MARTINGALES AND STOCHASTIC INTEGRALS FOR PROCESSES WITH A TWO-DIMENSIONAL PARAMETER

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Eugene Wong

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Eugene Wong

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

1. Introduction

For stochastic processes with a multidimensional parameter, both theory and application suffer from an underdeveloped theory of Markov processes and the absence of a martingale theory. Markovian properties for processes with a multidimensional parameter were introduced by Lévy in connection with his multiparameter Brownian motion, and have been studied to a limited extent. For processes parameterized by points on a lattice, Hammersley [7] has introduced the concept of "harness" as a generalization to martingales. However, this concept does not appear to carry over well to the continuous-parameter use. In this paper we develope the concept of a martingale as a random function parameterized by subsets of R^n. In special cases this reduces to a random function parameterized by points in R^n together with a partial ordering on the parameter. For a specific class of martingales, the Gaussian white noise, we shall define stochastic integrals, generalizing the Ito integral.

In view of the role that Brownian motion has played in the theory of Markov processes and martingales with a one-dimensional parameter, a

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reasonable first step would be to generalize the Brownian motion to multidimensional spaces. There are least two different generalizations of the Brownian motion that might be considered to be natural, each emphasizing a different aspect of the Brownian motion in one dimension. Lévy [10] defined a Brownian motion with parameter space R^n as a Gaussian process $\{B_z, z \in R^n\}$ with $E\,B_z = 0$ and

$$(1.1) \qquad E\,B_z B_{z_0} = \frac{1}{2}(|z| + |z_0| - |z-z_0|)$$

where $|z|$ denotes the Euclidean norm of z. Lévy conjectured [11] and McKean has proved [12] that for odd dimensional parameters the Brownian motion so defined had a Markovian character. The covaniance function in (1.1) is a special case of a general class of positive definite kernels on homogeneous spaces that Gangolli has studied [5,6]. Results thus far indicate that it would be interesting to study Brownian motion and other Markovian processes on certain classes of homogeneous spaces with the aid of harmonic analysis. Details of such a program do not appear to have been carried out, although some preliminary results in the direction have appeared [14].

A second natural way of generalizing the Brownian motion is to consider it as an integral of Gaussian white noise. Let $\overline{\mathcal{R}}^n$ denote the collection of all Borel sets in R^n having finite Lebesgue measure. Let $\{X_A, A \in \overline{\mathcal{R}}^n\}$ be a real Gaussian additive random set functions with

$$(1.2) \qquad EX_A = 0$$

$$E\,X_A X_B = \mathcal{L}(A \cap B)$$

where $\mathcal{L}$ denotes the Lebesgue measure. Intuitively, X_A can be thought of as the integral over A of a Gaussian white noise. We note that for n=1 $X_{[o,t]}$ is just the ordinary Brownian motion. In the multidimensional case the process

$$W_{(z_1,z_2,\ \ldots,\ z_n)} = X_{[o,z_1]\times[o,z_2]\times\ldots\times[o,z_n]} \tag{1.3}$$

is a sample-continuous process, and the probability measure that it induces on $C([0,1]^n)$ generalizes the Wiener measure. The process defined by (1.3), which we shall call Wiener process, has been studied by a number of authors [8,13,15]. In particular, results of the Cameron-Martin type on absolutely continuous affine transformations of the Wiener measure have been obtained [13].

Our interest is to develop a stochastic calculus of the Ito type for multi-parameter processes. The experience with stochastic integrals in one dimension makes it clear that the Ito calculus is a calculus of continuous-parameter martingales and local martingales [4,9]. Thus, a useful generalization of the stochastic integral must necessarily involve a generalization of the martingale property to multidimensional parameter spaces. From this point of view, it is natural to consider martingales as random functions parameterized by subsets of R^n rather than points in R^n. Set inclusion provides a partial ordering in terms of which the martingale property can be defined in a natural way. Martingales with a partially ordered parameter is not new [see e.g. 2]. However, they do not appear to have been studied with specific reference to multiparameter processes, nor has stochastic integral been defined.

2. Martingales

Let $\mathcal{S}$ be a directed set. That is, $\mathcal{S}$ is a nonempty set partially ordered by a binary relation $\prec$ satisfying the condition that for every pair x,y in $\mathcal{S}$ there is a $z \in \mathcal{S}$ such that $x \prec z$ and $y \prec z$. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A collection of σ-subalgebras $\{\mathcal{A}_s, s \in \mathcal{S}\}$ is said to be increasing if $s_1 \succ s_2 \Rightarrow \mathcal{A}_{s_1} \supseteq \mathcal{A}_{s_2}$. Given a family of random variables $\{X_s, s \in \mathcal{S}\}$ and an increasing collection $\{\mathcal{A}_s, s \in \mathcal{S}\}$, we shall say $\{X_s, \mathcal{A}_s, s \in \mathcal{S}\}$ is a martingale if $s \succ s_0$ implies

$$(2.1) \qquad E^{\mathcal{A}_{s_0}} X_s = X_{s_0} \quad , \quad \text{almost surely}$$

Let μ be a σ-finite Borel measure on R^n. Let $\overline{\mathcal{R}^n}$ denote the collection of all Borel sets of R^n which are μ-finite. Let $\{X_s, s \in \overline{\mathcal{R}^n}\}$ be a real Gaussian additive set functions with $EX_s = 0$ and

$$(2.2) \qquad EX_sX_{s'} = \mu(s \cap s')$$

If we take $\mathcal{S}$ to be any subcollection of $\overline{\mathcal{R}^n}$ which is a directed set with respect to set inclusion, and take $\mathcal{A}_s$ to be the σ-algebra generated by $\{X_{s'}, s' \subseteq s\}$, then $\{X_s, \mathcal{A}_s, s \in \mathcal{S}\}$ is a martingale. More generally, we can take $\{\mathcal{A}_s, s \in \mathcal{S}\}$ to be any increasing collection such that X_{s_0} is $\mathcal{A}_s$-measurable if $s_0 \subseteq s$, and $\mathcal{A}_s$-independent if s_0 and s are disjoint. It is customary to refer to $\{X_s, s \in \overline{\mathcal{R}^n}\}$ as a Gaussian white noise. Thus, we see that a Gaussian white noise has a natural interpretation as a martingale.

From (1.3) it is easy to see that the Wiener process W_z, $z \in R_+^n$, has a natural interpretation as a martingale with respect to the partial ordering defined by: $z \succ z' \Leftrightarrow z_i \geq z_i'$ for every i. Lévy's Brownian motion also has a natural interpretation as a martingale. The best way to see this is via the Chentsov construction [1]. Let R^n be given a polar coordinate system $(r,\theta) \in [0,\infty) \times S^{n-1}$, where S^{n-1} denotes the unit (n-1)-sphere. Let μ be a Borel measure on R^n defined by

$$\mu(A) = \int_A dr\, d\theta$$

where $d\theta$ denotes the uniform measure on S^{n-1}. Chentsov showed that Lévy's Brownian motion had a representation

$$B_z = \text{constant} \cdot X_{S_z}$$

where $\{X_A, A \in \overline{\mathcal{R}^n}\}$ is a Gaussian white noise corresponding to the μ-measure and S_z denotes the sphere in R^n having the origin and the point z as its two poles. It is clear that $\{B_z, z \in R^n\}$ is a martingale with respect to the partial ordering

$$\text{(2.3)} \qquad z \succ z' \Leftrightarrow S_z \supseteq S_{z'} \Leftrightarrow z = \alpha z' \quad (\alpha \geq 1)$$

It is interesting to observe in this connection that even in one dimension, a Brownian motion with a parameter space $(-\infty,\infty)$ is <u>not</u> a martingale with respect to the usual ordering of the real line, but only with respect to the partial ordering defined by (2.3).

3. Stochastic Integrals

We begin with the simplest extensions of stochastic integrals. Let A denote the unit square $[0,1]^2$ in the plane and $\mathcal{S}$ the collection of Borel subsets of A. Let μ be a finite measure on $(A,\mathcal{S})$, absolutely continuous with respect to the Lebesgue measure. Let $(\Omega,\mathcal{A},\mathcal{P})$ be a fixed probability space and $\{\mathcal{A}_s, s \in \mathcal{S}\}$ an increasing family of σ-subalgebras. Let $\{X_s, \mathcal{A}_s, s \in \mathcal{S}\}$ be a Gaussian white noise corresponding to μ-measure. That is, X_s, $s \in \mathcal{S}$, is a Gaussian family of random variables such that

(3.1) (a) X_s is $\mathcal{A}_{s'}$-measurable if $s' \supseteq s$

(b) X_s is $\mathcal{A}_{s'}$-independent if s and s' are disjoint

(c) $EX_s = 0$, $EX_sX_{s'} = \mu(s \cap s')$

Now, let $W_z = X_{[0,z_1]\times[0,z_2]}$ and $\mathcal{F}_z = \mathcal{A}_{[0,z_1]\times[0,z_2]}$. Then $\{W_z, \mathcal{F}_z, z \in A\}$ is a Wiener process and a martingale with respect to the partial ordering $z \prec z' \Leftrightarrow z_i \leq z_i'$, $i = 1, 2$. We shall investigate stochastic integrals of the form

$$I_1(\phi) = \int_A \phi_z \, W(dz_1,dz_2)$$

and

$$I_2(\psi) = \int_A \psi_z \, W(dz_1,z_2) \, W(z_1,dz_2)$$

for integrands ϕ and ψ satisfying the following conditions:

H_1: $\phi(\omega,z)$ and $\psi(\omega,z)$ are bimeasurable functions with respect to $\mathcal{A} \otimes \mathcal{S}$.

H_2: For each $z \in A$ ϕ_z and ψ_z are measurable with respect to $\mathcal{F}_z = \mathcal{A}_{[0,z_1]\times[0,z_2]}$.

H_3: $\int_A E\phi_z^2 \, \mu(dz) < \infty$

$$\int_A E\psi_z^2 \, \tilde{\mu}(dz) < \infty$$

In H_3 we have introduced the measure

$$\tilde{\mu}(dz_1,dz_2) = \mu(dz_1,[0,z_2])\, \mu([0,z_1],dz_2) \tag{3.2}$$

Definition of I_1 and I_2 follows a procedure similar to the one dimensional case. First, suppose that ϕ and ψ are simple, i.e., they are of the form

$$\phi_z = \phi_\nu,\ \psi_z = \psi_\nu,\ z \in \Delta_\nu\ ,\ \nu = 1, 2, \ldots, K, \tag{3.3}$$

$$\phi_z = \psi_z = 0, \qquad \text{elsewhere}$$

where $\Delta_\nu = [a_1^\nu,b_1^\nu) \times [a_2^\nu,b_2^\nu)$ are disjoint rectangles. For simple ϕ and ψ we set

$$I_1(\phi) = \sum_{\nu=1}^{K} \phi_\nu\, X_{\Delta_\nu} = \sum_{\nu=1}^{K} \phi_\nu\, \Delta_\nu W \tag{3.4}$$

$$I_2(\psi) = \sum_{\nu=1}^{K} \quad X_{[a_1^\nu, b_1^\nu)\times[0,a_2^\nu)} \; X_{[0,a_1^\nu)\times[a_2^\nu, b_2^\nu)}$$

$$= \sum_{\nu=1}^{K} \phi_\nu \left(W_{b_1^\nu, a_2^\nu} - W_{a_1^\nu, a_2^\nu}\right)\left(W_{a_1^\nu, b_2^\nu} - W_{a_1^\nu, a_2^\nu}\right)$$

$$= \sum_{\nu=1}^{K} \phi_\nu \; \Delta_1^\nu W \; \Delta_2^\nu W$$

where $\Delta_\nu W$, $\Delta_1^\nu W$ and $\Delta_2^\nu W$ are obvious simplifying notations.

<u>Lemma 3.1</u>. Let ϕ and ψ be <u>simple</u> processes satisfying the hypotheses $H_1 - H_3$. The integrals $I_1(\phi)$ and $I_2(\psi)$ defined by (3.4) satisfy the following conditions

P_1: I_i, $i = 1, 2$, are linear functions of the integrands.

P_2: $$EI_1^2(\phi) = \int_A E\phi_z^2 \; \mu(dz)$$

$$EI_2^2(\psi) = \int_A E\psi_z^2 \; \tilde{\mu}(dz)$$

$$EI_1(\phi) \; I_2(\psi) = 0$$

<u>Proof</u>. P_1 is obvious. For P_2, we write

$$EI_1^2(\phi) = E\left\{\sum_\nu \phi_\nu^2 (\Delta_\nu W)^2 + \sum_{\nu\neq\mu} \phi_\nu \; \phi_\mu \; \Delta_\nu W \; \Delta_\mu W\right\}$$

Now, ϕ_ν^2 is $\mathcal{F}_{a_\nu}$ -measurable while $E[(\Delta_\nu W)^2 | \mathcal{F}_{a_\nu}] = \mu(\Delta_\nu)$ so that

$$E \sum_{\nu} \phi_{\nu}^{2}(\Delta_{\nu}W)^{2} = E\left[\sum_{\nu} \phi_{\nu}{}^{2}\; E[(\Delta_{\nu}W)^{2} \mid \mathcal{F}_{a_{\nu}}]\right]$$

$$= E \sum_{\nu} \phi_{\nu}^{2}\; \mu(\Delta_{\nu})$$

$$= \int_{A} E\, \phi_{z}^{2}\; \mu(dz)$$

On the other hand, $\phi_{\nu}\, \phi_{\mu}$ is $\mathcal{F}_{a^{\nu} \vee a^{\mu}}$-measurable where $a \vee b$ $(\max(a_1, b_1), \max(a_2, b_2))$. Therefore,

$$E \sum_{\nu \neq \mu} \phi_{\nu}\, \phi_{\mu}\, \Delta_{\nu}W\, \Delta_{\nu}W = E \sum_{\nu \neq \mu} \phi_{\nu}\, \phi_{\mu}\, E(\Delta_{\nu}W\, \Delta_{\mu}W \mid \mathcal{F}_{a^{\nu} \vee a^{\mu}})$$

$$= 0$$

because the three rectangles Δ_{ν}, Δ_{μ} and $[0, \max(a_1^{\nu}, a_1^{\mu})) \times [0, \max(a_2^{\nu}, a_2^{\mu}))$ are disjoint. It follows that

$$E\; I_1^2(\phi) = \int_{A} E\, \phi_{z}^{2}\; \mu(dz)$$

The expectation $E\; I_2^2(\psi)$ is evaluated by a similar computation. We write

$$E\; I_2^2(\psi) = E\left\{\sum_{\nu} \phi_{\nu}^{2}\; E[(\Delta_1^{\nu}W)^2 (\Delta_2^{\nu}W)^2 \mid \mathcal{F}_{a^{\nu}}]\right.$$

$$\left. + \sum_{\nu \neq \mu} \phi_{\nu}\, \phi_{\mu}\; E[\Delta_1^{\nu}W\, \Delta_2^{\nu}W\, \Delta_1^{\mu}W\, \Delta_2^{\mu}W \mid \mathcal{F}_{a^{\nu} \vee a^{\mu}}]\right\}$$

For $\nu \neq \mu$, a^ν and a^μ differ in at least one coordinate (say $a_1^\nu > a_1^\mu$). Then, $\Delta_1^\nu W$ is independent of $\Delta_2^\nu W\ \Delta_1^\mu W\ \Delta_2^\mu W$ and $\mathcal{F}_{a^\nu \vee a^\mu}$ so that the second sum is equal to zero. Therefore,

$$E\, I_2^2(\phi) = E \sum_\nu \phi_\nu^2\, E[(\Delta_1^\nu W)^2 (\Delta_2^\nu W)^2 | \mathcal{F}_{a^\nu}]$$

$$= E \sum_\nu \phi_\nu^2\, E(\Delta_1^\nu W)^2\, E(\Delta_2^\nu W)^2$$

$$= \sum_\nu \phi_\nu^2\, \mu([a_1^\nu, b_1^\nu) \times [0, a_2^\nu)) \cdot \mu([0, a_1^\nu) \times [a_2^\nu, b_2^\nu))$$

$$= \int_A \phi_z^2\, \tilde{\mu}(dz)$$

Finally, the orthogonality of $I_1(\phi)$ and $I_2(\psi)$ is easily proved by noting that

$$E(\Delta_1^\nu W\ \Delta_2^\nu W\ \Delta_\mu W | \mathcal{F}_{a^\nu \vee a^\mu})$$

is always zero whether $\mu = \nu$ or not.

Lemma 3.2. Let $\mathcal{H}$ (resp. $\tilde{\mathcal{H}}$) denote the class of all processes (resp. ψ) satisfying $H_1 - H_3$. Let $\mathcal{H}_0$ ($\tilde{\mathcal{H}}_0$) denote the subclass of simple processes. Then $\mathcal{H}_0$ is dense in $\mathcal{H}$ with respect to the norm

$$\|\phi\|_1 = \sqrt{\int_A E\,\phi_z^2\,\mu(dz)}$$

and $\tilde{\mathcal{H}}_0$ is dense in $\tilde{\mathcal{H}}$ with respect to the norm

$$\|\psi\|_2 = \sqrt{\int_A E\,\psi_z^2\,\tilde{\mu}(dz)}$$

Proof. It is clear that we only need to prove the first case since μ is sufficiently general to include the case of $\tilde{\mu}$. It is also clear that the subset of bounded processes is dense in $\mathcal{H}$ so we only need to prove that every bounded ϕ in $\mathcal{H}$ can be approximated by elements of $\mathcal{H}_0$. For each positive integer k define a mapping $\alpha_k : R^2 \to R^2$ by

$$\alpha_k(z) = (\nu/2^k, \mu/2^k), \quad z \in [\frac{\nu}{2^k}, \frac{\nu+1}{2^k}) \times [\frac{\mu}{2^k}, \frac{\mu+1}{2^k})$$

$$\nu,\mu = 0, \pm 1, \pm 2, \ldots$$

Take a bounded ϕ in $\mathcal{H}$ and adopt the convention $\phi(\omega,z) \equiv 0$ for $z \notin A$. Then

$$\int_{R^2} |\phi(\omega,z+\zeta) - \phi(\omega,\alpha_k(z) + \zeta)|^2\,\mu(d\zeta) \xrightarrow[k\to\infty]{} 0$$

for every $z \in R^2$ and for almost all ω. It follows that

$$E \int_{R^2\times R^2} |\phi(\cdot,z+\zeta) - \phi(\cdot,\alpha_k(z) + \zeta)|^2\,\mu(dz)\,\mu(d\zeta) \xrightarrow[k\to\infty]{} 0$$

so that there is a subsequence

$$E \int_{R^2} |\phi(\cdot,z+\zeta) - \phi(\cdot,\alpha_{k_j}(z) + \zeta)|^2 \mu(dz)$$

converging to 0 for almost all ζ as $j \to \infty$. For each (k,ζ) set

$$\phi_{k,\zeta}(\omega,z) = \phi(\omega,\alpha_k(z-\zeta) + \zeta), \; z \in A$$

$$= 0 \qquad \text{elsewhere}$$

Since $\alpha_k(z-\zeta) + \zeta < z$, $\mathcal{F}_z \supseteq \mathcal{F}_{\alpha_k(z-\zeta)+\zeta}$ so that $\phi_{k,\zeta}(\cdot,z)$ is $\mathcal{F}_z$ - measurable for every (k,ζ). Since $\phi_{k,\zeta} \in \mathcal{H}_0$, every bounded $\phi \in \mathcal{H}$ can be approximated by a sequence in $\mathcal{H}_0$ and the proof is complete. ■

Now, it is clear how the stochastic integrals can be defined for integrands in $\mathcal{H}$ and $\tilde{\mathcal{H}}$. For $\phi \in \mathcal{H}$ lemma 3.2 implies the existence of a sequence $\{\phi_n\}$ in $\mathcal{H}_0$ such that

$$\int_A E(\phi_z - \phi_{n,z})^2 \mu(dz) \xrightarrow[n\to\infty]{} 0$$

which implies

$$\int_A E(\phi_{m,z} - \phi_{n,z})^2 \mu(dz) \xrightarrow[m,n\to\infty]{} 0$$

which in turn implies (lemma 3.1)

$$E[I_1(\phi_m) - I_1(\phi_n)]^2 \xrightarrow[m,n\to\infty]{} 0$$

so that $\{I_1(\phi_n)\}$ is a quadratic-mean convergent sequence. We define

(3.5) $$I_1(\phi) = \lim_{n \to \infty} \text{ in q.m. } I_1(\phi_n)$$

Similarly, for $\psi \in \tilde{\mathcal{H}}$ we take a sequence $\{\psi_n\}$ in $\tilde{\mathcal{H}}_0$ such that $\|\psi - \psi_n\|_2 \xrightarrow[n\to\infty]{} 0$ and define

(3.6) $$I_2(\psi) = \lim_{n \to \infty} \text{ in q.m. } I_2(\psi_n)$$

Theorem 3.1. Let the stochastic integrals

$$I_1(\phi) = \int_A \phi_z \, W(dz)$$

$$I_2(\psi) = \int_A \psi_z \, W(z_1,dz_2) \, W(dz_1,z_2)$$

be defined by (3.4) for $\phi \in \mathcal{H}_0$, $\psi \in \tilde{\mathcal{H}}_0$ and by (3.5) and (3.6) for $\phi \in \mathcal{H}$, $\psi \in \tilde{\mathcal{H}}$. Then, the following properties are satisfied.

(3.7) $$I_i(a\phi + b\psi) = a\, I_i(\phi) + b\, I_i(\psi)$$ (linearity)

(3.8) $$E\, I_i(\phi)\, I_j(\psi) = \delta_{ij} \int_A E\, \phi_\zeta\, \psi_\zeta\, \mu_i(d\zeta)$$ (inner product)

$$\mu_1(d\zeta) = \mu(d\zeta), \qquad \mu_2(d\zeta) = \tilde{\mu}(d\zeta)$$

(3.9) $$E[I_i(\phi) \,|\, \mathcal{F}_z] = \int_{\zeta \prec z} \phi_\zeta \, M_i(d\zeta)$$ (martingale)

$$M_1(d\zeta) = W(d\zeta), \qquad M_2(d\zeta) = W(d\zeta_1,\zeta_2)\ W(\zeta_1,d\zeta_2)$$

Proof. Linearity is trivial. (3.8) follows from Lemma 3.1 and the application of the Schwarz inequality. Hence, if $\{\phi_n\}$ and $\{\psi_n\}$ are approximating sequences for ϕ and ψ then

$$E\ I_i(\phi)\ I_j(\psi) = \lim_{n\to\infty} E\ I_i(\phi_n)\ I_j(\psi_n)$$

$$= \lim_{n\to\infty} \delta_{ij} \int_A E(\phi_{n,\zeta}\ \psi_{n,\zeta})\ \mu_i(d\zeta)$$

and (3.8) follows. To prove the martingale property, first suppose that ϕ is simple, and number the rectangles so that $\Delta_1, \Delta_2, \ldots, \Delta_m$ are in $[0,z_1) \times [0,z_2)$ while $\Delta_{m+1}, \ldots, \Delta_K$ are outside of it. Now,

$$I_i(\phi) = \sum_{\nu=1}^{m} \phi_\nu\ M_i(\Delta_\nu) + \sum_{\nu=m+1}^{K} \phi_\nu\ M_i(\Lambda_\nu)$$

The first term is $\mathcal{F}_z$-measurable while

$$E\left[\sum_{\nu=m+1}^{K} \phi_\nu\ M_i(\Delta_\nu)\middle|\mathcal{F}_z\right]$$

$$= E\left\{\sum_{m+1}^{K} E[M_i(\Delta_\nu)\,|\,\mathcal{F}_{a}^{\nu}]\middle|\mathcal{F}_z\right\}$$

$$= 0$$

Hence, the martingale property is true for a simple ϕ. For a general ϕ, write

$$E[I_i(\phi) \mid \mathcal{F}_z] = E[I_i(\phi_n) \mid \mathcal{F}_z] + E[I_i(\phi-\phi_n) \mid \mathcal{F}_z]$$

$$= \int_{\zeta \prec z} \phi_{n,\zeta} \, M_i(d\zeta) + E[I_i(\phi-\phi_n) \mid \mathcal{F}_z]$$

$$\xrightarrow[n \to \infty]{q.m.} \int_{\zeta \prec z} \phi_\zeta \, M_i(d\zeta)$$

and the proof is complete. ⌑

<u>Remarks</u>. (1) It is useful to interpret $I_1(\phi)$ and $I_2(\psi)$ as

$$I_1(\phi) = \int_A \phi_\zeta \frac{\partial^2}{\partial\zeta_1 \, \partial\zeta_2} W(\zeta_1,\zeta_2) \, d\zeta_1 \, d\zeta_2$$

$$I_2(\phi) = \int_A \phi_\zeta \frac{\partial W({}_1, {}_2)}{\partial\zeta_1} \frac{\partial W(\zeta_1,\zeta_2)}{\partial\zeta_2} \, d\zeta_1 \, d\zeta_2$$

(2) The necessity of introducing I_2 is clear if one wants to develop a stochastic differentiation rule. Even if W were differentiable (which it is not) we would have

$$\frac{\partial^2}{\partial z_1 \, \partial z_2} f(W_z) = f'(W_z) \frac{\partial^2 W_z}{\partial z_1 \, \partial z_2} + f''(W_z) \frac{\partial W_z}{\partial z_1} \frac{\partial W_z}{\partial z_2}$$

in which both $\frac{\partial^2 W}{\partial z_1 \, \partial z_2}$ and $\frac{\partial W}{\partial z_1} \frac{\partial W}{\partial z_2}$ appear.

(3) As the dimension of the parameter space increases, the number of

types of stochastic integrals that need to be introduced increases rather quickly. Thus, the stochastic calculus associated with multi parameter martingales becomes increasingly complicated as the dimension of the parameter space increases.

4. An Elementary Differentiation Formula

The Ito differentiation formula together with its generalizations form the cornerstone of the calculus of martingales with a one-dimensional parameter. Unfortunately, even in the two-dimensional case the corresponding formula is already much more complicated. In this section we shall develop a restricted version of such a differentiation formula. First, we need to generalize somewhat our definition of stochastic integrals.

Let ϕ and ψ be processes satisfying hypotheses H_1 and H_2 of the last section and instead of H_3 the following condition:

$$(H_3') \qquad \int_A \phi_z^2 \, \mu(dz) < \infty \qquad \text{almost surely}$$

$$\int_A \psi_z^2 \, \tilde{\mu}(dz) < \infty \qquad \text{almost surely}$$

Now define

$$z_{1n}(\omega) = \min\left\{a: \int_{[0,a]\times[0,1]} \phi_z^2(\omega)\mu(dz) \geq n\right\}$$

$$z_{2n}(\omega) = \min\left\{b: \int_{[0,1]\times[0,b]} \phi_z^2(\omega)\mu(dz) \geq n\right\}$$

and denote $z_n(\omega) = (z_{1n}(\omega),\ z_{2n}(\omega))$. If $\int_{[0,1]^2} \phi_j^2(\omega)\mu(dz) < n$ we set $z_n(\omega) = (1,1)$. If we define

$$\phi_z^{(n)}(\omega) = \phi_z(\omega), \qquad 0 \prec z \prec z_n(\omega)$$

then (H_3') implies $\phi_z^{(n)} \xrightarrow[n \to \infty]{\text{a.s.}} \phi_z$. Since for each n $\phi^{(n)}$ satisfies (H_3), $\int_A \phi_z^{(n)}\ W(dz)$ is well defined and

$$\mathcal{P}\left(\left| \int_A (\phi_z^{(n)} - \phi_z^{(m)})\ W(dz)\right| > 0\right)$$

$$\leq \mathcal{P}\left(\int_A \phi_z^2\ \mu(dz) \geq \min(m,n)\right)$$

$$\xrightarrow[m,n \to \infty]{} 0$$

Hence, $\int_A \phi_z^{(n)}\ W(dz)$ converges in probability as $n \to \infty$ and we can define

$$\int_A \phi_z\ W(dz) = \lim_{n \to \infty} \text{in prob.} \int_A \phi_z^{(n)}\ W(dz).$$

The integral $\int_A \psi_z\ W(dz)$ is defined in an analogous way. It is easy to see that $\phi_z^{(n)} \xrightarrow[n \to \infty]{\text{a.s.}} \phi_z$ for all $z \in A$ implies $\int_A \phi_z^{(n)}\ W(dz) \xrightarrow[n \to \infty]{\text{in prob.}}$ $\int_A \phi_z\ W(dz)$.

Theorem 4.1. Let $f(x,z)$, $x \in R$, $z \in [0,1]^2$, have continuous partial derivitives of the following order:

$$f'(x,z) = \frac{\partial f(x,z)}{\partial x}, \quad f_1(x,z) = \frac{\partial f(x,z)}{\partial z_1}, \quad f_2(x,z) = \frac{\partial f(x,z)}{\partial z_2}$$

$$f''(x,z) = \frac{\partial^2 f}{\partial x^2}, \quad f_1'(x,z) = \frac{\partial^2 f}{\partial x\, \partial z_1}, \quad f_2'(x,z) = \frac{\partial^2 f}{\partial x\, \partial z_2}$$

$$f_{12}(x,z) = \frac{\partial^2 f}{\partial z_1\, \partial z_2}$$

$$f'''(x,z) = \frac{\partial^3 f}{\partial x^3}, \quad f_1''(x,z) = \frac{\partial^3 f}{\partial x^2\, \partial z_1}, \quad f_2'' = \frac{\partial^3 f}{\partial x^2\, \partial z_2}$$

$$f''''(x,z) = \frac{\partial^4 f}{\partial x^4}$$

Then, for $(0,0) \prec a \prec z \prec (1,1)$, we have

$$(4.1) \qquad f(W_z,z) - f(W_{a_1,z_2};\, a_1,z_2) - f(W_{z_1,a_2} :\, z_1,a_2) + f(W_a,a)$$

$$= \int_{a<\zeta<z} [f'(W_\zeta,\zeta)\, W(d\zeta) + f''(W_\zeta,\zeta)\, W(d\zeta_1,\zeta_2)\, W(\zeta_1,d\zeta_2)]$$

$$+ \int_{a<\zeta<z} [f_2'(W_\zeta,\zeta)d\zeta_2 = W(d\zeta_1,\zeta_2) + \tfrac{1}{2} f'''(W_\zeta,\zeta)\, \mu(\zeta_1,$$

$$d\zeta_2)\, W(d\zeta_1,\zeta_2)]$$

$$+\int_{a<\zeta<z} [f_1'(W_\zeta,\zeta)d\zeta_1\ W(\zeta_1,d\zeta_2) + \frac{1}{2} f'''(W_\zeta,\zeta)\ \mu(d\zeta_1,\zeta_2)\ W(\zeta_1,d\zeta_2)]$$

$$+\int_{a<\zeta<z} [f_{12}(W_\zeta,\zeta)d\zeta_1\ d\zeta_2 + \frac{1}{2} f''(W_\zeta,\zeta)\ \mu(d\zeta) + \frac{1}{4} f''''(W_\zeta,\zeta)\ \tilde{u}(d\zeta)$$

$$+\frac{1}{2} f_1''(W_\zeta,\zeta)\ d\zeta_1\ \mu(\zeta_1,d\zeta_2) + \frac{1}{2} f_2''(W_\zeta,\zeta)d\zeta_2\ \mu(d\zeta_1,\zeta_2)]$$

Remark. The first term on the right hand side of (4.1) involves stochastic integrals of the two types that we have defined. The last term involves only ordinary integrals. However, the terms in between involve integrals of a mixed type, stochastic integral in one dimension and ordinary integral in the other. We have assumed that μ is absolutely continuous with respect to the Lebesgue measure (say $\frac{d\mu}{dz} = g$), hence the second term can be interpreted as

$$\int_{a_2}^{z_2} \left\{ \int_{a_1}^{z_1} [f_2'(W_\zeta,\zeta) + \frac{1}{2} f'''(W_\zeta,\zeta)\ g(\zeta_1,\zeta_2)]\ W(d\zeta_1,\zeta_2) \right\} d\zeta_2$$

where the inner integral is a stochastic integral of one-dimensional parameter. A similar interpretation can be given for the third term in (4).

Proof. It is clear that we only need to prove the case where the partial derivatives are not only continuous but also bounded. The rest follows by approximating f by functions with bounded continuous partials. For notational simplicity we shall only prove the case where f is a function only of W_z and not of z. The more general case imposes no additional difficulties.

Let the rectangle $[a_1,z_1] \times [a_2,z_2]$ be partitioned by a sequence of square subdivisions.

$$\Delta_\nu^{(n)} = \left[a_\nu^{(n)}, a_{\nu+1}^{(n)}\right) \times \left[b_\nu^{(n)}, b_{\nu+1}^{(n)}\right)$$

such that $a_{\nu+1}^{(n)} - a_\nu^{(n)} = b_{\nu+1}^{(n)} - b_\nu^{(n)}$ and $\lim_{n\to\infty} \max_\nu \left(a_{\nu+1}^{(n)} - a_\nu^{(n)}\right) = 0$. Let $\Delta_{\nu,n}$, $\delta_{\nu,n}^{(1)}$, $\delta_{\nu,n}^{(2)}$ and $W_{\nu,n}$ denote the following quantities:

$$\Delta_{\nu,n} = W_{a_{\nu+1}^{(n)},b_{\nu+1}^{(n)}} - W_{a_{\nu+1}^{(n)},b_\nu^{(n)}} - W_{a_\nu^{(n)},b_{\nu+1}^{(n)}} + W_{a_\nu^{(n)},b_\nu^{(n)}}$$

$$\delta_{\nu,n}^{(1)} = W_{a_{\nu+1}^{(n)},b_\nu^{(n)}} - W_{a_\nu^{(n)},b_\nu^{(n)}}$$

$$\delta_{\nu,n}^{(2)} = W_{a_\nu^{(n)},b_{\nu+1}^{(n)}} - W_{a_\nu^{(n)},b_\nu^{(n)}}$$

$$W_{\nu,n} = W_{a_\nu^{(n)},b_\nu^{(n)}}$$

We can write

$$f(W_z) - f(W_{a_1,z_2}) - f(W_{z_1,a_2}) + f(W_a)$$

$$= \sum_\nu \left[f\left(W_{a_{\nu+1}^{(n)},b_{\nu+1}^{(n)}}\right) - f\left(W_{a_{\nu+1}^{(n)},b_\nu^{(n)}}\right) - f\left(W_{a_\nu^{(n)},b_\nu^{(n)}}\right) + f\left(W_{a_\nu^{(n)},b_\nu^{(n)}}\right)\right]$$

$$= \sum_{\nu} \Bigg[f\left(\Delta_{\nu,n} + \delta_{\nu,n}^{(1)} + \delta_{\nu,n}^{(2)} + W_{\nu,n}\right) - f\left(\delta_{\nu,n}^{(1)} + W_{\nu,n}\right)$$

$$- f\left(\delta_{\nu,n}^{(2)} + W_{\nu,n}\right) + f(W_{\nu,n}) \Bigg]$$

$$= \sum_{\nu} \Bigg\{ f'(W_{\nu,n})\ \Delta_{\nu,n} + \frac{1}{2} f''(W_{\nu,n}) \left[\Delta_{\nu,n}^{2} + 2\delta_{\nu,n}^{(1)}\ \delta_{\nu,n}^{(2)}\right]$$

$$+ \frac{1}{2} f'''(W_{\nu,n}) \left[\delta_{\nu,n}^{(1)} \left(\delta_{\nu,n}^{(2)}\right)^2 + \left(\delta_{\nu,n}^{(1)}\right)^2 \delta_{\nu,n}^{(2)}\right]$$

$$+ \frac{1}{4} f''''(W_{\nu,n}) \left[\left(\delta_{\nu,n}^{(1)}\right)^2 \left(\delta_{\nu,n}^{(2)}\right)^2\right] \Bigg\}$$

$$+ \sum_{\nu} \Bigg\{ f''(W_{\nu,n})\left[\delta_{\nu,n}^{(1)} + \delta_{\nu,n}^{(2)}\right]\Delta_{\nu,n} + \frac{1}{3!} f'''(W_{\nu,n})$$

$$\left[(\Delta_{\nu,n})^3 + 3(\Delta_{\nu,n})^2 \left(\delta_{\nu,n}^{(1)} + \delta_{\nu,n}^{(2)}\right) + 3\Delta_{\nu,n}\left(\delta_{\nu,n}^{(1)} + \delta_{\nu,n}^{(2)}\right) \right]$$

$$+ \frac{1}{4!} f''''(\theta_{\nu,n}) \Bigg[(\Delta_{\nu,n})^4 + 4(\Delta_{\nu,n})^3 \left(\delta_{\nu,n}^{(1)} + \delta_{\nu,n}^{(2)} \right)$$

$$+ \binom{4}{2} (\Delta_{\nu,n})^2 \left(\delta_{\nu,n}^{(1)} + \delta_{\nu,n}^{(2)}\right)^2 + 4\ \Delta_{\nu,n} \left(\delta_{\nu,n}^{(1)} + \delta_{\nu,n}^{(2)}\right)^3\Bigg]$$

$$+ \frac{1}{4!} \left[f''''(\theta_{\nu,n}) - f''''(W_{\nu,n})\right] \binom{4}{2} \left(\delta_{\nu,n}^{(1)}\right)^2 \left(\delta_{\nu,n}^{(2)}\right)^2$$

$$+ \frac{1}{4!} f''''(\theta_{\nu,n}) \left[4 \left(\delta_{\nu,n}^{(1)}\right)^3 \left(\delta_{\nu,n}^{(2)}\right) + 4 \left(\delta_{\nu,n}^{(1)}\right)\left(\delta_{\nu,n}^{(2)}\right)^3\right]$$

$$+ \frac{1}{4!} \left[f''''(\theta_{\nu,n}) - f''''(\alpha_{\nu,n})\right] \left(\delta_{\nu,n}^{(1)}\right)^4$$

$$+ \frac{1}{4!}\left[f''''(\theta_{\nu,n}) - f''''(\beta_{\nu,n})\right]|\left(\delta_{\nu,n}^{(2)}\right)^4\Bigg\}$$

where $\theta_{\nu,n}$, $\alpha_{\nu,n}$, $\beta_{\nu,n}$ are W_z evaluated at some z in $\Delta^{(n)}$. If f has bounded continuous derivatives, the first sum

$$\sum_{\nu} \Bigg\{f'(W_{\nu,n})\Delta_{\nu,n} + \frac{1}{2} f''(W_{\nu,n})\left(\Lambda_{\nu,n}^2 + 2\,\delta_{\nu,n}^{(1)}\,\delta_{\nu,n}^{(2)}\right)$$

$$+ \frac{1}{2} f'''(W_{\nu,n}) \left[\delta_{\nu,n}^{(1)}\left(\delta_{\nu,n}^{(2)}\right)^2 + \left(\delta_{\nu,n}^{(1)}\right)^2\left(\delta_{\nu,n}^{(2)}\right)\right]$$

$$+ \frac{1}{4} f''''(W_{\nu,n})\left(\delta_{\nu,n}^{(1)}\right)^2\left(\delta_{\nu,n}^{(2)}\right)^2\Bigg\}$$

converges in quadratic mean to

$$\int_{0<\zeta<z} \Bigg\{f'(W_\zeta)\, W(d\zeta) + \frac{1}{2} f''(W_\zeta)\, [\mu(d\zeta) + 2\, W(d\zeta_1,\zeta_2)W(\zeta_1,d\zeta_2)$$

$$+ \frac{1}{2} f'''(W_\zeta)\, [W(d\zeta_1,\zeta_2)\,\mu(\zeta_1,d\zeta_2) + W(\zeta_1,d\zeta_2)\,\mu(d\zeta_1,\zeta_2)]$$

$$+ \frac{1}{4} f''''(W_\zeta)\,\mu(d\zeta_1,\zeta_2)\,\mu(\zeta_1,d\zeta_2)\Bigg\}$$

On the other hand the second sum converges in quadratic mean to zero. For example,

$$E\left[\sum_{\nu}\left[f''''(\theta_{\nu,n}) - f''''(\alpha_{\nu,n})\right]\left(\delta_{\nu,n}^{(1)}\right)^4\right]^2$$

$$\leq E\left\{\sup_{\nu}\ \sup_{\alpha,\beta\in\Delta_\nu^{(n)}} |f''''(W_\alpha) - f''''(W_\beta)|^2\left(\sum_{\nu}\left(\delta_{\nu,n}^{(1)}\right)^4\right)^2\right\}$$

$$\leq \left\{ E\left(\sum_{\nu}\left(\delta_{\nu,n}^{(1)}\right)^4\right)^4 E \sup_{\nu} \sup_{\alpha,\beta\in\Delta_{\nu}^{(n)}} |f''''(W_\alpha) - f''''(W_\beta)|^4 \right\}^{1/2}$$

It is easy to verify that $E[\sum_{\nu} (\delta_{\nu,n}^{(1)})^4]^4$ is bounded, and

$E \sup_{\nu,\alpha,\beta} |f''''(W_\alpha) - f''''(W_\beta)|^4 \xrightarrow[n\to\infty]{} 0$ by the bounded continuity of f''''. ■

Example 1. Let μ be the Lebesgue measure. Then

$$W_z^2 - z_1 z_2 = \int_{0<\zeta<z} z\, W_\zeta\, W(d\zeta) + 2 \int_{0<\zeta<z} W(d\zeta_1,\zeta_2)\, W(\zeta_1,d\zeta_2)$$

which yields an interesting relationship between the two types of stochastic integrals.

Example 2. Let $\mu(dz) = g(z_1,z_2)\, dz_1\, dz_2$, and take

$$F_z = e^{W_z - \frac{1}{2}\int_0^{z_1}\int_0^{z_2} g(\zeta_1,\zeta_2)d\zeta_1\, d\zeta_2}$$

Then,

$$F_z - 1 = \int_{0<\zeta<z} [F_\zeta\, W(d\zeta) + F_\zeta\, W(d\zeta_1,\zeta_2)\, W(\zeta_1,d\zeta_2)]$$

so that F_z is a positive martingale with $EF_z = 1$.

Example 3. Let X_z be a Wiener process corresponding to the Lebesgue measure and $W_z = \int_{0 \ \zeta \ z} g(\zeta)\ X(d\zeta)$. Then, W_z is a Wiener process with $\mu(dz) = g(z)dz$. Therefore, if we take

$$(4.2) \qquad F_z = e^{\int_{0\prec\zeta\prec z} g(\zeta)\ X(d\zeta) - \frac{1}{2}\int_{0\prec\zeta\prec z} g(\zeta)d\zeta}$$

then F_z is a positive martingale with $EF_z = 1$. If we introduce a new probability measure $\mathcal{P}'$ by

$$\frac{d\mathcal{P}'}{d\mathcal{P}} = F_{1,1}$$

then it is not hard to show that under $\mathcal{P}'$, $X_z - \int_{0\prec\zeta\prec z} g(\zeta)d\zeta$ is a Wiener process corresponding to the Lebesgue measure. This is obviously a generalization of the Cameron-Martin formula for translations of the Wiener measure. (c.f. [16])

5. Conclusion

The results of this paper are preliminary in several respects. First, there is a need for a general differentiation formula for $f(M_z,z)$ where M is a martingale of the form

$$(5.1) \qquad M_z = \int_{0 \ \zeta \ z} [\phi_\zeta\ W(d\zeta) + \psi_\zeta\ W(\zeta_1,d\zeta_2)\ W(d\zeta_1,\zeta_2)]$$

Second, there are reasons to believe that every martingale with respect to $\mathcal{F}_z = \mathcal{F}(W_\zeta, 0 < \zeta < z)$ can be represented in the form of (5.1). Such a representation theorem plays an important role in one dimension [3,9]. Finally, the exponential formula (4.2) represents only a very restricted class of absolutely continuous transformations of the Wiener measure. Complete characterization of absolutely continuous transformations of the Wiener measure would be an important achievement of the martingale theory.

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