A HISTORICAL SURVEY OF COMPUTATIONAL METHODS IN OPTIMAL CONTROL

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1. Introduction

The purpose of this paper is to trace the development of optimal control algorithms during the last twenty five years. This twenty five year period has produced an immense literature on optimal control: well in excess of 1000 papers and 100 books. Clearly, one can discuss only a small fraction of these within the scope of a survey paper. Consequently, rather than attempting the impossible in trying to pay tribute to all the major contributors to the literature, we have chosen the easier task of tracing, through the work of a small number of contributors, the development of a number of algorithms, from their humble beginnings as idealized methods for solving finite dimensional problems, to their current, highly sophisticated status.

In this survey, we restrict ourselves to algorithms for solving optimal control problems whose dynamics are described by ordinary differential equations. Although discrete time optimal control problems are not mentioned explicitly, it is hoped that the reader will recognize that these are solvable by the finite dimensional versions of the algorithms described in this paper. The reader who needs more details on this, will find them in [1.3] and in [1.12]. Problems with partial differential equations or difference-differential equations dynamics are so different

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in structure from ordinary optimal control problems that it was found inconvenient to include algorithms for their solution in this survey.

The paper is organized into sections, each dealing with a specific class of algorithms. Each of the algorithms treated is described in detail. However, in dealing with each algorithm, we have presented only the most general results available, and we have given these a broad brush treatment. Despite of this, we have made an effort to establish the origins of the various optimal control algorithms included and to exhibit their relationships with nonlinear programming algorithms. The references are organized into groups, paralleling the sections, each dealing with a specific class of algorithms, with the exception of the first group which consists of books that can be used for general reference work. Unfortunately, it was not feasible to compile a comprehensive bibliography. The papers cited are simply those with which the author is most familiar and which appear to be mentioned frequently.

We have used standard notation, e.g. $L_\infty^m [t_0, t_f]$ denotes the space of equivalence classes of measureable functions from $[t_0, t_f]$ into $\mathbb{R}^m$, with the essential sup norm, $L_2^m [t_0, t_f]$ denotes the space of equivalence classes of square integrable functions from $[t_0, t_f]$ into $\mathbb{R}^m$, $C^m [t_0, t_f]$ denotes the space of continuous functions from $[t_0, t_f]$ into $\mathbb{R}^m$, etc. We use the symbols $\|\cdot\|$ $\langle \cdot , \cdot \rangle$ for the norm and scalar product in Euclidean space, and $\|\cdot\|_B$, $\langle \cdot , \cdot \rangle_B$ in a Banach space B, and for $u, v \in L_2^m [t_0, t_f]$, $\|u\|_2^2 \triangleq \int_{t_0}^{t_f} \|u(t)\|^2 \, dt$, $\langle u, v \rangle_2 =$

$$\int_{t_0}^{t_2} \langle u(t), v(t) \rangle \, dt.$$

2. Gradient Methods

Gradient methods are used for solving unconstrained optimization problems of the form

$$2.1. \qquad \min \{f(u) \mid u \in B\}.$$

In (2.1), B is a Banach space, on which there is defined a scalar product $\langle \cdot, \cdot \rangle_B$ (B need not be a Hilbert space). The function $f: B \to \mathbb{R}$ is Frechet differentiable and it has the property that there exists a function $\nabla f: B \to B$ such that

$$2.2. \qquad \lim_{\|h\|_B \to 0} \frac{|f(u + h) - f(u) - \langle \nabla f(u), h \rangle_B|}{\|h\|_B} = 0$$

In addition, it is usual to assume, (i) that $f(\cdot)$ has a global minimum, and (ii) that $f(u + \lambda h)$ has a (finite) minimizer $\hat{\lambda}$ for any $u, h \in B$. In a gradient method, the successive iterates are defined by a relation of of the following form. Assuming that u_0 is given,

$$2.3 \qquad u_{i+1} = u_i - \lambda_i \nabla f(u_i), \; i = 0,1,2,\ldots.$$

where λ_i is a suitably chosen step size.

The first description of a gradient method of the form (2.3) is attributed to Cauchy [2.6], in 1847. In 1944, Curry [2.9] showed that when $B = \mathbb{R}^n$ and λ_i is chosen so as to minimize $f(u_i + \lambda \nabla f(u_i))$ over $\lambda_i \geq 0$, the method is convergent, in the sense that any accumulation point of the sequence $\{u_i\}_{i=0}^{\infty}$ must satisfy $\nabla f(u_i) = 0$. This particular version of the method (2.3) is usually referred to as steepest descent.

Basically, the steepest descent step length rule

$$2.4 \qquad \lambda_i = \arg \min_{\lambda \geq 0} f(u_i - \lambda \nabla f(u_i))$$

must be regarded as non-implementable whenever f is not convex, and highly expensive when f is convex. In the same paper, Curry observed that (2.4) can be relaxed to the extent that the method remains convergent (in the above indicated sense) when λ_i is chosen to be the first local minimum of $f(u_i - \lambda \nabla f(u_i))$, i.e.

$$2.5 \qquad \lambda_i = \min \{\lambda \geq 0 \mid \langle \nabla f(u_i), \nabla f(u_i - \lambda \nabla f(u_i)) \rangle_B = 0\}$$

The first practical and efficient step length rule is due to Goldstein [2.12] who, in 1962, showed that λ_i may be chosen according to the "two-line" rule:

$$2.6 \qquad -\lambda_i(1-\alpha) \langle \nabla f(u_i), \nabla f(u_i) \rangle_B \leq f(u_i - \lambda_i \nabla f(u_i)) - f(u_i)$$

$$\leq -\lambda_i \alpha \langle \nabla f(u_i), \nabla f(u_i) \rangle_B$$

where $\alpha \in (0, \frac{1}{2})$. The importance of Goldstein's invention lies in the fact that a λ_i satisfying (2.5) can be calculated in a finite number of iterations of a step length subprocedure, such as (2.1.33) in Polak [1.12], whereas λ_i as given by (2.4) or (2.5) cannot. A still simpler step length rule was justified by Armijo [2.1] in 1966. Armijo defined λ_i by a "one-line rule" which sets $\lambda_i = \lambda_0 \beta^k$, where $\lambda_0 > 0$, $\beta \in (0,1)$ are arbitrary, but fixed, and k is the smallest integer such that

2.7 $$f(u_i - \lambda_0\beta^k \nabla f(u_i)) - f(u_i) \leq -\lambda_0\beta^k \langle \nabla f(u_i), \nabla f(u_i) \rangle_B$$

The step length rules (2.5) - (2.7) are all "closed loop" in the sense that they involve a test which verifies that there has been an adequate decrease in cost. In 1963 Polyak [2.16] published a set of "open loop" step length rules which are simpler than any of the "closed loop" rules. Unfortunately, to define such a rule one needs information about f which is generally not available. Specifically, Polyak assumed that B is a Hilbert space, and considered the more general iterative process, which, for u_0 given, defines

2.8 $$u_{i+1} = u_i - \lambda_i h_i(u_i), \quad i = 0,1,2,\ldots .$$

He assumed that there were constants $K_1 > 0$, $K_2 > 0$ and $M > 0$ such that

2.9 $$\|h_i(u)\|_B \leq K_1 \|\nabla f(u)\|_B \quad \forall\, u \in B, \quad i = 0,1,2,\ldots$$

2.10 $$\langle \nabla f(u_i), h_i(u) \rangle_B \geq K_2 \|\nabla f(u)\|_B^2, \quad \forall\, u \in B, \; i = 0,1,2\ldots$$

2.11 $$\|\nabla f(u) - \nabla f(v)\|_B \leq M\|u - v\|_B, \quad \forall\, u, v \in B$$

and showed that if λ_i is chosen so that

2.12 $$0 < \varepsilon_1 \leq \lambda_i \leq \frac{2K_2}{MK_1} - \varepsilon_2, \quad \varepsilon_1 > 0, \; \varepsilon_2 > 0$$

then $f(u_{i+1}) < f(u_i)$, $i = 0,1,2,\ldots$, and $\nabla f(u_i) \to 0$ as $i \to \infty$. In (2.12) ε_1, ε_2 are arbitrary. In spirit, Polyak's result has much more in common

with the classical approach to the solution of systems of nonlinear equations than with optimization theory. Since K_1, K_2 and M are usually not available, even when they exist, the main merit of (2.12) is to justify the common practice of using a fixed, heuristically determined, λ_i in solving problems where function evaluations are prohibitively expensive.

Now consider the fixed time optimal control problem

$$2.13 \qquad \min \int_{t_0}^{t_f} h^0(x(t),u(t),t)dt + L(x(t_j))$$

subject to

$$2.14 \qquad \frac{d}{dt} x(t) = h(x(t),u(t),t),\ x(t_0) = x_0,$$

where h: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^n$, h^0: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^1$, and L: $\mathbb{R}^n \to \mathbb{R}^1$ are continuous in all arguments and their partial derivatives, $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial u}$, $\frac{\partial h^0}{\partial x}$, $\frac{\partial h^0}{\partial u}$ and $\frac{\partial L}{\partial x}$ are also continuous in all arguments. The problem (2.13), (2.14) can be rewritten as

$$2.15 \qquad \min \{f(u) \mid u \in L_\infty^m [t_0,t_f]\},$$

with

$$2.16 \qquad f(u) \triangleq \phi(u,x(u))$$

where ϕ: $L_\infty^m [t_0,t_f] \times C \to \mathbb{R}^1$ is defined by

$$2.17 \qquad \phi(u,x) \triangleq \int_{t_0}^{t_f} h^0(x(t,u),u(t),t)dt$$

and $x(t,u)$ denotes the solution of (2.14) for the given u. Thus, (2.15) is actually a problem of the form

$$2.18 \qquad \min \{\phi(x,u) \,|\, r(x,u) = 0,\ x \in C[t_0,t_f],\ u \in L_\infty[t_0,t_f]\}$$

and r is such that for every u, x(u) is the unique solution of $r(x,u) = 0$. (We denote $x(\cdot,u)$ by $x(u)$ here).

A highly nontrivial difficulty in applying a gradient method to (2.15) was encountered in the need for obtaining formulas for the gradient of $f(\cdot)$ in (2.15). In particular, the fact that the classical calculus of variations was always formulated in terms of first and second variations (weak and strong), rather than in terms of derivatives, proved to be a substantial conceptual obstacle in this work. It appears that this difficulty was resolved in classified projects some time before any results were published. In the open literature we find in, 1959, a description by Breakwell [2.4] of the following approach based on the use of multiplier functions. Proceeding formally in the framework of (2.18), and reasoning as if the problem were defined on $\mathbb{R}^n$, he set
defined on $\mathbb{R}^n$, he set

$$2.19 \qquad H(x,u,p) \triangleq \phi(x,u) + \langle p, r(x,u) \rangle$$

and observed that if $\nabla f(u)$ for (2.15) exists, then it is given by

$$2.20 \qquad \nabla f(u) = \nabla_u H(x(u),u,p)$$

with p determined by

2.21 $$\nabla_x H(x(u),u,p) = 0.$$

In terms of the problem (2.13), (2.14), he concluded that (2.19) - (2.21) must assume the form

2.22 $$H(x,u,p,t) = h^0(x,u,t) - \langle p, h(x,u,t) \rangle$$

2.23 $$\nabla f(u)(t) = \left[\frac{\partial}{\partial u} h^0(x(t,u),u(t),t) \right]^T - \left[\frac{\partial}{\partial u} h(x(t,u),u(t),t) \right]^T p(t)$$

$$t \in [t_0,t_p]$$

and

2.24 $$\frac{d}{dt} p(t) = \left[\frac{\partial}{\partial x} H(x(t,u),u(t),t) \right]^T$$

$$= - \left[\frac{\partial}{\partial x} h(x(t,u),u(t),t) \right]^T p(t) + \left[\frac{\partial}{\partial x} h^0(x(t,u),u(t),t) \right]^T$$

2.24a $$p(t_f) = - \left[\frac{\partial}{\partial x} L(x(t_f,u)) \right]^T.$$

Whereas the validity of (2.23), (2.24), for $u \in L_\infty^m [t_0,t_f]$ and under the assumptions stated earlier, is generally recognized, it is difficult to find a text or a paper where these are explicitly established. Theorem (10.7.3) in Dieudonné [2.10] seems to come closest to filling this need.

The first use of the formulas (2.23) - (2.24) in a gradient method, is generally attributed to Bryson and Denham [2.5] (1962). Also in 1962, Shatrovskii [2.17] and Kelley [2.14] published their versions of a gradient method for solving the problem (2.13) and (2.14). None of this work

contained a proof of convergence. It was deemed sufficient to show that the cost will decrease at each iteration. Inspite of its lack of mathematical sophistication, this early work was of very great practical significance, because it resulted in the solution of a number of important aeronautical problems and demonstrated beyond a doubt that an optimal control approach can yield valuable results. In conjunction with the Pontryagin Maximum Principle [1.13], this work eventually provided the impetus to a veritable flood of papers on optimal control.

It is interesting to note that due to a lack of communications, the adoption of demonstrably convergent step size rules such as (2.6) and (2.7) has been slow in the optimal control algorithm user community. It is only in 1971 (see Sec. 2.5 in [1.12]) that we find these step size rules explicitly associated with optimal control algorithms.

3. Newton-Raphson Method.

Newton's method (which is also refered to as the Newton-Raphson method) was originally proposed as a method for solving equations of the form $g(u) = 0$, with $g: \mathbb{R}^1 \to \mathbb{R}^1$ continuously differentiable. It constructed iterates according to

$$3.1 \qquad u_{i+1} = u_i - \left[\frac{\partial}{\partial u} g(u_i) \right]^{-1} g(u_i), \quad i = 0,1,2,\ldots$$

which obviously requires that $\left[\frac{\partial}{\partial u} g(u_i) \right]^{-1}$ exist. The method has an obvious generalization to the case where $g: \mathbb{R}^n \to \mathbb{R}^n$, i.e. (3.1) remains the iteration formula. Eq. (3.1) can be justified heuristically as resulting from the solution of $g(u_i + \delta u) = 0$ to first order terms. In the 1930's, extensions of the Newton-Raphson method to the case where $g: B \to B$, with B a Banach space, were considered by such well known numerical analysts as Ostrowski and Kantorovich. The most famous early result was published by Kantorovich [3.14] in 1949. He showed, under the assumption that the operator $\left[\frac{\partial}{\partial u} g(u) \right]^{-1}$ is well defined for all $u \in B$, and that it satisfies a Lipstitz condition, that there exist constants $K_1 > 0$, $K_2 > 0$ such that if $\hat{u}$ satisfies $g(\hat{u}) = 0$, and $\|u_0 - \hat{u}\|_B$ is sufficiently small, then the sequence defined by (3.1) converges to $\hat{u}$ and satisfies

$$3.2 \qquad \|u_i - \hat{u}\|_B \leq K_1 \sum_{k=i}^{\infty} \delta^{2^i}, \quad i = 0, 1, 2, \ldots\ldots$$

with

$$3.3 \qquad \delta = K_2 \|u_1 - u_0\|_B$$

Thus, if u_0 is such that $\delta < 1$, then $u_i \to \hat{u}$ superlinearly, i.e. $\|u_i - \hat{u}\|_B \to 0$

faster than $K\theta^i$, for any $K > 0$ and any $\theta \in (0,1)$. In fact, it can be deduced from (3.3), that for some $K_3 > 0$,

$$3.4 \qquad \|u_{i+1} - \hat{u}\| \leq K_3 \|u_i - \hat{u}\|^2, \quad i = 0,1,2,..$$

Relation (3.4) gave rise to the definition of "quadratic convergence". Kantorovich's result has been sharpened and refined over the years in various ways. However, we shall not dwell on this work.

As far as the problem (2.1) is concerned, if we set $g(u) = \nabla f(u)$, then, since an optimal $\hat{u}$ for (2.1) satisfies $\nabla f(\hat{u}) = 0$, we can hope to compute such a $\hat{u}$ as the limit point of a sequence $\{u_i\}_{i=0}^{\infty}$ generated by the Newton-Raphson formula

$$3.5 \qquad u_{i+1} = u_i - [\frac{\partial^2}{\partial u^2} f(u_i)]^{-1} \nabla f(u_i), \quad i = 0,1,2,...$$

(since $\frac{\partial}{\partial u} \nabla f(u) = \frac{\partial^2}{\partial u^2} f(u)$).

The iteration formula (3.5) can also be justified by reasoning that given a u_i, we calculate u_{i+1} by minimizing the quadrate approximation

$$3.6 \qquad f_i(u) \equiv f(u_i) + \langle \nabla f(u_i), u - u_i \rangle_B + \frac{1}{2} \langle u - u_i, \frac{\partial^2}{\partial u^2} f(u_i)(u - u_i) \rangle_B$$

to $f(u)$. As far as the problem (2.1) is concerned, the first major breakthrough in the application of Newton's method is due to Goldstein [3.10] (1965). He assumed (for $B = \mathbb{R}^n$) that there exist constants

$0 < m \leq M$ such that

$$3.7 \qquad m\|v\|_B^2 \leq \langle v, \frac{\partial^2}{\partial u^2} f(u)v \rangle_B \leq M \|v\|_B^2 \quad \forall\ u,\ v \in B.$$

Then for, u_0 arbitrary, he defined the iteration process

$$3.8 \qquad u_{i+1} = u_i - \lambda_i \left[\frac{\partial^2}{\partial u^2} f(u_i) \right]^{-1} \nabla f(u_i),\ i = 0,1,2\ldots .$$

$$3.9 \qquad -\lambda_i(1-\alpha) \langle \nabla f(u_i), \left[\frac{\partial^2}{\partial u^2} f(u_i) \right]^{-1} \nabla f(u_i) \rangle_B$$

$$\leq f(u_{i+1}) - f(u_i) \leq -\lambda_i\alpha \langle \nabla f(u_i), \left[\frac{\partial^2}{\partial u^2} f(u_i) \right]^{-1} \nabla f(u_i) \rangle_B$$

$$i = 0,1,2\ldots.$$

Under the above assumption, Goldstein showed that a sequence $\{u_i\}$ constructed according to (3.9) always satisfies (i) $f(u_{i+1}) < f(u_i)$ for $i = 0,1,2\ldots$, (ii) $u_i \to \hat{u}$ as $i \to \infty$, where $\hat{u}$ is the minimizer of $f(\cdot)$, and (iii) $\lambda_i = 1$ for all $i \geq N$ for some N. It is not difficult to see that these conclusions remain valid whenever f has a finite number of local minima, provided (3.2) holds for u in neighborhood of every local minimizer, except that $\hat{u}$ will be a local minimizer.

Thus, he showed that, by using a "feedback" type step size rule, one can considerably extend the range of convergence of the Newton-Raphson method, without sacrificing rate of convergence. Previous attempts to achieve the same, usually, crossbred the Newton-Raphson method with a constant step size gradient method. These attempts were not anywhere near as successful as Goldstein's invention. Incidentally, it is not

difficult to see that the somewhat cumbersome "two-line" rule (3.9) can be replaced by the Armijo "one-line rule" (2.7), suitably modified, where it is necessary to set $\alpha \in (0, \frac{1}{3})$. (see [1.12] Section 6.2).

The latest significant extension of the Newton-Raphson method was developed in 1970-71, independently be Pshenichnyi [3.23] and by Robinson [3.24], [3.25]. Robinson considered a system of equations and inequalities

$$3.10 \qquad g_1(u) = 0,\ g_2(u) \leqq 0$$

where $g = (g_1, g_2)$ is a differentiable map from a Banach space B into $X \subset B$. Given u_0, he defines the successive iterates as solutions of

$$3.11 \qquad \min \{\|u\|_B \mid g_1(u_i) + \frac{\partial}{\partial u} g_1(u_i)(u-u_i) = 0,$$

$$g_2(u_i) + \frac{\partial}{\partial u} g_2(u_i)(u-u_i) \leqq 0\}.$$

We see that when $g_2 \equiv 0$ and g_1: $B \to B$, (3.10) reduces to the ordinary Newton method, provided $[\frac{\partial}{\partial u} g_1(u)]^{-1}$ exists. Robinson showed that a sequence $\{u_i\}$ constructed according to (3.11) satisfies the Kantowich bound (3.2). The importance of this result to optimal control lies in the fact that it opens up the possibility of obtaining a superlinearly convergent algorithm for solving the boundary value problems, with inequality side conditions, which are a consequence of the Pontryagin maximum principle [1.13] for optimal control problems with inequality constraints on the initial and terminal state.

In applying the Newton-Raphson method to the optimal control problem

(2.13) - (2.14), there was the question as to how one calculates the operators needed in (3.5). Although formulas for calculating the second variation in terms of multiplier functions had been known and used in one manner or another in optimal control algorithms since the early sixties by Breakwell et al. [3.7], McReinolds and Bryson [3.21], Kelley et al. [3.17], and others, it was not until 1966 that Mitter [3.22] stated correctly that to apply the Newton-Raphson method to the problem (2.13) - (2.14) by minimizing the approximation (3.6), we need to solve the problem below for $\delta u_i(\cdot)$, $\delta x_i(\cdot)$

$$3.12 \quad \min \{ \int_{t_0}^{t_f} ([\frac{\partial}{\partial u} H_i(t)]\delta u(t) + \frac{1}{2} \langle \delta u(t), [\frac{\partial^2}{\partial u}$$

$$+ \frac{1}{2} \langle \delta x(t), [\frac{\partial^2}{\partial x^2} H_i(t)]\delta x(t) \rangle + \langle \delta x(t), [\frac{\partial^2}{\partial u \partial x} H_i(t)]\ \delta u(t) \rangle) \, dt \mid$$

$$\frac{d}{dt} \delta x(t) = [\frac{\partial}{\partial x} h(x(t,u_i)u_i(t),t)]\delta x(t) + [\frac{\partial}{\partial u} h(x(t,u_i),u_i(t),t)]$$

$$\delta u(t),$$

$$t \in [t_0,t_f],\ \delta x(t_0) = 0.\}$$

where

$$3.13 \quad H_i(t) = h^0(x(t,u_i),u_i(t),t) - \langle p_i(t), h(x(t,u_i),u_i(t),t) \rangle,$$

and $p_i(\cdot)$ is determined by (2.24) for $u = u_i$. Once δu_i is computed, one sets $u_{i+1} = u_i + \delta u_i$, calculates $x(t,u_{i+1})$ and, replacing i by i+1, one returns to 3.12.

Since the Newton-Raphson method is basically a method used for solving equations, it can obviously be used for solving the Pontryagin optimality condi-

tions for the optimal control problem defined by the cost (2.13), the dynamics (2.14), with added boundary conditions $g_0(x(t_0)) = 0$, $g_f(x(t_f)) = 0$, where $g = (g_0, g_f)$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$ and has a nonsingular Jacobian everywhere. We recall that according to the Pontryagin Maximum principle, if $\hat{u}(\cdot)$ is optimal for this problem, then there exists a multiplier function $\hat{p}(\cdot)$ satisfying the adjoint differential equation (2.24) for $u = \hat{u}$, but with boundary condition $\hat{p}(t_f) = -\left[\frac{\partial}{\partial x} L(x(t_f,\hat{u}))\right]^T + \left[\frac{\partial}{\partial x} g_f(x(t_f,\hat{u})\right]^T \psi_f$, $\hat{p}(t_0) = \left[\frac{\partial}{\partial x} g_0(x(\hat{u})(t_0)\right]^T \psi_0$, for some vectors ψ_f, ψ_0 and, in addition,

3.14 $$\frac{\partial}{\partial u} H(x(t,\hat{u}),\hat{u}(t),\hat{p}(t),t)) \equiv 0 \qquad \text{a.e.}$$

where H is defined as in (3.13), with $\hat{u}$, $\hat{p}$ replacing u_i, p_i.
In the case when (3.14) can be used to eliminate u(t) in terms of $x(t,\hat{u})$ and $\hat{p}(t)$, we are left with a boundary value problem of the form

3.15 $$\frac{d}{dt} x(t) = h(x(t),u(x,p)(t),t)$$

3.16 $$\frac{d}{dt} p(t) = -\frac{\partial}{\partial x}\left[h(x(t),u(x,p)(t),t)\right]^T p(t) - \left[\frac{\partial}{\partial x} h^0(x(t),u(x,p)(t),t)\right]^T,$$

3.17 $$g_0(x(t_0)) = 0,\ g_f(x(t_f)) = 0,\ G_0(x(t_0))p(t_0) = \psi_0,\ G_f(x(t_f))p(t_f) = \psi_f.$$

There are currently two different ways of applying the Newton-Raphson method to the solution of systems such as (3.15) - (3.17), and it appears that these two variations of the Newton-Raphson method were known for some time prior to their publication in the open literature. Consequently, it is difficult to establish precedents. However, the following is clear, in 1956, Goodman and Lance [3.12] showed that the Newton-Raphson method

can be used to solve boundary value problems of the form

3.18 $$\frac{d}{dt} z(t) = k(z(t),t) \quad t \in [t_0,t_f]$$

3.18a $$g_0(z(t_0)) = 0,\ g_f(z(t_f)) = 0$$

with $g = (g_0,g_f)$ mapping $\mathbb{R}^n$ into $\mathbb{R}^n$, by solving the equation

3.19 $$q(\xi) = 0$$

where $\xi \in \mathbb{R}^n$ and $q(\xi) = (g_0(\xi),g_f(z(t_f,\xi)))$.

In this approach the successive iterates ξ_i are initial values and are constructed by solving the linear system

3.20a $$[\frac{\partial}{\partial z} g_0(\xi_i)](\xi_{i+1} - \xi_i) = - g_0(\xi_i)$$

3.20b $$[\frac{\partial}{\partial z} g_f(z(t_f,\xi_i))]\ \Phi_i(t_f)(\xi_{i+1} - \xi_i) = - g_f(z(t_f,\xi_i)),$$

where $\Phi_i(t_f)$ is the $n \times n$ matrix solution at $t = t_f$ of

3.21 $$\frac{d}{dt} \Phi(t) = [\frac{\partial}{\partial z} k(z(t,\xi_i),t)]\ \Phi(t),\ \Phi(t_0) = I.$$

An alternative approach, named by Bellman quasilinearization, was described by McGill and Kenneth [3.20] and Isaev and Sonin [3.13] in 1963 and by Bellman and Kalaba [3.6] in 1965. It iterates on the entire trajectory $z(t)$, rather than on the initial boundary value, and is defined, by analogy with (3.1), as follows. Given $z_0(\cdot)$, calculate $z_{i+1}(\cdot)$

from $z_i(\cdot)$ by solving the linear boundary value problem

3.22 $$\frac{d}{dt}(z_{i+1}(t)-z_i(t)) - [\frac{\partial}{\partial z} k(z_i(t),t)]((z_{i+1}(t)-z_i(t)) = -(\frac{d}{dt} z_i(t) - k(z_i(t),t))$$

3.23 $$[\frac{\partial}{\partial z} g_0(z_i(t_0)](z_{i+1}(t_0)-z_i(t_0)) = - g_0(z_i(t_0))$$

3.24 $$\frac{\partial}{\partial z} g_f(z_i(t_f))(z_{i+1}(t_f)-z_i(t_f)) = - g_f(z_i(t_f))$$

Obviously, (3.22) - (3.24) simplify out to a linear boundary value problem in $z_{i+1}(t)$.

Incidentally, even when (3.14) cannot be used to eliminate $\hat{u}(\cdot)$ from the optimality conditions, the quasilinearization approach can still be used for solving the system consisting of (3.14) - (3.17) where in (3.15), (3.16) u must be considered as independent of x and p, see [1.12], [3.18]. This is not true of the Goodman-Lance version. It is also interesting to note that, for some unknown reason the Goldstein or Armijo step size rules are generally not used by the optimal control users community. The solution of (3.20a), (3.20b) or of (3.22)-(3.26) may be quite difficult because of ill conditioning. A clever method for improving the conditioning of these systems was proposed by Abramov [3.2] in 1961, see also [1.12]. This method seems to be virtually unknown in the U.S.. Instead, following Kalman's [3.13a], 1960 work on the linear-quadratic regulator, the approach in this country has been to set $x_i(t) = K_i(t)p_i(t) + q_i(t)$, in (3.22) (recall, $z_i(t) = (x_i(t),p_i(t))$ for the optimal control problem), where $K_i(t)$ and

$q_i(t)$ are computed as solutions to matrix differential equations of the Riccati type. This approach also gets around the ill conditioning of (3.22) - (3.24. It involves less numerical work than the Abramov method, and it may do a better job of reducing ill conditioning, but it suffers from the disadvantage that the Riccati equation defining $K_i(f)$ may not have a solution, which leads to numerical difficulties. For a detailed exposition of these methods see [1.12].

4. Conjugate Gradient Methods.

A conjugate gradient method was first proposed by Hestenes and Stiefel [4.6] in 1952 as an algorithm for solving large systems of linear equations in a finite number of variables. The method was then extended in 1962 by Antosiewicz and Reinboldt [4.1] to operator equations in linear spaces. In 1964, Fletcher and Reeves [4.5] observed, formally, that the Hestenes-Stiefel method could be adopted for the solution of problems of the form min $\{f(u) \mid u \in \mathbb{R}^n\}$ with $f: \mathbb{R}^n \to \mathbb{R}^1$ continuously differentiable. They showed that their method minimizes a quadrate function on $\mathbb{R}^n$ in no more than n iterations, but gave no proofs for more general functions. This method became quite popular because empirically it was found to have a much faster rate of convergence than gradient methods.

It later turned out that Zoutendijk (1968) had an unpublished proof of convergence for this method. This proof is sketched out in [1.12].

In 1969, a slightly different adaptation of the Hestenes-Stiefel method was published, independently, by Polak and Ribiere [4.11], Polyak [4.12], and Sorenson [4.14]. The first two of these papers contained proofs of convergence for convex functions. Finally, in 1970, Cohen [4.2], using some results of Daniel [4.3], showed that when all these methods are re-started every n-iterations, they converge n-step quadratically (i.e., they achieve in n steps a reduction of error comparable to the reduction obtained by Newton's method in one iteration). An extension of the Fletcher-Reeves method to optimal control problems of the form (2.13) - (2.14) was described by Lasdon, Mitter and Warren [4.9] in 1967, and their experiments indicated that the Fletcher-Reeves conjugate gradient method

converges considerably faster than steepest descent on optimal control problems, as well. A more recent study by O'Kane [4.10], 1972, corroborated the Lasdon-Mitter-Warren observations and, in addition, it showed that a straightforward adaptation of the Klessig-Polak [4.8] (1971) implementation of the Polak-Ribiere [4.11] method performed considerably better than the usual adaptation of the Fletcher-Reeves method, on all the optimal control problems which he tested.

To solve problem (2.15) by conjugate gradient methods one sets $B = L_\infty^m [t_0,t_f]$, $\langle \cdot,\cdot \rangle_B$ to be the L_2 scalar product, and one computes gradients according to (2.23) - (2.24). Then, given u_0, one sets $h_0 = - \nabla f(u_0)$ and, for $i = 0,1,2, \ldots$

$$4.1 \qquad u_{i+1} = u_i + \lambda_i h_i$$

$$4.2 \qquad \lambda_i = \arg \min_{\lambda \geq 0} f(u_i + \lambda h_i)$$

$$4.3 \qquad h_{i+1} = - \nabla f(u_{i+1}) + \gamma_i h_i$$

$$4.4a \qquad \gamma_i = \frac{\langle \nabla f(u_{i+1}), \nabla f(u_{i+1}) \rangle_B}{\langle \nabla f(u_i), \nabla f(u_i) \rangle_B}$$

$$4.4b \qquad \gamma_i = \frac{\langle \nabla f(u_{i+1}) - \nabla f(u_i), \nabla f(u_{i+1}) \rangle_B}{\langle \nabla f(u_i), \nabla f(u_i) \rangle_B}$$

where (4.4a) is the Fletcher-Reeves formula for γ_i and (4.4b) is the Polak-Ribiere-Polyak-Sorenson formula for γ_i. The major difficulty in

using these methods, as stated, lies in the need to calculate a good approximation to λ_i, The Fletcher-Reeves method, in particular, is known to be quite sensitive to an accumulation of errors due to poor approximations to λ_i. In the finite dimensional case, one gets rid of the accumulated errors by restarting the process (4.1) - (4.4) every n iterations. As we have already stated, according to Cohen [4.2], the restarted algorithms are superlinearly convergent. However, for the control problem, n is effectively infinite and hence the restart strategy is not obvious. In 1971, Klessig and Polak [4.8] showed that the algorithm using (4.4b) remains superlinearly convergent when a finite operations type step size rule is substituted for (4.2). The robustness with respect to step size approximations of the algorithm using (4.4b) most probably accounts for why O'Kane [4.10] found that the adaptation of this algorithm performs considerably better than the adaptation of the Fletcher Reeves method. Furthermore, he found that the secant procedure used in the step size rule in [4.8] is considerably less costly in optimal control problem applications than cubic interpollation type procedures for approximating (4.2).

A heuristic extension of Cohen's results would lead one to suspect that the rate of convergence of conjugate gradient methods for optimal control is probably only linear, but with an exponential bound $K\theta^i$, $i = 0, 1, 2...$, where $\theta \in [0,1]$ is much smaller than for any gradient method. The available empirical data is too scant to conclude much more than that.

5. Methods of Feasible Directions, Centers and Gradient Projection.

Feasible directions algorithms are best known in the context of finite dimensional nonlinear programming problems. The earliest method in this class was proposed in 1956 by Frank and Wolfe [5.11]. Its range of application is limited to problems with affine constraints. Zoutendijk [5.31] developed several general methods of feasible directions, some of which are clearly related to the Frank-Wolfe method. In retrospect, his work constitutes one of the principal contributions to the theory of nonlinear programming. Other methods of feasible directions were described by Zukhovitskii-Polyak-Primak [5.32] (1963), Huard [5.6], [5.12] (1966) (who called his methods "methods of centers"), Topkis and Veinott [5.28] (1967). Polak [5.23] (1969) [1.12] (1971), and Pironneau-Polak [5.20] (1971). Although in their final form Huard's methods of centers are quite similar to some of Zoutendijk's methods of feasible directions, they were derived in an entirely different manner, hence their name. The rate of convergence of methods of centers was studied by Tremolieres [5.29] (1968) Lootsma [5.18] (1970), Mifflin [5.19] (1971), and that of methods of centers and of feasible directions by Pironneau-Polak [5.21], [5.22] (1972). Of all the above mentioned nonlinear programming algorithms, only the Frank-Wolfe and the Pironneau-Polak methods are readily adaptable to the solution of certain optimal control problems.

The original Frank-Wolfe method solves problems of the form

$$5.1 \qquad \min \{f^0(u) \mid u \in \Omega\}$$

with f^0: $\mathbb{R}^n \to \mathbb{R}^1$, strictly convex and continuously differentiable, and

$\Omega \subset \mathbb{R}^n$ convex and compact. Given $u_0 \in \Omega$, the successive iterates are given by

$$5.2 \qquad u_{i+1} = u_i + \lambda_i(\bar{u}_i - u_i),\ i = 0, 1, 2\ldots,$$

where $\bar{u}_i$ is any solution of the problem

$$5.3 \qquad \min \{\langle \nabla f^0(u_i), u-u_i \rangle \mid u \in \Omega\}$$

and

$$5.4 \qquad \lambda_i = \arg \min \{f^0(u_i + \lambda(\bar{u}_i - u_i)) \mid \lambda \in [0,1]\}.$$

In the period 1964-1965, Demyanov and Rubinov [1.6] published a number of applications of their extension of the Frank-Wolfe method, which they renamed as the conditional gradient method. In particular, they showed that the Frank-Wolfe algorithm remains well defined and convergent when Ω is a convex, weakly compact subset of a Banach space B, and f^0: $B \to \mathbb{R}^1$ is continuously Frechet differentiable.

Obviously, the direction finding subproblem (5.3) can only be solved for special cases of Ω. For example, following Demyanov and Rubinov [1.6], consider the optimal control problem,

$$5.5 \qquad \min \{f^0(u) \mid u \in \Omega \subset L_\infty^m [t_0, t_f]\}$$

where

$$5.6a \qquad f^0(u) \equiv \int_{t_0}^{t_f} h^0(x(t,u), u(t), t)\,dt$$

with $x(t,u)$ defined as the solution of

5.6b $$\frac{d}{dt}\, x(t) = A(t)x(t) + B(t)u(t),\ x(t_0) = x_0,\ t \in [t_0,t_f],$$

and

5.7 $$\Omega = \{u \mid |u^j(t)| \leq 1,\ j = 1,2,..,m,\ t \in [t_0,t_f]\}.$$

In (5.5), (5.6), h^0: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^n$ is continuously differentiable, and the matrices $A(\cdot)$, $B(\cdot)$ are both continuous. Referring to (2.23), (2.24), we see that for this case, (5.3) becomes

5.8 $$\min \{ \int_{t_0}^{t_f} ([\frac{\partial}{\partial u} h^0(x(t,u_i),u_i(t),t)](u(t) - u_i(t))$$

$$- \langle B(t)^T p_i(t), u(t)-u_i(t) \rangle) dt \mid |u^j(t)| \leq 1,$$

$$t \in [t_0,t_f]\}$$

where $p_i(t)$ is determined by

5.9 $$\frac{d}{dt}\, p_i(t) = - A(t)^T p_i(t) + [\frac{\partial}{\partial x} h^0(x(t,u_i),u_i(t),t)]^T,\ t \in [t_0,t_f],$$

$$p_i(t_f) = 0.$$

The problem (5.8) is obviously easy to solve. Among the various sets Ω for which the problem (5.3) remains solvable when f^0 is defined by (5.6a), (5.6b), Demyanov and Rubinov considered the cases $\Omega = \{u \mid \|u(t)\| \leq 1,$

$t \in [t_0,t_f]\}$ and $\Omega = \{u \mid \int_{t_0}^{t_f} \|u(t)\|^2\, dt \leq 1\}$. For an extensive treatment see their book [1.6].

In their paper, Frank and Wolfe showed that any sequence $\{u_i\}$ constructed according to (5.2) converged to a $\hat{u}$, which is optimal for (5.1). Furthermore, they showed that $\|u_i - \hat{u}\| \leq K/i$, for some $K > 0$. This bound on the speed of convergence is considerably slower than that on the previously discussed gradient methods which converge linearly (i.e. $\|u_i - \hat{u}\| \leq K\theta^i$, with $\theta \in (0,1)$). It was shown in 1970 by Canon and Cullum [5.7] that the bound K/i cannot be improved upon. To get around this difficulty, at least for optimal control problems, also in 1970, Barnes [5.2], [5.3] added a quadratic term in u(t) (based on a second order expansion to the integrand in (5.8). This made no difference to the ease with which the "direction finding" subproblem ((5.8) modified) could be solved, but, under certain reasonable assumptions, produced linear convergence for the resulting method.

In 1969 Jacobson and Mayne [1.7a,5.13-15] proposed a series of closely related algorithms which can, in no way, be viewed as an extension of an algorithm for finite dimensional problems. In spite of this, they do bear an interesting similarity to the Frank-Wolfe method. To illustrate, in the simplest of these algorithms, when applied to the problem (5.5) - (5.7), they calculate $\bar{u}_i(t)$ by minimizing the Hamiltonian at $u_i(\cdot)$, i.e.

$$5.10 \qquad \bar{u}_i(t) = \arg\min \{h^0(x(t,u_i),v,t) - \langle p_i(t), B(t)v \rangle \mid$$

$$|v^j| \leq 1,\ j = 1,2\ldots,m\},\ t \in [t_0,t_f],$$

where $p_i(t)$ is given by (5.9). Then they set

5.11 $$u_{i+1}(t) = \begin{cases} u_i(t) & t_0 \leq t < t_i \\ \bar{u}_i(t) & t_i \leq t \leq t_f \end{cases}$$

where $t_i \in (t_0, t_f)$ is computed so as to minimize the cost. There is an obvious relationship between the direction finding subproblems (5.10) and (5.8), but there is no relationship between the step length rules (5.4) and (5.11). While they have not been able to produce a proof of convergence, Jacobson and Mayne have shown that their algorithms decrease the cost at each iteration and that they have a number of computationally attractive features. Also, these algorithms are qualitatively consistent with existing general convergence theorems, such as those given in Section 1.3 of [1.12]. Experimental results show that these algorithms are very well behaved.

In 1971 Pironneau and Polak [5.20], [5.21] described a family of dual methods of feasible directions for solving the problem

$$\min \{f^0(z) \mid f^j(z) \leq 0,\ j = 1,2\ldots,m\}$$

where $f^j: B \to \mathbb{R}^1$ are continuously, Frechet differentiable functions. They showed that their methods were convergent and that under the usual assumptions they converged linearly. The reason for inventing a "dual" type method was that when existing "primal" type algorithms are extended to Banach spaces, they contain direction finding subproblems which are as difficult to solve as the original problem.

For $B = \mathbb{R}^n$, the simplest version of the Pironneau-Polak algorithm proceeds as follows: given u_i, calculate multipliers $\mu^j(u_i)$, $j = 0,1,2\ldots,m$,

by solving the finite dimensional quadratic programming problem,

$$5.12 \qquad \max_{\mu} \{ \sum_{i=1}^{m} \mu^j f^j(z_i) - \frac{1}{2} \| \sum_{j=0}^{m} \mu^j \nabla f^j(z_i) \|^2 \mid \mu \geq 0, \sum_{j=0}^{m} \mu^j = 1 \}$$

Then, set

$$5.13 \qquad h(z_i) = - \sum_{j=0}^{m} \mu^j(z_i) \nabla f^j(z_i),$$

and, finally, set

$$z_{i+1} = z_i + \lambda_i h(z_i),$$

where $\lambda_i = \beta^k \lambda_0$, with $\beta \in (0,1)$, $\lambda_0 > 0$ and k defined as the smallest integer and that $f^j(z_{i+1}) \leq 0$, j = 1,2..., m and an obvious modification of the Armijo rule (2.7) is satisfied. This algorithm can be used for solving optimal control problems of the form

$$5.14 \qquad \min \int_{t_0}^{t_f} h^0(x(t),u(t),t)dt$$

subject to

$$5.15 \qquad \frac{d}{dt} x(t) = h(x(t),u(t),t)$$

$$5.16a \qquad q_0^j(\xi) \leq 0, \; j = 1,2..., m_0, \; x(t_0) = \xi,$$

5.16b $\quad q_f^j(x(t_f)) \leq 0,\ j = 1,2\ldots,\ m_f$

where all the functions are continuously differentiable. The control $u(\cdot)$ is assumed to be in $L_\infty^m[t_0,t_f]$, the initial state $\xi \in \mathbb{R}^n$, and the variable z is taken to be (ξ,u). The functions f^j are defined by relations of the form: $f^0(z) = \int_{t_0}^{t_f} h^0(x(t,\xi,u),u(t),t)dt$, $f^j(z) = q_0^j(\xi)$, $j = 1,2,\ldots,m_0$, $f^{j+m_0}(z) = q_f^j(x(t_f,\xi,u))$, $j = 1,2\ldots,m_f$. The required gradients are calculated by solving adjoint equations, essentially as in the gradient methods, and the norm $\|\cdot\|^2$ in (5.12) is replaced by $\|\cdot\|^2 + \|\cdot\|_2^2$, where $\|\cdot\|_2$ denotes the L_2 norm. With this, the subproblem (5.12) remains finite dimensional and the algorithm convergent.

More sophisticated versions of this algorithm use only "ε-active" constraint functions f^j, $j \in \{1,2,\ldots,m\}$, in the direction finding subproblem (5.12) (i.e. only those f^j for which $f^j(z_i) \geq -\varepsilon$ are used). This makes (5.12) easier to solve, but, what is more important in the optimal control case, it reduces substantially the number of adjoint equations that need to be integrated so as to compute the coefficients for (5.12). The values of ε to be used at each iteration are determined by a test, similar to the one first used by Zoutendijk [5.31] in 1960.

The first gradient projection algorithm is attributed to Rosen [5.26], 1960. This method is particularly well suited for solving problems of the form $\min\{f^0(z) \mid f^i(z),\ j = 1,2\ldots,m\}$, where $f^0\colon \mathbb{R}^n \to \mathbb{R}^1$, is continuously differentiable and the $f^j\colon \mathbb{R}^n \to \mathbb{R}^1$, $j = 1,2\ldots,m$, are affine. The efficiency of Rosen's algorithm depends greatly on a clever utilization of the affinity of the constraint functions. There was some question

as to the convergence of Rosen's original method, which led Rosen to publishing an extension on [5.27]. Another modification of Rosen's original method was published by Polak [5.23] in 1969. None of these methods have seen direct application to continuous optimal control problems.

As far as optimal control problems are concerned, we find that in 1962 Bryson and Denham [2.5] published a heuristic gradient projection method for problems with equality terminal state constraints. At about the same time, Kelley [2.14] proposed his version of a heuristically justified gradient projection method for the same class of problems.

In 1965 Demyanov [1.6] described a gradient projection method which is not particularly attractive for nonlinear programming problems, but which is easy to use for certain optimal control problems. The convergence and rate of convergence of this algorithm, for a class of step length rules, was established by Polyak [5.25] in 1966. A related, Newton direction projection method was shown to converge superlinearly by Levitin and Polyak in their well known 1966 survey paper [5.17]. The original Demyanov algorithm was designed for solving problems of the form

$$\min \{f^0(u) \mid u \in \Omega\} \tag{5.17}$$

where Ω is a weakly compact, convex subset of a Banach space B and $f^0: B \to \mathbb{R}^1$ is continuously Frechet differentiable, with gradient $\nabla f^0(\cdot)$, as already explained for the gradient methods. Given $u_i \in \Omega$, Demyanov proposed to construct $u_{i+1} \in \Omega$ according to

$$u_{i+1} = u_i + \omega_{i\lambda_i}, \quad i = 0,1,2.... \tag{5.18}$$

where for $\lambda \geq 0$, $\omega_{i\lambda} \in \Omega$ is defined by

$$5.19 \qquad \|u_i - \lambda\nabla f^0(u_i) - \omega_{i\lambda}\|_B = \min \{\|u_i - \lambda\nabla f^0(u_i) - \omega\|_B \mid \omega \in \Omega\}$$

and, in the simplest case, λ_i is chosen so that

$$5.20 \qquad f^0(u_i + \omega_{i\lambda_i}) = \min_{\lambda \geq 0} f^0(u_i + \omega_{i\lambda})$$

Levitin and Polyak [5.17] showed that when $f^0(\cdot)$ is twice continuously differentiable and strictly convex, the algorithm (5.18) remains convergent when λ_i is taken to be any value in a certain interval, $[a,b]$, with $0 < a < b$ depending on bounds on the second derivative of $f^0(\cdot)$.

As an application to optimal control, consider again the problem (5.5) - (5.7). For this case, given u_i, $\nabla f^0(u_i)$ is as in (5.8), i.e.,

$$5.21 \qquad \nabla f^0(u_i)(t) = \left[\frac{\partial}{\partial x} h^0\,(x(t,u_i),u_i(t),t)\right]^T + \\ - \left[\frac{\partial}{\partial x} h(x(t,u_i),\, u_i(t),t)\right]^T p_i(t),\ t \in [t_0,t_f]$$

and $\omega_{i\lambda}(t)$, the projection of $u_i - \lambda\nabla f^0(u_i)$ on $\Omega \equiv \{u(\cdot) \mid |u^j(t)| \leq 1,\ j = 1,2\ldots,m\}$, is obviously given by

$$5.22 \qquad \omega^j_{i\lambda}(t) = \text{sat}\,(u^j_i(t) - \lambda\nabla f^0(u_i)^j(t)),\ t \in [t_0,t_f],\ j = 1,2,\ldots,m,$$

where sat $x = x$ for $|x| \leq 1$ and sat $x = x/|x|$ for $|x| > 1$. For further examples, see the book by Demyanov and Rubinov [1.6].

6. Feedback Solutions.

During the early 1950's Bellman [6.1,6.2] developed the concept of the "principle of optimality" into a conceptual method for solving dynamic optimization problems. This method constitutes the first extension of the classical calculus of variations to problems with inputs. The Bellman method is known as dynamic programming, and it can be derived as follows. Consider a typical optimal control problem: minimize

$$6.1 \qquad \int_{t_0}^{t_f} h^0(x,u,t)dt + L(x(t_f))$$

subject to

$$6.2 \qquad \frac{d}{dt}\, x(t) = h(x(t),u(t),t) \quad t \in [t_0,t_f],\ x(t_0) = x_0,$$

$$6.3 \qquad u(t) \in U,\ t \in [t_0,t_f],$$

with h^0, h and L continuously differentiable. For any initial state x and initial time t, Bellman introduced the optimal return function

$$V^0(x,t) = \int_{t}^{t_f} h^0(\hat{x}(s),\hat{u}(s),s)ds + L(\hat{x}(t_f)),$$

where $\hat{x}(\cdot)$ and $\hat{u}(\cdot)$ are the optimal trajectory and control, respectively, for this starting point. Assuming that $V^0(\cdot,\cdot)$ was continuously differentiable, he reasoned as follows. Suppose that for the first Δt seconds we apply an arbitrary, constant, control u which takes us to the state $x + \Delta x$, followed by the control which is optimal for the initial point $(x + \Delta x,$

$t + \Delta t)$. Then, clearly, we must have

$$6.4 \qquad V^0(x,t) = \min_u \{h^0(x,u,t)\Delta t + V^0(x+\Delta x, t(+\Delta t)\}$$

$$\approx \min_u \{h^0(x,u,t)\Delta t + V^0(x,t) + \frac{\partial V^0}{\partial x}(x,t)\Delta x + \frac{\partial V^0}{\partial x}(x,t)\Delta t)\}$$

Letting $\Delta t \to 0$ and rearranging terms, we then obtain the Hamilton-Jacobi-Bellman equation

$$6.5 \qquad -\frac{\partial V^0}{\partial t}(x,t) = \min_u \{h^0(x,u,t) + \frac{\partial V^0}{\partial x}(x,t)h(x,u,t)\},$$

with the boundary condition

$$6.6 \qquad V^0(x(t_f),t_f) = L(x(t_f)),$$

where in (6.5) we have made use of (6.2).

While (6.5) is not always valid without qualifications (see Boltyanskii [6.6]), it shows that whenever it does hold, the optimal control $u(t)$ is a function of the state $x(t)$, i.e. $u(t) = u(x(t))$. This feedback form of the control was immediately recognized as being very important by control engineers, who, in the past, had accumulated years of experience with non-optimal feedback control and came to appreciate the many advantages it offered. Unfortunately, eqn. (6.5) turned out to be impossible to solve in most cases of interest, due to the "curse of dimensionality". To overcome this difficulty, in 1968, Larson [6.14], proposed a discretized version of dynamic programming, which he called "Incremental Dynamic

Programming". In spite of these attempts to make dynamic programming a practical algorithm, it cannot be considered to be particularly successful in this role, at least as far as deterministic problems are concerned. The main use of dynamic programming has been as a conceptual tool, particularly, before the Pontryagin maximum principle was well understood. For example, the algorithms proposed by McReynolds [6.22], Mitter [3.22], Jacobson and Mayne [1.7a] and many others have all been reasoned out in terms of dynamic programming.

One of the most significant applications of dynamic programming is due to Kalman [6.13] 1960, who used it to solve the so called Linear-Quadratic Regulator problem, which is,

$$6.7 \qquad \min \{ \frac{1}{2} \int_{t_0}^{t_f} [\langle x(t), Q(t)x(t) \rangle + \langle u(t), R(t)u(t) \rangle] dt + \frac{1}{2} \langle x(t_f), Sx(t_f) \rangle \}$$

subject to

$$6.8 \qquad \frac{d}{dt} x(t) = A(t)x(t) + B(t)\, u(t),\ x(t_0) = x_0,$$

where A, B, Q and R are continuous matrices, and Q R and S are symmetric and positive definite. Guessing that the solution of (6.5), (6.6) for this case must be of the form

$$6.9 \qquad V^0(x,t) = \frac{1}{2} \langle x, K(t)x \rangle$$

where K(t) is a symmetric matrix, Kalman obtained from (6.5) that

6.10 $$u(t) = - R^{-1}(t)B^{T}(t)K(t)x(t),\ t \in [t_0,t_f]$$

Substituting back for u(t) into (6.5) and making use of (6.6), he concluded (since the resulting equation had to hold for any x) that K(t) must be given by

6.11 $$\frac{d}{dt} K(t) = - K(t)A(t) - A^{T}(t)K(t) + K(t)F(t)K(t) - Q(t);\ K(t_f) = S,$$

where $F = BR^{-1}B^{T}$. Consequently, once K(t) is calculated, the optimal control u is given by (6.10).

In 1967 Bucy [6.8] gave a set of conditions on the matrices in the problem (6.7) (6.8) for K(t) as defined by the Riccati equation (6.11) to exist and to be positive definite, as required for (6.10) to hold. Further elaborations of this work can be found in Brockett [6.7], 1970, and, for the discrete case, in Polak [1.12], 1971.

Probably, the first modern optimal control problem to be considered was not the linear-quadratic regulator problem but the <u>minimum time</u> regulator problem. This problem was motivated by a desire to eliminate disturbance effects in a system as quickly as possible and was the catalyst which eventually brought the Pontryagin team to the now celebrated maximum principle [1.13]. The minimum time regulator problem and the possibility of a relay or bang-bang type solution were considered by McDonald

[6.21] in 1950, Hopkin [6.11] in 1950 and in 1953-1955, by Lerner [6.19,6.20], Bogner and Kazda [6.4-6.5], Kalman [6.12], and many others. For an exhaustive bibliography and discussion of this early work, the reader is refered to Oldenburger [6.24]. The general structure of the solution to this problem, for the case of linear dynamics, was studied in 1956 by Bellman, Glicksberg and Gross [6.3], by Bushaw [6.9] in 1958 and by LaSalle [6.15-6.16] in 1960. Bushaw's work was particularly interesting in that he showed that for scalar, second order systems of the form $\ddot{x} + a\dot{x} + bx = u$, with the constraint $|u(t)| \leq 1$, where it was necessary to transfer an initial state $(x(0),\dot{x}(0))$ at $t = 0$ to the origin $(0,0)$, in the shortest possible time, the phase plane could be divided by a single curve into two disjoint parts, R_1, R_2, with the property that if $(x(t),\dot{x}(t)) \in R_1$, then the optimal control $\hat{u}(\cdot)$ satisfies $\hat{u}(t) = +1$ and if $(x(t),\dot{x}(t)) \in R_2$, then $\hat{u}(t) = -1$. For a detailed exposition of this work, see for example, Athans and Falb [1.1]. Thus, Bushaw succeeded in defining a feedback control law for this class of problems. His work was later extended to nonlinear second order systems by Lee and Markus [6.17,6.18], (1961) and, in 1961, to sampled data systems by Desoer and Wing [6.10], Nelson [6.23] and Polak [6.25]. Similarly, it was found that simple minimum fuel or minimal energy regulator problems with second order dynamics could also be solved by dividing the phase space in to control value regions (see [1.1] for a detailed exposition and bibliography). Following the early success with second order systems, there was a great deal of rather unsuccessful work directed toward constructing practical computers for implementing the switching surfaces which defined the "bang-bang" regions of the state space.

Virtually none of this work seems to have survived the test of time. Most of this work is listed in Chapter 1 of Oldenburger [6.24].

At the present time to solve free time optimal control problems, such as the minimum time regulator problem, one uses one of several possible transformations to convert these problems into fixed time problems and then uses one of the methods discussed in sections 2-5. It is not clear who introduced these transformations first; one of the earliest uses of them appears in Warga [6.26]. In any event, they form part of our scientific folklore. For a typical example, see [1.12], p. 72. Obviously, a solution to a minimum time problem obtained, after transformation, through the use of the earlier described algorithms, is not a feedback solution. At present, the only practical way for obtaining an optimal feedback solution, seems to lie in the use of high speed computers in sampled-data mode, i.e. in recomputing the optimal control every T seconds, with T small.

7. Iterative Methods for Minimum Time, Minimum Fuel and Minimum Energy Optimal Control Problems with Linear Dynamics.

Consider problems of the form:

$$7.1 \qquad \min \int_{t_0}^{t_f} h^0(u(t))dt$$

subject to

$$7.2 \qquad \frac{d}{dt} x(t) = A(t)x(t) + B(t)u(t)$$

$$7.3 \qquad x(t_0) = x_0,\ x(t_f) \in \Omega \subset \mathbb{R}^n;\ |u^j(t)| \leq 1,\ j = 1,2\ldots,m,$$

where A and B are continuous matrices, t_f is either given or to be found, Ω is a convex, compact subset of $\mathbb{R}^n$ and h^0 is of one of the following three forms: $h^0(u) \equiv 1$ (minimum time), $h^0(u) = \sum_{j=1}^{m} |u^j|$ (minimum fuel), $h^0(u) = \sum_{j=1}^{m} (u^j)^2$ (minimum energy). There are several related algorithms for solving these problems. Some of the first of these algorithms were introduced by Krassovskii [7.6-7.7] and Kulikowski [7.8-7.10] in 1959, as an outgrowth of the Krein L-problems of moments. Krassovskii's algorithm was extended by Neustadt [7.12] (1960). Then, Neustadt's method was independently rediscovered by Eaton [7.3] (1962). This was followed by an algorithm, related to the conditional gradient method, by Barr [7.1] and Barr and Gilbert [7.2], (1966), and finally by two algorithms due to Polak [7.19] (1969) and Meyer and Polak [7.4] (1970).

All these methods have been proved to be convergent. Basically, the feature which is common to all these algorithms is that they de-

compose the original problem (7.1) - (7.3) into a sequence of problems of the same form as (7.1) - (7.3), except that Ω is replaced by a tangent hyperplane. The following method is the simplest representative of this class (see p. 219[1.12]). Suppose that $t_0 = 0$, t_f is given, that $h^0(u) = \|u\|^2$, and that $\Omega = \{x \mid \|x\| \leq 1\}$, with $x_0 \notin \Omega$. Given $\nu_i \in \partial\Omega$ (the boundary of Ω), replace Ω by the hyperplane

$$7.4 \qquad P_i = \{\nu \mid \langle \nu - \nu_i, \nu_i \rangle = 0\}$$

Then, applying the Pontryagin maximum principle, we find that the optimal co-state $p_i(\cdot)$, for the i-th iteration, satisfies the differential equation

$$7.5 \qquad \frac{d}{dt} p_i(t) = - A(t)^T p_i(t), \; t \in [0,t_f], \; p_i(t_f) = \lambda_i \nu_i,$$

where λ_i is to be determined, and the corresponding optimal control is given by

$$7.6 \qquad u_i(t) = \arg \max_{|u^j| \leq 1} \{- \|u\|^2 + B^T(t) p_i(t) u\},$$

Thus we find that $u_i(t)$ depends on the unknown parameter λ_i in (7.5). Substituting from (7.6) into (7.2), we then determine λ_i from the boundary condition $x(t_f,u_i) \in P_i$. If $x(t_f,\hat{u}_i) = \xi_i$, it is clear that u_i is optimal for the original problem. Otherwise, we set $\xi_{i+1} = \xi_i + \mu_i(x(t_f,u_i) - \nu_i)$, where μ_i is such that $\nu_{i+1} \in \partial\Omega$ and the corresponding cost, $\int_0^{t_f} \|u_{i+1}(t)\|^2 dt$ is minimized, (with u_{i+1} and p_{i+1} calculated according to (7.5), (7.6), with $p_{i+1}(t_f) = \lambda_{i+1}\nu_{i+1}$).

Before we leave this class of problems, it may be interesting to observe that these problems cannot be solved directly by any of the algorithms described in the preceding sections. To apply the Pironneau-Polak method [5.20], for example, one must use the substitutions $u^j(t) = \cos v^j(t)$, $j = 1,2,\ldots,m$, and consider the $v^j(\cdot)$ as the controls. It was shown in [5.20] that in conjunction with some small modifications to the original Pironneau-Polak method, these transformations will not result in jamming at spurious solutions.

8. Penalty Function Methods.

Penalty function methods are distinguished into two classes: exterior and interior. Exterior penalty function methods were first described by Courant [8.6] in 1945, as a tool for solving boundary value problems. The convergence properties of Courant's method were established by Butler and Martin [8.2] in 1962. Since then, a large literature grew up dealing with the use of penalty function methods in nonlinear programming. A comprehensive survey on this subject can be found in Lootsma [8.18]. As far as optimal control problems are concerned, in 1962, Chang [8.4] used exterior penalty functions as a device for obtaining optimality conditions for problems with state space constraints and, in the same year, Kelly [8.15] used them as a heuristically justified computational tool. In 1965, Russel [8.22] showed that, at least for problems whose dynamics are linear in the control, penalty functions can be used to remove trajectory constraints, while in 1969 Cullum [8.7] showed that in the more general case, the use of penalty functions may lead to a solution of the relaxed optimal control problem, or to no solution at all, rather than to a solution of the original problem. We shall now elaborate on the above. For the moment, suppose that f^j: $\mathbb{R}^n \to \mathbb{R}^1$, $j = 0,1,\ldots,m$ and r: $\mathbb{R}^n \to \mathbb{R}^s$ are continuously differentiable, and consider the constrained optimization problem

$$8.1 \qquad \min \{f^0(u) \mid f^j(u) \leq 0,\ j = 1,2\ldots,m;\ r(u) = 0\}.$$

Next, for any $\varepsilon > 0$, define

$$8.2 \qquad f^0_\varepsilon(u) = f^0(u) + \frac{1}{2\varepsilon}\left[\|r(u)\|^2 + \sum_{j=1}^{m} (\max\{0, f^j(u)\})^2\right],$$

and consider the family of unconstrained optimization problems

$$8.3 \qquad \min\{f^0_\varepsilon(u) \mid u \in \mathbb{R}^n\},\ \varepsilon > 0.$$

The term $\frac{1}{2\varepsilon}$ [---] in (8.2) is the penalty function, and it is strictly positive whenever u does not satisfy the constraints in (8.1). It is not difficult to show on the basis of Butler and Martin's results (e.g., see[1.12] for an exposition)that if $u(\varepsilon)$ is a solution of (8.3), and, if for $\varepsilon \downarrow 0, u(\varepsilon) \to \hat{u}$, then $f^0(u(\varepsilon)) \uparrow f^0(\hat{u})$, $f^j(\hat{u}) \leq 0$ for $j = 1,2\ldots,m$, $r(\hat{u}) = 0$, and $f^0(\hat{u})$ is the minimum value for (8.1), i.e. $\hat{u}$ solves (8.1). This result could easily be anticipated, since the penalty in (8.3), for not satisfying the constraints of (8.1), goes to infinity as $\varepsilon \downarrow 0$.

Now consider the optimal control problem

$$8.3 \qquad \min\left\{\int_{t_0}^{t_f} h^0(x(t),u(t),t)dt \;\middle|\; \frac{d}{dt}\, x(t) = h(x(t),u(t),t),\right.$$

$$\left. x(t_0) = x_0;\ q_1(x(t)) \leq 0,\ q_2(u(t)) \leq 0,\ t \in [t_0,t_f];\ r(x(t_f)) = 0,\right.$$

where q_1: $\mathbb{R}^n \to \mathbb{R}^\ell$, q_2: $\mathbb{R}^r \to \mathbb{R}^m$, r: $\mathbb{R}^n \to \mathbb{R}^s$ are all continuously differentiable. Next, consider the unconstrained family of problems

$$8.4 \qquad \min \{ \int_{t_0}^{t_f} [h^0(x(t),u(t),t) + \frac{1}{2\varepsilon} \sum_{j=1}^{\ell} (\max \{0,q_1^j(x(t))\})^2$$

$$+ \sum_{j=1}^{m} (\max \{0,q^j(u(t))\})^2] \, dt + \frac{1}{\varepsilon} \| r(x)t_f) \|^2 |$$

$$\frac{d}{dt} x(t) = h(x(t),u(t),t), \; x(t_0) = x_0\}$$

The obvious conclusion as to what happens when $\varepsilon \downarrow 0$ in (8.4) turns out to be false. The facts are as follows. It was shown by Russel [8.22], under certain mild assumptions, that when h^0 and h are linear in the control u, the following holds. If $u_\varepsilon(\cdot)$ is a solution of (8.4) and $x_\varepsilon(\cdot)$ the corresponding trajectory, and $\varepsilon_i \downarrow 0$, as $i \to \infty$, a subsequence of $\{u_{\varepsilon_i}\}$ converges weakly to an optimal control $\hat{u}(\cdot)$ for (8.3) and the corresponding subsequence of trajectories $\{x_{\varepsilon_i}\}$ converges strongly to the corresponding optimal trajectory $\hat{x}(\cdot)$. It was then shown by Cullum [8.7] that when Russel's assumptions are relaxed, none of this may be true. In fact, the costs of (8.4) may converge to a lower value than the optimal value for (8.3), while the corresponding trajectories $x_\varepsilon(\cdot)$ may converge to a trajectory $\hat{x}(\cdot)$ which is not admissible for (8.3), in the sense that there is no admissible (measurable) control $\hat{u}(\cdot)$, for (8.3), which results in $\hat{x}(\cdot)$. Thus, the situation in the case of optimal control problems is much more complex than in the case of nonlinear programming problems.

Interior penalty functions were introduced by Carrol [8.3] in 1961 for nonlinear programming problems. His results were subsequently considerably developed by Fiacco and McCormak [8.10] (1964). An application to an optimal control problem was described by Lasdon, Warren and Rice [8.17], in 1967. To illustrate their use, suppose that $r \equiv 0$ in (8.1), i.e. it is not there, and consider the family of problems

8.5 $\qquad \min \{\overline{f}_\varepsilon{}^0(u) \,|\, u \in \Omega\}$

where $\Omega = \text{int}\,\{u \,|\, f^j(u) \leq 0,\ j = 1,2\ldots,m\}$ and for $\varepsilon > 0$

$$8.6 \qquad \overline{f}_\varepsilon{}^0(u) = f^0(u) - \varepsilon \sum_{j=1}^{m} \frac{1}{f^j(u)} .$$

Here the penalty term acts as a barrier for (8.5), keeping its solution $\overline{u}_\varepsilon$ away from the boundary of Ω.

The problems (8.5) are essentially unconstrained, but to solve them one needs a starting point $\tilde{u} \in \Omega$. The computation of such a $\tilde{u}$ can be anything but trivial. It is this fact that restricts the use of interior penalty functions in the solution of optimal control problems.

As far as the nonlinear programming problem (8.1) is concerned, with $r \equiv 0$, it was shown by Fiacco and McCormack [8.11] that for a general class of interior penalty functions (the one in (8.6) being one example), the solutions u_ε have the following property. Suppose $\varepsilon \to 0$ and $u_\varepsilon \to \hat{u}$. Then $\overline{f}_\varepsilon{}^0(u_\varepsilon) \downarrow f^0(\hat{u})$, and $\hat{u}$ is optimal for (8.1). There seem to be no parallel results in the literature for general optimal control problems of the form (8.3).

Finally, it should be pointed out that a combination of interior and exterior penalty functions can be used in solving problems such as (8.1). For details, see Fiacco and McCormack [8.11], Lootsma [8.19], or Polak [1.12].

Another interesting application of penalty functions, in conjunction with Ritz type expansions, has been in the removal of dynamic constraints. This permits the substitution of quadrature for the integration of

differential equations in the calculation of gradients, and tends to reduce computing time in solving optimal control problems. The idea of combining penalty functions with the Ritz method, to remove differential equation constraints, is attributed to Kononenko (unpublished work prior to 1968). Lions [8.18] (1968) used it for solving optimization problems with partial differential equations, Balakrishnan [8.1] (1968) used it as a convenient way of rederiving the Pontryagin maximum principle. Since then, largely due to efforts by Balakrishnan, his students and others [8.9] [8.13], [8.14], this method, which has become known as "the Balakrishnan ε-method", has been gaining acceptance as an important computational tool.

A number of technical questions regarding its efficiency, convergence and rate of convergence still remain open. So far, it was shown by Balakrishnan [8.1] that the optimal values of the penalized problems will converge to the optimal value of the relaxed problem. (In the relaxation of problem (8.4) (see [8.23], for example) the original dynamics are replaced by $\frac{d}{dt} x(t) \in \text{co } h(x(t), U, t)$, where $U = \{u | q_2(u) \leq 0\}$ and co denotes the convex null.) Basically, the conclusion to be drawn from the Cullum [8.7] and Balakrishnan [8.1] results is that optimal control problems are frequently ill posed, or in a sense, unstable. It appears that stability can often be regained by slightly relaxing the constraints.

We shall now illustrate the use of the "ε-method". Consider the problem

$$8.7 \qquad \min \int_{t_0}^{t_f} h^0(x,u)dt$$

subject to

8.8 $$\frac{d}{dt}\, x(t) = h(x(t),u(t)),\ x(t_0) = x_0,\ x(t_f) = x_f,$$

where h^0, h are both continuously differentiable, and assume that $u \in L_\infty^m\,[t_0,t_f]$, $x(t) \in \mathbb{R}^n$. Now, set

8.9 $$x_N(t) = x_0 + \left(\frac{t - t_0}{t_f - t_0}\right)(x_f - x_0) + \sum_{i=1}^{N} \alpha_i \sin i\,\omega t,$$

where $\omega = 2\pi/(t_f - t_0)$ and $\alpha_1,\alpha_2,\ldots,\alpha_N$ are undetermined vector coefficients. Next, consider the family of unconstrained optimization problems, indexed by $\varepsilon > 0$, $N = 1,2,3,\ldots$,

8.10 $$\min\{f^0_{\varepsilon,N}(u,\alpha) \mid u \in L_\infty^m\,[t_0,t_f],\ \alpha \in \mathbb{R}^{Nn}\}$$

where

8.11 $$f^0_{\varepsilon,N}(u,\alpha) = \int_{t_0}^{t_f} [h^0(x_N(t),u(t))dt + \frac{1}{2\varepsilon} \left\| \frac{d}{dt}\, x_N(t) - h(x_N(t),u(t),t) \right\|^2]dt.$$

As $\varepsilon \to 0$ and $N \to \infty$, one may expect that a subsequence of the optimal solutions $\hat{x}_N(\cdot)$ of (8.10) will converge strongly to an optimal trajectory of the relaxation of (8.7). However, this has still not been proven. It may yet turn out that it is necessary to drive ε and N to their limits in a correlated manner.

The obvious difficulty with the ε-method is that it is not possible to solve (8.10) for N large. However, it has been found empirically that a good approximation to a desired solution can often be obtained for fairly low values of N.

9. Discretizations.

The numerical solution of an optimal control problem such as (8.3) requires some form of discretization, at least to the extent that all numerical methods for integrating differential equations involve discretization. However, one may introduce discretizations for other reasons as well, the primary one being that it makes the problem solvable by some specific technique. Among the earliest discretizations are those due to Rosen [9.13], [9.14] who, between 1964 and 1967, showed that various discretizations of optimal control problems can be solved by linear and nonlinear programming techniques. Rosen's approach was to discretize the differential equation in the problem. By contrast, Canon, Cullum and Polak [9.2] (1966) proposed using a standard sampled-data discretization of the control, which restricts the control to a class of piecewise constant functions with a fixed number of discontinuities. The intention of this discretization was to make the problem finite dimensional and hence solvable by various nonlinear programming algorithms. The inevitable question to be asked was, what happens as the discretizations are further and further refined? Among the first answers came those of Cullum [9.3-9.6] who, in the period 1966-1971, showed (a) that only under reasonably restrictive assumptions can one be sure that the solutions of discretized problems will converge, in some sense, to the solutions of the original problem, and (b) that it mattered considerably how the discretizations were performed. For example, she showed that if one discretized a free time problem, then one was almost certain that the solutions of the discretized problem will not approximate a solution of

the original problem. However, if one transformed the free time problem into a fixed time problem and then discretized, then, at least in some cases, one could be sure that the solutions of the discretized problem would approximate a solution of the original problem. Similarly, she showed that state space trajectories must be transformed into affine constraints by the introduction of additional variables in order to make the problem stable with respect to discretization. In addition she showed that it may be necessary to relax constraints for the discretized problem to be well defined. Other interesting results in this area are due to Kirillova [9.8] (1963), Budak et al. [9.1] (1969) and Daniel [9.7,1.5] (1979) In particular, Daniel showed that Rosen's approximations will converge in a Sobolev space under certain assumptions.

An entirely different aspect of discretization was explored by Polak [9.11], [9.12] and Klessig and Polak [9.10] in the period 1969-1971. Their work was motivated by a desire to reduce the great amount of time that is used up in integrating differential equations at each iteration of the various methods discussed in Sections 2-8 of this paper. In appendix A of [1.12], we find a theory of adaptive approximations. The gyst of this theory consists of a demonstration that there are a number of schemes for approximating values of functions which can be used in an optimization algorithm, without affecting its convergence properties. The nature of these schemes is that they use coarse approximations in the early iterations and refine the precision only when a test, build into the method, indicates that this is necessary. Thus, only the last few iterations need to be carried out with great precision and hence one obtains considerable savings in computing time. In [9.10] we find an analysis of such a scheme to be used in

conjunction with gradient methods, while in [9.12] the Pironneau-Polak method of feasible directions [5.20] is extended to utilize adaptive approximations.

Thus we have seen work on discretization evolve through three stages. The first was the use of discretizations as the only means for implementing an optimal control algorithm on a digital computer. In the second stage we find answers as to whether progressively refined discretization leads to a better and better approximation to the desired solution. In this stage the successive discretizations are viewed as a Ritz method. Finally, we encounter the last stage, where discretization methods are constructed with a view of reducing the total work needed to obtain a solution of pre-specified precision.

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