant way [31-32]. Reciprocity has been defined for nonlinear n-port resistors of dimension "n" via differential geometric concepts [1,11]. In this paper, we propose a more general definition which is applicable to all algebraic n-ports of arbitrary dimension. Contrary to the common assertion that all one-ports are reciprocal, our definition shows that the nullator and the norator are non-reciprocal one-ports! This classification seems to be more appropriate in view of the fact that neither nullator nor norator can be modeled by reciprocal elements alone. Another reason for introducing our definition is that it reduces to the well-known Lorentz reciprocity relation [4,6] when the n-port is linear.

Definition 6. An algebraic n-port $\mathcal{N}(k,m)$ $(k \geq 1, 0 \leq m \leq n)$ represented by the parametric equations (2) is said to be reciprocal if, for each $\rho \in R^m$,

$$\langle (d\xi(\rho))', (d\eta(\rho))'' \rangle = \langle (d\xi(\rho))'', (d\eta(\rho))' \rangle \tag{6}$$

where $[(d\xi(\rho))', (d\eta(\rho))']$ and $[(d\xi(\rho))'', (d\eta(\rho))'']$ are any two elements in the tangent space $T_\rho(R^m)$ attached to the point $[\xi(\rho), \eta(\rho)] \in R^{2n}$ [33]. The n-port is said to be non-reciprocal if it is not reciprocal. In particular, it is said to be anti-reciprocal if, for each $\rho \in R^m$,

$$\langle (d\xi(\rho))', (d\eta(\rho))'' \rangle = -\langle (d\xi(\rho))'', (d\eta(\rho))' \rangle \tag{7}$$

<u>Theorem 3</u>. Reciprocity Criterion[3]

<u>An algebraic n-port $\mathcal{N}(k,m)$ represented by the parametric form (2) with $k \geq 1$ is reciprocal {anti-reciprocal} if, and only if, its associated reciprocity matrix</u>

$$R(\rho) \triangleq \left[\frac{\partial \eta(\rho)}{\partial \rho} \right]^T \left[\frac{\partial \xi(\rho)}{\partial \rho} \right] \tag{8}$$

<u>is symmetric {skew-symmetric}</u>.

<u>Proof</u>. Since $[(d\xi(\rho))', (d\eta(\rho))']$ and $[(d\xi(\rho))'', (d\eta(\rho))'']$ are elements of the tangent space $T_\rho(R^m)$ attached to the point $[\xi(\rho), \eta(\rho)] \in R^{2n}$, it follows that:

$$(d\xi(\rho))' = \frac{\partial \xi(\rho)}{\partial \rho} (d\rho)', \quad (d\eta(\rho))' = \frac{\partial \eta(\rho)}{\partial \rho} (d\rho)'$$

$$(d\xi(\rho))'' = \frac{\partial \xi(\rho)}{\partial \rho} (d\rho)'', \quad (d\eta(\rho))'' = \frac{\partial \eta(\rho)}{\partial \rho} (d\rho)''$$

where $(d\rho)'$ and $(d\rho)''$ are any two vectors in R^m. Hence,

$$\langle (d\xi(\rho))', (d\eta(\rho))'' \rangle = \langle \frac{\partial \xi(\rho)}{\partial \rho} (d\rho)', \frac{\partial \eta(\rho)}{\partial \rho} (d\rho)'' \rangle = \langle R(\rho)(d\rho)', (d\rho)'' \rangle \tag{9}$$

$$\langle (d\xi(\rho))'', (d\eta(\rho))' \rangle = \langle R(\rho)(d\rho)'', (d\rho)' \rangle = \langle [R(\rho)]^T (d\rho)', (d\rho)'' \rangle \tag{10}$$

Substituting (9) and (10) into (6) and (7), respectively, we obtain $R(\rho) = [R(\rho)]^T$ and $R(\rho) = - [R(\rho)]^T$. Q.E.D.

Applying the reciprocity criterion (8) to a nullator and a norator shows that they are both non-reciprocal. In the common case where the n-port is of dimension n and admits of a hybrid representation, we have:

Corollary. An algebraic n-port which admits a C^1-hybrid representation $y = h(x)$, where $[x,y] = \Sigma [\xi,\eta]$ as defined by (3) is reciprocal {anti-reciprocal} if, and only if, its associated hybrid reciprocity matrix

$$R_H(x) \triangleq [B+AJ_h(x)]^T \; [A+BJ_h(x)] \tag{11}$$

is symmetric {skew-symmetric}.

Proof. Since $\Sigma^{-1} = \Sigma$, we have $\xi(x) = Ax + Bh(x)$ and $\eta(x) = Bx + Ah(x)$. Substituting $\partial\eta(x)/\partial x = B + AJ_h(x)$ and $\partial\xi(x)/\partial x = A + BJ_h(x)$ into (6) and (7), respectively, we obtain $R_H(x) = [R_H(x)]^T$ and $R_H(x) = - [R_H(x)]^T$.

Q.E.D.

The necessary and sufficient conditions for an n-port represented by a hybrid or conjugate hybrid representation to be reciprocal {anti-reciprocal} can now be easily determined by deriving the corresponding reciprocity matrix $R_H(x)$ and applying the preceding corollary. The result is summarized in the first 8 rows of Table 1. The conditions on the last 2 rows are derived using (8) of theorem 3 directly.

Theorem 4.

Let $\mathcal{N}(k,n)$ be an n-port with a C^{ℓ}-hybrid representation $y = h(x)$, where $k > 2$, $2 < \ell < k$. If $\mathcal{N}(k,n)$ is anti-reciprocal, then it is necessarily an affine n-port in the sense that

$$y = h(x) = Hx + c$$

where H is an $n \times n$ constant matrix and c is a constant n-vector.

Proof. From Table 1, $\mathcal{N}(k,n)$ is anti-reciprocal implies that $J_h(x)$ is skew-symmetric. If we let $a_{ij}(x)$ denote the ijth element of $J_h(x)$, then $a_{ij}(x) = - a_{ji}(x)$, $i, j = 1,2,\cdots,n$. Now $\ell \geq 2$ implies that:

$$\frac{\partial}{\partial x_j}[a_{ik}(x)] = -\frac{\partial}{\partial x_j}[a_{ki}(x)] = -\frac{\partial}{\partial x_i}[a_{kj}(x)] = \frac{\partial}{\partial x_i}[a_{jk}(x)] \tag{12}$$

$$\frac{\partial}{\partial x_j}[a_{ik}(x)] = \frac{\partial}{\partial x_k}[a_{ij}(x)] = -\frac{\partial}{\partial x_k}[a_{ji}(x)] = -\frac{\partial}{\partial x_i}[a_{jk}(x)] \tag{13}$$

Eqs. (12) and (13) imply that $\frac{\partial}{\partial x_j}[a_{ik}(x)] = 0 \quad \forall\, i, j, k = 1,2,\cdots,n$.

Hence $J_h(x)$ must be a constant matrix. Q.E.D.

Closely related to the concept of reciprocity are the potential functions which we now consider.

Definition 7. Let S be a convex subset of R^n. A C^1-function $f\colon\ S \to R^n$ is said to be a state function if the Jacobian matrix $J_f(x)$ is symmetric

$\forall\, x \in S$. In this case, the line integral of $f(\cdot)$ between any two points in S depends only on these endpoints and is therefore independent of the path of integration [35].

Lemma 2. Let $\mathcal{N}(1,m)$ be a reciprocal n-port represented by a C^2-parametric function $\xi = \xi(\rho)$ and $\eta = \eta(\rho)$, $\rho \in R^m$. Then $f(\cdot)$ and $f^*(\cdot)$ as defined in (14) and (15) below are both state functions on R^m:

$$f(\rho) \triangleq \left[\frac{\partial \eta(\rho)}{\partial \rho} \right]^T \xi(\rho) \tag{14}$$

$$f^*(\rho) \triangleq \left[\frac{\partial \xi(\rho)}{\partial \rho} \right]^T \eta(\rho) \tag{15}$$

Proof. It suffices to prove that $f(\cdot)$ is a state function. Differentiating (14), we obtain

$$J_f(\rho) = \left[\frac{\partial \eta(\rho)}{\partial \rho} \right]^T \left[\frac{\partial \xi(\rho)}{\partial \rho} \right] + S(\rho) \triangleq R(\rho) + S(\rho)$$

where $R(\rho)$ is defined in (8), and $S(\rho)$ is a symmetric matrix whose kj th element is given by:

$$s_{kj}(\rho) \triangleq \left\langle \frac{\partial}{\partial \rho_k} \left[\frac{\partial \eta(\rho)}{\partial \rho_j} \right], \xi(\rho) \right\rangle$$

$$= \left\langle \frac{\partial}{\partial \rho_j} \left[\frac{\partial \eta(\rho)}{\partial \rho_k} \right], \xi(\rho) \right\rangle = s_{jk}(\rho) \tag{16}$$

Since $\mathcal{N}(1,m)$ is reciprocal, it follows from theorem 3 that $R(\rho)$ is also symmetric. Hence $J_f(\rho)$ is symmetric. Q.E.D.

Definition 8. Let $\mathcal{N}(1,m)$ be a reciprocal n-port represented by a C^2-parametric function $\xi = \xi(\rho)$ and $\eta = \eta(\rho)$ $\rho \in R^m$. We define the generalized potential function $\Omega(\rho,\rho_0)$ and the generalized co-potential function $\Omega^*(\rho,\rho_0)$ to be the following line integrals [35]:

$$\Omega(\rho,\rho_0) \triangleq \int_{\Gamma[\eta(\rho_0),\eta(\rho)} \langle \xi(\rho), d\eta(\rho) \rangle + k_A(\rho_0)$$

$$= \int_{\Gamma[\rho_0,\rho]} \langle f(\rho), d\rho \rangle + k_A(\rho_0) \tag{17}$$

$$\Omega^*(\rho,\rho_0) \triangleq \int_{\Gamma[\xi(\rho_0),\xi(\rho)} \langle \eta(\rho), d\xi(\rho) \rangle + k_A^*(\rho_0)$$

$$= \int_{\Gamma[\rho_0,\rho]} \langle f^*(\rho), d\rho \rangle + k_A^*(\rho_0) \tag{18}$$

where $\Gamma[\rho_0,\rho]$ denotes any path of integration from ρ_0 to ρ, and $k_A(\rho_0)$, $k_A^*(\rho_0)$ are constants (depending on ρ_0) such that

$$k_A(\rho_0) + k_A^*(\rho_0) = \langle \xi(\rho_0), \eta(\rho_0) \rangle \tag{19}$$

Remarks 3.

(i) In view of Lemma 2, the line integrals defined in (17) and (18) are independent of path of integration.

(ii) In most cases, it is possible to choose $\xi(\rho_0)$ and $\eta(\rho_0)$ such that $\langle \xi(\rho_0), \eta(\rho_0) \rangle = 0$. For example, one can choose either $\xi(\rho_0) = 0$ or

$\eta(\rho_0) = 0$. In this case, we can set $k_A(\rho_0) = k_A^*(\rho_0) = 0$ in (17) and (18).

(iii) In the case where $\mathcal{N}(1,n)$ admits an η-controlled representation $\xi = \xi(\eta)$ {ξ-controlled representation $\eta = \eta(\xi)$}, where $\xi(\cdot)$ and $\eta(\cdot)$ are C^1 functions on R^n, then (17) and (18) reduce to the form given by Millar and Cherry [36-37] as follows:

$$\Omega(\eta) = \int_{\Gamma[0,\eta]} \langle \xi(\eta), d\eta \rangle \tag{20}$$

$$\Omega^*(\xi) = \int_{\Gamma[0,\xi]} \langle \eta(\xi), d\xi \rangle \tag{21}$$

In the special case where $\mathcal{N}(1,m)$ is an n-port resistor, inductor, capacitor, or memristor, the generalized potential and co-potential functions reduce to the familiar forms listed in Table 2.

V. POWER AND ENERGY RELATED CHARACTERIZATION OF ALGEBRAIC N-PORTS

Our objective in this section is to investigate several fundamental properties of algebraic n-ports which are related to power and energy. In particular, various criteria will be derived which guarantee a resistive, inductive, capacitive, or memristive n-port is passive, non-energic, or lossless. The observation that reciprocity plays an important role in determining these properties is somewhat surprising.

Definition 9. A pair of port voltage and current n-vector-valued time functions $(v(t),i(t))$ is said to be an admissible pair of an n-port $\mathcal{N}$ if it satisfies the constitutive relation $R(\xi,\eta) = 0$ of $\mathcal{N}$, $\forall\, t \in R^1$.

If $\mathcal{N}$ is parametrizable, then each admissible pair $(v(t),i(t))$ gives rise to at least one parametric waveform $\rho(t)$ relative to each prescribed parametric equation $\xi = \xi(\rho)$ and $\eta = \eta(\rho)$. In our subsequent results, it is crucial that $\rho(t)$ be uniquely determined. This motivates our next lemma:

Lemma 3. Let $\mathcal{N}(k,m)$, $k > 1$, be characterized by $\xi = \xi(\rho)$ and $\eta = \eta(\rho)$. If $\mu(\cdot) \triangleq [\xi(\cdot),\eta(\cdot)]$ is an injection on R^m, then there exists a unique parametric waveform $\rho(t)$ associated with each admissible pair $(v(t),i(t))$. Conversely, if $\mu(\cdot)$ is not injective, then there exists an admissible pair of $\mathcal{N}(k,m)$ which gives rise to more than one parametric waveforms $\rho(t)$.

Proof. The proof is easy though quite lengthy [38] and is therefore omitted.

Definition 10. An n-port $\mathcal{N}$ is said to be non-energic if $\langle v(t),i(t) \rangle = 0$ $\forall\, t \in R^1$ and $\forall$ continuous admissible pairs $(v(t),i(t))$ of $\mathcal{N}$.

Definition 11. An n-port $\mathcal{N}$ is said to be lossless if the average power

$$P(v(t),i(t)) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle v(t),i(t) \rangle \, dt = 0 \qquad (22)$$

$\forall$ bounded continuous admissible pair $(v(t),i(t))$ of $\mathcal{N}$. Otherwise, it is said to be lossy.

Theorem 5. Lossless Criteria for n-port Inductors and Capacitors

Let $\mathcal{L}(1,m)$ $\{\mathcal{C}(1,m)\}$ be a reciprocal n-port inductor {capacitor} represented by $\mu \triangleq [\phi,i]$ $\{\mu \triangleq [v,q]\}$: $R^m \to R^{2n}$. If:

(1) $\quad \mu(\cdot)$ is injective

(2) $$\lim_{\|\rho\| \to \infty} \|i(\rho)\| = \infty \quad \{\lim_{\|\rho\| \to \infty} \|v(\rho)\| = \infty\} \tag{23}$$

<u>Then $\mathcal{L}(1,m)$ $\{\mathcal{C}(1,m)\}$ is lossless.</u>

<u>Proof</u>. It suffices to prove the inductor case. When $m = 0$, $\mathcal{L}(1,0)$ is represented by a single point $[\phi(\rho^0), i(\rho^0)] \in R^n \times R^n$. Hence, the only bounded continuous admissible pair of $\mathcal{L}(1,0)$ is $(v(t),i(t)) = (0,i(\rho^0))$. Hence (22) implies $P(0,i(\rho^0)) = 0$ and $\mathcal{L}(1,0)$ is lossless. Now suppose $m > 0$. Let $(v(t),i(t))$ be a bounded continuous pair of $\mathcal{L}(1,m)$. By Lemma 3, $\exists$ a unique parametric waveform $\rho(t)$ such that

$$v(t) = \frac{d\phi(\rho(t))}{dt}, \quad i(t) = i(\rho(t)) \tag{24}$$

Clearly, $\rho(t)$ is continuous since $i(\rho)$ and $i(t)$ are continuous and $\rho(t)$ is unique. Moreover, (23) implies that $\rho(t)$ is bounded since $i(t)$ is bounded. Taking the time derivative of the inductor energy function as defined in Table 2, we obtain:

$$\frac{dW_L(\rho(t),\rho_0)}{dt} = \frac{\partial}{\partial \rho}\left[\int_{\Gamma\rho_0,\rho(t)} \langle i(\rho), d\phi(\rho) \rangle\right] \frac{d\rho(t)}{dt}$$

$$= \langle i(\rho(t)), \frac{d\phi(\rho(t))}{dt} \rangle = \langle i(t), v(t) \rangle \tag{25}$$

Substituting (25) into (22), we obtain

$$P(v(t),i(t)) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dW_L(\rho(t),\rho_0)}{dt}\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} [W_L(\rho(T),\rho_0) - W_L(\rho(0),\rho_0)].$$

Since $\rho(t)$ is bounded and $W_L(\cdot,\rho_0)$ is a continuous function of ρ, there is a constant K, $0 < k < \infty$, such that $|W_L(\rho(t),\rho_0)| \leq K \quad \forall\ t \in R^1$. Hence $P(v(t),i(t)) = 0$. Since $(v(t),i(t))$ is an arbitrary admissible pair, $\mathcal{L}$ (1,m) is lossless. Q.E.D.

The following examples show that the conditons of theorem 5 are as sharp as possible:

Example 1. Let $\mathcal{N}$ be a 1-port inductor represented by $i = i(\rho) = 1$ and $\phi = \phi(\rho) = \rho$; i.e., a constant current source. Then $\mathcal{N}$ is reciprocal and $\mu \triangleq [\phi,i]$ is injective. However, $\mathcal{N}$ is lossy since $P(v(t),i(t)) = P(\sin^2 t,1) = \frac{1}{2}$ and $(\sin^2 t,1)$ is obviously a bounded continuous admissible pair. Hence only (23) is violated in this example.

Example 2. Let $\mathcal{N}$ be a 1-port inductor represented by:

$$i(\rho) = \frac{1}{\sqrt{2}}(\rho+1) - \sqrt{2},\ \phi(\rho) = \frac{1}{\sqrt{2}}(\rho-1) + \sqrt{2},\ \rho < 0$$

$$i(\rho) = \cos(\rho - \tfrac{\pi}{4}),\ \phi(\rho) = \sin(\rho - \tfrac{\pi}{4}),\ 0 \leq \rho < 2\pi$$

$$i(\rho) = \frac{1}{\sqrt{2}}(\rho-2\pi+1) - \sqrt{2},\ \phi(\rho) = \frac{1}{\sqrt{2}}(\rho-2\pi-1) + \sqrt{2},\ \rho \geq 2\pi$$

Clearly, $\mathcal{N}$ is reciprocal and (23) is satisfied. However, $\mu(\cdot) \triangleq [\phi(\cdot),i(\cdot)]$ is not injective since $\mu(0) = \mu(2\pi)$. Indeed, $\mathcal{N}$ is lossy since $P(v(t),i(t)) = P(\cos t, \cos t) = \frac{1}{2}$ and $(v(t),i(t)) = (\cos t, \cos t)$ is a bounded continuous admissible pair.

To show that reciprocity also plays an important role in theorem 5, we offer the following:

<u>Lemma 4</u>. <u>Let $\mathcal{L}$ $\{\mathcal{C}\}$ be a linear, non-reciprocal ϕ-controlled $\{q\text{-controlled}\}$ n-port inductor $\{$capacitor$\}$ characterized by $i = L^{-1}\phi$ $\{v = C^{-1}q\}$ where $L^{-1} \triangleq [\ell_{jk}]$ $\{C^{-1} \triangleq [c_{jk}]\}$ is an $n \times n$ constant matrix such that</u> $\lim\limits_{\|\phi\|\to\infty} \|L^{-1}\phi\| = \infty$ $\{\lim\limits_{\|q\|\to\infty} \|C^{-1}q\| = \infty\}$. <u>Then $\mathcal{L}$ $\{\mathcal{C}\}$ is lossy</u>.

<u>Proof</u>. It suffices to prove the inductor case. Let $\mathcal{L}$ be represented by the injective function $\mu \triangleq [\phi,i]$: $R^n \to R^{2n}$, where $\phi(\rho) = \rho$, $i(\rho) = L^{-1}\rho$. Let $(v(t),i(t))$ be any bounded continuous admissible pair. Let $\rho(t) = \phi(t)$ be the unique associated parametric waveform such that $v(t) = \dot{\phi}(t)$ and $i(t) = L^{-1}\phi(t)$. Clearly, $\phi(\cdot)$ is also bounded and continuous. Consider

$$\int_0^T \langle v(t),i(t)\rangle dt = \int_0^T \sum_{j,k=1}^{n} \dot{\phi}_j(t)\ell_{jk}\,\phi_k(t)dt$$

$$= \sum_{j=1}^{n} \ell_{jj} \int_0^T \phi_j(t)\dot{\phi}_j(t)dt + \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \int_0^T [\ell_{jk}\dot{\phi}_j(t)\phi_k(t)-\ell_{kj}\dot{\phi}_k(t)\phi_j(t)]dt$$

$$= \gamma(T) + \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \int_0^T [\ell_{jk}-\ell_{kj}]\dot{\phi}_j(t)\phi_k(t)dt \qquad (26)$$

where

$$\gamma(T) \triangleq \sum_{j=1}^{n} \frac{1}{2} \ell_{jj}[\phi_j^2(T)-\phi_j^2(0)] + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \ell_{kj}[\phi_k(T)\phi_j(T)-\phi_k(0)\phi_j(0)]$$

is bounded for all $T \in R^1$ since $\phi(T)$ is bounded $\forall\ T \in R^1$. Hence, $\exists$ a constant K such that

$$\sup_{T\in R^1} |\gamma(T)| \leq K \tag{27}$$

Notice that (26) and (27) are valid for any bounded continuous admissible pair.

Since $\mathcal{L}$ is non-reciprocal and linear, $\exists$ a pair of integers r and s such that

$$\ell_{rs} - \ell_{sr} \neq 0 \tag{28}$$

In order to show that $\mathcal{L}$ is lossy, it suffices to exhibit a bounded continuous admissible pair $(v(t),i(t))$ such that $P(v(t),i(t)) \neq 0$. Consider the admissible pair:

$$\begin{aligned}
&v_j(t) = 0, \text{ for } j = 1,2,\cdots,n,\ j \neq r, \text{ and } j \neq s \\
&v_r(t) = \sin t \\
&v_s(t) = \cos t \\
&i_j(t) = -\ell_{jr}\cos t + \ell_{js}\sin t, \text{ for } j = 1,2,\cdots,n
\end{aligned} \tag{29}$$

and substituting it into (26), we obtain

$$\int_0^T \langle v(t),i(t)\rangle dt = \frac{1}{2} T\ [\ell_{rs}-\ell_{sr}] + \gamma_1(T) \qquad (30)$$

where $\gamma_1(T) \triangleq \gamma(T) + [\ell_{rs}-\ell_{sr}][\frac{1}{4}(\sin 2T - \sin 0)]$ is bounded $\forall\ T \in R^1$ in view of (27). Hence, $P(v(t),i(t)) = \frac{1}{2}[\ell_{rs}-\ell_{sr}] \neq 0$; i.e., $\mathcal{L}$ is lossy.

Q.E.D.

Corollary. Every linear non-reciprocal n-port inductor or capacitor characterized by a non-singular inductance or capacitance matrix is lossy.

Theorem 6. Lossless Criteria for memristors.

Every anti-reciprocal ϕ-controlled or q-controlled memristor is non-energic, and hence lossless.

Proof. It suffices to consider the q-controlled case. Let the memristor be represented by $\phi(\rho) = r(\rho)$ and $q(\rho) = \rho$. Hence $\mu(\cdot) \triangleq [\phi(\cdot),q(\cdot)]$ is injective. Let $(v(t),i(t))$ be a continuous admissible pair. Then $\exists$ a unique parametric waveform $\rho(t) = q(t)$ such that $v(t) = dr(q(t))/dt$ and $i(t) = dq(t)/dt$. Hence

$$\langle v(t),i(t)\rangle = \langle J_r(q(t))\dot{q}(t),\dot{q}(t)\rangle = 0 \qquad (31)$$

since the anti-reciprocity condition implies that $J_r(\cdot)$ is skew-symmetric by Table 2.

Q.E.D.

Definition 12. An n-port is said to be initially relaxed at the initial time $t = t_0$ if it has no stored energy at t_0.

Remarks 5.

(i) Both n-port resistor and memristor are always initially relaxed since neither can store energy.

(ii) An n-port inductor is initially relaxed if, and only if, $\phi(t_0) = 0$. Observe that the initial inductor current need not be zero.

(iii) An n-port capacitor is initially relaxed if, and only if, $q(t_0) = 0$. Observe that the initial capacitor voltage need not be zero.

Definition 13. An initially relaxed n-port $\mathcal{N}$ is said to be passive[4] if

$$w(t) \triangleq \int_{t_0}^{t} \langle v(\tau), i(\tau) \rangle d\tau \geq 0, \ \forall\, t \geq t_0 \tag{32}$$

$\forall$ admissible pairs $(v(t), i(t))$ in which (32) is integrable. $\mathcal{N}$ is said to be active if it is not passive.

Theorem 7. Passivity criteria for n-port resistors.

Let $\mathcal{N}$ be an n-port resistor represented by $\mu \triangleq [v,i]: R^m \to R^{2n}$. Then $\mathcal{N}$ is passive if, and only if $\langle v(\rho), i(\rho) \rangle \geq 0 \ \forall\, \rho \in R^m$. In particular, if $\mathcal{N}$ admits a C^0-hybrid representation $y = h(x)$, then $\mathcal{N}$ is passive if $h(\cdot)$ is non-decreasing on R^n and $h(0) = 0$.

Proof. The first half of Theorem 7 is obvious. Hence it suffices to prove the second half. Since $h(\cdot)$ is non-decreasing on R^n, we have

$$\langle h(x^a) - h(x^b), x^a - x^b \rangle \geq 0 \quad \forall\, x_a, x_b \in R^n \tag{33}$$

In particular, let $x_b = 0$ and $x_a = x$. Since $h(0) = 0$, (33) reduces to

$\langle h(x),x\rangle = \langle y,x\rangle = \langle v,i\rangle \geq 0.$ Q.E.D.

Theorem 8. Passivity Criteria for n-port inductors and capacitors. Let $\mathcal{L}(1,m)$ $\{\mathcal{C}(1,m)\}$ be a reciprocal n-port inductor {capacitor} represented by $\mu \triangleq [\phi,i]: R^m \to R^{2n}$ $\{\mu \triangleq [v,q]: R^m \to R^{2n}\}$, where $\mu(\cdot)$ is an injection on R^m. Suppose $\exists$ a point $\rho_0 \in R^m$ such that $\phi(\rho_0) = 0$ $\{q(\rho_0) = 0\}$. Then $\mathcal{L}(1,m)$ $\{\mathcal{C}(1,m)\}$ is passive if, and only if the associated inductor energy $W_L(\rho_1\rho_0)$ {capacitor energy $W_C(\rho_1\rho_0)$} is non-negative $\forall\ \rho \in R^m$. In particular, if $\mathcal{L}$ $\{\mathcal{C}\}$ is a reciprocal non-decreasing ϕ-controlled n-port inductor {q-controlled n-port capacitor} such that $i(\phi) = 0$ when $\phi = 0$ $\{v(q) = 0$ when $q = 0\}$, then $\mathcal{L}$ $\{\mathcal{C}\}$ is passive.

Proof. It suffices to consider the inductor case. Let $(v(t),i(t))$ be a continuous admissible pair which is consistent with the assumption that $\phi(t_0) = 0$. Lemma 3 implies that $\exists$ a parametric waveform $\rho(t)$ such that

$$v(t) = \frac{d\phi(\rho(t))}{dt},\ i(t) = i(\rho(t)),\ \text{and}\ \phi(\rho(t_0)) = 0 = \phi(\rho_0).$$

Clearly, $\rho(t)$ is continuous and unique. Hence,

$$w(t) \triangleq \int_{t_0}^{t} \langle v(\tau),i(\tau)\rangle\, d\tau = \int_{t_0}^{t} \frac{d}{d\tau}\,[W_L(\rho(\tau),\rho_0)]d\tau$$

$$= W_L(\rho(t),\rho_0) - W_L(\rho(t_0),\rho_0)$$

Now,

$$W_L(\rho(t_0),\rho_0) = \int_{\Gamma[\phi(\rho_0),\phi(\rho(t_0))]} \langle i(\rho),d\phi(\rho)\rangle$$

$$= \int_{\Gamma[0,0]} \langle i(\rho), d\phi(\rho) \rangle = 0.$$

Hence, $w(t) = W_L(\rho(t),\rho_0) \geq 0$ and $\mathcal{L}$ is passive.

In the special case where $\mathcal{L}$ can be represented by $i = i(\phi)$, we have

$$W_L(\phi) = \int_{\Gamma[0,\phi]} \langle i(\phi), d\phi \rangle$$

and $i(\phi) = \nabla W_L(\phi)$. But $i(\cdot)$ is non-decreasing and hence $W_L(\cdot)$ is a convex function on R^n [26]. Since $\nabla W_L(0) = i(0) = 0$, $\phi = 0$ is a global minima of the convex function $W_L(\phi)$. Hence $W_L(\phi) \geq 0 \ \forall \ \phi \in R^n$. Thus $\mathcal{L}$ is passive. Q.E.D.

To show that reciprocity plays a crucial role in the passivity of n-port inductors and capacitors, we offer the following:

Lemma 5.

Every anti-reciprocal linear n-port inductor {capacitor} is active.

Proof. We prove only the inductor case. Let $\mathcal{L}$ be characterized by $i = L^{-1}\phi$, where $L^{-1} \triangleq [\ell_{jk}]$ is an $n \times n$ non-zero constant matrix. Let $(v(t),i(t))$ be a continuous admissible pair of $\mathcal{L}$ which is consistent with the assumption $\phi(t_0) = 0$. Then $\exists$ a unique parametric waveform $\rho(t) = \phi(t)$ such that $v(t) = \dot{\phi}(t)$, $i(t) = L^{-1}\phi(t)$, and $\phi(t_0) = 0$. Since $v(\cdot)$ is continuous, $\phi(\cdot)$ is C^1 on R^1. Hence,

$$w(t) = \int_{t_0}^{t} \langle v(\tau), i(\tau) \rangle d\tau = \int_{t_0}^{t} \sum_{j,k=1}^{n} \ell_{jk}\dot{\phi}_j(\tau)\phi_k(\tau)d\tau$$

$$= \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \int_{t_0}^{t} [\ell_{jk}\phi_k(\tau)\dot{\phi}_j(\tau) + \ell_{kj}\phi_j(\tau)\dot{\phi}_k(\tau)]d\tau$$

$$+ \sum_{j=1}^{n} \ell_{jj} \int_{t_0}^{t} \phi_j(\tau)\dot{\phi}_j(\tau)d\tau \qquad (34)$$

To show $\mathcal{L}$ is active, it suffices to exhibit a continuous admissible pair such that $\phi(t_0) = 0$ and $w(t_1) < 0$ for some time $t_1 > t_0$. Since L^{-1} is a non-zero matrix and $\mathcal{L}$ is anti-reciprocal, $\exists$ exists a pair of integers r and s such that $\ell_{rs} = -\ell_{sr} \neq 0$. Consider the admissible pair:

$$v_j(t) = 0, \forall j = 1,2,\cdots,n, j \neq r \text{ and } j \neq s$$

$$v_r(t) = e^t$$

$$v_s(t) = \alpha e^t[\sin t + \cos t] \qquad (35)$$

$$i_j(t) = \ell_{jr}[e^t - e^{t_0}] + \ell_{js}\alpha[e^t \sin t - e^{t_0} \sin t_0],$$

$$\forall j = 1,2,\cdots,n$$

where the constant α is to be assigned later. The associated parametric waveform $\phi(t)$ is given by:

$$\phi_j(t) = 0, \forall j = 1,2,\cdots,n, j \neq r \text{ and } j \neq s$$

$$\phi_r(t) = e^t - e^{t_0}$$

$$\phi_s(t) = \alpha[e^t \sin t - e^{t_0} \sin t_0] \qquad (36)$$

Substituting (35) and (36) into (34), we obtain

$$w(t) = \int_{t_0}^{t} 2\, \ell_{rs} e^{t} \alpha [e^{\tau} \sin \tau - e^{t_0} \sin t_0] d\tau$$

$$- \ell_{rs} [e^{t} - e^{t_0}][\alpha (e^{t} \sin t - e^{t_0} \sin t_0)]$$

$$= \ell_{rs} \alpha \{ \frac{2}{5} e^{2\tau} (2 \sin \tau - \cos \tau)] \Big|_{t_0}^{t} - 2e^{t_0} \sin t_0 [e^{t} - e^{t_0}]$$

$$- [e^{t} - e^{t_0}][e^{t} \sin t - e^{t_0} \sin t_0]\} \qquad (37)$$

Setting $t = t_1 = t_0 + 2\pi$, (37) reduces to:

$$w(t_0 + 2\pi) = \frac{1}{5} \alpha \ell_{rs}\, e^{2t_0} [\sin t_0 + 2 \cos t_0](1 - e^{4\pi}) \qquad (38)$$

If $\sin t_0 + 2 \cos t_0 \neq 0$, we can let $\alpha = \sin t_0 + 2 \cos t_0$ and $w(t_0 + 2\pi) < 0$. Hence $\mathcal{L}$ is active. If $\sin t_0 + 2 \cos t_0 = 0$, then $\sin t_0 = - 2 \cos t_0 \neq 0$ and we can let $t_2 = t_0 + \pi$, and $w(t_0 + \pi) = \alpha \ell_{rs} e^{2t_0} \cos t_0 [\frac{23}{5} - 4e^{\pi}]$. Hence, if we let $\alpha = - \cos t_0 [\frac{23}{5} - 4e^{\pi}]$, then $w(t_0 + \pi) < 0$ and $\mathcal{L}$ is active. Q.E.D.

Theorem 9. Passivity criteria for memristors.

Every non-decreasing q-controlled or ϕ-controlled n-port memristor is passive.

Proof. Let $\mathcal{N}$ be charactorized by $q = q(\phi)$ and let $(v(t), i(t))$ be a continuous admissible pair. Let $\rho(t) = \phi(t)$ be the associated unique parametric waveform. Then

$$\langle v(t),i(t)\rangle = \langle \dot{\phi}(t),J_g(\phi(t))\dot{\phi}(t)\rangle \geq 0 \qquad (39)$$

since $q(\cdot)$ is non-decreasing. Q.E.D.

VI. CONCLUDING REMARKS

A unified theory of algebraic n-ports has been presented via the parametric representation (2). The dimension of an n-port is introduced and shown to be a logical tool for classifying and separating regular n-ports from such pathological elements as nullators, norators, nullors, etc. A new definition of reciprocity is proposed which led to the logical conclusion that every one-port of dimension 1 is reciprocal, and that the nullator and the norator are both non-reciprocal one-ports. Several surprising results have been obtained: (1) contrary to the well-known result that every linear n-port can be decomposed into a reciprocal and an anti-reciprocal n-port, theorem 4 shows that no such generalization is possible with nonlinear n-ports. (2) Contrary to the common belief that reciprocity is an independent local property, Lemmas 4 and 5, as well as theorems 5 and 8, show that this property plays a significant role in determining the losslessness and passivity of n-ports.

It is hoped that the global and local characterizations in Sections III and IV will provide a foundation for the synthesis of algebraic n-ports. The basic philosophy would be to decompose a prescribed n-port into an appropriate interconnection of component n-ports chosen from among the subclasses defined in this paper.

FOOTNOTES:

1. To economize on symbols, we use the same generic index "n" for different "n" ports. We will also assume that whenever necessary, our n-ports are provided with internal isolation transformers so that arbitrary interconnections will not introduce circulation currents [2].
2. An $n \times n$ matrix A is said to be a P-matrix if all its principal submatrices have positive determinants [14].
3. This reciprocity criterion was first derived in this coordinate-free form for linear n-ports in [34]. This criterion had also been derived for nonlinear n-ports by Brayton [1].
4. The definition of passivity as presented by Youla et al., [3] has been shown to be unsatisfactory when <u>measurable</u> admissible pairs are allowed [39-40]. However, since we restrict our admissible pairs to be <u>continuous</u> time functions, no such difficulty arises in our case.

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Table 2. Generalized potential and co-potential functions Associated with algebraic n-ports.

Type of N-Port				Corresponding Terminologies			
Reciprocal algebraic n-port	$\mathcal{N}$	ξ	η	Generalized Potential		Generalized Co-potential	
				$\Omega(\rho,\rho_0)$	$\Omega(\eta)$	$\Omega^*(\rho,\rho_0)$	$\Omega^*(\xi)$
resistor	$\mathcal{R}$	v	i	content		co-content	
				$G(\rho,\rho_0)$	$G(i)$	$G^*(\rho,\rho_0)$	$G^*(v)$
inductor	$\mathcal{L}$	ϕ	i	inductor co-energy		inductor energy	
				$W_L^*(\rho,\rho_0)$	$W_L^*(i)$	$W_L(\rho,\rho_0)$	$W_L(\phi)$
capacitor	$\mathcal{C}$	v	q	capacitor energy		capacitor co-energy	
				$W_C(\rho,\rho_0)$	$W_C(q)$	$W_C^*(\rho,\rho_0)$	$W_C^*(v)$
memriston	$\mathcal{N}$	ϕ	q	action		co-action	
				$A_M(\rho,\rho_0)$	$A_M(q)$	$A_M^*(\rho,\rho_0)$	$A_M^*(\phi)$

Table 1. Representation of Algebraic N-Ports and Their Criteria for Reciprocity and Anti-reciprocity.

Mode of Representation	Defining Equations $\xi = \begin{bmatrix}\xi_a \\ \xi_b\end{bmatrix}$, $\eta = \begin{bmatrix}\eta_a \\ \eta_b\end{bmatrix}$	Jacobian Matrix	Necessary and Sufficient Conditions for Reciprocity	Necessary and Sufficient Conditions for Anit-Reciprocity
1. ξ-controlled Representation	$\eta = g(\xi)$	$J_g = \frac{\partial g}{\partial \xi}$	J_g is symmetric	J_g is skew-symmetric
2. η-controlled Representation	$\xi = r(\eta)$	$J_r = \frac{\partial r}{\partial \eta}$	J_r is symmetric	J_r is skew-symmetric
3. Hybrid I Representation	$\eta_a = h_a^1(\xi_a, \eta_b)$ $\xi_b = h_b^1(\xi_a, \eta_b)$	$J_{h^1} = \begin{bmatrix} \frac{\partial h_a^1}{\partial \xi_a} & \frac{\partial h_a^1}{\partial \eta_b} \\ \frac{\partial h_b^1}{\partial \xi_a} & \frac{\partial h_b^1}{\partial \eta_b} \end{bmatrix}$	(a) $\partial h_a^1/\partial \xi_a$ is symmetric (b) $\partial h_b^1/\partial \eta_b$ is symmetric (c) $\partial h_a^1/\partial \eta_b = -[\partial h_b^1/\partial \xi_a]^T$	J_{h^1} is skew-symmetric
4. Hybrid II Representation	$\xi_a = h_a^2(\eta_a, \xi_b)$ $\eta_b = h_b^2(\eta_a, \xi_b)$	$J_{h^2} = \begin{bmatrix} \frac{\partial h_a^2}{\partial \eta_a} & \frac{\partial h_a^2}{\partial \xi_b} \\ \frac{\partial h_b^2}{\partial \eta_a} & \frac{\partial h_b^2}{\partial \xi_b} \end{bmatrix}$	(a) $\partial h_a^2/\partial \eta_a$ is symmetric (b) $\partial h_b^2/\partial \xi_b$ is symmetric (c) $\partial h_a^2/\partial \xi_b = -[\partial h_b^2/\partial \eta_a]^T$	J_{h^2} is skew-symmetric
5. Conjugate Hybrid I Representation	$\eta_a = h_a^{*1}(\xi_a, \eta_b^*)$ $\xi_b = h_b^{*1}(\xi_a, \eta_b^*)$ $\eta_b^* \equiv -\eta_b$	$J_{h^{*1}} = \begin{bmatrix} \frac{\partial h_a^{*1}}{\partial \xi_a} & \frac{\partial h_a^{*1}}{\partial \eta_b^*} \\ \frac{\partial h_b^{*1}}{\partial \xi_a} & \frac{\partial h_b^{*1}}{\partial \eta_b^*} \end{bmatrix}$	$J_{h^{*1}}$ is symmetric	(a) $\partial h_a^{*1}/\partial \xi_a$ is skew-symmetric (b) $\partial h_b^{*1}/\partial \eta_b^*$ is skew-symmetric (c) $\partial h_a^{*1}/\partial \eta_b^* = [\partial h_b^{*1}/\partial \xi_a]^T$
6. Conjugate Hybrid II Representation	$\xi_a = h_a^{*2}(\eta_a, \xi_b)$ $\eta_b^* = h_b^{*2}(\eta_a, \xi_b)$ $\eta_b^* \equiv -\eta_b$	$J_{h^{*2}} = \begin{bmatrix} \frac{\partial h_a^{*2}}{\partial \xi_a} & \frac{\partial h_a^{*2}}{\partial \xi_b} \\ \frac{\partial h_b^{*2}}{\partial \eta_a} & \frac{\partial h_b^{*2}}{\partial \xi_b} \end{bmatrix}$	$J_{h^{*2}}$ is symmetric	(a) $\partial h_a^{*2}/\partial \eta_a$ is skew-symmetric (b) $\partial h_b^{*2}/\partial \xi_b$ is skew-symmetric (c) $\partial h_a^{*2}/\partial \xi_b = [\partial h_b^{*2}/\partial \eta_a]^T$
7. Conjugate Hybrid III Representation	$\eta_a = h_a^{*3}(\xi_a^*, \eta_b)$ $\xi_b = h_b^{*3}(\xi_a^*, \eta_b)$ $\xi_a^* \equiv -\xi_a$	$J_{h^{*3}} = \begin{bmatrix} \frac{\partial h_a^{*3}}{\partial \xi_a^*} & \frac{\partial h_a^{*3}}{\partial \eta_b} \\ \frac{\partial h_b^{*3}}{\partial \xi_a^*} & \frac{\partial h_b^{*3}}{\partial \eta_b} \end{bmatrix}$	$J_{h^{*3}}$ is symmetric	(a) $\partial h_a^{*3}/\partial \xi_a^*$ is skew-symmetric (b) $\partial h_b^{*3}/\partial \eta_b$ is skew-symmetric (c) $\partial h_a^{*3}/\partial \eta_b = [\partial h_b^{*3}/\partial \xi_a^*]^T$
8. Conjugate Hybrid IV Representation	$\xi_a^* = h_a^{*4}(\eta_a, \xi_b)$ $\eta_b = h_b^{*4}(\eta_a, \xi_b)$ $\xi_a^* \equiv -\xi_a$	$J_{h^{*4}} = \begin{bmatrix} \frac{\partial h_a^{*4}}{\partial \eta_a} & \frac{\partial h_a^{*4}}{\partial \xi_b} \\ \frac{\partial h_b^{*4}}{\partial \eta_a} & \frac{\partial h_b^{*4}}{\partial \xi_b} \end{bmatrix}$	$J_{h^{*4}}$ is symmetric	(a) $\partial h_a^{*4}/\partial \eta_a$ is skew-symmetric (b) $\partial h_b^{*4}/\partial \xi_b$ is skew-symmetric (c) $\partial h_a^{*4}/\partial \xi_b = [\partial h_b^{*4}/\partial \eta_a]^T$
9. Transmission I Representation	$\xi_a = t_\xi^1(\xi_b, \eta_b^*)$ $\eta_a = t_\eta^1(\xi_b, \eta_b^*)$ $\eta_b^* \equiv -\eta_b$	$J_{t^1} = \begin{bmatrix} \frac{\partial t_\xi^1}{\partial \xi_b} & \frac{\partial t_\xi^1}{\partial \eta_b^*} \\ \frac{\partial t_\eta^1}{\partial \xi_b} & \frac{\partial t_\eta^1}{\partial \eta_b^*} \end{bmatrix}$	(a) $\left[\frac{\partial t_\xi^1}{\partial \xi_b}\right]^T \frac{\partial t_\eta^1}{\partial \xi_b}$ is symmetric (b) $\left[\frac{\partial t_\xi^1}{\partial \eta_b^*}\right]^T \frac{\partial t_\eta^1}{\partial \eta_b^*}$ is symmetric (c) $\left[\frac{\partial t_\xi^1}{\partial \xi_b}\right]^T \frac{\partial t_\eta^1}{\partial \eta_b^*} - \left[\frac{\partial t_\eta^1}{\partial \xi_b}\right]^T \frac{\partial t_\xi^1}{\partial \eta_b^*} = I_n$	(a) $\left[\frac{\partial t_\xi^1}{\partial \xi_b}\right]^T \frac{\partial t_\eta^1}{\partial \xi_b}$ is skew-symmetric (b) $\left[\frac{\partial t_\xi^1}{\partial \eta_b^*}\right]^T \frac{\partial t_\eta^1}{\partial \eta_b^*}$ is skew-symmetric (c) $\left[\frac{\partial t_\xi^1}{\partial \xi_b}\right]^T \frac{\partial t_\eta^1}{\partial \eta_b^*} + \left[\frac{\partial t_\eta^1}{\partial \xi_b}\right]^T \frac{\partial t_\xi^1}{\partial \eta_b^*} = I_n$
10. Transmission II Representation	$\xi_b = t_\xi^2(\xi_a, \eta_a^*)$ $\eta_b = t_\eta^2(\xi_a, \eta_a^*)$ $\eta_a^* \equiv -\eta_a$	$J_{t^2} = \begin{bmatrix} \frac{\partial t_\xi^2}{\partial \xi_a} & \frac{\partial t_\xi^2}{\partial \eta_a^*} \\ \frac{\partial t_\eta^2}{\partial \xi_a} & \frac{\partial t_\eta^2}{\partial \eta_a^*} \end{bmatrix}$	(a) $\left[\frac{\partial t_\xi^2}{\partial \xi_a}\right]^T \frac{\partial t_\eta^2}{\partial \xi_a}$ is symmetric (b) $\left[\frac{\partial t_\xi^2}{\partial \eta_a^*}\right]^T \frac{\partial t_\eta^2}{\partial \eta_a^*}$ is symmetric (c) $\left[\frac{\partial t_\xi^2}{\partial \xi_a}\right]^T \frac{\partial t_\eta^2}{\partial \eta_a^*} - \left[\frac{\partial t_\eta^2}{\partial \xi_a}\right]^T \frac{\partial t_\xi^2}{\partial \eta_a^*} = I_n$	(a) $\left[\frac{\partial t_\xi^2}{\partial \xi_a}\right]^T \frac{\partial t_\eta^2}{\partial \xi_a}$ is skew-symmetric (b) $\left[\frac{\partial t_\xi^2}{\partial \eta_a^*}\right]^T \frac{\partial t_\eta^2}{\partial \eta_a^*}$ is skew-symmetric (c) $\left[\frac{\partial t_\xi^2}{\partial \xi_a}\right]^T \frac{\partial t_\eta^2}{\partial \eta_a^*} + \left[\frac{\partial t_\eta^2}{\partial \xi_a}\right]^T \frac{\partial t_\xi^2}{\partial \eta_a^*} = I_n$