SHORT-TERM GENERAL EQUILIBRIUM IN SPACE

by

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Abstract

A general equilibrium model is presented where space is treated as a continuum rather than a finite number of discrete points. "Short-term" is intended to mean that some of the capital goods may be fixed in location and supply, that there are a finite number of market-places fixed in location, and that migration opportunities for households and firms may be limited. Firms may have non-convex production sets provided that the fixed factors show non-increasing marginal products. Similarly, households may have non-concave utility functions so long as the marginal utilities of the fixed factors are non-increasing. The transportation technology is linear. Under these conditions a Pareto-efficient competitive equilibrium exists. Furthermore, every Pareto-efficient allocation is sustainable as a competitive equilibrium.

1. Introduction and Summary

The models of Muth [1], Alonso [2], and Mills [3] are theoretically limited if we use them to analyze land-use and land-rents in an urban area. In the first place they are partial equilibrium models, and important rent and investment flows are simply treated as leakages. Also the models are highly stylized and it is not obvious how to extend them if there are many market places or if the urban area is interacting with other urban areas. Secondly, the stock and spatial distribution of locationally fixed capital, such as buildings, derived using these models, is required to be in equilibrium; and locational rents are derived simultaneously with this equilibrium stock. This equilibrium solution depends upon the state of the technology, and the size of the population as well as its income, taste etc. Now building structures deteriorate very slowly and permit very little ex post substitution, so that the annual change in this capital stock is quite small. Therefore, if we suppose that the technology or the population composition is constantly changing, it is unlikely that the current distribution of this capital stock is in "equilibrium" in the sense of these models.

This paper is an attempt to overcome the two limitations mentioned above. It is a short-term general equilibrium model of a system of cities. "Short-term" means that some of the capital stock may be fixed in quantity and location and no changes in this stock is permitted. (However, one may consider long-term equilibrium, in the sense of the models discussed above, simply by taking land as the only fixed factor.)

Migration of households and firms is permitted but in a very rudimentary manner: any household or firm is allowed to choose, without cost, any location within a pre-specified area. Commodities which are not locationally fixed can be transported anywhere. The transport technology is linear.

The production sets of firms are required to show non-increasing marginal products in the fixed factors only. Similarly households must have non-increasing marginal utilities in these factors. (No distinction is made between a capital good and the services delivered by this good and used either for consumption or as a factor of production.) The precise statements of these conditions is given in the next Section.

Section 3 consists of the main welfare theorem. It roughly states that every Pareto-efficient allocation can be sustained as a competitive equilibrium, that is, there is a redistribution of endowments among households and a system of prices at market-places and locational rents for the fixed factors under which the allocation is sustained assuming profit-maximizing firms and utility-maximizing households. The proof of this theorem is straightforward and standard in the literature.

In Section 4 we show that there exists a competitive equilibrium which is compatible with the prespecified distribution of endowments. The proof of this result is somewhat complicated. The reason is that the capital good at different locations must be treated differently so that in effect we have an infinite-dimensional commodity space. Nevertheless the outline of proof is standard. We follow the version presented in [4]. The technical difficulties are overcome by using some of the results presented in [5], [6], [7].

An interesting by-product of these results is the existence of locational rents for the fixed factors, whether or not the disbribution of these factors is in equilibrium. It also turns out that the rents can be interpreted as the value of the marginal product of these factors. Presumably in a dynamic model, which permits changes in building stock, these rents can serve as signals which guide building investment. However, this possibility is not explored in the paper.

The model permits zoning regulations of a simple kind whereby firms and households may be restricted as to where they may locate. But since we do not allow household utilities or production possibilities of firms to be affected directly by the location of other economic activities, therefore the use of zoning as a tool for internalizing locational externalities is not explored.

2. The Model

Notation. E^n is the n-dimensional real vector space. If $x = (x_1, \ldots, x_n) \in E^n$, $x \geqq 0$ means $x_i \geqq 0$ for all i, $x \geq 0$ means $x \geqq 0$ and $x \neq 0$, $x > 0$ means $x_i > 0$ for all i. $E^n_+ = \{x \in E^n | x \geqq 0\}$. $|x| = \sum_i |x_i|$. $1_n = (1, \ldots, 1)$.

2.1. The commodity space

a) Fixed commodities. Economic activity is distributed over a compact subset $S \subset E^2$. Points $s \in S$ are called locations. There are M market places in S, fixed in location. The distance from location s to the mth market place is given by a non-negative bounded measurable function $\delta_m(s)$. Land at different locations is developed differently. Thus there may be residential homes of different kinds, different kinds of office buildings etc., or the land may be undeveloped (which is considered as one type of developed land). In all there are L (> 0) different types of developed land each of which yields a different type of service which may be used for consumption or production. The services given by a piece of developed land can be used only at the location of this piece. Hence these services are called fixed commodities. The amounts and spatial distribution of these fixed commodities is given exogenously, and the function $\bar{x}^L$: $S \to E^L_+$ gives the amount of these commodities per unit area, i.e., in any region $R \subset S$ the total supply of the fixed commodities is $\int_R \bar{x}^L(s)ds$.

Assumption L. $\bar{x}^L : S \to E^L_+$ is a bounded measurable function. Further, there exists $\delta > 0$ such that $\bar{x}^L(s) \geqq \delta 1_L$ for all $s \in S$.

The positive lower bound is imposed to guarantee that the equilibrium rents for fixed commodities are finite.

b) Non-fixed commodities. Besides the L fixed commodities noted above, the economy deals in N other goods or services, and these can be transported from one location to another. They are called mobile or non-fixed commodities. Transportation of these mobile commodities consumes some of these mobile commodities. (The transportation technology is described later.) Whereas the fixed commodities are traded at every location, the mobile commodities can be traded only at the marketplaces.

2.2. Firms and households

The assumptions below are somewhat restrictive but permit great simplification in the analysis.

a) Firms. There are F firms. The production possibilities open to firm f are given by its input-output technology set $\mathcal{Y}_f \subset E^{2N+L}$. An element of $\mathcal{Y}_f$ is denoted by $y_f = (y^N_{fo}, y^N_{fi}, y^L_f)$ with the interpretation that $y^N_{fi} \in E^n$, $y^L_f \in E^L$ are the inputs of the mobile and fixed commodities respectively, and $y^N_{fo} \in E^N$ is the output of mobile commodities. Thus firms can produce mobile commodities only.

Assumption F1. $\mathcal{Y}_f$ is a compact subset of E_+^{2N+L} and $0 \in \mathcal{Y}_f$.

Each firm can operate simultaneously at different locations, and it can employ different input-output techniques at different locations.[1] Thus the decisions to be made by each firm include making a choice of locations, of an input-output technique at each location, and of the amount of trading to be conducted between each location and each market-place. As mentioned in the previous section, each firm has limited opportunities of migration. This is formalized as follows.

Assumption F2. f can locate only in a prespecified measurable subset $S_f \subset S$.

Now suppose that f adopts technique $(y^N_{fo}, y^N_{fi}, y^L_f)$ at $s \in S_f$, and suppose that f divides the mobile outputs and inputs as $y^N_{fo} = \sum_1^M y^{Nm}_{fo}$, $y^N_{fi} = \sum_1^M y^{Nm}_{fi}$, such that y^{Nm}_{fo} is bought at and y^N_{fi} is sold at the mth

[1]If we require that each firm must adopt the same input-output technique at every location where the firm operates, then there may be no equilibrium.

marketplace. Thus $(y_{fo}^{Nm} + y_{fi}^{Nm})$ must be transported over a distance $\delta_m(s)$. This transportation itself consumes some mobile commodities, the amounts of which are given by $\delta_m(s)T(y_{fo}^{Nm} + y_{fi}^{Nm})$.

Assumption T. The transportation technology is described by a fixed non-negative N x N matrix T.

The production plans of f can now be formally defined.

Definition 1. The set of production plans of f is the set of all non-negative, measurable functions $(y_{fo}^{N1}, \ldots, y_{fo}^{NM}, y_{fi}^{N1}, \ldots, y_{fi}^{NM}, y_f^L)$ defined on S_f and such that for all $s \in S_f$

$$(\sum_m y_{fo}^{Nm}(s), \sum_m y_{fi}^{Nm}(s), y_f^L(s)) \in \mathcal{Y}_f \tag{1}$$

Corresponding to each such production plan is the supply vector at the market-places $Y_f^N = (Y_f^{N1}, \ldots, Y_f^{NM}) \in E^{NM}$ given by

$$Y_f^{Nm} = \int_{S_f} [y_{fo}^{Nm}(s) - y_{fi}^{Nm}(s) - \delta_m(s)T(y_{fo}^{Nm}(s) + y_{fi}^{Nm}(s))]ds$$

$$= \int_{S_f} [T_-^m(s)y_{fo}^{Nm}(s) - T_+^m(s)y_{fi}^{Nm}(s)]ds,$$

where $T_+^m(s) = [I + \delta_m(s)T]$, $T_-^m(s) = [I - \delta_m(s)T]$, I = identity matrix.

b) Households. There are H types of household with $\bar{\Pi}_h > 0$

households of the <u>h</u>th type, each of which has the same initial endowment $\bar{x}_h^N \geqq 0$ of non-fixed commodities, and the same consumption set C_h and utility function U_h. The utility function may depend upon location.

<u>Assumption H1</u>. C_h is a closed subset of E_+^{N+L}. (i) There is $\delta > 0$ such that if $c_h = (c_h^N, c_h^L) \in C_h$ then $|c_h^L| > \delta$. (ii) U_h is a continuous function defined on $S \times C_h$. (iii) For all $s \in S$, $c_h \in C_h$, the set $\{c_h' \in C_h | U_h(s,c_h') \leqq U_h(s,c_h)\}$ is compact. (iv) For all $s \in S$, $c_h = (c_h^N, c_h^L) \in C_h$, $\delta > 0$, there exist $c_h' = (c_h'^N, c_h'^N) \in C_h$, $0 < \varepsilon < 1$, such that $|c_h'^N - c_h^N| < \delta$, $c_h'^L < \varepsilon\, c_h^L$ and $U_h(s,c_h') > U_h(s,c_h)$.

(i) is used to guarantee that the density of households is bounded, (iii) and (iv) together imply that an excess supply of all the non-fixed commodities can be distributed so as to guarantee a minimum increase in every household's utility.

Corresponding to Assumption F2 we have H2.

<u>Assumption H2</u>. A household of type h can locate only in a prespecified measurable subset $S_h \subset S$.

Now suppose that a household of type h locates at $s \in S_h$, and suppose it consumes amount $c_h^L(s) = x_h^L(s)$ of the fixed commodities. Let its consumption of non-fixed commodities be $c_h^N(s) = x_{ho}^{N0} + x_{hi}^{N1} + \dots + x_{hi}^{NM}$, where $x_{ho}^{N0} (\leqq \bar{x}_h^N)$ is the portion of the initial endowment retained for consumption and $x_{hi}^{Nm} \geqq 0$ is purchased at the <u>m</u>th market-place. The remainder $(\bar{x}_h^N - x_{ho}^{N0})$ of the initial endowment is delievered for sale to the various market-places, with x_{ho}^{Nm} going to the <u>m</u>th market place. The household also uses the transport technology T. If we let $\pi_h(s)$ be the number of households of type h per unit area located at s, then we can define consumption plans as follows.

Definition 2. The set of consumption plans of households of type h is the set of all non-negative, measurable functions $(x_{ho}^{N0}, x_{ho}^{N1}, \ldots, x_{ho}^{NM}, x_{hi}^{N1}, \ldots, x_{hi}^{NM}, x_h^L)$ defined on S_h and such that for all $s \in S_h$

$$c_h(s) = (x_{ho}^{N0}(s) + \sum_m x_{hi}^{Nm}(s),\ x_h^L(s)) \in C_h, \tag{2}$$

$$\bar{x}_h^N \geqq \sum_{m=0}^{M} x_{ho}^{Nm}(s), \tag{3}$$

and

$$\bar{\Pi}_h = \int_{S_h} \pi_h(s)ds, \tag{4}$$

and such that the integral in (5) below exists.

Corresponding to each consumption plan is the demand vector at the market-places $X_h^N = (X_h^{N1}, \ldots, X_h^{NM}) \in E^{NM}$ given by

$$X_h^{Nm} = \int_{S_h} [T_+^m(s)x_{hi}^{Nm}(s) - T_-^m(s)x_{ho}^{Nm}(s)]\pi_h(s)ds, \tag{5}$$

and the utility distribution

$$U_h(s) = U_h(s, c_h(s)).$$

c) Diminishing marginal products and utilities. The two assumptions below imply respectively that production sets have non-increasing marginal products in the fixed factors and households have non-increasing marginal utilities in the fixed commodities.

Notation. Let $Q \subset E^{K+L}$ and let its elements be denoted $q = (q^K, q^L)$. Then $[Q]$ denotes the convex hull of Q and $[Q]^L = \{\sum_i \lambda_i (q_i^K, q^L) \mid (q_i^K, q^L) \in Q, \lambda_i \geqq 0$, all i, with $\sum_i \lambda_i = 1\}$. For any set A, $c\ell\, A$ denotes the closure of A.

Assumption F3. (Recall that $\mathcal{Y}_f \subset E^{2N+L}$). $[\mathcal{Y}_f]^L = c\ell[\mathcal{Y}_f]$.

Assumption H3. For all $s \in S$, $\mu_h \in E^1$, $[C_h(s,\mu_h)]^L = c\ell[C_h(s,\mu_h)]$ where $C_h(s,\mu_h) = \{c_h \in C_h | U_h(s,c_h) \geqq \mu_h\}$.

The model is completed by describing the ownership of firms and the fixed commodities.

Assumption 01. The share of ownership in firm f of a household of type h is α_{hf}. The α_{hf} are non-negative, and $\sum_{h=1}^{H} \alpha_{hf}\, \bar{\Pi}_h = 1$, $f = 1, \ldots, F$.

The fixed factors are owned by K real-estate firms. The kth such firm owns all the fixed commodities located in area S_k. In turn these firms are owned by the households.

Assumption 02. S is partitioned into K measurable subsets $S_1, \ldots, S_K$, with the kth real-estate firm owning all fixed commodities located in S_k. The share of ownership in firm k of a household of type h is β_{hk}.

The β_{hk} are non-negative, and $\sum_{h=1}^{H} \beta_{hk}\, \bar{\Pi}_h = 1$, $k = 1, \ldots, K$.

3. The Welfare Theorem

The main definitions and ideas of proof are borrowed from Reference [4, Chapter 4].

3.1. Preliminary results

Definition 3. For $s \in S$, $\mu = (\mu_1, \ldots, \mu_H) \in E^H$, let $W(s,\mu) \subset E_+^{2NM+H}$ be the set of all vectors $w = w(s) = (y^N(s), x^N(s), \pi(s))$ where $y^N(s) = (y^{N1}, \ldots, y^{NM})$, $x^N(s) = (x^{N1}, \ldots, x^{NM})$, $\pi(s) = (\pi_1, \ldots, \pi_M)$, such that there exist

(i) for each f, a vector $(\xi_f, y_f^L) = (y_{fo}^{N1}, \ldots, y_{fo}^{NM}, y_{fi}^{N1}, \ldots, y_{fi}^{NM}, y_f^L)$ which satisfies (1) if $s \in S_f$, and $(\xi_f, y_f^L) = 0$ if $s \notin S_f$,

(ii) for each h, a vector $(\theta_h, x_h^L) = (x_{ho}^{NO}, \ldots, x_{ho}^{NM}, x_{hi}^{N1}, \ldots, x_{hi}^{NM}, x_h^L)$ which satisfies (2), (3) and $c_h \in C_h(s,\mu_h)$ if $s \in S_h$, and $(\theta_h, x_h^L) = 0$ if $s \notin S_h$,

(iii) for each h, a number $\pi_h \geqq 0$ if $s \in S_h$, and $\pi_h = 0$ if $s \notin S_h$,

and such that

$$\text{(iv)} \quad \bar{x}^L(s) \geqq \sum_f y_f^L + \sum_h x_h^L \pi_h, \tag{6}$$

$$\text{(v)} \quad y^{Nm}(s) = \sum_f (T_-^m(s) y_{fo}^{Nm} - T_+^m(s) y_{fi}^{Nm}), \tag{7}$$

$$\text{(vi)} \quad x^{Nm}(s) = \sum_h (T_+^m(s) x_{hi}^{Nm} - T_-^m(s) x_{ho}^{Nm}) \pi_h(s). \tag{8}$$

Let $\Xi_f(s)$ and $\Theta_h(s,\mu_h)$ respectively denote the set of all vectors

(ξ_f, y_f^L) and (θ_h, x_h^L) which satisfy conditions (i) and (ii) of Definition 3 above. Proposition 1 follows readily from Assumptions F3, H3.

<u>Proposition 1</u>. For $s \in S_f$, $c\ell[\Xi_f(s)] = [\Xi_f(s)]^L$

$= \{(y_{fo}^{N1}, \ldots, y_{fo}^{NM}, y_{fi}^{N1}, \ldots, y_{fi}^{NM}, y_f^L) \geqq 0 \,|\, (\sum_m y_{fo}^{Nm}, \sum_m y_{fi}^{Nm}, y_f^L) \in [\mathcal{Y}_f]^L\}$.

For $s \in S_h$, $c\ell[Ⓗ_h(s,\mu_h)] = [Ⓗ_h(s,\mu_h)]^L$

$= \{(x_{ho}^{N0}, \ldots, x_{ho}^{NM}, x_{hi}^{N1}, \ldots, x_{hi}^{NM}, x_h^L) \geqq 0 \,|\, \sum_m x_{ho}^{Nm} \leqq \bar{x}_h^N, (x_{ho}^{N0} + \sum_m x_{hi}^{Nm}, x_h^L) \in$

$[C_h(s,\mu_h)]^L\}$.

For $s \in S$, $\mu \in E^H$,

$[W(s,\mu)] = \{(y^N, x^N, \pi) \,|\, (\xi_f, y_f^L) \in [\Xi_f(s)]^L, (\theta_h, x_h^L) \in [Ⓗ_h(s,\mu_h)]^L,$

$\pi_h \geqq 0$ if $s \in S_h$, $\pi_h = 0$ if $S \notin S_h$, and (6)-(8) hold$\}$.

The next two lemmas are crucial in establishing rents for the fixed commodities.

<u>Lemma 1</u>. Let $\mu \in E^H$ be fixed, and for each $s \in S$ let $\phi_h = \phi_h(s)$, $\eta_f = \eta_f(s)$, $\lambda = \lambda(s)$ be vectors of appropriate dimension. Suppose that an optimal solution of the programming problem (9)-(11) exists.

$$\text{Max} \sum_f \langle \eta_f, \xi_f \rangle - \sum_h (\langle \phi_h, \theta_h \rangle - \lambda_h)\pi_h, \tag{9}$$

subject to

$$(\xi_f, y_f^L) \in \Xi_f(s), \ (\theta_h, x_h^L) \in Ⓗ_h(s,\mu_h), \ \pi_h \geqq 0 \text{ if } s \in S$$

$$\pi_h = 0 \text{ if } s \notin S_h, \tag{10}$$

and

$$\sum_f y_f^L + \sum_h x_h^L \pi_h \leq \bar{x}^L(s). \tag{11}$$

Then there exists a closed convex set $R^L = R^L(s,\phi,\eta,\lambda) \subset E_+^L$ such that for every $r^L \in R^L$ each optimal solution of (9)-(11) is also an optimal solution for the following problem:

$$\max \sum_f (\langle \eta_f,\xi_f \rangle - \langle r^L,y_f^L \rangle) - \sum_h (\langle \phi_h,\theta_h \rangle + \langle r^L,x_h^L \rangle - \lambda_h)\pi_h$$
$$+ \langle r^L,\bar{x}^L(s) \rangle$$

subject to (10);

and furthermore the two optimal values are the same.

Proof. See Appendix.

Lemma 2. If in Lemma 1 the functions $\phi(s)$, $\eta(s)$, $\lambda(s)$ are measurable in s, then $r^L(s) \in R^L(s,\phi(s),\eta(s),\lambda(s))$ can be chosen to be measurable.

Proof. See Appendix.

Notation: Let Γ be a (point-to-set) correspondence from S into subsets of E^n. Then we denote by $\mathcal{F}\Gamma$ the set of all integrable functions $\gamma : S \to E^n$ such that $\gamma(s) \in \Gamma(s)$ for almost all s; also denote $\int \Gamma = \left\{ \int_S \gamma(s)ds \,\middle|\, \gamma \in \mathcal{F}\Gamma \right\}$.

Definition 4. Let $\mathcal{W}_\mu = \mathcal{F}W(\cdot,\mu)$, $W_\mu = \int \mathcal{W}_\mu$. The elements of W_μ are denoted (Y^N,X^N,Π). An element $w(\cdot)$ of $\mathcal{W}_\mu$ is called a μ-possible allocation.

Proposition 2 is elementary, whereas Proposition 3 is immediate from the "measurable choice theorem" of [7].

Proposition 2. $W(\cdot,\mu)$ is a measurable correspondence.

Proposition 3. If $w(\cdot) \in \mathcal{W}_\mu$ then the functions $\xi_f(\cdot)$, $y_f^L(\cdot)$, $\theta_h(\cdot)$, $x_h^L(\cdot)$, $\pi_h(\cdot)$ corresponding to $w(\cdot)$ in Definition 3 can be chosen to be measurable.

Proposition 4. $W_\mu = \int \mathcal{W}_\mu$ is convex.

Proof. Since $\mathcal{Y}_f$, δ_m, and x_{ho}^{Nm} are bounded, and since π_h is bounded by Assumptions H1 (i), L and (6), it follows from Definition 3 that the correspondance $W(\cdot,\mu)$ is bounded from below. The result now follows from [5, Theorem 3].

Lemma 3. Suppose $W = (Y^N, X^N, \Pi) \in W_\mu$ with $\Pi > 0$. Then there exists $W' = (Y'^N, X'^N, \Pi') \in W_\mu$ with $Y'^N = Y^N$, and $\Pi' - \Pi > 0$.

Proof. Let $w(.) = (y^N(\cdot), x^N(.), \pi(\cdot)) \in \mathcal{W}_\mu$ be such that $W = \int w$. Let $(\xi_f(\cdot),$ $y_f^L(.), \theta_h(\cdot)$, $x_h^L(.)$, $\pi_h(.))$ correspond to $w(.)$. By hypothesis, $U_h(s, c_h(s)) \geqq \mu_h$, $s \in S_h$. By Assumption H1 (iv), there exist $(\theta_h'(\cdot), x_h'^L(\cdot))$ and $0 < \varepsilon_h(\cdot) < 1$ such that $U_h(s, c_h'(s)) > U_h(s, c_h(s))$, $s \in S_h$, and such that $x_h'^L(s) \leqq \varepsilon_h(s)\, x_h^L(s)$. Let $w'(.)$ correspond to $(\xi_f(\cdot), y_f^L(\cdot), \theta_h'(\cdot), x_h'^L(\cdot),$ $\pi_h'(.))$ where $\pi_h'(s) = \frac{1}{\varepsilon_h(s)} \pi_h(s)$. Then $\sum_f y_f^L(s) + \sum_h x_h'^L(s)\, \pi_h'(s) \leqq$ $\sum_f y_f^L(s) + \sum_h x_h^L(s)\, \pi_h(s) \leqq \bar{x}^L(s)$ so that $w'(.) \in \mathcal{W}_\mu$. Also $\Pi_h' - \Pi_h =$ $\int \left(\frac{1}{\varepsilon_h(s)} - 1\right) \pi_h > 0$ since $\int \pi_h = \Pi_h > 0$.

Definition 6. Let $w(\cdot) \in \mathcal{W}_\mu$, and let $(Y^N, X^N, \Pi) = \int w$. $w(\cdot)$ is said to be a <u>μ-feasible allocation</u> if $Y^N - X^N \geqq 0$ and $\Pi = \bar{\Pi}$. Let $\hat{\mathcal{W}}_\mu$ be the set of μ-feasible allocations and let $\hat{W}_\mu = \{\int w \mid w \in \hat{\mathcal{W}}_\mu\}$.

<u>Definition 7</u>. Let $\hat{\mathcal{U}} = \{\mu \in E^H \mid \hat{\mathcal{W}}_\mu \neq \phi\}$. $\hat{\mathcal{U}}$ is the set of <u>feasible utility allocations</u>. $\mu \in \hat{\mathcal{U}}$ is said to be <u>dominated</u> if there exists $\mu' \in \hat{\mathcal{U}}$ such that $\mu' > \mu$. $\mu \in \hat{\mathcal{U}}$ is said to be <u>Pareto-efficient</u> if it is not dominated.

<u>Lemma 4</u>. Let $w(\cdot) \in \hat{\mathcal{W}}_\mu$ and let $(Y^N, X^N, \bar{\Pi}) = \int w$. Suppose that $Y^N - X^N > 0$. Then μ is dominated.

<u>Proof</u>. Let $(\xi_f, y_f^L, \theta_h, x_h^L, \pi_h)$ correspond to the μ-feasible allocation w. Let $\delta > 0$. Then using Assumption H1 it is easy to see that there exist $\varepsilon > 0$ and a measurable θ_h' with $|\theta_h'(s) - \theta_h(s)| \leq \delta$ for all s such that the allocation $w'(\cdot)$ corresponding to $(\xi_f, y_f^L, \theta_h', x_h^L, \pi_h)$ is in $\mathcal{W}_{\mu+\varepsilon 1_H}$. Then

$$\int w' = (Y^N, X'^N, \bar{\Pi})$$

and $X'^N \leqq X^N + \delta D 1_{NM}$ where $D > 0$ in a constant independent of δ. Since $Y^N - X^N > 0$ it follows that if we choose $\delta > 0$ sufficiently small then $w' \in \hat{\mathcal{W}}_{\mu+\varepsilon 1_H}$ so that μ is dominated.

<u>Lemma 5</u>. Let μ be a Pareto-efficient utility allocation. Then

$$\{(Y^N - X^N, \Pi) \mid (Y^N, X^N, \Pi) \in W_\mu\} \cap \{(Z, \bar{\Pi}) \mid Z > 0\} = \phi, \tag{12}$$

and there exist $p \in E^{NM}$, $\lambda \in E^H$, with $(p, \lambda) \geq 0$, such that

$$\langle p, Y^N - X^N \rangle - \langle \lambda, \bar{\Pi} - \Pi \rangle \leqq 0 \text{ for } (Y^N, X^N, \Pi) \in W_\mu, \tag{13}$$

$$\langle p, Y^N - X^N \rangle = 0 \qquad \text{for } (Y^N, X^N, \bar{\Pi}) \in \hat{W}_\mu. \tag{14}$$

Furthermore if $(p,\lambda) \geq 0$ satisfies (13), (14) then $p \geq 0$, $\lambda \geqq 0$.

Proof. (12) is an immediate consequence of Lemma 4. Since W_μ is convex by Proposition 4, it follows that the first set in (12) is convex and so by the Separation Theorem for convex sets there exists a non-zero vector $(p,\lambda) \in E^{NM+H}$ such that

$$\langle p, Y^N - X^N \rangle + \langle \lambda, \Pi \rangle \leqq \langle p, Z \rangle + \langle \lambda, \bar{\Pi} \rangle \quad \text{for } (Y^N, X^N, \Pi) \in W_\mu,\ Z > 0. \tag{15}$$

Since $Z > 0$ can be chosen arbitrarily (15) implies that $p \geqq 0$ and then of course (14) follows from (13).

Now it is clear that if $(Y^N, X^N, \bar{\Pi}) \in \hat{W}_\mu$ then certainly $(Y^N, X^N, \Pi) \in W_\mu$ for $0 \leqq \Pi \leqq \bar{\Pi}$, so that (15) can hold only if $\lambda \geqq 0$.

Finally, suppose $p = 0$ in contradiction of the assertion. Then $\lambda \geq 0$ and (13) simplifies to

$$\langle \lambda, \bar{\Pi} - \Pi \rangle \geqq 0 \text{ for } (Y^N, X^N, \Pi) \in W_\mu. \tag{16}$$

By Lemma 3 there exists $(Y'^N, X'^N, \Pi') \in W_\mu$ such that $\Pi' > \bar{\Pi}$, so that since $\lambda \geq 0$ we get

$$\langle \lambda, \bar{\Pi} - \Pi' \rangle < 0$$

which contradicts (16). Hence $p \neq 0$ and the lemma is proved.

Theorem 1. Suppose $\mu = (\mu_1, \ldots, \mu_H)$ is a Pareto-efficient allocation. Then there exist (i) (market prices) $p = (p^1, \ldots, p^M) \in E^{NM}$, $p \geq 0$, (ii) (money incomes) $\lambda = (\lambda_1, \ldots, \lambda_H) \in E^H$, $\lambda \geqq 0$, and (iii) (rents) measurable function $r^L : S \to E^L$, $r^L(s) \geqq 0$, with the following property:

for every measurable $\hat{\omega} = (\hat{\xi}_f, \hat{y}_f^L, \hat{\theta}_h, \hat{x}_h^L, \hat{\pi}_h)$ such that the corresponding $\hat{w} = (\hat{y}^N, \hat{x}^N, \hat{\pi})$ is a μ-feasible allocation we must have

(I) (firms maximize profits) for each f, $(\hat{\xi}_f, \hat{y}_f^L)$ maximizes profit

$$P_f(\xi_f, y_f^L) = \int_{S_f} \left[\sum_m \langle p^m, T_-^m(s) y_{fo}^{Nm}(s) - T_+^m(s) y_{fi}^{Nm}(s) \rangle - \langle r^L(s), y_f^L(s) \rangle \right] ds$$

over the set of all possible production plans,

(II) (households minimize expenditures) for each h,

$$\lambda_h = \sum_m \langle p^m, T_+^m(s) \hat{x}_{hi}^{Nm}(s) - T_-^m(s) \hat{x}_{ho}^{Nm}(s) \rangle + \langle r^L(s), \hat{x}_h^L(s) \rangle \text{ for}$$

almost all $s \in S_h$ for which $\hat{\pi}_h(s) > 0$,

$$\lambda_h \leq \sum_m \langle p^m, T_+^m(s) x_{hi}^{Nm}(s) - T_-^m(s) x_{ho}^{Nm}(s) \rangle + \langle r^L(s), x_h^L(s) \rangle \text{ for}$$

almost all $s \in S_h$ and for all (θ_h, x_h^L) for which $\sum_m x_{ho}^{Nm} \leqq \bar{x}_h^N(s)$ and

$c_h = (x_{ho}^{NO} + \sum_m x_{hi}^{Nm}, x_h^L) \in C_h(s, \mu_h)$,

and finally

(III) (social budget is balanced)

$$\sum_h \lambda_h \bar{\Pi}_h = \sum_h \left[\sum_f \alpha_{hf} P(\hat{\xi}_f, \hat{y}_f^L) + \sum_k \beta_{hk} \int_{S_k} \langle r^L(s), \bar{x}^L(s) \rangle \, ds \right].$$

Proof. By Lemma 5 there exist $p = (p^1, \ldots, p^M) \geq 0$ and $\lambda = (\lambda_1, \ldots, \lambda_H) \geqq 0$ such that

$$\langle \lambda, \bar{\Pi} \rangle = \int_S [\langle p, \hat{y}^N(s) - \hat{x}^N(s) \rangle + \langle \lambda, \hat{\pi}(s) \rangle] ds, \tag{17}$$

$$\langle \lambda, \bar{\Pi} \rangle = \mathrm{Max}\left\{ \int_S [\langle p, y^N(s) - x^N(s) \rangle + \langle \lambda, \pi(s) \rangle] ds \,\middle|\, (y^N, x^N, \pi) \in \mathcal{U}_\mu \right\}. \tag{18}$$

By [6, Appendix: Theorem C], the right-hand side of (18) is equal to

$$\int_S [\sup\{\langle p, y^N - x^N \rangle + \langle \lambda, \pi \rangle \mid (y^N, x^N, \pi) \in W(s,\mu)\}] ds,$$

so that for almost all $s \in S$

$$\langle p, \hat{y}^N(s) - \hat{x}^N(s) \rangle + \langle \lambda, \hat{\pi}(s) \rangle = \mathrm{Sup}\{\langle p, y^N - x^N \rangle + \langle \lambda, \pi \rangle \mid (y^N, x^N, \pi) \in W(s,\mu)\}. \tag{19}$$

Therefore, for almost all $s \in S$ $\hat{\omega}(s)$ is an optimum solution of the programming problem (20)-(24).

$$\mathrm{Max}\left\{ \sum_f \sum_m \langle p^m, T_-^m(s) y_{fo}^{Nm} - T_+^m(s) y_{fi}^{Nm} \rangle - \sum_h \left(\sum_m \langle p^m, T_+^m(s) x_{hi}^{Nm} - T_-^m(s) x_{ho}^{Nm} \rangle - \lambda_h \right) \pi_h \right\} \tag{20}$$

subject to

$$(y_{fo}^{N1}, \ldots, y_{fo}^{NM}, y_{fi}^{N1}, \ldots, y_{fi}^{NM}, y_f^L) \in \Xi_f(s), \tag{21}$$

$$(x_{ho}^{N0}, \ldots, x_{ho}^{NM}, x_{hi}^{N1}, \ldots, x_{hi}^{NM}, x_h^L) \in \text{Ⓗ}_h(s,\mu_h), \tag{22}$$

$$\pi_h(s) \geqq 0 \quad \text{if} \quad s \in S_h, \qquad \pi_h = 0 \quad \text{if} \quad s \notin S_h, \tag{23}$$

$$\sum_f y_f^L + \sum_h x_h^L \pi_h \leqq \bar{x}^L(s). \tag{24}$$

By Lemmas 1 and 2, there exists a measurable (rent) function $r^L : S \to E_+^L$ such that for almost all $s \in S$ $\hat{\omega}(s)$ is an optimum solution for the following equivalent problem.

$$\text{Max}\left\{\sum_f \left(\sum_m \langle p^m, T_-^m(s)y_{fo}^{Nm} - T_+^m(s)y_{fi}^{Nm} \rangle - \langle r^L(s), y_f^L \rangle\right)\right.$$

$$\left. - \sum_h \left(\sum_m \langle p^m, T_+^m(s)x_{hi}^{Nm} - T_-^m(s)x_{ho}^{Nm} \rangle + \langle r^L(s), x_h^L \rangle - \lambda_h\right)\pi_h\right\}$$

$$+ \langle r^L(s), \bar{x}^L(s)\rangle \tag{25}$$

subject to (21)-(23).

Hence for almost all $s \in S$

$$\langle p, \hat{y}^N(s) - \hat{x}^N(s) \rangle + \langle \lambda, \hat{\pi}(s) \rangle$$

$$= \sum_f \text{Sup}\left\{\sum_m \langle p^m, T_-^m(s)y_{fo}^{Nm} - T_+^m(s)y_{fi}^{Nm} \rangle - \langle r^L(s), y_f^L \rangle \,\middle|\, (\xi_f, y_f^L) \in \Xi_f(s)\right\} \tag{26}$$

$$- \sum_h \text{Min}\left\{ \left(\sum_m \langle p^m, T_+^m(s) x_{hi}^{Nm} - T_-^m(s) x_{ho}^{Nm} \rangle + \langle r^L(s), x_h^L \rangle - \lambda_h \right) \pi_h \right.$$

$$\left. \mid (\theta_h, x_h^L) \in \textcircled{H}_h(s, \mu_h),\ \pi_h \geqq 0 \text{ if } s \in S_h,\ \pi_h = 0 \text{ if } S \notin S_h \right\} \tag{27}$$

$$+ \langle r^L(s), \bar{x}^L(s) \rangle .$$

The conclusions (I) and (II) follow from (27) and (26) respectively, whereas integrating both sides with respect to s and using (14) gives

$$\langle \lambda, \bar{\Pi} \rangle = \sum_f P(\hat{\xi}_f, \hat{y}_f^L) + 0 + \int_S \langle r^L(s), \bar{x}^L(s) \rangle \, ds$$

which is the same as (III). The theorem is proved.

Note that if $\lambda \neq 0$, and the ownership shares are chosen such that $\alpha_{hf} = \beta_{hk} = \dfrac{\lambda_h}{\sum_h \lambda_h \bar{\Pi}_h}$ for all f, k, then in fact all the individual household budgets would be balanced.

4. Existence of Equilibrium

4.1. Preliminary results

$[W(s,\mu)]$ need not be closed. However the following result still holds.

Proposition 5. If $(y^N, x^N, \pi) \in c\ell[W(s,\mu)]$ then there exists $(y^N, x'^N, \pi) \in [W(s,\mu)]$ with $x'^N \leqq x^N$. Also $(y^N, x^N, \pi) \in [W(s,\mu)]$ if $\pi > 0$.

Proof. Let $(y^N, x^N, \pi) = \lim_k (y^N(k), x^N(k), \pi(k))$ where $w(k) = (y^N(k), x^N(k),$

$\pi(k)) \in [W(s,\mu)]$ for every k. By Proposition 1 there exist $(\xi_f(k), y_f^L(k)) \in [\Xi_f(s)]^L$, $(\theta_h(k), x_h^L(k)) \in [\text{Ⓗ}_h(s,\mu_h)]^L$ such that

$$y^{Nm}(k) = \sum_f (T_-^m(s)y_{fo}^{Nm}(k) - T_+^m(s)y_{fi}^{Nm}(k)), \tag{28}$$

$$x^{Nm}(k) = \sum_h (T_+^m(s)x_{hi}^{Nm}(k) - T_-^m(s)x_{ho}^{Nm}(k))\pi_h(k), \tag{29}$$

and

$$\bar{x}^L(s) \geqq \sum_f y_f^L(k) + \sum_h x_h^L(k)\pi(k). \tag{30}$$

Since $[\Xi_f(s)]^L$ is compact we can assume, taking subsequences if necessary, that $(\xi_f(k), y_f^L(k))$ converges to $(\xi_f, y_f^L) \in [\Xi_f(s)]^L$. Therefore

$$y^{Nm} = \sum_f (T_-^m(s)y_{fo}^{Nm} - T_+^m(s)y_{fi}^{Nm}).$$

Next let h be fixed. If $\pi_h = 0$, we set $(\theta_h', x_f'^L) = 0$. On the other hand if $\pi_h > 0$, then $\pi_h(k) > \varepsilon$ for some $\varepsilon > 0$, and large k. Since $0 \leqq x_{ho}^{Nm}(k) \leqq \bar{x}_h^N$ for all k, it follows from (29) that the set of vectors $x_{hi}^{Nm}(k)$, $x_{ho}^{Nm}(k)$ is bounded. Similarly from (30) we conclude that the vectors $x_h^L(k)$ lie in a bounded set. Hence the collection $(\theta_h(k), x_h^L(k))$ is bounded. But by Proposition 1 $[\text{Ⓗ}_h(s,\mu_h)]^L$ is closed so that, taking subsequences if needed, we can assume that $(\theta_h(k), x_h^L(k))$ converges to $(\theta_h', x_h'^L) \in [\text{Ⓗ}_h(s,\mu_h)]^L$.

Evidently

$$x^{Nm} \geqq \sum_h (T_+^m(s)x_{hi}^{'Nm} - T_-^m(s)x_{ho}^{'Nm})\pi,$$

and

$$\bar{x}^L(s) \geqq \sum_f y_f^L + \sum_h x_h^{'L}\pi.$$

The assertion is proved.

<u>Notation</u>. We will be dealing with subsets of the space L_1 of all integrable functions $Y : S \to E^n$. By the <u>strong</u> topology on L_1 we mean the usual Banach space topology on L_1. By the <u>weak</u> topology on L_1 we mean the topology on L_1 induced by the continuous linear functionals L_∞. The notions of strong and weak convergence and closure are derived from these two topologies.

<u>Definition 8</u>. Let $\tilde{\mathcal{W}}_\mu = \mathcal{F}(c\ell[W(\cdot,\mu)])$. Let $\hat{\tilde{\mathcal{W}}}_\mu$ be the set of all $w \in \tilde{\mathcal{W}}_\mu$ such that if $\int w = (Y^N, X^N, \Pi)$ then $Y^N - X^N \geqq 0$ and $\Pi = \bar{\Pi}$.

<u>Proposition 6</u>. $\hat{\tilde{\mathcal{W}}}_\mu$ is a convex weakly compact subset of L_1.

<u>Proof</u>. It is clear that $\hat{\tilde{\mathcal{W}}}_\mu$ is convex. It is also strongly closed because if w_k is a sequence in $\hat{\tilde{\mathcal{W}}}_\mu$ converging strongly to w in L_1 then there is a subsequence (taken to be w_k itself) such that almost surely for $s \in S$

$$\lim_k w_k(s) = w(s),$$

and so $w(s) \in c\ell[W(s,\mu)]$. Hence $w \in \tilde{\mathcal{W}}_\mu$. Also

$$\lim_k \int w_k = \lim_k (Y_k^N, X_k^N, \Pi_k) = \int w = (Y^N, X^N, \Pi)$$

and since $Y_k^N - X_k^N \geqq 0$, $\Pi = \bar{\Pi}$ for all k, therefore $Y^N - X^N \geqq 0$ and $\Pi = \bar{\Pi}$. Thus $w \in \hat{\tilde{\mathcal{W}}}_\mu$.

Therefore to prove weak compactness it is enough to show that $\hat{\tilde{\mathcal{W}}}_\mu$ is bounded in L_1. First of all from Assumptions L and H1 (i), and from (6) we see that there exists $\bar{\pi}_h$ such that

$$0 \leqq \pi_h(s) \leqq \bar{\pi}_h \tag{31}$$

for all $(y^N, x^N, \pi) \in \tilde{\mathcal{W}}_\mu$. From the boundedness of $\mathcal{Y}_f$ and (7) we can conclude that there exists $\underline{y}^N$, $\bar{y}^N$ such that

$$\bar{y}^N \geqq y^N(s) \geqq \underline{y}^N \tag{32}$$

for all $(y^N, x^N, \pi) \in \tilde{\mathcal{W}}_\mu$. Finally from (31), (8) and the restriction $0 \leqq x_{ho}^{Nm} \leqq \bar{x}_h^N$ we conclude that there exists $\underline{x}^N$ such that

$$\underline{x}^N \leqq x^N(s) \tag{33}$$

for all $(y^N, x^N, \pi) \in \tilde{\mathcal{W}}_\mu$.

Now let $w = (y^N, x^N, \pi) \in \hat{\tilde{\mathcal{W}}}_\mu$, so that

$$\int y^N \geqq \int x^N \tag{34}$$

From (32) we get

$$\int |y^N| \leqq \int |\bar{y}^N| + \int |\underline{y}^N|. \tag{35}$$

From (33) we get

$$\int|x^N| \leqq \int|\underline{x}^N| + \left|\int x^N - \underline{x}^N\right|$$

$$\leqq \int|\underline{x}^N| + \left|\int y^N - \underline{x}^N\right| \quad \text{from (34)}$$

$$\leqq 2\int|\underline{x}^N| + \int|\bar{y}^N| + \int|\underline{y}^N| \quad \text{from (35).}$$

Hence

$$\int\left(|y^N| + |x^N| + |\pi|\right) \leqq 2\int\left(|\underline{x}^N| + |\bar{y}^N| + |\underline{y}^N|\right) + |\bar{\pi}|$$

so that $\hat{\tilde{\mathcal{W}}}_\mu$ is indeed a bounded subset of L_1 and the assertion is proved.

<u>Proposition 7</u>. Let μ^k be a sequence of feasible utility allocations converging to μ. Then

$$\limsup_k \int\hat{\tilde{\mathcal{W}}}_{\mu^k} \subset \int\hat{\tilde{\mathcal{W}}}_\mu .$$

<u>Proof</u>. Let $w_k \in \hat{\tilde{\mathcal{W}}}_{\mu^k}$ be such that $W_k = \int w_k$ converges to some vector W. Let $\underline{\mu} \in E^H$ be such that $\underline{\mu} \leqq \mu^k$ for all k. Then certainly $w_k \in \hat{\tilde{\mathcal{W}}}_{\underline{\mu}}$ for all k and so by Proposition 6 we can assume, taking a subsequence if needed, that w_k converges weakly to some $w \in \hat{\tilde{\mathcal{W}}}_{\underline{\mu}}$. It remains to show that $w \in \hat{\tilde{\mathcal{W}}}_\mu$. By [9, V.3.14] there is a sequence of convex combinations, call it $\{\gamma_n\}$, of the elements w_k which converges strongly to w. Hence there is a subsequence of $\{\gamma_n\}$, denoted again $\{\gamma_n\}$, such that for almost all s

$$\lim_n \gamma_n(s) = w(s).$$

Since γ_n is a convex combination of the elements w_k it follows that for almost all s

$$w(s) = \lim_n \gamma_n(s) \in \Gamma(s)$$

where $\Gamma(s)$ is the closure of the limit points of convex combinations of the sequence $\{w_k(s)\}$. Since $w_k(s) \in c\ell[W(s,\mu^k)]$ we conclude that for almost all s

$$w(s) \in \limsup_k c\ell[W(s,\mu^k)].$$

But from Definition 3 it follows that

$$\lim_k W(s,\mu^k) = W(s,\mu),$$

and therefore, for almost all s

$$w(s) \in c\ell[W(s,\mu)].$$

Since w_k converges weakly to w, $W = \int w$. The proof is completed.

<u>Lemma 6</u>. The set $\hat{\mathcal{U}} \subset E^H$ of feasible utility allocations is closed.

<u>Proof</u>. Let μ^k be a sequence in $\hat{\mathcal{U}}$ converging to $\mu \in E^H$. By Proposition 7 there exists $w = (y^N, x^N, \pi) \in \hat{\tilde{\mathcal{W}}}_\mu$. Hence

$$\int y^N - \int x^N \geqq 0, \quad \int \pi = \bar{\Pi}.$$

By Proposition 5, there exists a function $w' = (y^N, x'^N, \pi) \in \mathcal{F}([W(\cdot,\mu)])$ such that $\int x'^N \leqq \int x^N$. But then by [5, Theorem 3] there exists

$\hat{w} = (\hat{y}^N, \hat{x}^N, \hat{\pi}) \in \mathcal{W}_\mu$ such that $\int \hat{w} = \int w'$. It follows that

$$\int \hat{y}^N - \int \hat{x}^N = \int \hat{y}^N - \int \hat{x}'^N \geqq \int y^N - \int x^N \geqq 0,$$

and

$$\int \hat{\pi} = \int \pi = \bar{\Pi}.$$

Hence $\hat{w} \in \mathcal{W}_\mu$ i.e., μ is a feasible utility allocation.

Corollary 1. For every $\mu \in E^H$ the set $\{\mu' \in \hat{\mathcal{U}} | \mu' \geqq \mu\}$ is compact.

Proof. It is trivial that there exists $\bar{\mu} \in E^H$ such that $\mu' \leqq \bar{\mu}$ for all $\mu' \in \hat{\mathcal{U}}$. Hence $\{\mu' \in \hat{\mathcal{U}} | \mu' \geqq \mu\}$ is bounded, and this set is closed by Lemma 6.

4.2. Existence of equilibrium.

Let $\mathcal{U}^e$ be the set of all Pareto-efficient utility allocations. Let $\sum = \{(p,\lambda) \in E_+^{NM+H} \big| |p| + |\lambda| = 1\}$.

Definition 9. For $\mu \in \mathcal{U}^e$ let $\mathcal{P}(\mu)$ be the set of all (p,λ,r^L) such that $(p,\lambda) \in \sum$, $r^L : S \to E_+^L$ is a measurable function and such that (p,λ,r^L) has the property described in Theorem 1.

Proposition 8. There exists a number $R < \infty$ such that if $(p,\lambda,r^L) \in \mathcal{P}(\mu)$ then $\int |r^L| \leqq R$.

Proof. Since $|\lambda| \leqq 1$, from condition (III) of Theorem 1 we can deduce that

$$|\bar{\Pi}| \geqq \langle \lambda, \bar{\Pi} \rangle \geqq \int_S \langle r^L(s), \bar{x}^L(s) \rangle ds.$$

Using Assumption L we get $\langle r^L(s), \bar{x}^L(s) \rangle \geqq \delta |r^L(s)|$. Hence $|\bar{\Pi}| \geqq \delta \int |r^L|$ and the result follows. Thus

$$\mathcal{P}(\mu) \subset \sum \times \mathcal{R} = \sum \times \{r^L : S \to E_+^L \mid \int |r^L| \leqq R\}$$

Proposition 9. $\mathcal{P}(\mu)$ is a convex, weakly compact subset of $\sum \times L_1$.

Proof. We first prove convexity. Let $\hat{\omega} = (\hat{\xi}_f, \hat{y}_f^L, \hat{\theta}_h, \hat{x}_h^L, \hat{\pi}_h)$ be such that the corresponding $\hat{w} = (\hat{y}^N, \hat{x}^N, \hat{\pi}) \in \hat{\mathcal{W}}_\mu$. Let $(p_i, \lambda^i, r_i^L) \in \mathcal{P}(\mu)$, i = 1, 2. Define vectors $\eta_f^i(s)$, $\phi_h^i(s)$ so that

$$\sum_f \langle \eta_f^i(s), \xi_f \rangle - \sum_h (\langle \phi_h^i(s), \theta_h \rangle - \lambda_h^i)\pi_h =$$

$$\sum_f \sum_m \langle p_i^m, T_-^m(s) y_{fo}^{Nm} - T_+^m(s) y_{fi}^{Nm} \rangle - \sum_h \Big(\sum_m \langle p_i^m, T_+^m(s) x_{hi}^{Nm} - T_-^m(s) x_{ho}^{Nm} \rangle - \lambda_h^i \Big) \pi_h \tag{36}$$

Then we know that for almost all $s \in S$, and for each i = 1, 2, $\hat{\omega}(s)$ is an optimal solution for the programming problem (37)-(39).

$$\text{Max} \sum_f (\langle \eta_f^i(s), \xi_f \rangle - \langle r_i^L(s), y_f^L \rangle) - \sum_h (\langle \phi_h^i(s), \theta_h \rangle + \langle r_i^L(s), x_h^L \rangle - \lambda_h^i)\pi_h \tag{37}$$

subject to

$$(\xi_f, y_f^L) \in \Xi_f(s), \quad (\theta_h, x_h^L) \in \Theta_h(s, \mu_h), \tag{38}$$

$$\pi_h \geqq 0 \quad \text{if} \quad s \in S_h, \quad \pi_h = 0 \quad \text{if} \quad s \notin S_h \tag{39}$$

Let $0 \leqq \gamma \leqq 1$. Since the maximand in (37) is a linear function of η_f^i, r_i^L, ϕ_h^i, λ^i, and since the constraints (38), (39) do not depend on i, it follows that $\hat{\omega}(s)$ is also an optimal solution for the programming problem (37)-(39) when η_f^i, r_i^L, ϕ_h^i, λ^i are replaced respectively by $\gamma\eta_f^1 + (1-\gamma)\eta_f^2$, $\gamma r_1^L + (1-\gamma)r_2^L$, $\gamma\phi_h^1 + (1-\gamma)\phi_h^2$, $\gamma\lambda^1 + (1-\gamma)\lambda^2$. The convexity of $\mathcal{P}(\mu)$ now follows immediately.

By Proposition 8 $\mathcal{P}(\mu)$ is a bounded subset of $\Sigma \times L_1$. Hence to prove weak compactness it is enough to show that $\mathcal{P}(\mu)$ is strongly closed in $\sum \times L_1$. To this end let (p_i,λ^i,r_i^L) i = 1, 2, ... be a sequence in $\mathcal{P}(\mu)$, and let (p,λ,r^L) be such that $|p_i - p| \to 0$, $|\lambda^i - \lambda| \to 0$, and $\int |r_i^L - r^L| \to 0$ as $i \to \infty$. Taking subsequences if necessary we can suppose that $|r_i^L(s) - r^L(s)| \to 0$ for almost all s. Now for almost all s $\hat{\omega}(s)$ is an optimal solution of (37)-(39) for each i = 1, 2, ... It is then trivial that $\hat{\omega}(s)$ is an optimal solution when in the maximand in (37) we have $\eta_f = \lim \eta_f^i$, etc. It then follows that $(p,\lambda,r^L) \in \mathcal{P}(\mu)$. The assertion is proved.

<u>Proposition 10</u>. The point-to-set mapping $\mu \to \mathcal{P}(\mu)$ defined on $\mathcal{M}^e$ is upper semi-continuous i.e., if the sequence μ^i in $\mathcal{M}^e$ converges to μ in $\mathcal{M}^e$ and if the sequence $(p_i,\lambda^i,r_i^L) \in \mathcal{P}(\mu^i)$ converges weakly to (p,λ,r^L), then $(p,\lambda,r^L) \in \mathcal{P}(\mu)$.

<u>Proof</u>. By Lemma 5, for every i

$$\langle p_i, Y^N - X^N \rangle - \langle \lambda^i, \Pi - \bar{\Pi} \rangle \leqq 0 \quad \text{for} \quad (Y^N, X^N, \Pi) \in W_{\mu^i}. \tag{40}$$

Now let $(Y^N, X^N, \Pi) \in W_\mu$. From Assumption H1 it follows readily that for every $\delta > 0$ there exists $\varepsilon > 0$ such that $(Y^N, X'^N, \Pi) \in W_{\mu+\varepsilon 1_H}$ for some X'^N with $|X'^N - X^N| \leq \delta$. Since μ^i converges to μ it follows that $\mu^i \leqq \mu + \varepsilon 1_H$, and hence $W_{\mu+\varepsilon 1_H} \subset W_{\mu^i}$, for all i sufficiently large, say $i > I(\varepsilon)$. Therefore from Lemma 5 we conclude that

$$\langle p_i, Y^N - X'^N \rangle - \langle \lambda^i, \Pi - \bar{\Pi} \rangle \leqq 0 \quad \text{for all } i > I(\varepsilon).$$

Taking limits as $i \to \infty$ in the above yields

$$\langle p, Y^N - X'^N \rangle - \langle \lambda, \Pi - \bar{\Pi} \rangle \leqq 0.$$

Since $|X'^N - X^N|$ can be taken arbitrarily small, we have proved that

$$\langle p, Y^N - X^N \rangle - \langle \lambda, \Pi - \bar{\Pi} \rangle \leqq 0 \quad \text{for all } (Y^N, X^N, \Pi) \in W_\mu. \tag{41}$$

Next, by Lemma 1, for almost all s

$$r_i^L(s) \in R^L(s, \phi^i, \eta^i, \lambda^i), \quad i = 1, 2, \ldots \tag{42}$$

where the vectors ϕ^i, η^i are such that (36) holds. By hypothesis r_i^L converges weakly to r^L, so that if we follow the same argument as in the proof of Proposition 7 we will conclude that for almost all s

$$r^L(s) \in \limsup_i R^L(s, \phi^i, \eta^i, \lambda^i). \tag{43}$$

By Lemma A-1 of the Appendix, (43) implies that

$$r^L(s) \in R^L(s, \phi, \eta, \lambda), \tag{44}$$

where ϕ, η correspond to p. But now it is easy to see, using (41) and

(44), that $(p,\lambda,r^L) \in \mathcal{P}(\mu)$ and the assertion is proved.

Definition 9. For $(p,\lambda,r^L) \in \sum \times \mathcal{R}$ define the profit function P_f and money-savings function M_h by

$$P_f(p,\lambda,r^L) = \mathrm{Max}\left\{\int_{S_f}\left[\sum_m \langle p^m, T_-^m(s)y_{fo}^{Nm}(s) - T_+^m\, y_{fi}^{Nm}(s)\rangle - \langle r^L(s), y_f^L(s)\rangle\right] ds \,\middle|\, (y_{fo}^{N1}, \ldots, y_{fo}^{NM}, y_{fi}^{N1}, \ldots, y_{fi}^{NM}, y_f^L) \text{ is a production plan for } f\right\}, \tag{45}$$

and

$$M_h(p,\lambda,r^L) = \sum_f \alpha_{hf}\, P_f(p,\lambda,r^L) + \sum_k \beta_{hk} \int_{S_k} \langle r^L(s), \bar{x}^L(s)\rangle - \lambda_h.$$

Proposition 11. Let (p^i,λ^i,r_i^L), $i = 1, 2, \ldots$ be a sequence in $\sum \times \mathcal{R}$ converging weakly to (p,λ,r^L). Then

$$\lim_i P_f(p^i,\lambda^i,r_i^L) = P_f(p,\lambda,r^L), \tag{46}$$

$$\lim_i M_h(p^i,\lambda^i,r_i^L) = M_h(p,\lambda,r^L). \tag{47}$$

Proof. Since (47) follows immediately from (46) it is necessary only to prove (46). But (46) follows from the fact that the set of all production plans is a bounded subset of L_∞ since $\mathcal{Y}_f$ is bounded and the fact that the maximand in (45) is linear in (p,λ,r^L).

The next definition is borrowed from [4, p. 108].

Definition 10. A set of prices $(p,\lambda,r^L) \in \sum \times \mathcal{R}$, together with a set of production plans (ξ_f, y_f^L), $f = 1, .., F$, and a set of consumption plans (θ_h, x_h^L, π_h), $h = 1, ..., H$, constitute a <u>compensated equilibrium</u> if

a) $p \geq 0$,

and there exists a utility allocation $\mu = (\mu_1, .., \mu_H)$ such that

b) $w(\cdot) \in \hat{\mathcal{W}}_\mu$ where w is the allocation corresponding to (ξ_f, y_f^L), (θ_h, x_h^L, π_h),

c) (ξ_f, y_f^L) maximizes f's profit, i.e., satisfies condition (I) of Theorem 1,

d) (θ_h, x_h^L) minimizes expenditures of households of type h, i.e., satisfies condition (II) of Theorem 1,

e) the household budget is balanced i.e., $M_h(p, \lambda, r^L) = 0$.

To prove the existence of a compensated equilibrium we need to assume that there exists at least one utility allocation which can be attained by households using only their own endowment. In our model the situation is somewhat complicated because every household needs some fixed commodities but they do not own these commodities directly. This leads us to the following, somewhat awkward, assumption.

<u>Assumption U1</u>. The utility allocation 0 is attainable i.e., for every h and every $r^L \in \mathcal{R}$ there exist measurable functions $\tilde{\pi}_h : S_h \to E^1$, $\tilde{c}_h = (\tilde{c}_h^N, \tilde{c}_h^L) : S_h \to E_+^{N+L}$ such that

i) $\int \tilde{\pi}_h > 0$

ii) $\tilde{c}_h(s) \in C_h(s,0)$ for almost all $s \in S_h$ for which $\tilde{\pi}_h > 0$,

iii) $\tilde{c}_h^N(s) \leqq \bar{x}_h^N$ for all $s \in S_h$

iv) $$\int_{S_h} \langle r^L(s), \tilde{c}_h^L(s) \rangle \tilde{\pi}_h(s)ds \leqq \left(\int_{S_h} \tilde{\pi}_h\right)\left(\sum_k \beta_{hk} \int_{S_k} \langle r^L(s), \bar{x}^L(s) \rangle ds\right)$$

<u>Assumption U2</u>. The utility allocation 0 is dominated, i.e., $0 \notin \mathcal{U}^e$.

<u>Definition 11</u>. Let $\sum_H = \{(\sigma_1, \ldots, \sigma_H) | \sigma_i \geqq 0, |\sigma| = 1\}$. Let $\hat{\mathcal{U}}_0 = \{\mu | \mu$ is feasible, $\mu \geqq 0\}$. Let $\mathcal{U}_0^e = \hat{\mathcal{U}} \cap \mathcal{U}^e$.

For $\sigma \in \sum_H$ let $\gamma(\sigma) = \max \{\gamma | \gamma \geqq 0, \gamma\sigma \in \hat{\mathcal{U}}_0\}$. Since $\hat{\mathcal{U}}_0$ is compact by Corollary 1, $\gamma(\sigma)$ is well-defined, and since 0 is dominated by assumption, $\gamma(\sigma) > 0$ for all σ. Let $\mu(\sigma) = \gamma(\sigma)\sigma$. It is easy to see that $\mu(\sigma) \in \mathcal{U}_0^e$ because if it does not, there exists $\mu \in \hat{\mathcal{U}}_0$, $\mu > \mu(\sigma)$. But then there exists $\mu' \in \hat{\mathcal{U}}_0$ such that $\mu'=\gamma'\sigma$ and $\gamma' > \gamma(\sigma)$ thereby contradicting the definition of $\gamma(\sigma)$.

The proof of the next proposition is identical to that of [4, Lemma 5.3], hence it is omitted.

<u>Proposition 12</u>. The function $\sigma \to \mu(\sigma)$ is a continuous mapping of $\sum_H$ onto $\mathcal{U}_0^e$, furthermore, $\mu_h(\sigma) = 0$ if and only if $\sigma_h = 0$.

<u>Theorem 2</u>. There exists a compensated equilibrium.

<u>Proof</u>. Consider the convex set $\mathcal{C} = \sum_H \times \sum \times \mathcal{R}$ endowed with the weak topology so that it is (weakly) compact. Define the point-to-set mapping τ on $\mathcal{C}$ by

$$\tau(\sigma,p,\lambda,r^L) = \tau_1(p,\lambda,r^L) \times \mathcal{P}(\mu(\sigma))$$

where

$$\tau_1(p,\lambda,r^L) = \sum_H \cap \{\sigma | \sigma_h = 0 \quad \text{if} \quad M_h(p,\lambda,r^L) < 0\}. \tag{48}$$

Evidently τ maps points of $\mathcal{C}$ into convex, (weakly) compact subsets of $\mathcal{C}$. From Propositions 10, 11 it follows easily that τ is upper semi-continuous. By K. Fan's extension [10] of Kakutani's fixed point theorem we can conclude that there exists $(\sigma,p,\lambda,r^L) \in$ such that

$$(p,\lambda,r^L) \in \mathcal{P}(\mu) \quad \text{where } \mu = \mu(\sigma), \tag{49}$$

$$\sigma \in \tau_1(p,\lambda,r^L). \tag{50}$$

From (49) we know, using Theorem 1, that there exist production plans (ξ_f, y_f^L) and consumptions plans (θ_h, x_h^L, π_h) such that conditions a) - d) of Definition 10 are satisfied, and further,

$$\sum_h M_h(p,\lambda,r^L) = 0$$

Hence it is enough to show that $M_h(p,\lambda,r^L) \geqq 0$ for each h. Suppose in contradiction that $M_h(p,\lambda,r^L) < 0$ for some h. From (48) and (50) we conclude that $\sigma_h = 0$ and so $\mu_h = 0$ by Proposition 12. Now by Assumption U1,

$$\int_{S_h} \langle r^L(s), \tilde{c}_h^L(s) \rangle \tilde{\pi}_h(s)ds \leqq \left(\int_{S_h} \tilde{\pi}_h \right) \left(\sum_k \beta_{hk} \int_{S_k} \langle r^L(s), \bar{x}^L(s) \rangle ds \right) \tag{51}$$

And from condition d) of Definition 10 we know that

$$\int_{S_h} \langle r^L(s), \tilde{c}_h^L(s) \rangle \tilde{\pi}_h(s)ds \geqq \lambda_h\left(\int_{S_h} \tilde{\pi}_h\right). \tag{52}$$

From (51) and (52) we conclude that

$$\lambda_h \leqq \sum_k \beta_{hk} \int_{S_k} \langle r^L(s), \bar{x}^L(s) \rangle ds,$$

but then from Definition 9 we get $M_h(p,\lambda,r^L) \geqq 0$ which contradicts the assumption. The theorem is proved.

Various sets of additional assumptions can be made, each of which guarantees that a compensated equilibrium is in fact a competitive equilibrium. But since these schemes are well-known we refrain from presenting them. For example see [4, p. 116-119].

Appendix

1. Proof of Lemma 1. Suppose that $\hat{\omega} = (\hat{\xi}_f, \hat{y}_f^L, \hat{\theta}_h, \hat{x}_h^L, \hat{\pi}_h)$ is an optimum solution. First of all since the maximand in (9) is linear in all the variables except π_h, it follows from Proposition 1 that $\hat{\omega}$ is also an optimum solution of the programming problem (9)-(11) with $\Xi_f(s)$, $\Theta_h(s, \mu_h)$ replaced respectively by $[\Xi_f(s)]^L$ and $[\Theta_h(s,\mu_h)]^L$. In terms of the variables $\tilde{\theta}_h = \pi_h \theta_h$, $\tilde{x}_h^L = \pi_h x_h^L$ the hypothesis now asserts that $(\hat{\xi}_f, \hat{y}_f^L)$, $(\hat{\tilde{\theta}}_h, \hat{\tilde{x}}_h^L) = \hat{\pi}_h(\hat{\theta}_h, \hat{x}_h^L)$, $\hat{\pi}_h$ is an optimum solution for the following problem.

$$\text{Maximize } \sum_f \langle \eta_f, \xi_f \rangle - \sum_h (\langle \phi_h, \tilde{\theta}_h \rangle - \lambda_h \pi_h) \tag{A1}$$

subject to

$$(\xi_f, y_f^L) \in [\Xi_f(s)]^L, \ \frac{1}{\pi_h} (\tilde{\theta}_h, \tilde{x}_h^L) \in [\Theta_h (s, \mu_h)]^L, \ \pi_h \geqq 0 \tag{A2}$$

$$\sum_f y_f^L + \sum_h \tilde{x}_h^L \leqq \bar{x}^L(s). \tag{A3}$$

Note that the objective function is linear in all the variables. It can also be verified that the constraint set defined by (A2), (A3) is convex.

Let $V(x) = V(x, \phi, \eta, \lambda)$ be the maximum value of the problem above when $\bar{x}^L(s) = x$, and let $\Omega(x)$ be the corresponding set of feasible solutions defined by (A2), (A3). It is clear that $\Omega(x) \neq \phi$ for $x > 0$. It can be checked that for $(\alpha_i, x_i) > 0$, $i = 1, 2$, with $\alpha_1 + \alpha_2 = 1$, $\Omega(\alpha_1 x_1 + \alpha_2 x_2) \supset \alpha_1 \Omega(x_1) + \alpha_2 \Omega(x_2)$, and then it follows that V is a concave function in x. Since $\bar{x}^L(s) > 0$ by Assumption L, there exists a supergradient r^L of the function $V(\cdot,\phi,\eta,\lambda)$ at $\bar{x}^L(s)$ (see [8, Lemma 7, p. 99]), that is,

$$V(x,\phi,\eta,\lambda) \leqq V(\bar{x}^L(s),\phi,\eta,\lambda) - \langle r^L, x - \bar{x}^L(s) \rangle \text{ for all } x \tag{A4}$$

and since $V(x) \geqq V(\bar{x}^L(s))$ for $x \geqq \bar{x}^L(s)$ it follows that $r^L \geqq 0$. Let $R^L = R^L(s,\phi,\eta,\lambda)$ be the set of all r^L which satisfy (A4). Since R^L is described by a collection of linear inequalities R^L is a closed convex set.

Finally, it can be readily checked using (A4) that every optimum solution of (A1)-(A3) is also optimum for the following problem, and the optimum values are equal.

$$\text{Maximize } \sum_f \langle \eta_f, \xi_f \rangle - \sum_h (\langle \phi_h, \tilde{\theta}_h \rangle - \lambda_h) \pi_h - \langle r^L, \sum_f y_f^L$$

$$+ \sum_h \tilde{x}_h^L \pi_h - \bar{x}^L(s) \rangle$$

subject to (A2).

The lemma is proved.

2. Proof of Lemma 2. Let $V(x,s) = V(x,\phi(s),\eta(s),\lambda(s))$ denote the maximum value used in the proof above, thereby making explicit the dependence on the parameter s. It is clear that V is a measurable function. The correspondance $R^L(s)$ defined by

$$R^L(s) = \{r^L | V(x,s) \leqq V(\bar{x}^L(s),s) - \langle r^L, x - x^L \rangle \text{ for all } x\}$$

can be shown to be measurable. But then any measurable selection $r^L(s) \in R^L(s)$ suffices to prove the lemma.

3. Lemma A-1. Let η^i, ϕ^i, λ^i be a sequence of vectors converging to $\bar{\eta}$, $\bar{\phi}$, $\bar{\lambda}$, and for each i let $r_i^L \in R^L(s,\phi^i,\eta^i,\lambda^i)$ be such that r_i^L converges to $\bar{r}^L \in E_+^L$. Then $\bar{r}^L \in R^L(s,\bar{\phi},\bar{\eta},\bar{\lambda})$.

Proof. The maximand in (A1) is linear in the variables η, ϕ, λ so that $V(x,\phi,\eta,\lambda)$ is convex, and hence continuous, in (ϕ,η,λ). Now let $x \in E^L$ be fixed. By hypothesis

$$V(x,\phi^i,\eta^i,\lambda^i) \leqq V(\bar{x}^L(s),\phi^i,\eta^i,\lambda^i) - \langle r_i^L, x - \bar{x}^L(s) \rangle ,$$

so that taking limits as $i \to \infty$ we conclude that

$$V(x,\bar{\phi},\bar{\eta},\bar{\lambda}) \leqq V(\bar{x}^L(s),\bar{\phi},\bar{\eta},\bar{\lambda}) - \langle \bar{r}^L, x - \bar{x}^L(s) \rangle$$

and the assertion is proved.

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