# FUNCTIONALS AND MARTINGALES OF WIENER PROCESS WITH A TWO-DIMENSIONAL PARAMETER

by

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Memorandum No. ERL-M361

18 September 1972

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Research sponsored by the U. S. Army Research Office -- Durham, Contract DAHC-04-67-C-0046 and the Naval Electronic Systems Command, Contract N00039-71-C-0255.

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## 1. Introduction

In this paper we continue the development of a theory of martingales and stochastic integrals for processes with a two-dimensional parameter which was initiated in [1]. With little loss of generality we shall take as parameter space the unit square $T = [0,1]^2$ and define a partial ordering for the points in T by

$$a \prec b \Leftrightarrow a_i \leq b_i, \ i = 1,2.$$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a fixed probability space. A family of $\sigma$-subfields $\{\mathcal{F}_z, z \in T\}$ is said to be increasing if $z \succ z' \Rightarrow \mathcal{F}_z \supset \mathcal{F}_{z'}$. Given an increasing family $\{\mathcal{F}_z, z \in T\}$, we say a process $\{X_z, z \in T\}$ is a martingale with respect to it, (or $\{X_z, \mathcal{F}_z, z \in T\}$ is a martingale) if

$$z \succ z' \Rightarrow E^{\mathcal{F}_{z'}} X_z = X_{z'} \qquad \text{almost surely}$$

A Gaussian random function $\{W_z, z \in T\}$ is said to be a Wiener process if it satisfies the conditions.

$$EW_z = 0, \quad z \in T$$

$$EW_{(x,y)}W_{(x',y')} = \min(x,x') \min(y,y')$$

Let $\{W_z, z \in T\}$ be a Wiener process and denote by $\mathcal{W}_z$ the $\sigma$-field generated by $W_\zeta$, $\zeta \prec z$. Then $\{W_z, \mathcal{W}_z, z \in T\}$ is a martingale. The martingale property of a Wiener process is obvious if we view $W_z$ as the integral over the rectangle $\zeta \prec z$ of a Gaussian white noise [1]. More generally, we shall say the pair $\{W_z, \mathcal{F}_z, z \in T\}$ is a Wiener process if it is a martingale and

W is a Wiener process. Clearly, we must have $\mathcal{F}_z \supset \mathcal{W}_z$.

In this paper our objective is to study functionals of a Wiener process and $\mathcal{W}_z$-martingales by means of a pair of stochastic integrals

$$\int_T \phi_z \, W(dz)$$

$$[\int_{T\times T}] \psi(z_1,z_2)W(dz_1)W(dz_2)$$

where $\phi$ and $\psi$ are random functions satisfying appropriate measurability and integrability conditions. Integrals of the first type and a special case of the second type were introduced in [1]. Our main result in this paper is that if X is a functional of Wiener process $\{W_z, z \in T\}$ and if $E|X|^2 < \infty$ then X admits a representation of the form

$$X = \int_T \phi_z W(dz) + [\int_{T\times T}] \psi(z_1,z_2)W(dz_1)W(dz_2)$$

It then follows from a martingale property of the stochastic integrals that every $\mathcal{W}_z$-martingale is of the form

$$M_z = \int_{\zeta \prec z} \phi_\zeta W(d\zeta) + [\int_{\substack{\zeta_i \prec z \\ i=1,2}}] \psi_{\zeta_1,\zeta_2} W(d\zeta_1)W(d\zeta_2)$$

Ito [2] introduced the concept of a multiple Wiener integral (more appropriately named multiple <u>Ito-Wiener integral</u>)

$$\int_{T^n} h(z_1,z_2,\cdots,z_n)W(dz_1)\cdots W(dz_n)$$

where h is non-random, and showed that every functional of a Wiener process can be represented in a series of multiple Ito-Wiener integrals. Clearly, there is relationship between the two integrals that we introduce in this paper and the multiple Ito-Wiener integrals. This relationship will be explored in some detail.

## 2. Martingales on Increassing Paths.

We define a path in $T = [0,1]^2$ as a continuous function $\theta = [0,1] \to T$. We shall say a path is increasing if $\alpha > \beta \Rightarrow \theta(\alpha) \succ \theta(\beta)$, and smooth if $\theta$ has a continuous derivative on $(0,1)$. Let $\{M_z, \mathcal{F}_z, z \in T\}$ be a martingale and $\theta(\cdot)$ an increasing path. Clearly, $\{M_{\theta(t)}, \mathcal{F}_{\theta(t)}, t \in [0,1]\}$ is a one-parameter martingale. Therefore, a two-parameter martingale defines a one-parameter martingale on every increasing path. Conversely, a two-parameter process which is a one-parameter martingale on every increasing path is a martingale. This is because if $z \succ z'$ then we can take the path $\theta(t) = z' + (z-z')t$ and find

$$E^{\mathcal{F}_{z'}} M_z = E^{\mathcal{F}_{\theta(0)}} M_{\theta(1)} = M_{\theta(0)} = M_{z'}$$

The characterization of two-parameter martingales as one-parameter martingales on increasing paths allows one to make use of results in one-parameter martingale theory.

Let $\{M_z, \mathcal{F}_z, z \in T\}$ be a martingale such that almost all sample functions are continuous. Then for every increasing path $\theta$, $\{M_{\theta(t)}, \mathcal{F}_{\theta(t)}, 0 \le t \le 1\}$ is a sample continuous martingale. As such, it is necessarily locally square integrable, and then exists a unique continuous increasing function $A_t$ such that

$$M^2_{\theta(t)} - A_t$$

is an $\mathcal{F}_{\theta(t)}$ - local martingale [3]. We shall say $\{M_z, \mathcal{F}_z, z \in T\}$ is <u>path-independent</u> if for every increasing path $\theta$, $A_1$ depends only on the end-points $\theta(0)$ and $\theta(1)$ and not on the points in between. For a path-independent martingale M we can define a function $\langle M,M \rangle_z$, $z \in T$ as the increasing function $A_1$ for all paths connecting the points $\theta(0) = (0,0)$ and $\theta(1) = z$. It will then follow that $\{M^2_z - \langle M,M \rangle_z, \mathcal{F}_z, z \in T\}$ is a martingale if $EM^2_z < \infty$ and otherwise a local martingale. Here, we define a two-parameter <u>local martingale</u> as a process which is a local martingale on every increasing path. We can call $\langle M,M \rangle$ the <u>increasing process</u> of M, since $z \succ z' \Rightarrow \langle M,M \rangle_z \geq \langle M,M \rangle_{z'}$. Conversely, a sample continuous martingale M is necessarily path-independent if we can find an increasing process $\langle M,M \rangle$ such that $M^2 - \langle M,M \rangle$ is a local martingale. It is easy to verify that a Wiener process is a path-independent martingale with

(2.1) $$\langle W,W \rangle_z = \text{Area}(\zeta < z).$$

## 3. <u>Stochastic Integrals</u>.

Let $\{W_z, \mathcal{F}_z, z \in T\}$ be a Wiener process. Integrals of the form

$$I_1(\phi) = \int_T \phi(z) W(dz)$$

have been defined in [1], and will be referred to as <u>stochastic integrals of the first type</u>. The definition and some properties of these integrals are summarized below.

Let $\phi(\omega,z)$ satisfy the following conditions:

$H_1$: $\phi(\omega,z)$ is a bimeasurable function of $(\omega,z)$ with respect to $\mathcal{F} \otimes \mathcal{B}$ where $\mathcal{B}$ denotes the $\sigma$-algebra of Borel sets in T.

$H_2$: For each $z \in T$, $\phi_z$ is $\mathcal{F}_z$-measurable.

$H_3$: $\int_T E\phi_z^2 < \infty$

For the case where $\phi_z$ is simple (i.e. $\phi_z$ is of the form $\phi_z = \phi_\nu$, $z \in \Delta_\nu$, $\nu = 1, 2, \cdots, k$ and zero elsewhere, where $\Delta_\nu = [a_1^\nu,b_1^\nu)\times[a_2^\nu,b_2^\nu)$ are disjoint rectangles), $I_1(\phi)$ is defined as

$$I_1(\phi) = \sum_{\nu=1}^{k} \phi_\nu \Delta_\nu W$$

where

$$\Delta_\nu W = W(b_1^\nu,b_2^\nu) + W(a_1^\nu,a_2^\nu) - W(b_1^\nu,a_2^\nu) - W(a_1^\nu,b_2^\nu).$$

The definition of $I_1(\phi)$ is then extended to $\phi_z$ satisfying $H_1 - H_3$ by a standard completion argument. The main properties of $I_1(\phi)$ are:

(a) linearity: $I_1(\alpha\phi+\beta\psi) = \alpha I_1(\phi) + \beta I_1(\psi)$

(b) inner product: $EI_1(\phi)\cdot I_1(\psi) = \int_T \phi_z\psi_z dz$

(c) martingale: $E(I_1(\phi)|\mathcal{F}_z) = \int_{\zeta \prec z} \phi(\zeta)W(d\zeta)$.

We consider now what we will call the stochastic integral of the second type and will denote it by:

$$\int\!\!\int_{T\times T}\psi(z_1,z_2)W(dz_1)W(dz_2)$$

The motivation for introducing the stochastic integral of the second type is as follows. Let $\phi(z)$ be a square integrable nonrandom function of z and consider

$$X_z = \int_{\zeta \prec z} \phi(\zeta)W(d\zeta)$$

which can be interpreted as either a Wiener integral [2] or a stochastic integral of the first type. Consider now $X_z^2$. By partitioning the rectangle $[0,x] \times [0,y]$ we have, roughly,

$$X_z^2 \cong \sum_{i,j,k,\ell} \phi(i\Delta,j\Delta)\Delta_{ij}W\cdot\phi(k\Delta,\ell\Delta)\Delta_{k\ell}W$$

where

$$\Delta_{ij}W = W((i+1)\Delta,(j+1)\Delta) + W(i\Delta,j\Delta) - W((i+1)\Delta,j\Delta) - W(i\Delta,(j+1)\Delta).$$

(namely $\Delta_{ij}W$ is the white noise integral of the $\Delta\times\Delta$ square starting at $(i\Delta,j\Delta)$). The summation in the above expression are of three types:

(a) $i = k$, $j = \ell$, by known properties of the quadratic variation of Brownian motion in one dimensional time we expect that the terms of this type sum up to $\iint \phi^2(x,y)\,dxdy$.

(b) terms where $(i\Delta,j\Delta) \succ (k\Delta,\ell\Delta)$ or $(i\Delta,j\Delta) \prec (k\Delta,\ell\Delta)$; namely, ordered points. By the definition of the stochastic integral of the first type we expect these terms to add up to $2\int_{s\prec z}\phi(s)\int_{\zeta\prec s}\phi(\zeta)W(d\zeta)W(ds)$

(c) terms for which $(i\Delta, j\Delta)$ and $(k\Delta, \ell\Delta)$ are a pair of unordered points. The stochastic integral of the second type will be defined so as to collect such terms, and as will be seen in section 5 the two types of stochastic integrals suffice to represent all functionals and martingales of the two dimensional Wiener process $W_z$.

We turn, now, to the definition of the stochastic integral. For two points in A $z_1$, $z_2$ which are unordered we will use $z_1 \vee z_2$ to denote the smallest z satisfying $z \succ z$, $z \succ z_2$. In other words if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are unordered then $z_1 \vee z_2 = (\max(x_1, x_2), \max(y_1, y_2))$. Let $G \subset T \times T$ be such that $(z_1, z_2) \in G$ if $z_1$ and $z_2$ are unordered, let $h_G(z_1, z_2)$ be the indicator function of this set. Let $\psi(\omega, z_1, z_2)$ be a random function on $T \times T$ satisfying

$H_1'$: $\psi(\omega, z_1, z_2)$ is jointly measurable with respect to $\mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}$

$H_2'$: for each unordered pair $z_1$, $z_2$ the function $\psi(\omega, z_1, z_2)$ is measurable with respect to $\mathcal{F}_{z_1 \vee z_2}$

$H_3'$: $E\iint\limits_{TT} \psi^2(z_1, z_2) dz_1 dz_2 < \infty$

Let $\psi(z_1, z_2)$ be a simple function of $z_1$, $z_2 \in T$ i.e. $\psi$ is of the form

$$\psi(z_1, z_2) = \alpha, \; z_1 \in \Delta_1, \; z_2 \in \Delta_2$$

$$\psi(z_1, z_2) = 0 \quad \text{elsewhere}$$

where $\Delta_\nu = [a_1^\nu, b_1^\nu) \times [a_2^\nu, b_2^\nu)$, $\nu = 1, 2$ are a pair of non overlapping rectangles. Assume first that $\Delta_1$ and $\Delta_2$ are such that for any $z_1 \in \Delta_1$, $z_2 \in \Delta_2$ the pair $z_1$, $z_2$ are unordered. In this case we define

$$I_2(\psi) = [\int\!\!\!\!\int]_{T\times T} \psi(z_1,z_2)W(dz_1)W(dz_2) = \alpha\Delta_1 W\Delta_2 W$$

where $\quad \Delta_\nu W = W(b_1^\nu,b_2^\nu) + W(a_1^\nu,a_2^\nu) - W(a_1^\nu,b_2^\nu) - W(a_2^\nu,b_1^\nu).$

Without the assumption that for any $z_1 \in \Delta_1$, $z_2 \in \Delta_2$ the pair $z_1$, $z_2$ are unordered, we define $I_2(\psi)$ as follows. Let us define an $\varepsilon$ lattice on T. Let $[z]^\varepsilon$ denote the lattice point nearest to z from below and to the left of z. The lattice on T defines a lattice on T × T. Let $i^\varepsilon$, $j^\varepsilon$ denote points of the lattice on T × T. We define, now,

$$I_2^\varepsilon(\psi) = \sum_{i^\varepsilon,j^\varepsilon} \psi(i^\varepsilon,j^\varepsilon)h_G(i^\varepsilon,j^\varepsilon)\cdot\Delta_{i^\varepsilon}W\cdot\Delta_{j^\varepsilon}W$$

where the summation is over all lattice points or, what is the same because of $h_G$, over all unordered lattice pairs. $\psi$ is still a simple function. Let $\varepsilon_2$ be a subpartition of $\varepsilon_1$ then

$$E\left[I_2^{\varepsilon_1} - I_2^{\varepsilon_2}\right]^2 = E\psi^2 \cdot \iint_{TT}\left[h_G([z_1]^{\varepsilon_1},[z_2]^{\varepsilon_1}) - h_G([z_1]^{\varepsilon_1},[z_2]^{\varepsilon_2})\right]^2 dz_1 dz_2$$

It follows from the result of the appendix that $E\left[I_2^{\varepsilon_1} - I_2^{\varepsilon_2}\right]^2 \to 0$ as $\varepsilon_1$, $\varepsilon_2 \to 0$. Therefore $I_2^\varepsilon(\psi)$ converges in quadratic mean. We define

$$I_2(\psi) = \underset{\varepsilon\to 0}{\text{lim q.m.}}\ I_2^\varepsilon(\psi).$$

It is easily verified that this definition is consistent with the definition given earlier for the case where for any $z_1 \in \Delta_1$ and $z_2 \in \Delta_2$, are $z_1$ and $z_2$ unordered. We now extend the definition of $I_2(\psi)$ for functions which are the $L_2$ closure of linear combinations of simple functions in the usual way. It follows that $I_2(\psi)$ is defined for all random functions satisfying $H_1' - H_3'$ and inherits from $I_2^\varepsilon$ the following properties

1) $I_2(a_1\psi_1 + a_2\psi_2) = a_1 I_2(\psi_1) + a_2 I_2(\psi_2)$

2) $I_2(\psi) = I_2(h_G\psi)$

3) $EI_2(\psi_1)I_2(\psi_2) = \int_T\int_T h_G(z_1,z_2)\psi_1(z_1,z_2)\psi_2(z_1,z_2)dz_1\, dz_2$

4) $EI_1(\psi)I_2(\psi) = 0$

5) $E(I_2(\psi)|\mathcal{F}_z) = [\int_{\zeta_1,\zeta_2 < z}]\ \psi(\zeta_1,\zeta_2)W(d\zeta_1)W(d\zeta_2)$

These properties are easily verified for $I_2^\varepsilon(\psi)$ and extended to $I_2^\varepsilon(\psi)$ by standard arguments.

Both $I_1$ and $I_2$ can be extended to integrands $\phi$ and $\psi$ which do not satisfy $H_3$ and $H_3'$, but instead the conditions

$$\int_T \phi_z^2 dz < \infty \qquad \text{almost surely}$$

$$\int_T\int_T \psi^2(z_1,z_2)dz_1\, dz_2 \qquad \text{almost surely}$$

The extension is by approximating $\phi$ (resp. $\psi$) by a sequence of bounded function $\phi_n(\psi_n)$ converging almost surely to $\phi$ (resp. $\psi$) at every point z (resp. every pair $(z_1,z_2)$). $I_1$ and $I_2$ can then be defined as

$$I_1(\phi) = \lim_{n\to\infty} \text{ in prob. } I_1(\phi_n)$$

$$I_2(\psi) = \lim_{n\to\infty} \text{ in prob. } I_2(\psi_n)$$

So defined, $I_1$ and $I_2$ retain most of the properties, except they need not be square integrable and need not have the martingale property.

## 4. Relation between stochastic integrals and multiple Wiener integrals.

In this section we consider the n-th order Wiener integral [2]

$$(4.1) \qquad J_n(g) = \int_T \cdots \int_T g(t_1,\cdots,t_n)W(dt_1)\cdots W(dt_n)$$

where $g(\underset{\sim}{t})$ is a non-random function and $t_2$ is a non-random function and $t_i$ is a two dimensional parameter. The following theorem and corollary will be proved.

<u>Theorem 4.1</u> Every Wiener integral can be represented as the sum of two stochastic integrals:

$$(4.2) J_n(g) = \int_T \phi(t)W(dt) + [\int_{T\times T}] \psi(t_1,t_2)W(dt_1)W(dt_2)$$

It follows from this theorem that:

<u>Corollary</u>. Every square integrable functional of the two dimensional

Brownian motion can be represented as

$$(4.3)\quad F(W_0^T) = E(F(W_0^T)) + \int_T \phi(t)W(dt) + [\int\!\!\int]_{T\times T} \psi(t_1,t_2)W(dt_1)W(dt_2)$$

and every square integrable $\mathcal{W}_z$-martingale has the representation

$$(4.4)\quad m_z = m_0 + \int_{\zeta<z} \phi(\zeta)W(d\zeta) + [\int\!\!\int]_{\zeta_1,\zeta_2<z} \psi(\zeta_1,\zeta_2)W(d\zeta_1)W(d\zeta_2)$$

Two proofs of these results will be given. The first proof is based on section 3 and the appendix. The second proof is based on a differentiation formula and will be given in section 5. The second proof also clarifies the relation between stochastic integrals and the Hermite polynomial representation of Wiener integrals.

First Proof: From the way multiple Wiener integrals are defined in [2], it follows immediately that it is sufficient to prove the theorem for the case where $g(t_1,\cdots,t_n) = g(\underset{\sim}{t})$ is a simple function and $t_i \varepsilon T$ where T is the unit rectangle. Let $\Delta_\nu = [a_1^\nu,b_1^\nu) \times [a_2^\nu,b_2^\nu)$, $\nu = 1, \cdots, n$ be a set of non-overlapping rectangles and let $g(\underset{\sim}{t})$ be the indicator function of this set. Let $\tilde{g}(\underset{\sim}{t})$ denote the symmetrized version of $g(\underset{\sim}{t})$ [2].

$$(4.5)\qquad \tilde{g}(\underset{\sim}{t}) = \frac{1}{n!}\sum_{\pi} g(t_{i_1},t_{i_2},\cdots,t_{i_n})$$

where the summation is over all permutations of $1, \cdots, n$. Let F denote the set in $T^n$ for which $t_n > t_i$, $i = 1, \cdots, n-1$ and let $h_F(\underset{\sim}{t})$ denote the indicator function of this set. Consider now the multiple Wiener integral

$$(4.6)\quad J_n(h_F g) = \int_T \cdots \int_T h_F(\underset{\sim}{t})\tilde{g}(\underset{\sim}{t})W(dt_1)\cdots W(dt_n).$$

Let m be an integer and consider the lattice on T defined by the lattice points $(im^{-1}, jm^{-1})$ i, j = 1, 2, $\cdots$, m. For each $t = (x,y) \in T$ we define $[t]^m$ as the lattice point nearest to t from the left and below t. $[\underset{\sim}{t}]^m = ([t_1]^m, \ldots [t_n]^m)$.

By the properties of the multiple Wiener integral [2] and the results of the appendix to this paper we can approximate $J_n(\tilde{g}h_F)$ arbitrarily closely in the $L^2$ sense by

$$(4.7)\quad J^{(m)}(h\tilde{g}) = \int_T \cdots \int_T h_F([t]^m)\tilde{g}([t]^m)W(dt_1)\cdots W(dt_m)$$

for sufficiently large m. Note that the integrand is simple and

$$(4.8)\quad J^{(m)}(h_F\tilde{g}) = \sum h_F(i_1^m,\cdots,i_n^m)\tilde{g}(i_1^m,\cdots,i_n^m)\Delta_{i_1^m}W\cdots\Delta_{i_n^m}W$$

where $i^m = (\alpha^m, \beta^m)$ denote the lattice points on T,

$$(4.9)\quad \Delta_{i^m} = W(\alpha^m+m^{-1},\beta^m+m^{-1}) + W(\alpha^m,\beta^m)$$

$$- W(\alpha^m+m^{-1},\beta^m) - W(\alpha^m,\beta^m+m^{-1})$$

and the summation is over all the lattice points in $T^m$ induced by the lattice on T.

Consider now the type I stochastic integral

$$(4.10)\quad I_1(h_F\tilde{g}) = \iint_T\left[\int_{t_1<t_n} \cdots \int_{t_{n-1}<t_n} h_F(t_1,\cdots,t_n)\tilde{g}(t_1,\cdots,t_n)W(dt_1)\cdots W(dt_{n-1})\right] W(dt_n)$$

where the integrand in the integration with respect to $W(dt_n)$ is a Wiener integral of order (n-1). Let

$$(4.11) I_1^m(h_F\tilde{g}) = \int_T \left[ \int_{t_1<[t_n^m]} \cdots \int_{t_{n-1}<[t_n^m]} h([\underset{\sim}{t}]^m)\tilde{g}([\underset{\sim}{t}]^m) W(dt_1)\cdots W(dt_{n-1}) \right] W(dt_n)$$

For the Wiener integral we have [2]

$$(4.12) E(J_n(g_1) - J_n(g_2))^2 = \int_T \cdots \int_T (g_1(\underset{\sim}{t}) - g_2(\underset{\sim}{t}))^2 dt_1 \cdots dt_n$$

By this result, the result of the appendix to this paper and the properties of the stochastic integral of the first type, we have

$$E(I_1(h_F\tilde{g}) - I_1^m(h_F\tilde{g}))^2 \to 0$$

as $m \to \infty$. On the other hand we have by the definition of the Wiener integral of order (n-1) $\left(\text{the integrand of the expression for } I_1^m(h_F\tilde{g})\right)$

$$(4.13) \int_{t_1<[t_n^m]} \cdots \int_{t_{n-1}<[t_n^m]} h([\underset{\sim}{t}]^m)\tilde{g}([\underset{\sim}{t}]^m) W(dt_1)\cdots W(dt_{n-1})$$

$$= \sum h_F(i_1^m, \cdots, i_n^m)\tilde{g}(i_1^m, \cdots, i_n^m) \Delta_{i_1^m} W \cdots \Delta_{i_{n-1}^m} W$$

where the notation and summation is as in (4.8). By the definition of the stochastic integral of the first type it follows that $I_1^m(h_F\tilde{g})$ is the same as $J^m(h_F\tilde{g})$. Therefore, by standard completion arguments, for any non-random square integrable $g(t)$, the Wiener integral $J_n(hg)$ can be expressed as a stochastic integral of the first type.

Consider now the set G in $T^n$ for which $t_{n-1}$ and $t_n$ are unordered and $t_{n-1} \curlyvee t_n \succ t_i$, $i = 1, \cdots, n-2$. Let $h_G(\underset{\sim}{t})$ denote the indicator function of this set. Consider the multiple Wiener integral

$$(4.14) \qquad J_n(h_G \tilde{g}) = \int_T \cdots \int_T h_G(\underset{\sim}{t}) \tilde{g}(\underset{\sim}{t}) W(dt_1) \cdots W(dt_n)$$

and the stochastic integral of the second type

$$(4.15) \; I_2(h_G \tilde{g}) = \underset{T \times T}{[\int\!\!\int]} \left[ \int_{t_1 \prec t_n \curlyvee t_{n-1}} \cdots \int_{t_{n-2} \prec t_n \curlyvee t_{n-1}} h_G(\underset{\sim}{t}) \tilde{g}(\underset{\sim}{t}) W(dt_1) \cdots W(dt_{n-2}) \right] W(dt_{n-1}) W(dt_n)$$

where the integrand is a (n-2) order multiple Wiener integral with $t_{n-1}$, $t_n$ as parameters. It follows by the same arguments as before that $I_2(h_G \tilde{g}) = J_n(h_G \tilde{g})$ and, therefore, by the standard completion argument the same equality holds for any square integrable $g(\underset{\sim}{t})$.

Now, let $F_i$ denote the subset of $T^n$ for which $t_i \succ t_j$, $j = 1, \cdots, n$, and let $G_{ij}$ denote the set in $T^n$ such that $t_i$, $t_j$ are unordered and $t_i \curlyvee t_j > t_\ell$, $\ell = 1, \cdots, n$. Obviously $G_{ij} = G_{ji}$. Given any point $t_1, \ldots, t_n$ if one of the $t_i$ say $t_{i_0}$ is ordered with respect to all others and $t_{i_0} \succ t_j$, $j = 1 \cdots n$ then $(t_1, \cdots, t_n) \in F_{i_0}$. If there is no such $t_{i_0}$ then consider the components $t_i = (x_i, y_i)$ let $x_{j_0}$ be largest number among $x_1, \cdots, x_n$ and let $y_{k_0}$ be the largest among $y_1, \cdots, y_k$ then obviously $t_{j_0} \curlyvee t_{k_0} \succ t_i$ and $(t_1, \cdots, t_n) \in G_{j_0 k_0}$. Therefore

$$\bigcup_{i=1}^{n} F_i + \bigcup_{i<j} G_{ij} = T^n$$

Note that $I_1(h_{F_i}\tilde{g})$ is independent of i since $\tilde{g}$ is symmetric in $t_1, \cdots, t_n$ and similarly $I_2(h_{G_{ij}}\tilde{g})$ is independent of (i,j). By the definition of the multiple Wiener integral we can set $\tilde{g}(\underset{\sim}{t}) = 0$ whenever $t_i = t_j$, $i \neq j$, without changing the value of $I_n(\tilde{g})$. Under this assumption $F_i \cap F_j = \phi$ for $i \neq j$ and $F_i \cap G_{jk} = \phi$. The sets $G_{ij} = G_{ji}$ are not disjoint since, for example $t_1 = (\frac{1}{3}, 1)$, $t_2 = (\frac{1}{2}, 1)$, $t_3 = (1, \frac{1}{2})$ belong to both $G_{1,3}$ and $G_{2,3}$. However, by an argument similar to the one given in the appendix it follows immediately that

$$\int_T \cdots \int_T |h_{G_{ij}}(\underset{\sim}{t}) - h_{G_{ke}}(\underset{\sim}{t})| \, dt_1, \cdots, dt_n = 0$$

whenever $G_{ij} \neq G_{k\ell}$. Therefore

$$J_n(\tilde{g}) = \sum_i J_n(h_{F_i}\tilde{g}) + \sum_{i<j} J_n(h_{G_{ij}}\tilde{g})$$

$$= nJ_n(h_F\tilde{g}) + \frac{n(n-1)}{2} J_n(h_G\tilde{g})$$

which proves the theorem. In order to prove the corollary we use the result that any zero mean $L_2$ functional $\xi$ of $W_z$ is expressible as:

$$\xi = \sum_n I_n(\tilde{g}_n).$$

Therefore, by the theorem just proved

$$\xi = \sum_i \int_T \phi_i(z) W(dz)$$

$$+ \sum_i [\int\!\!\int_{T\times T}] \, \psi_i(z_1, z_2) W(dz_1) W(dz_2)$$

By the orthogonality property of Wiener integrals of different order, the orthogonality between type I and type II stochastic integrals and since it was assumed that $E\xi^2 < \infty$, $\sum \phi_i(z)$ and $\sum \psi_i(z_1,z_2)$ converge in quadratic mean (to, say, $\phi(z)$ and $\psi(z_1 z_2)$ respectively) and

$$\xi - E\xi = \int_T \phi(z)W(dz) + [\int]_{T\times T} \psi(z_1,z_2)W(dz_1)W(dz_2)$$

Furthermore if $m_t$ is a square integrable $\mathcal{W}_t$-martingale, set $m_{(1,1)} = \xi$ then

$$\begin{aligned} m_t &= E\left( \int_T \phi(z)W(dz) \,|\, \mathcal{F}_t\right) + E\left([\int]_{T\times T} \psi(z_1,z_2)\; W(dz_1)W(dz_2) \,|\, \mathcal{F}_t\right) \\ &= \int_{z<t} \phi(z)W(dz) + [\int]_{z_1,z_2<t} \psi(z_1,z_2)W(dz_1)W(dz_2) \end{aligned}$$

which completes the proof of the corollary.

5. Differentiation Formula and Hermite Functionals.

For path-independent martingales a differentiation formula can be established almost immediately by using the differentiation rule for one-parameter martingales on increasing paths.

Let $M_z = (M_{1z}, M_{2z}, \cdots, M_{nz})$ be a set of sample-continuous path-independent martingales with respect to a fixed increasing family of σ-fields $\{\mathcal{F}_z, z \in T\}$. Since both $M_{iz} + M_{jz}$ and $M_{iz} - M_{jz}$ are path-independent sample-continuous martingales we can define an inner product process

(5.1) $$\langle M_i, M_j \rangle_z = \frac{1}{4} [\langle M_i+M_j, M_i+M_j \rangle_z - \langle M_i-M_j, M_i-M_j \rangle_z]$$

Let $f(u,z)$, $u \in R^n$, $z \in T$, be a real or complex valued function, having continuous mixed second partials with respect to the components of u and a continuous gradient with respect to z. We adopt the notation

$$f^i(u,z) = \frac{\partial f(u,z)}{\partial u_i}$$

$$f^{ij}(u,z) = \frac{\partial^2 f(u,z)}{\partial u_i \partial u_j}$$

$$\nabla f(u,z) = \operatorname{grad}_z f(u,z)$$

Let $\theta(t)$, $0 \leq t \leq 1$, be an increasing path. Since $M_{i\theta(t)}$, $0 \leq t \leq 1$, are one-parameter continuous martingales the familiar differentiation formula of Ito and Kunita-Watanabe [3] yields

$$(5.2) \qquad f(M_{\theta(t)},\theta(t)) - f(M_{\theta(0)},\theta(0)) = \sum_i \int_0^t f^i(M_{\theta(s)},\theta(s))dM_{i\theta(s)}$$

$$+ \int_0^t \Big[ \frac{1}{2} \sum_{i,j} f^{ij}(u,z)\nabla\langle M_i,M_j\rangle_z + \nabla f(u,z)\Big]_{\substack{z=\theta(s)\\ u=M_{\theta(s)}}} \cdot d\theta(s)$$

Equation (5.2) can be expressed in a simpler and more suggestive way as

$$(5.3) \qquad \operatorname{grad} f(M_z,z) = \sum_i f^i(M_z,z)\nabla M_z$$

$$+ \frac{1}{2} \sum_{i,j} f^{ij}(M_z,z)\nabla\langle M_i,M_j\rangle_z + \nabla f(M_z,z)$$

the precise meaning of which is given by (5.2).

Suppose $\langle M_i, M_j \rangle_z$ are nonrandom functions, which we shall denote by $V_{ij}(z)$. For example, this is the case if

$$M_{iz} = \int_{0<\zeta<z} \phi_i(\zeta) W(d\zeta)$$

where $\phi_i$ are nonrandom functions, in which event $V_{ij}$ are given by

$$V_{ij}(z) = \int_{0<\zeta<z} \phi_i(\zeta)\phi_j(\zeta)d\zeta$$

If $\langle M_i, M_j \rangle = V_{ij}$ are nonrandom and if f is a function satisfying

$$(5.4) \qquad \frac{1}{2}\sum_{i,j} f^{ij}(u,z)\nabla V_{ij}(z) + \nabla f(u,z) = 0$$

then (5.2) yields the result

$$(5.5) \qquad f(M_z,z) - f(M_{z_0},z_0) = \sum_i \int_0^t f^i(M_{\theta(s)},\theta(s))dM_{i\theta(s)}$$

where $\theta$ is any increasing path such that $\theta(0) = z_0$ and $\theta(t) = z$. Since the right hand side is a local martingale for any increasing path $\theta$, we have proved that $f(M_z,z)$ is a local martingale provided that f satisfies (5.4). Theorem 5.1 shows that (5.4) has many solutions.

<u>Theorem 5.1</u>. For every $m = (m_1, m_2, \ldots, m_n)$, $m_i$ being integers, there is a polynomial in u

$$(5.6) \qquad f_m(u,z) = \sum_{k\leq m} a_{mk}(z) u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n}$$

which satisfies (5.4) with $a_{mm}(z) = 1$. (We have used $k \preceq m$ to denote $\{k_i \leq m_i$ for every $i\}$, and $k \prec m$ will stand for $\{k \preceq m$ and $k_i < m_i$ for at least one $i\}$.).

Proof: We first observe that for any real values $\alpha_1, \alpha_2, \cdots, \alpha_n$, the function

$$(5.7) \qquad f(u,z;\alpha) = \exp\{i\sum_j \alpha_j u_j + \frac{1}{2}\sum_{j,k} \alpha_j\alpha_k V_{jk}(z)\}$$

is always a solution of (5.4). Therefore, we can take

$$f_m(u,z) = (-i)^{m_1+m_2+\cdots+m_n}\left[\frac{\partial^{m_1+m_2+\cdots m_n}}{\partial\alpha_1^{m_1}\partial\alpha_2^{m_2}\cdots\partial\alpha_n^{m_n}} f(n,z;\alpha)\right]\Bigg|_{\substack{\alpha_j=0 \\ j=1,2,\cdots,n}}$$

which is of the form (5.6) and satisfies (5.4). ¤

Equation (5.7) allows us to generate solutions to (5.4) almost at will. By itself, it also allows us to prove the following important result.

Theorem 5.2. Let $\{X_z, z \in T\}$ be a path-independent sample continuous martingale with increasing process

$$V(z) = \text{area } (\zeta \prec z)$$

Then X is a Wiener process.

Proof: Take a set of points $a_1, a_2, \cdots, a_n$ in T and define

$$M_{iz} = X_{a_i \wedge z}$$

Then we have

$$\langle M_i, M_j \rangle_z = V(a_i \wedge a_j \wedge z)$$

and $M_{iz}$ are sample-continuous path-independent martingales. Therefore, (5.7) and (5.5) yield the result

$$E \exp\{i \sum_j \alpha_j M_{jz} + \frac{1}{2} \sum_{j,k} \alpha_j \alpha_k V_{jk}(z)\} = 1$$

Putting $z = (1,1)$ yields:

$$E \exp\{i \sum_i \alpha_j X_{a_j}\} = e^{-\frac{1}{2} \sum_{j,k} \alpha_j \alpha_k V(a_j \wedge a_k)}$$

which proves the theorem. ¤

Let $W_z$, $z \in T$, be a Wiener process, and let $\{\phi_\nu(z), z \in T\}$ be a complete orthonormal system of real-valued nonrandom functions. For each $\nu$ and each $z \in T$,

$$M_{\nu z} = \int_{\zeta < z} \phi_\nu(\zeta) W(d\zeta) \tag{5.8}$$

is well defined both as a Wiener integral and as a type-I stochastic integral. $\{M_{\nu z}, z \in T\}$ is a collection of sample-continuous martingales with

$$\langle M_\nu, M_\mu \rangle_z = V_{\nu\mu}(z) = \int_{\zeta < z} \phi_\nu(\zeta) \phi_\mu(\zeta) d\zeta \tag{5.9}$$

A celebrated result of Cameron and Martin [4] states that every square-integrable functional of the Wiener process $\{W_z, z \in T\}$ can be represented in a series of Hermite functionals, a Hermite functional being a product of the form

$$\prod_{i=1}^{n} H_{p_\nu}\left(\int_T \phi_{q_\nu}(\zeta)W(d\zeta)\right)$$

where $H_p$ are Hermite polynomials.

For each $p = (p_1, p_2, \cdots, p_n)$, $\prod_{\nu=1}^{n} H_{p_\nu}(u_\nu)$ is a polynomial in $u = (u_1, u_2, \cdots, u_n)$ of degree p. Theorem 5.1 implies that we can write

$$\prod_{\nu=1}^{n} H_{p_\nu}(u_\nu) = \sum_{k \leq p} \beta_{pk} f_k(u,(1,1)) \tag{5.10}$$

when $f_k$ satisfy (5.4). It follows that there is a function

$$f(u,z) = \sum_{k \leq p} \beta_{pk} f_k(u,z) \tag{5.11}$$

such that $f(M_z, z)$, $z \in T$, is a martingale and

$$\prod_{\nu=1}^{n} H_{p_\nu}\left(\int_T \phi_\nu(\zeta)W(d\zeta)\right) = f(M_z,z)\Big|_{z=(1,1)} \tag{5.12}$$

$$= \sum_i \int_0^1 f^i(M_{\theta(s)}, \theta(s))\, dM_{i\theta(s)}$$

where $\theta$ is any increasing path connecting (0,0) and (1,1).

Theorem 5.3. Let $\{W_z, \mathcal{F}_z, z \in T\}$ be a Wiener process and define

(5.13) $$M_{iz} = \int_{\zeta \prec z} \phi_i(\zeta) W(d\zeta)$$

where $\phi_i$ are nonrandom functions. Let $f(u,z)$ be a function satisfying (5.4) with continuous partials with respect to the components of u up through the third order. Then

(5.14) $$f(M_z,z) = f(0,0) + \int_{\zeta \prec z} \sum_i f^i(M_\zeta,\zeta)\phi_i(\zeta)W(d\zeta)$$

$$+ \frac{1}{2} \left[\int \int_{\zeta,\zeta' \prec z}\right] \sum_{i,j} f^{ij}(M_{\zeta\vee\zeta'},\zeta\vee\zeta')\phi_i(\zeta)\phi_j(\zeta')W(d\zeta)W(d\zeta')$$

Remark: Unlike (5.2), (5.14) is truly a differentiation formula of two-parameter stochastic calculus since it involves stochastic integrals of both types.

Proof: It is clear that we only need to prove (5.14) for the case z = (1,1), since the general case follows from the martingale property of both sides. Now, let the unit square T be partitioned by a sequence of square subdivisions. It is convenient to take the squares to be of the same size (say $\delta_k$) in each partition and we assume

$$\delta_k \xrightarrow[k \to \infty]{} 0$$

We can order the lattice points of each partition in some arbitrary way and denote them by

$$z_{k\nu} = (x_{k\nu}, y_{k\nu})$$

We can now write

$$f(M(1,1),(1,1)) - f(M(1,0),(1,0)) - f(M(0,1),(0,1)) + f(M(0,0),(0,0))$$

$$= \sum_{\nu} \{ f(M(x_{k\nu}+\delta_k, y_{k\nu}+\delta_n),(x_{k\nu}+\delta_k, y_{k\nu}+\delta_k))$$

$$- f(M(x_{k\nu}+\delta_k, y_{k\nu}),(x_{k\nu}+\delta_k, y_{k\nu}))$$

$$- f(M(x_{k\nu}, y_{k\nu}+\delta_k),(x_{k\nu}, y_{k\nu}+\delta_k))$$

$$+ f(M(x_{k\nu}, y_{k\nu}),(x_{k\nu}, y_{k\nu})) \}$$

Since f satisfies (5.4), we can use (5.5) for the bracketed terms and write

$$f(M(1,1),(1,1)) - f(M(1,0),(1,0)) - f(M(0,1),(0,1)) + f(M(0,0),(0,0))$$

$$= \sum_{\nu} \sum_{i} \int_0^1 \{ f^i [M(x_{k\nu}+\delta_k, y_{k\nu}+s\delta_k),(x_{k\nu}+\delta_k, y_{k\nu}+s\delta_k)]$$

$$\cdot M_i(x_{k\nu}+\delta_k, y_{k\nu}+\delta_k ds)$$

$$- f^i [M(x_{k\nu}, y_{k\nu}+s\delta_k),(x_{k\nu}, y_{k\nu}+s\delta_k)] \cdot M_i(x_{k\nu}, y_{k\nu}+\delta_k ds) \}$$

$$= \sum_{\nu} \sum_{i} \int_{y_{k\nu}}^{y_{k\nu}+\delta_k} \{ f^i [M(x_{k\nu}+\delta_k, y),(x_{k\nu}+\delta_k, y)] M_i(x_{k\nu}+\delta_\nu, dy)$$

$$- f^i [M(x_{k\nu}, y),(x_{k\nu}, y)] M_i(x_{k\nu}, dy) \}$$

Because of the forward-difference nature of one-parameter stochastic integrals, we can write

$$f(M(1,1),(1,1)) - f(M(1,0),(1,0)) - f(M(0,1),(0,1)) + f(M(0,0),(0,0))$$

$$= \lim_{k\to\infty} \text{in prob.} \sum_{\nu}\sum_{i}\{f^{i}[M(x_{k\nu}^{+},y_{k\nu}),(x_{k\nu}^{+},y_{k\nu})][M_i(x_{k\nu}^{+},y_{k\nu}^{+}) - M_i(x_{k\nu}^{+},y_{k\nu})]$$

$$- f^{i}[M(x_{k\nu},y_{k\nu}),(x_{k\nu},y_{k\nu})][M_i(x_{k\nu},y_{k\nu}^{+}) - M_i(x_{k\nu},y_{k\nu})]\}$$

where $x_{k\nu}^{+} = x_{k\nu} + \delta_k$ and $y_{k\nu}^{+} = y_{k\nu} + \delta_k$.

Rearranging terms and using (5.5) for the difference

$$f^{i}[M(x_{k\nu}^{+},y_{k\nu}),(x_{k\nu}^{+},y_{k\nu})] - f^{i}[M(x_{k\nu},y_{k\nu}),(x_{k\nu},y_{k\nu})]$$

$$= \int_{x_{k\nu}}^{x_{k\nu}+\delta} \sum_{j} f^{ij}[M(x,y_{k\nu}),(x,y_{k\nu})]M_j(dx,y_{k\nu})$$

we find

$$f[M(1,1),(1,1)] - f[M(1,0),(1,0)] - f[M(0,1),(0,1)] + f[M(0,0),(0,0)]$$

$$= \lim_{k\to\infty} \text{in prob.} \left\{\sum_{\nu}\sum_{i} f^{i}[M(x_{k\nu},y_{k\nu}),(x_{k\nu},y_{k\nu})]\Delta_{k\nu}M_i\right.$$

$$\left. + \sum_{\nu}\sum_{i,j} f^{ij}[M(x_{k\nu},y_{k\nu}),(x_{k\nu},y_{k\nu})]\cdot(\delta_{k\nu}^{1}M_i)(\delta_{k\nu}^{2}M_j)\right\}$$

where we have adopted the notations

$$\Delta_{k\nu}M_i = M_i(x^+_{k\nu},y^+_{k\nu}) - M_i(x_{k\nu},y^+_{k\nu}) - M_i(x^+_{k\nu},y_{k\nu}) + M_i(x_{k\nu},y_{k\nu})$$

$$\delta^1_{k\nu}M_i = M_i(x^+_{k\nu},y_{k\nu}) - M_i(x_{k\nu},y_{k\nu})$$

$$\delta^2_{k\nu}M_i = M_i(x_{k\nu},y^+_{k\nu}) - M_i(x_{k\nu},y_{k\nu})$$

From (5.13) we have

$$\Delta_{k\nu}M_i \sim \phi_i(x_{k\nu},y_{k\nu})\Delta_{k\nu}W$$

Therefore,

$$\lim_{k\to\infty} \text{ in prob. } \sum_{\nu} f^i[M(x_{k\nu},y_{k\nu}),(x_{k\nu},y_{k\nu})]\,\Delta_{k\nu}M_i$$

$$= \int_T f^i(M_\zeta,\zeta)\phi_i(\zeta)W(d\zeta)$$

Now, we observe that for any function g

$$\sum_{\nu}\sum_{\substack{\mu\\ \nu\neq\mu}} g(x_{k\nu}\vee x_{k\mu},y_{k\nu}\vee y_{k\mu})\Delta_{k\nu}M_i\Delta_{k\mu}M_j$$

$$= \sum_{\nu} g(x_{k\nu},y_{k\nu})[\delta^1_{k\nu}M_i\delta^2_{k\nu}M_j + \delta^2_{k}M_i\delta^1_{k\nu}M_j]$$

Since $f^{ij} = f^{ji}$, we have

$$\lim_{k\to\infty} \text{ in prob. } \sum_{i,j}\sum_{\nu} f^{ij}[M(x_{k\nu},y_{k\nu}),(x_{k\nu},y_{k\nu})](\delta^1_{k\nu}M_i)(\delta^2_{k\nu}M_j)$$

$$= \frac{1}{2} \lim_{k \to \infty} \text{in prob.} \sum_{i,j} \sum_{\nu \neq \mu} f^{ij}[M(z_{k\nu} \vee z_{k\mu}), z_{k\nu} \vee z_{k\mu}] \Delta_{k\nu} M_i \Delta_{k\mu} M_j$$

$$= \frac{1}{2} \left[ \int_{T \times T} \right] f^{ij}(M_{\zeta \vee \zeta'}, \zeta \vee \zeta') \phi_i(\zeta) \phi_j(\zeta') W(d\zeta) W(d\zeta')$$

The proof of theorem 5.3 is now complete. ◻

We now observe that as a corollary of theorem 5.3 we have the following:

Corollary: (c.f. corollary of theorem 4.1) Let X be a square integrable functional of $\{W_z, z \in T\}$. Then X has a representation of the form

$$X = \int_T \phi_\zeta W(d\zeta) + \left[ \int_{T \times T} \right] \psi_{\zeta,\zeta'} W(d\zeta) W(d\zeta') + EX \tag{5.15}$$

Proof: First, we observe that from (5.12) every Hermite functional has a representation

$$\prod_{\nu=1}^{n} H_{p_\nu}\left( \int_T \psi_\nu(\zeta) W(d\zeta) \right) = \text{constant} + \int_T \sum_i f^i(M_{\zeta,\zeta}) \phi_i(\zeta) W(d\zeta) \tag{5.16}$$

$$+ \frac{1}{2} \left[ \int_{T \times T} \right] \sum_{i,j} f^{ij}(M_{\zeta \vee \zeta'}, \zeta \vee \zeta') \phi_i(\zeta) \phi_j(\zeta') W(d\zeta) W(d\zeta')$$

The assertion of the corollary now follows from completeness of Hermite functionials in the space of square-integrable Wiener functionals and from q.m. closure of stochastic integrals. Thus, we have provided a second proof of theorem 4.1. ◻

Results of this section provide still another link between the stochastic integrals introduced in section 3 and multiple Ito-Wiener integrals. In [2] Ito proved the formula

$$\int_T \cdots \int_T \phi_1(z_1)\phi_1(z_2) \cdots \phi(z_{p_1}) \cdots \cdot \phi_n(z_{p_1+p_2+\cdots+p_{n-1}+1}) \cdots \phi_n(z_{p_1+p_2+\cdots p_n})$$

$$= \prod_{\nu=1}^{n} \frac{H_{p_\nu}\left(\frac{1}{\sqrt{2}} \int_T \psi_\nu(z)W(dz)\right)}{(\sqrt{2})^{p_\nu}}$$

where the left hand side is a multiple Ito-Wiener integral, $\{\phi_1,\cdots,\phi_n\}$ is an orthonomial system and $H_p$ are Hermite polynomials. It follows that if we denote $M_{iz} = \int_{\zeta<z} \phi_i(\zeta)W(d\zeta)$ then there exists a polynomial in u f(u,z) satisfying (5.4) such that

$$(5.17)\quad \int_T \cdots \int_T \phi_1(z_1) \cdots \phi_n(z_{p_1+p_2+\cdots+p_n})W(dz_1) \cdots W(dz_{p_1+p_2+\cdots+p_n})$$

$$= \int_T \sum_i f^i(M_{\zeta,\zeta})\phi_i(\zeta)W(d\zeta)$$

$$+ \frac{1}{2}\left[\int_{T\times T}\right] \sum_{i,j} f^{ij}(M_{\zeta\vee\zeta'},\zeta\vee\zeta')\phi_i(\zeta)\phi_j(\zeta')W(d\zeta)W(d\zeta')$$

$$+ \text{constant}.$$

## Appendix

In this appendix we state and prove a lemma which is referred to in sections 3 and 4.

Let $\Delta_\nu = [a_1^\nu, b_1^\nu) \times [a_2^\nu, b_2^\nu)$, $\nu = 1, 2, \cdots, n$ be a set of n rectangles and let $g(t_1, \cdots, t_n) = g(\underset{\sim}{t})$ denote the indicator function of this set. Let T denote the rectangle $[0,1] \times [0,1]$, and let F denote the set of points in $T^n$ such that $t_n \succ t_i$, $i = 1, 2, \cdots, n-1$. Let G be the set of points in $T^n$ such that $t_{n-1}$, $t_n$ are unordered and $t_n \vee t_{n-1} \succ t_i$, $i = 1, 2, \cdots, n-2$. Let $h_F(t_1, \cdots, t_n) = h_F(\underset{\sim}{t})$ and $h_G(\underset{\sim}{t})$ denote the characteristic functions of F and G respectively. Let m be an integer and consider the lattice on $T^n$ defined by the lattice points $(im^{-1}, jm^{-1})$, $i, j = 1, 2, \cdots, m$. For each $t = (x,y) \in T$ we define $[t]^m$ as the lattice point nearest to t from below and to the left of t, $[\underset{\sim}{t}]^m = ([t_1]^m, \cdots, [t_n]^m)$. $d\underset{\sim}{t}$ will denote $dt_1 dt_2 \cdots dt_n$ $(= dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n)$.

**Lemma**

$$\int_T \cdots \int_T (g(\underset{\sim}{t}) H_F(\underset{\sim}{t}) - g([\underset{\sim}{t}]^m) h_F([\underset{\sim}{t}]^m))^2 \, d\underset{\sim}{t} \to 0$$

as $n \to 0$,

$$\int_T \cdots \int_T (g(\underset{\sim}{t}) h_G(\underset{\sim}{t}) - g([\underset{\sim}{t}]^m) h_G(]\underset{\sim}{t}]^m))^2 \, d\underset{\sim}{t} \to 0$$

as $n \to 0$, where the integrals are n-fold integrals over $T^n$.

Proof: Since the proof for $h_F$ and $h_G$ is almost identical we will use h to denote both $h_F$ and $h_G$. Adding and subtracting $h(t)g([t])$:

$$\int_T \cdots \int_T (g(\underset{\sim}{t})h(\underset{\sim}{t}) - g([\underset{\sim}{t}]^m)h([\underset{\sim}{t}]^m))^2 d\underset{\sim}{t} \leq$$

$$\leq 2 \int \cdots \int h^2(\underset{\sim}{t})[g(\underset{\sim}{t}) - g([\underset{\sim}{t}]^m)]^2 d\underset{\sim}{t}$$

$$+ 2 \int \cdots \int g^2([\underset{\sim}{t}]^m)[h(\underset{\sim}{t}) - h([\underset{\sim}{t}]^m)]^2 d\underset{\sim}{t}$$

$$\leq 2 \int \cdots \int [g(\underset{\sim}{t}) - g([\underset{\sim}{t}]^m)]^2 d\underset{\sim}{t} + 2 \int \cdots \int [h(\underset{\sim}{t}) - h([\underset{\sim}{t}]^m)]^2 d\underset{\sim}{t}$$

The first integral obviously tends to zero as $m \to \infty$. The integrand in the second integral is either one or zero. Consider now the case $h = h_F$. In order that $|h_F(\underset{\sim}{t})-h_F([t])| = 1$ we must have for some i $t_n \nsucc t_i$ and $[t_n] \succ [t_i]$ ($t_n \succ t_j$ and $[t_n] \nsucc [t_j]$ is impossible). Therefore a necessary condition for $|h_F(\underset{\sim}{t})-h_F([t])| = 1$ is that for at least one $t_i$ $(i \neq n)$ should differ from $t_n$ in one of its coordinates by no more than $m^{-1}$, namely $|x_i - x_n| < m^{-1}$ or $|y_i - y_n| < m^{-1}$. We now overbound the lebesgue measure of the set in $A \times \cdots \times A$ for which $|y_n - y_i| < m^{-1}$ for some $i \neq n$ by the following argument. Assume that points are placed on $[0,1] \times [0,1]$ at random with a uniform probability distribution. A first random sample gives $x_n$, $y_n$, a second and independent shot gives $x_1, y_1$. The probability that $|x_1 - x_n| \leq m^{-1}$ is upper bounded by $2m^{-1}$. The probability that $|x_i - x_n| \leq m^{-1}$ or $|y_i - y_n| \leq m^{-1}$ for at least one $i \neq n$ is upper bounded by $4(n-1)m^{-1}$. Therefore, as $m \to \infty$, the second integral in the last inequality goes to zero. The case where $|h_G(\underset{\sim}{t})-h_G([\underset{\sim}{t}]^m)| = 1$ follows along very similar lines and we ommit the details.

Note: The results of the lemma hold for $z(\underset{\sim}{t})$ continuous on $T^n$.

## REFERENCES


1. E. Wong, "Martingales and Stochastic Integrals for Processes with a Two-Dimensional Parameter," Tech. Memo. ERL-M365, Electronics Research Laboratory, University of California, Berkeley, 1972.
2. K. Ito, "Multiple Wiener Integral," J. Math. Soc. Japan, 3 (1965), 157 - 169.
3. H. Kunita and S. Watanabe, "On Square Integrable Martingales," Nagoya Math. J. 30 (1967), 209 - 245.
4. R. H. Cameron and W. T. Martin, "The Orthogonal Development of Non-linear Functionals in a Series of Fourier Hermite Functions," Ann. of Math. 48 (1947), 385 - 392.