CANCELLATIONS IN MULTIVARIABLE CONTINUOUS-TIME AND DISCRETE-TIME FEEDBACK SYSTEMS

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Memorandum No. ERL-M346

31 July 1972

ELECTRONICS RESEARCH LABORATORY

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August 7, 1972

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ABSTRACT

This paper considers a multivariable system with proper rational matrix transfer functions G_0 and G_f in the forward and feedback branches, resp. Strictly algebraic procedures lead to polynomials whose zeros are the poles of the matrix transfer functions from input to output (H_y), and from input to error (H_e). The role of the assumption $\det[I + G_f(\infty)G_0(\infty)] \neq 0$ and the relation between the zeros of $\det[I + G_f G_0]$ and the poles of H_y and H_e are indicated. The implications for stability analysis of continuous-time as well as discrete-time systems are obvious.

Research Sponsored by the Joint Services Electronics Program, Contract F44620-71-C-0087.

I. Introduction

The main result of this paper (see (32) and (33), below) is an algorithmic method for generating two polynomials whose zeros constitute a complete list of the poles of the two transfer functions of the feedback system under consideration. The methods of the paper apply equally to continuous-time systems (Laplace transform methods) or to discrete-time systems (z-transform methods). In our exposition we use s to label the complex variable.

II. Notation and Preliminaries

Let $\mathbb{R}$, ($\mathbb{C}$), denote the field of real, (complex, resp.) numbers. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, Let $\mathbb{R}[s]$, ($\mathbb{R}(s)$), be the set of all polynomials, (rational functions, resp.), in the complex variable s with real coefficients. Let $\mathbb{R}[s]^{m\times m}$, ($\mathbb{R}(s)^{m\times m}$), be the set of all m×m matrices with elements in $\mathbb{R}[s]$, ($\mathbb{R}(s)$, resp.). Let N and D be matrices with elements in $\mathbb{R}[s]$; a matrix M is said to be a <u>common left divisor</u> of N and D iff there exist matrices $\tilde{N}$ and $\tilde{D}$ such that $N=M\tilde{N}$ and $D=M\tilde{D}$ where M, $\tilde{N}$, and $\tilde{D}$ have elements in $\mathbb{R}[s]$; both N and D are said to be <u>right multiples</u> of M; a matrix L with elements in $\mathbb{R}[s]$ is said to be a <u>greatest common left divisor</u> (g.c.l.d.) of N and D iff (i) it is a common left divisor of N and D, and (ii) it is a right multiple of every common left divisor of N and D. When a g.c.l.d. L is unimodular (i.e., det L = constant ≠ 0) then the polynomial matrices N and D are said to be <u>left coprime</u>. We define similarly a greatest common right divisor (g.c.r.d.) and right coprime. Given any $G \in \mathbb{R}(s)^{m\times m}$ it can easily be written as ND^{-1} or $\overline{D}^{-1}\overline{N}$ where $N,\overline{N},D,\overline{D} \in \mathbb{R}[s]^{m\times m}$. By a standard procedure ([1] - [5]) a g.c.r.d. of N and D and a g.c.l.d. of $\overline{N}$ and $\overline{D}$ can be extracted so that

1 $$G = N_r D_r^{-1} = D_\ell^{-1} N_\ell$$

where (a) $N_r, N_\ell, D_r, D_\ell \in \mathbb{R}[s]^{m\times m}$; (b) N_r and D_r are right coprime, (c) N_ℓ and D_ℓ are left coprime. It is well known ([1]) that N_ℓ and D_ℓ are left coprime if and only if there exist $P_\ell, Q_\ell \in \mathbb{R}[s]^{m\times m}$ such that

2 $$N_\ell P_\ell + D_\ell Q_\ell = I$$

For completeness, we prove the well known fact: if

3 $$G = D_\ell^{-1} N_\ell$$

where (i) $N_\ell, D_\ell \in \mathbb{R}[s]^{m\times m}$; (ii) N_ℓ and D_ℓ are left coprime, then

4 $p \in \mathbb{C}$ is a pole of G if and only if p is a zero of det D_ℓ

<u>Proof</u> :

$\Leftarrow$ Premultiply (2) by D_ℓ^{-1} and use (3) to obtain

5 $$GP_\ell + Q_\ell = D_\ell^{-1}$$

since, by assumption, $p \in \mathbb{C}$ is a zero of det D_ℓ and since $\det[D_\ell^{-1}] = 1/\det D_\ell$, p is a pole of the r.h.s. of (5). Since P_ℓ, Q_ℓ (being polynomial matrices) are bounded at p, G must have a pole at p.

$\Rightarrow$ Use Cramer's rule in (3) to obtain

6 $$G = N_\ell (\text{Adj } D_\ell)/\det D_\ell$$

By assumption, $p \in \mathbb{C}$ is a pole of G. Since N_ℓ, Adj $D_\ell \in \mathbb{R}[s]^{m\times m}$ (hence have no finite poles) we must have from (6) that $\det D_\ell(p) = 0$. Q.E.D.

Notes: (a) Similarly, if

7 $$G = N_r D_r^{-1}$$

where (i) $N_r, D_r \in \mathbb{R}[s]^{m\times m}$; (ii) N_r and D_r are right coprime, then

8 $p \in \mathbb{C}$ is a pole of G if and only if p is a zero of det D_r.

(b) It can be shown [2] that if R = [A,B,C,D] is <u>any</u> minimal realization of G, then $\det[sI-A] = k\cdot\det D_r$ where k is the nonzero constant such that the polynomial $k\cdot\det D_r$ is monic.

(c) If $T, U \in \mathbb{R}[s]^{m\times m}$ and are right or left coprime, it does not follow that the polynomials det T and det U are coprime. To wit: $T(s) = \text{diag}[s-1, s-2]$, $U(s) = \text{diag}[s-2, s-1]$.

III. <u>Description of the System</u>

Consider the continuous-time, linear, time-invariant, multivariable feedback system S described by $y = G_0 e$, $e = u - G_f y$ where u,e,y are the input, the error and the output, resp.; also let $G_0, G_f \in \mathbb{R}(s)^{m\times m}$ and are proper (i.e., bounded at infinity). We perform the following factorizations:

9 $$G_0 = N_0 D_0^{-1}$$

where $N_0, D_0 \in \mathbb{R}[s]^{m\times m}$ and are right coprime, and $\det D_0 \not\equiv 0$.

10 $$G_f = D_f^{-1} N_f$$

where $N_f, D_f \in \mathbb{R}[s]^{m\times m}$ and are left coprime, and $\det D_f \not\equiv 0$.

In case the feedback system is discrete-time, u, e, and y are interpreted to be the z-transforms of the input-, error-, and output-sequences and the matrix-valued rational functions G_0 and G_f are the matrix z-transform functions (see [6]). The mathematical derivation which follows applies equally well to continuous-time and discrete-time systems.

We assume

11 $$\det[I + G_f(\infty)G_0(\infty)] \neq 0.$$

Let the transfer function from u to e be H_e and that from u to y (the closed-loop transfer function) be H_y. From the system description

12 $$H_e = [I + G_f G_0]^{-1}$$

13 $$H_y = G_0 [I + G_f G_0]^{-1}$$

Comment on assumption (11): It is easy to show that:

Whenever $G_0, G_f \in \mathbb{R}(s)^{m\times m}$ and are proper, then $\det[I + G_f(\infty) G_0(\infty)] \neq 0$ if and only if H_e and H_y are proper (i.e., bounded at infinity).

By substituting (9) and (10) in (12) and (13) we obtain

14 $$H_e = D_0 \, \Omega^{-1} \, D_f$$

15 $$H_y = N_0 \, \Omega^{-1} \, D_f$$

where

16 $$\Omega \triangleq (D_f \, D_0 + N_f \, N_0).$$

$\det \Omega \neq 0$ because (11) requires that H_e tend to a nonsingular matrix at infinity. We will now extract greatest common right-and left-divisors from the products in (14) and (15). Let L be a g.c.l.d. of Ω and D_f, then

17 $$\Omega = L\tilde{\Omega}$$

18 $$D_f = L\tilde{D}_f$$

$\left.\right\}$ where $L, \tilde{\Omega}, \tilde{D}_f \in \mathbb{R}[s]^{m\times m}$.

Since $\tilde{D}_f$ and $\tilde{\Omega}$ are left coprime there exist $\tilde{P}, \tilde{Q} \in \mathbb{R}[s]^{m\times m}$ such that

19 $$\tilde{D}_f \, \tilde{P} + \tilde{\Omega} \, \tilde{Q} = I$$

Substituting (17) and (18) in (14) we obtain

$$H_e = D_0 \tilde{\Omega}^{-1} \tilde{D}_f \tag{20}$$

Let R_e be a g.c.r.d. of D_0 and $\tilde{\Omega}$, then

$$\left.\begin{aligned} D_0 &= \tilde{D}_0 R_e && (21) \\ \tilde{\Omega} &= \Omega_e R_e && (22) \end{aligned}\right\} \text{ where } R_e, \Omega_e, \tilde{D}_0 \in \mathbb{R}[s]^{m\times m}$$

Since $\tilde{D}_0$ and Ω_e are right coprime there exist $P_e, Q_e \in \mathbb{R}[s]^{m\times m}$ such that

$$P_e \tilde{D}_0 + Q_e \Omega_e = I \tag{23}$$

Substituting (22) in (19) we obtain

$$\tilde{D}_f \tilde{P} + \Omega_e R_e \tilde{Q} = I \Rightarrow \tilde{D}_f \text{ and } \Omega_e \text{ are left coprime.} \tag{24}$$

Substituting (21) and (22) in (20) we obtain

$$\boxed{H_e = \tilde{D}_0 \Omega_e^{-1} \tilde{D}_f} \tag{25}$$

We now go through a similar procedure for (15). Substituting (17) and (18) in (15) we obtain

$$H_y = N_0 \tilde{\Omega}^{-1} \tilde{D}_f \tag{26}$$

Let R_y be a g.c.r.d. of N_0 and $\tilde{\Omega}$, then

$$\left.\begin{aligned} N_0 &= \tilde{N}_0 R_y && (27) \\ \tilde{\Omega} &= \Omega_y R_y && (28) \end{aligned}\right\} \text{ where } R_y, \Omega_y, \tilde{N}_0 \in \mathbb{R}[s]^{m\times m}.$$

Since $\tilde{N}_0$ and Ω_y are right coprime there exist $P_y, Q_y \in \mathbb{R}[s]^{m\times m}$ such that

$$P_y \tilde{N}_0 + Q_y \Omega_y = I \tag{29}$$

Substituting (28) in (19) we obtain

$$\tilde{D}_f \tilde{P} + \Omega_y R_y \tilde{Q} = I \Rightarrow \tilde{D}_f \text{ and } \Omega_y \text{ are left coprime.} \tag{30}$$

Substituting (27) and (28) in (26) we obtain

31 $$\boxed{H_y = \tilde{N}_0\, \Omega_y^{-1}\, \tilde{D}_f}$$

Now that we have (25) and (31) we can state the

Theorem: For system S, with assumption (11) and the notation above, we have

32 $p_e \in \overline{\mathbb{C}}$ is a pole of H_e if and only if p_e is a zero of det Ω_e

33 $p_y \in \overline{\mathbb{C}}$ is a pole of H_y if and only if p_y is a zero of det Ω_y.

Proof: Since (11) implies that H_e and H_y are bounded at ∞, all of their poles are necessarily finite, so we need only consider finite poles.

Proof of (32):

$\Leftarrow$ Multiply (23) on the right by Ω_e^{-1} and then by $\tilde{D}_f$ to obtain

34 $$P_e\, \tilde{D}_0\, \Omega_e^{-1}\, \tilde{D}_f + Q_e\, \tilde{D}_f = \Omega_e^{-1}\, \tilde{D}_f$$

Using (25), (34) becomes

35 $$P_e\, H_e + Q_e\, \tilde{D}_f = \Omega_e^{-1}\, \tilde{D}_f.$$

Now, by assumption det $\Omega_e(p_e) = 0$. Since $\tilde{D}_f$ and Ω_e are left coprime by (24), the r.h.s. of (35) has a pole at p_e by (4); therefore, the l.h.s. of (35) has a pole at p_e. Since $P_e, Q_e, \tilde{D}_f \in \mathbb{R}[s]^{m\times m}$ it follows that p_e must be a pole of H_e.

$\Rightarrow$ Use Cramer's rule in (25) to obtain

36 $$H_e = \tilde{D}_0(\text{Adj } \Omega_e)\, \tilde{D}_f/\det \Omega_e.$$

By assumption p_e is a pole of H_e. Since $\tilde{D}_0$, Adj Ω_e, $\tilde{D}_f \in \mathbb{R}[s]^{m\times m}$ (hence have no finite poles) we must have from (36) that det $\Omega_e(p_e) = 0$. Q.E.D.

The proof of (33) is similar and will not be given.

Remark: From (12), (25), and (32) we obtain

37 $$\det[I + G_f G_0] = \frac{\det \Omega_e}{\det \tilde{D}_0 \cdot \det \tilde{D}_f} = \frac{\prod_i (s-p_{ei})}{\left[\prod_k (s-p_{0k})\right]\left[\prod_j (s-p_{fj})\right]}$$

where (i) p_{ei} are the poles of H_e, counting multiplicities; (ii) p_{oi}, (p_{fi}), are the zeros of $\tilde{D}_0$, ($\tilde{D}_f$, resp.), counting multiplicities. Since cancellations may occur in (37), some pole of H_e, say, p_{ek}, might not be a zero of $\det[I + G_f G_0]$. Hence (37) implies that

38 $$\{\text{zeros of } \det[I + G_f G_0]\} \subset \{\text{poles of } H_e\}.$$

Similarly by (9), (13), (26), and (28), we obtain

39 $$\det[I + G_f G_0] = \frac{\det \Omega_y \cdot \det R_y}{\det D_0 \cdot \det \tilde{D}_f}.$$

Hence (33) and (39) imply that

40 $$\{\text{zeros of } \det[I + G_f G_0]\} \subset \{\text{poles of } H_y\} \cup \{\text{zeros of } \det R_y\}.$$

Note that by (27), any zero of $\det R_y$ is a zero of $\det N_0$, but not conversely. By (38) and (40) we have that neither the stability of H_e nor that of H_y can be determined by only checking the zeros of $\det[I + G_f G_0]$.

Examples

The examples below are purposefully simple; they illustrate statements (32), (33), (38) and (40), and the fact that the stability of the feedback system requires consideration of H_y and H_e.

Example 1: H_e unstable, H_y stable. Let

$$G_0 = \begin{bmatrix} \frac{s-1}{s(s+2)} & 0 \\ 0 & \frac{s-2}{s+1} \end{bmatrix} \quad ; \quad G_f = \begin{bmatrix} -\left(\frac{s+2}{s-1}\right) & 0 \\ 0 & -2\left(\frac{s+1}{s(s-2)}\right) \end{bmatrix}$$

$$G_0 = N_0 D_0^{-1} = \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} s(s+2) & 0 \\ 0 & s+1 \end{bmatrix}^{-1}$$

$$G_f = D_f^{-1} N_f = \begin{bmatrix} s-1 & 0 \\ 0 & s(s-2) \end{bmatrix}^{-1} \begin{bmatrix} -(s+2) & 0 \\ 0 & -2(s+1) \end{bmatrix}$$

$$\det [I + G_f G_0] = (s-1)(s-2)/s^2 \quad ; \quad \det [I + G_f(\infty) G_0(\infty)] = 1$$

$$H_e = [I + G_f G_0]^{-1} = \begin{bmatrix} \frac{s}{s-1} & 0 \\ 0 & \frac{s}{s-2} \end{bmatrix} \quad ; \quad \Omega_e = \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}$$

$$H_y = G_o [I + G_f G_0]^{-1} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix} \quad ; \quad \Omega_y = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$$

<u>Example</u> 2 : H_e stable, H_y unstable. Let

$$G_0 = \begin{bmatrix} \frac{s+2}{s(s-1)} & 0 \\ 0 & \frac{s+1}{s-2} \end{bmatrix} \quad ; \quad G_f = \begin{bmatrix} \frac{2(s-1)}{s+2} & 0 \\ 0 & \frac{s-2}{s(s+1)} \end{bmatrix}$$

$$G_0 = N_0 D_0^{-1} = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} s(s-1) & 0 \\ 0 & s-2 \end{bmatrix}^{-1}$$

$$G_f = D_f^{-1} N_f = \begin{bmatrix} s+2 & 0 \\ 0 & s(s+1) \end{bmatrix}^{-1} \begin{bmatrix} 2(s-1) & 0 \\ 0 & s-2 \end{bmatrix}$$

$$\det[I + G_f G_0] = (s+1)(s+2)/s^2 \quad ; \quad \det[I + G_f(\infty) G_0(\infty)] = 1$$

$$H_e = [I + G_f G_0]^{-1} = \begin{bmatrix} \frac{s}{s+2} & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix} ; \quad \Omega_e = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$$

$$H_y = G_0[I + G_f G_0]^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{s}{s-2} \end{bmatrix} ; \quad \Omega_y = \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}$$

IV. Conclusion

Since there are well known algorithmic methods, [1] - [5], for writing transfer function matrices in the form $N_0 D_0^{-1}$ and $D_f^{-1} N_f$, and for extracting greatest common left-or right-divisors, statements (32) and (33) give an algorithm for listing all the poles of H_e and H_y. The reasons why system stability cannot be guaranteed by considering only the zeros of $\det[I + G_f G_0]$ are exhibited by (38), (40), and the examples. It should be stressed that if we consider the minimal realizations of G_0 and G_f then provided all the poles of H_e and H_y are in the open left half plane, (open unit disc for the discrete-time case), the state trajectories corresponding to any bounded input are bounded. More precisely, if these states are called x_0 and x_f, resp., then the map from $u(\cdot)$ to $(x_0(\cdot), x_f(\cdot))$ is L_p-stable, for $1 \leq p \leq \infty$.

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