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ZEROS AND POLES OF MATRIX TRANSFER FUNCTIONS AND THEIR DYNAMICAL INTERPRETATION

by

C. A. Desoer and J. D. Schulman

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Zeros and Poles of Matrix Transfer Functions and their Dynamical Interpretation

by C.A. Desoer and J.D. Schulman

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

The given rational matrix transfer function $H(\cdot)$ is viewed as a network function of a multiport. The $n_0 \times n_i$ matrix $H(s)$ is factored into $D(s)^{-1}N(s) = \tilde{N}(s)\tilde{D}(s)^{-1}$, where $D(\cdot)$, $N(\cdot)$, $\tilde{N}(\cdot)$, $\tilde{D}(\cdot)$ are polynomial matrices of appropriate size, with $D(\cdot)$ and $N(\cdot)$ left coprime and $\tilde{N}(\cdot)$ and $\tilde{D}(\cdot)$ right coprime. For $n_0 \geq n_i$, $(n_0 \leq n_i)$, a zero of $H(\cdot)$ is a point z where the local rank of $\tilde{N}(\cdot)$, $(N(\cdot)$, resp.), drops below the normal rank. The theorems make precise the intuitive concept that a multiport blocks the transmission of signals proportional to e^{zt} if and only if z is a zero of $H(\cdot)$. Classical analysis defines the concept of a pole. We show that p is a pole of $H(\cdot)$ if and only if some "singular" input creates a zero-state response of the form re^{pt}, for $t > 0$. Although these results have state-space interpretation, they are derived by purely algebraic techniques, independently of state-space techniques. Consequently with appropriate modifications, these results apply to the sampled-data case.

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Introduction

There is no widely accepted definition of a zero of a matrix transfer function. We propose a definition based on the factorization of a rational matrix into a product of a polynomial matrix and the inverse of another polynomial matrix [1-6]. Such a factorization has been successfully used in realization problems [4,5,6], in design problems [4-7], in the study of the cancellation problem in feedback systems [8], and in deriving the necessary and sufficient conditions for the input-output stability of an open-loop unstable <u>distributed</u> multivariable feedback system [9].

A number of authors have addressed themselves to the problem of calculating the "zeros of large systems" or the "zeros of multivariable systems" [15,16,17]. These authors define such zeros to be zeros of the scalar rational functions which are elements of the matrix transfer function. Except for the special case of a diagonal matrix transfer function, such zeros have nothing to do with the concept of zeros introduced here.

Section I reviews the properties of the zeros of a scalar transfer function. Sections II and III develop appropriate definitions of the zeros of rational $n \times n$ matrices and Section IV considers rectangular matrices. It is shown that our definition of zeros retains essentially the main dynamical properties of zeros for the scalar case. Section V characterizes the poles in terms of their dynamical properties.

<u>Notation</u>: $\mathbb{R}$, ($\mathbb{C}$), denotes the field of real, (complex, resp.), numbers. $\mathbb{R}[s]$, ($\mathbb{R}[s]^{p\times q}$), denotes the ring of polynomials, ($p\times q$ polynomial matrices, resp.), in the complex variable s with real coefficients. $\mathbb{R}(s)$, ($\mathbb{R}(s)^{p\times q}$),

denotes the field of rational functions ($p\times q$ matrices of rational functions, resp.), in the complex variable s with real coefficients. $\mathbb{C}(s)$ and $\mathbb{C}[s]$ are similarly defined. For an exposition on the terminology and basic algebraic facts see [1,3,6,10,11]. Whenever we consider both a time function and its Laplace transform, we use ^ to distinguish the Laplace transform; e.g., $u(t)$ and $\hat{u}(s)$. θ_m denotes the zero vector in $\mathbb{C}^m$, and I_n denotes the $n\times n$ identity matrix. The superscript ' denotes the transpose.

Section I. Zeros of a Scalar Transfer Function

The properties of zeros of a scalar transfer function are well known [12,13].

Theorem I: Given a rational function, $h(s)$, $s \in \mathbb{C}$, with $h(s) = n(s)/d(s)$ where $n(s)$ and $d(s)$ are polynomials in s and are coprime, then,

(a) $z \in \mathbb{C}$ is a zero of $h(s) \Leftrightarrow z \in \mathbb{C}$ is a pole of $h(s)^{-1} = d(s)/n(s)$;

(b) if $z \in \mathbb{C}$ and $h(z) = 0$, (or equivalently $n(z) = 0$), then by choosing an appropriate initial state x_0, the complete response has the property $y(t,0,x_0,1(t)e^{zt}) = 0$, $\forall t \geq 0$;

(c) if $h(z) \neq 0$ and $h(\cdot)$ does not have a pole at z, then by choosing an appropriate initial state x_0, the complete response has the property $y(t,0,x_0,1(t)e^{zt}) = h(z)e^{zt}$, $\forall t \geq 0$.

Comment: It is obvious that one could have started with the system in the zero-state at $t = 0^-$ and then apply a suitable linear combination of $\delta(t)$, $\delta'(t)$, $\delta''(t)$, etc. at $t = 0$ to kick the system into the appropriate initial state at $t = 0^+$. By doing this the only difference in Theorem I would be that (b) and (c) would be valid only for all $t > 0$.

Section II. Zeros of $H(s) \in \mathbb{R}(s)^{n\times n}$

An important difference between the scalar case and the matrix case is that in the matrix case "zeros" can coincide with poles. Consider

$$H(s) = \begin{bmatrix} \frac{s+1}{s-1} & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}, \tag{1}$$

clearly $s = 1$ and $s = -1$ are poles of $H: \mathbb{C} \to \mathbb{C}^{2\times 2}$. However $s = -1$, $(s = 1)$, should, by any reasonable definition of "zeros", be called a zero of $H(s)$ since for that value of s there is no transmission (in the sense of Theorem I (b)) from the first, (second, resp.), input to the two outputs. Since the concept of a zero of a polynomial is unambiguous, we use it as a basis for our definition.

Definition I. (i) Given $H(s) = D(s)^{-1}N(s)$ where $N(s), D(s) \in \mathbb{R}[s]^{n\times n}$ and are left coprime, then $z \in \mathbb{C}$ is called a zero of $H(s)$ iff $\det N(z) = 0$.

(ii) Given $H(s) = \tilde{N}(s)\tilde{D}(s)^{-1}$ where $\tilde{N}(s), \tilde{D}(s) \in \mathbb{R}[s]^{n\times n}$ and are right coprime, then $z \in \mathbb{C}$ is called a zero of $H(s)$ iff $\det \tilde{N}(z) = 0$.

Note that the factorization described above is also valid when $H(\cdot)$ is a rectangular matrix. It is well known [1,2,6,10] that for the case where $H(s) \in \mathbb{R}(s)^{n_o\times n_i}$, with n_0 not necessarily equal to n_i, that

(I) $N(\cdot) \in \mathbb{R}[s]^{n_o\times n_i}$ and $D(\cdot) \in \mathbb{R}[s]^{n_o\times n_o}$ are left coprime if and only if there are polynomial matrices $P(\cdot) \in \mathbb{R}[s]^{n_i\times n_o}$ and $Q(\cdot) \in \mathbb{R}[s]^{n_o\times n_o}$ such that

$$N(s)P(s) + D(s)Q(s) = I_{n_o} \quad \forall s \in \mathbb{C}. \tag{2}$$

(II) $\tilde{N}(\cdot) \in \mathbb{R}[s]^{n_o \times n_i}$ and $\tilde{D}(\cdot) \in \mathbb{R}[s]^{n_i \times n_i}$ are <u>right coprime</u> if and only if there are polynomial matrices $\tilde{P}(\cdot) \in \mathbb{R}[s]^{n_i \times n_o}$ and $\tilde{Q}(\cdot) \in \mathbb{R}[s]^{n_i \times n_i}$ such that

$$\tilde{P}(s)\tilde{N}(s) + \tilde{Q}(s)\tilde{D}(s) = I_{n_i} \quad \forall s \in \mathbb{C}. \tag{3}$$

Finally note that

$$N(s)\tilde{D}(s) = D(s)\tilde{N}(s) \quad \forall s \in \mathbb{C}. \tag{4}$$

<u>Lemma II</u>: For the square matrix H(s), Definition I, (i) and (ii), above are equivalent since

$$\det N(z) = 0 \Leftrightarrow \det \tilde{N}(z) = 0. \tag{5}$$

<u>Proof</u>: $\Rightarrow$ From (2) and the two factorizations of H(s),

$$P(s) + \tilde{D}(s)\tilde{N}(s)^{-1}Q(s) = N(s)^{-1}. \tag{6}$$

At $s = z$, the r.h.s. of (6) has a pole, hence $\det \tilde{N}(z) = 0$.

$\Leftarrow$ Similar argument on

$$\tilde{P}(s) + \tilde{Q}(s)N(s)^{-1}D(s) = \tilde{N}(s)^{-1}. \tag{7}$$

Q.E.D.

<u>Theorem II</u>: Given a rational matrix $H(s) \in \mathbb{R}(s)^{n \times n}$ with

$$H(s) = D(s)^{-1}N(s), \tag{8}$$

where $N(s), D(s) \in \mathbb{R}[s]^{n\times n}$ and are right coprime, then,

(a) $z \in \mathbb{C}$ is a zero of $H(s) \Leftrightarrow z \in \mathbb{C}$ is a pole of $s \mapsto H(s)^{-1}$;

(b) if $z \in \mathbb{C}$ is a zero of $H(s)$ then $\exists$ $\{m_\alpha\}$, $g \in \mathbb{C}^n$, $g \neq \theta_n$, $\alpha \in$ some index set, such that, for the input

$$u(t) = 1(t)e^{zt}g + \sum_\alpha m_\alpha \delta^{(\alpha)}(t),$$

the zero-state response has the property

$$y(t,0,\theta_n,u(\cdot)) = \theta_n \quad \forall t > 0;$$

(c) if $v \in \mathbb{C}$ is neither a zero nor a pole of $H(s)$, (i.e., $\det N(v) \neq 0$, $\det D(v) \neq 0$), and if $k \neq \theta_n$ is <u>any vector in $\mathbb{C}^n$</u> then $\exists$ $\{m_\alpha\} \in \mathbb{C}^n$, $\alpha \in$ index set, such that, for the input

$$u(t) = 1(t)e^{vt}k + \sum_\alpha m_\alpha \delta^{(\alpha)}(t),$$

the zero-state response has the property

$$y(t,0,\theta_n,u(\cdot)) = H(v)ke^{vt} \quad \forall t > 0.$$

<u>Proof of II(a)</u>: $\Rightarrow$ From (6) we have

$$P(s) + H(s)^{-1}Q(s) = N(s)^{-1} \tag{9}$$

Since by assumption $\det N(z) = 0$, (9) implies that $H(s)^{-1}$ must have a pole at z.

$\Leftarrow$ Follows directly from (9). Q.E.D.

Proof of II(b): Since by assumption $\det N(z) = 0, \exists g \in \mathbb{C}^n$ such that

$$N(z)g = \theta_n, \; g \neq \theta_n. \tag{10}$$

In other words, there exists a nonzero vector, g, which is an element of the null space of N(z). Taking Laplace transforms of the input given in II(b) we have

$$\hat{u}(s) = g/(s-z) + m(s) \text{ where } m(s) \triangleq \sum_\alpha m_\alpha s^\alpha.$$

Hence

$$\hat{y}(s) = H(s)\hat{u}(s) = D(s)^{-1}N(s)g/(s-z) + D(s)^{-1}N(s)m(s). \tag{11}$$

We will show that m(s) can be so chosen that the r.h.s. of (11) is a polynomial vector. Since the Laplace transform of $\delta^{(n)}$ is s^n, the zero-state response will then be identically zero for positive t (at t = 0, the zero-state response might contain impulses and derivatives of impulses).

Note that

$$p(s) \triangleq N(s)g/(s-z) \tag{12}$$

is a polynomial vector since the residue of p(s) at s = z is $N(z)g = \theta_n$. So (11) becomes

$$\hat{y}(s) = D(s)^{-1}[p(s) + N(s)m(s)]. \tag{13}$$

Choose $m(s) = -P(s)p(s)$, so that

$$\hat{y}(s) = D(s)^{-1}[I - N(s)P(s)]p(s). \tag{14}$$

Using (2), eq. (14) becomes $\hat{y}(s) = Q(s)p(s)$ which is a polynomial vector, and the conclusion II(b) follows. Q.E.D.

Proof of II(c): Since v is not a pole, $\det D(v) \neq 0$ as can be seen from (33) below. Taking Laplace transform of the input given in II(c) we obtain the Laplace transform of the corresponding zero-state response

$$\hat{y}(s) = H(s)k/(s-v) + H(s)\cdot(\sum_{\alpha} m_{\alpha}s^{\alpha}) \tag{15}$$

$$= H(v)k/(s-v) + D(s)^{-1}[N(s)k/(s-v)-D(s)D(v)^{-1}N(v)k/(s-v)] + D(s)^{-1}N(s)m(s), \tag{16}$$

where $m(s) \triangleq \sum_{\alpha} m_{\alpha}s^{\alpha}$; the bracketed term has no pole at $s = v$, and hence is a polynomial vector in s, say $q(s)$. Thus,

$$\hat{y}(s) = H(v)k/(s-v) + D(s)^{-1}[q(s) + N(s)m(s)]. \tag{17}$$

Choose $m(s) = -P(s)q(s)$ and substitute into (17) to obtain

$$\hat{y}(s) = H(v)k/(s-v) + D(s)^{-1}[I - N(s)P(s)]q(s). \tag{18}$$

Using (2) again we obtain

$$\hat{y}(s) = H(v)k/(s-v) + Q(s)q(s). \tag{19}$$

Taking the inverse Laplace transform of (19), conclusion II(c) follows.

Q.E.D.

Remarks: (i) Note that in the proof of II(c) we never used the fact that $H(s)$ was a square matrix. Therefore, provided $D(v)$ is nonsingular, this proof is valid for cases where $H(s)$ is rectangular. (ii) In the present case, since $\det N(v) \neq 0$, for <u>any</u> $k \neq \theta_n$, $H(v)k \neq \theta_n$; hence, $y(t)$ is <u>not identically zero for $t > 0$</u>.

<u>Section III. Zeros of $H(s) \in \mathbb{R}(s)^{n\times n}$; State-Space Method</u>

We again consider an $n\times n$ matrix $H(s)$ with elements in $\mathbb{R}(s)$. As we shall see, by appropriately choosing a non-zero initial state vector x_0, the statements in Theorem II(b) and (c) now hold for all nonnegative t (i.e., this will eliminate any steps, impulses and derivatives of impulses that previously might have occurred in the zero-state response at $t = 0$).

<u>Theorem III</u>: Given a rational proper (i.e., $H(\cdot)$ bounded at infinity) matrix $H(s) \in \mathbb{R}(s)^{n\times n}$ with

$$H(s) = \tilde{N}(s)\tilde{D}(s)^{-1} \quad \text{where } \tilde{N}(s), \tilde{D}(s) \in \mathbb{R}[s]^{n\times n} \text{ and are right coprime;}$$

given any time-invariant representation $R = (A, B, C, E)$ where $A, B, C, E \in \mathbb{R}^{n\times n}$ such that

$$\tilde{N}(s)\tilde{D}(s)^{-1} = C(sI-A)^{-1}B + E,$$

then

(a) $z \in \mathbb{C}$ is a zero of $H(s) \Leftrightarrow z \in \mathbb{C}$ is a pole of $s \mapsto H(s)^{-1}$.

(b) if $z \in \mathbb{C}$ is a zero of $H(s)$ and not an eigenvalue of A then there exists an input

$$u(t) = 1(t)\tilde{D}(z)e^{zt}g \quad (\text{where } \tilde{D}(z) \in \mathbb{C}^{n\times n},\ g \in \mathbb{C}^n,\ g \neq \theta_n)$$

and an initial state $x_0 \in \mathbb{C}^n$ such that the corresponding output has the property

$$y(t,0,x_0,1(t)\tilde{D}(z)e^{zt}g) = \theta_n \quad \forall t \geq 0.$$

(c) if $v \in \mathbb{C}$ is neither a zero nor a pole of $H(s)$ then for any non-zero $k \in \mathbb{C}^n$, and for any input of the form $u(t) = 1(t)e^{vt}k$, there exists an $x_0 \in \mathbb{C}^n$ such that the output has the property $y(t,0,x_0,1(t)e^{vt}k) = H(v)e^{vt} \quad \forall t \geq 0$.

<u>Proof of III(a)</u>: Identical with that of Theorem II(a).

<u>Proof of III(b)</u>: Using the representation (A, B, C, E) and the following identity [12,13] which holds for any s and z that are not eigenvalues of A

$$(sI-A)^{-1}/(s-z) = (zI-A)^{-1}/(s-z) - (sI-A)^{-1}(zI-A)^{-1}, \tag{20}$$

then by standard techniques, with the initial state x_0 chosen as

$$x_0 = (zI-A)^{-1}B\tilde{D}(z)g \tag{21}$$

the conclusion III(b) follows.

<u>Proof of III(c)</u>: Again using standard techniques with the initial state chosen as

$$x_0 = (vI-A)^{-1}Bk \tag{22}$$

the conclusion follows.

<u>Section IV: Zeros of $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$</u>.

Let n_o, (n_i), denote the number of outputs, (inputs, resp.), of the linear time-invariant multivariable system represented by the possibly rectangular transfer function matrix H(s). We consider the following factorizations of H(s) throughout Section IV:

$$H(s) = D(s)^{-1}N(s) = \tilde{N}(s)\tilde{D}(s)^{-1} \tag{23}$$

where

$$N(s) \in \mathbb{R}[s]^{n_o \times n_i},\ D(s) \in \mathbb{R}[s]^{n_o \times n_o},\ \tilde{N}(s) \in \mathbb{R}[s]^{n_o \times n_i},\ \tilde{D}(s) \in \mathbb{R}[s]^{n_i \times n_i}, \tag{24}$$

$$\begin{aligned} &N(s) \text{ and } D(s) \text{ are left coprime.} \\ &\tilde{N}(s) \text{ and } \tilde{D}(s) \text{ are right coprime.} \end{aligned} \tag{25}$$

Since we are dealing with <u>rectangular</u> matrices Definition I has to be generalized. First we recall some well established terms [14].

<u>Definition</u>[1] (i) For any $z \in \mathbb{C}$, the rank of $N(z)$ is called the <u>local rank of $N(\cdot)$ at z</u> and is denoted by $\rho_N(z)$.

(ii) For any $z \in \mathbb{C}$, the rank of $\tilde{N}(z)$ is called the <u>local rank of $\tilde{N}(\cdot)$ at z</u> and is denoted by $\rho_{\tilde{N}}(z)$.

(iii) $\max_{z \in \mathbb{C}} \rho_N(z) \triangleq \bar{\rho}_N$ is called the <u>normal rank of $N(\cdot)$</u>.

---

1 The local rank of $N(\cdot)$, $(\tilde{N}(\cdot))$, at z is simply the rank of the matrix $N(z)$, $(\tilde{N}(z))$, where the elements are complex <u>numbers</u>. The normal rank of $N(\cdot)$, $(\tilde{N}(\cdot))$, is in fact the rank of the matrix $N(\cdot)$, $(\tilde{N}(\cdot)$, resp.), provided that when we calculate minors we view their elements as members of $\mathbb{C}[s]$. Equivalently, in determining linear independence, we view the rows, (columns), as elements of the module $(\mathbb{C}[s])^{n_i}$, $((\mathbb{C}[s])^{n_o}$, resp.), over $\mathbb{C}[s]$.

(iv) $\max\limits_{z \in \mathbb{C}} \rho_{\tilde{N}}(z) \triangleq \bar{\rho}_{\tilde{N}}$ is called the <u>normal rank of $\tilde{N}(\cdot)$</u>.

Part 1: $n_o \geq n_i$

We assume in Part 1 that the normal rank of $N(\cdot)$ and $\tilde{N}(\cdot)$ is equal to n_i.

Since $N(\cdot)$, $(\tilde{N}(\cdot))$, is a polynomial matrix, $\rho_N(z) = \bar{\rho}_N$, $(\rho_{\tilde{N}}(z) = \bar{\rho}_{\tilde{N}}$, resp.), except at a finite number of points. With $n_o \geq n_i$, the condition $\rho_N(z) < \bar{\rho}_N$, $(\rho_{\tilde{N}}(z) < \bar{\rho}_{\tilde{N}})$, means that the columns of $N(z)$, $(\tilde{N}(z)$, resp.), are linearly dependent.

<u>Definition IV.1</u>: $z \in \mathbb{C}$ is called <u>a zero of $H(\cdot)$</u> $\in \mathbb{R}(s)^{n_o \times n_i}$, with $n_o \geq n_i$, iff $\rho_{\tilde{N}}(z) < \bar{\rho}_{\tilde{N}} = n_i$.

<u>Lemma IV.1</u>: $\rho_{\tilde{N}}(z) < \bar{\rho}_{\tilde{N}} = n_i \Rightarrow \rho_N(z) < \bar{\rho}_N = n_i$.

<u>Proof of Lemma IV.1</u>

Since $\rho_{\tilde{N}}(z) < n_i$, $\exists$ $c \in \mathbb{C}^{n_i}$ such that

$$\tilde{N}(z)c = \theta_{n_o}, \quad c \neq \theta_{n_i} \tag{27}$$

Multiply (3) on the right by c and use (27) to obtain

$$\tilde{Q}(z)\tilde{D}(z)c = c \neq \theta_{n_i}, \tag{28}$$

and therefore

$$\tilde{D}(z)c \neq \theta_{n_i} . \qquad (29)$$

Multiply (4) on the right by c, let $s \to z$, and use (27) to obtain

$$N(z)\tilde{D}(z)c = \theta_{n_o} . \qquad (30)$$

Hence, in view of (29), $\rho_N(z) < n_i$ Q.E.D.

<u>Comment on poles, zeros and local rank</u>. (i) If we use (23) in (2) and (3) we obtain

$$H(s)P(s) + Q(s) = D(s)^{-1} \qquad (31)$$

$$\tilde{P}(s)H(s) + \tilde{Q}(s) = \tilde{D}(s)^{-1} \qquad (32)$$

From (31) and (32) we observe that

$$p \in \mathbb{C} \text{ is a pole of } H(\cdot) \Leftrightarrow \det D(p) = 0 \Leftrightarrow \det \tilde{D}(p) = 0. \qquad (33)$$

(ii) If $z \in \mathbb{C}$ is a zero of $H(\cdot)$, but <u>not</u> a pole of $H(\cdot)$, then

$$\rho_N(z) = \rho_{\tilde{N}}(z) < n_i \qquad (34)$$

<u>Proof of (34)</u>: From (4) we have

$$N(z) = D(z)\tilde{N}(z)\tilde{D}(z)^{-1}, \qquad (35)$$

and

$$\tilde{N}(z) = D(z)^{-1}N(z)\tilde{D}(z). \qquad (36)$$

The conclusion follows since $D(z)$ and $\tilde{D}(z)$ are nonsingular. Q.E.D.

<u>Theorem IV.1</u> Let $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$, $n_o \geq n_i$, with the factorizations (23), and $n_i = \overline{\rho}_N$. Under these conditions,

(a) $z \in \mathbb{C}$ is a zero of $H(\cdot) \Rightarrow z$ is a pole of <u>any</u> left-inverse[2], $H^L(\cdot)$, of $H(\cdot)$;

(b) If $z \in \mathbb{C}$ is a zero of $H(\cdot)$, then $\exists\ \{m_\alpha\}$, $g \in \mathbb{C}^{n_i}$ with $g \neq \theta_{n_i}$, such that for the input

$$u(t) = 1(t)e^{zt}g + \sum_\alpha m_\alpha \delta^{(\alpha)}(t), \quad \alpha \in \text{some index set},$$

the zero-state response has the property

$$y(t,0,\theta_n,u(\cdot)) = \theta_{n_o} \quad \forall t > 0;$$

(c) if $v \in \mathbb{C}$ is neither a zero nor a pole of $H(\cdot)$ and if $k \neq \theta_{n_i}$ is <u>any</u> vector in $\mathbb{C}^{n_i}$, then $\exists\ \{m_\alpha\} \in \mathbb{C}^{n_i}$, such that for the input

$$u(t) = 1(t)e^{vt}k + \sum_\alpha m_\alpha \delta^{(\alpha)}(t), \quad \alpha \in \text{some index set},$$

the zero-state response has the property

$$y(t,0,\theta_n,u(\cdot)) = H(v)ke^{vt} \quad \forall t > 0.$$

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2 Any <u>left-inverse</u>, $H^L(\cdot) \in \mathbb{R}(s)^{n_i \times n_o}$, of $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$ has the property

$$H^L(s)\ H(s) = I_{n_i} \quad \forall s \in \mathbb{C}. \tag{37}$$

A candidate for $H^L(\cdot)$ is $H^L(s) = [H(s)'\ H(s)]^{-1}\ H(s)'$

Remark: Since in Theorem IV.1(c), $v \in \mathbb{C}$ is neither a zero, nor a pole of $H(\cdot)$, $D(v)$ is nonsingular and the column rank of $N(v)$ is n_i; therefore the column rank of $H(v)$ is n_i. Hence, for any $k \in \mathbb{C}^{n_i}$, $k \neq \theta_{n_i}$, the zero-state response is not identically zero for $t > 0$.

Proof of IV.1(a): By contradiction. Suppose that $s \mapsto H^L(s)$ is analytic at z, then $H^L(z) \in \mathbb{C}^{n_i \times n_o}$. Define c as in (27) and use (4) and (29) to obtain,

$$H(z)\tilde{D}(z)c = \tilde{N}(z)c = \theta_{n_o} , \tag{38}$$

where $c \neq \theta_{n_i}$, $\tilde{D}(z)c \neq \theta_{n_i}$. Using (37) we obtain

$$H^L(s)H(s)\tilde{D}(z)c = \tilde{D}(z)c \neq \theta_{n_i} \qquad \forall s \in \mathbb{C} . \tag{39}$$

Letting $s \to z$ in (39) and noting (38) we obtain the contradiction,

$$H^L(z)\theta_{n_o} = \tilde{D}(z)c \neq \theta_{n_i} . \tag{40}$$

Q.E.D.

Comment: From (38) we see that even if z is a pole of $H(\cdot)$ (equivalently, $\det \tilde{D}(z) = 0$) there is a linear combination of the columns of $H(\cdot)$ which sum to the zero-vector θ_{n_o}. In that sense we could say that, at z, the rank of $H(\cdot)$ is smaller than n_i.

Comment: The converse of IV(a) is not true. To wit:

$$H^L(s) = \begin{bmatrix} 1 & 0 & \frac{1}{s-p} \\ 0 & 1 & \frac{1}{s-p} \end{bmatrix} , \quad H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{41}$$

where indeed, $\forall p \in \mathbb{C}$

$$H^L(s)H(s) = I \quad \forall s \in \mathbb{C}.$$

Proof of IV.1(b): By assumption $z \in \mathbb{C}$ is a zero of $H(\cdot)$. From Definition IV.1 and Lemma IV.1 it follows that $\rho_N(z) < n_i$. Hence $\exists\ g \in \mathbb{C}^{n_i}$ such that

$$N(z)g = \theta_{n_o}, \quad g \neq \theta_{n_i} \ . \tag{42}$$

The remainder of the proof is identical with that of II(b), above, except that H(s) and N(s) are rectangular and the vectors are n_o- or n_i- dimensional.

Q.E.D.

Proof of IV.1(c): The proof of Theorem II(c) establishes IV.1(c). The only difference is that H(s) and N(s) are rectangular and the dimensions of the vectors change accordingly.

Part 2: $n_o \leq n_i$

We assume in Part 2 that $\bar{\rho}_N = \bar{\rho}_{\tilde{N}} = n_o \leq n_i$. Consequently, we have to modify Definition IV.1.

Definition IV.2: $z \in \mathbb{C}$ is called a zero of $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$, with $n_o \leq n_i$, iff $\rho_N(z) < \bar{\rho}_N = n_o$.

Lemma IV.2: $\rho_N(z) < \bar{\rho}_N = n_o \Rightarrow \rho_{\tilde{N}}(z) < \rho_{\tilde{N}} = n_o$.

Proof of Lemma IV.2: By assumption $\exists$ a row vector $c' \in \mathbb{C}^{n_o}$ such that

$$c'N(z) = \theta'_{n_i}, \quad c' \neq \theta'_{n_o}. \tag{43}$$

By (2) we conclude that

$$c'D(z) \neq \theta'_{n_o}. \tag{44}$$

Hence, by (4) we have

$$c'D(z)\tilde{N}(z) = \theta'_{n_i}. \tag{45}$$

Therefore, $\rho_{\tilde{N}}(z) < n_o$. Q.E.D.

Observation: Again by using (4) we can show a dual statement to (34); viz., if $z \in C$ is a zero of $H(\cdot)$ but not a pole of $H(\cdot)$, then $\rho_N(z) = \rho_{\tilde{N}}(z) < n_o$.

Comment: We cannot have a theorem identical with Theorem IV.1. Indeed, for $n_o < n_i, \forall\, v \in \mathbb{C}$, there is a non-zero vector $g \in \mathbb{C}^{n_i}$ such that $N(v)g = \theta_{n_o}$. Thus by the proof of Theorem IV.1(b), there is an input of form given in Theorem IV.1(b) which produces a zero-state response which is identically zero for $t > 0$. Intuitively, since $n_o < n_i$, we can use the "surplus" of inputs to compensate their collective actions at every output.

Theorem IV.2: Let $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$, $n_o \leq n_i$, with the factorization (23), and $n_o = \overline{\rho}_N$. Under these conditions,

(a) $z \in \mathbb{C}$ is a zero of $H(\cdot) \Rightarrow z$ is a pole of any right-inverse[3], $H^R(\cdot)$, of $H(\cdot)$;

(b) If $z \in \mathbb{C}$ is a zero of $H(\cdot)$, then there is a linear combination of the zero-state response, viz., $\psi(t) = c'D(z)y(t)$, (where $c'N(z) = \theta'_{n_i}$, $c' \neq \theta'_{n_o}$), such that for <u>any input</u> of the form

$$u(t) = 1(t)ge^{zt} + \sum_\alpha m_\alpha \delta^{(\alpha)}(t),$$

(where $g \in \mathbb{C}^{n_i}$, $\alpha \in$ some index set, and the m_α's are appropriate vectors in $\mathbb{C}^{n_i}$ which depend on g), that linear combination, $\psi(t)$, is identically zero for $t > 0$;

(c) If $v \in \mathbb{C}$ is neither a zero nor a pole of $H(\cdot)$ and if $k \neq \theta_{n_i}$ is <u>any</u> vector in $\mathbb{C}^{n_i}$, then for some $\{m_\alpha\} \in \mathbb{C}^{n_i}$, the input

$$u(t) = 1(t)e^{vt}k + \sum_\alpha m_\alpha \delta^{(\alpha)}t, \ (\alpha \in \text{some index set}),$$

generates a zero-state response of the form

$$y(t,0,\theta_n,u(\cdot)) = H(v)ke^{vt} \quad \forall t > 0 .$$

<u>Remark IV.2</u>: In the special case where $n_o < n_i$ then concerning <u>Theorem</u>

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3 Any <u>right-inverse</u>, $H^R(\cdot) \in \mathbb{R}(s)^{n_i \times n_o}$, of $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$ has the property

$$H(s)H^R(s) = I_{n_o} \quad \forall s \in \mathbb{C}. \tag{46}$$

A candidate for $H^R(\cdot)$ is $H(s)'[H(s)H(s)']^{-1}$.

IV.2(c): (i) there are some $k \in \mathbb{C}^{n_i}$ with $k \neq \theta_{n_i}$ such that $H(v)k = \theta_{n_o}$ and the zero-state response is identically zero for all $t > 0$; (ii) since $\det D(v) \neq 0$ and $\rho_N(v) = \rho_{\tilde{N}}(v) = n_o$, for any non-zero row-vector c' there is a vector $k \in \mathbb{C}^{n_i}$ such that $c'H(v)k \neq 0$, $k \neq \theta_{n_i}$; hence, for any linear combination of the zero-state response, $c'y(t)$, there is an input of the form specified in Theorem IV.2(c) for which this linear combination is not identically zero for all $t > 0$. Intuitively, there is no "zero of transmission" at the frequency v.

Proof of IV.2(a): By contradiction. Suppose that $s \mapsto H^R(s)$ is analytic at z, then $H^R(z) \in \mathbb{C}^{n_i \times n_o}$. Define c' as in eq. (43) and use (4) and (44) to obtain

$$c'D(z)H(z) = c'D(z)D(z)^{-1}N(z) = \theta'_{n_i}, \tag{47}$$

where $c' \neq \theta'_{n_o}$, $c'D(z) \neq \theta'_{n_o}$. Using (46) we obtain

$$c'D(z)H(s)H^R(s) = c'D(z) \neq \theta'_{n_o} \quad \forall s \in \mathbb{C}. \tag{48}$$

Letting $s \to z$ in (48) and noting (47) we obtain the contradiction,

$$\theta'_{n_i} H^R(z) = c'D(z) \neq \theta'_{n_o}.$$

Q.E.D.

Proof of IV.2(b): We define $\psi(t) \triangleq c'D(z)y(t)$ where $c'N(z) = \theta'_{n_i}$, $c' \neq \theta'_{n_o}$ (by assumption), and $c'D(z) \neq \theta'_{n_o}$ (from (2)). Taking Laplace transform of $\psi(t)$ and using the given input we obtain,

$$\hat{\psi}(s) = c'D(z)H(s)\hat{u}(s) \tag{49}$$

$$= c'D(z)D(s)^{-1}N(s)[g/(s-z) + m(s)] \tag{50}$$

where $m(s) \triangleq \sum_{\alpha} m_{\alpha} s^{\alpha}$.

Pick $m(s) = g\mu(s)$ where $\mu(s) \in \mathbb{R}[s]$ and substitute in (50) to obtain

$$\hat{\psi}(s) = c'D(z)D(s)^{-1}N(s)g[1/(s-z) + \mu(s)], \tag{51}$$

where $c'D(z)D(s)^{-1}N(s)g \triangleq h(s) \in \mathbb{R}(s)$ has a zero at $s = z$ by (47). Hence, we can write $h(s)$ as

$$h(s) = \frac{n(s)}{d(s)} (s-z), \tag{52}$$

where $n,d \in \mathbb{R}[s]$ are coprime and $d(z) \neq 0$. Substitute (52) in (51) to obtain

$$\hat{\psi}(s) = \frac{n(s)}{d(s)} [1 + (s-z)\mu(s)]. \tag{53}$$

Since at $s = z$, the bracket term is equal to 1, and $d(z) \neq 0$, we can choose $\mu(s) \in \mathbb{R}[s]$ so that the bracket term is a nonzero constant, say β, times $d(s)$. (For example, we could set $\mu(s) = [\beta d(s)-1]/(s-z)$, where β is such that $\beta d(z) = 1$).

Therefore, with $\mu(s) \in \mathbb{R}[s]$ chosen so that

$$[1 + (s-z)\mu(s)] = \beta d(s), \quad \beta \in \mathbb{R}, \ \beta \neq 0, \tag{54}$$

we have from (53) that

$$\hat{\psi}(s) = \beta n(s). \tag{55}$$

Hence, $\psi(t) = c'D(z)y(t) = 0 \quad \forall t > 0.$ Q.E.D.

Proof of IV.2(c): Same as proof of II(c), with of course, the necessary changes in the dimensions of the matrices and vectors involved.

Section V. Poles of H(s)

With $H(s) \in \mathbb{R}(s)^{n_o \times n_i}$, classical analysis [18] defines a pole of the rational matrix $H(\cdot)$; viz., $p \in \mathbb{C}$ is called a pole if and only if some element of $H(\cdot)$ has a pole at p. In Section V, the relative magnitude of n_o and n_i is of no consequence.

Theorem V: Let $H(s) \in \mathbb{R}(s)^{n_o \times n_i}$ with factorization (23). Under this condition, $p \in \mathbb{C}$ is a pole of $H(\cdot) \Leftrightarrow \exists$ an input,

$$u(t) = \sum_{\alpha} u_{\alpha} \delta^{(\alpha)}(t), \tag{56}$$

where $u_{\alpha} \in \mathbb{C}^{n_i}$, $\alpha \in$ some index set, such that the corresponding zero-state response has property that

$$y(t) = re^{pt} \quad \forall t > 0, \tag{57}$$

where r is a nonzero vector in $\mathbb{C}^{n_o}$.

In other words, a "singular" input $u(\cdot)$ of the form (56) kicks the system from its zero state at $t = 0^-$, to a state at $t = 0^+$ which results in the purely exponential output for all $t > 0$ if and only if $p \in \mathbb{C}$ is a pole of $H(\cdot)$.

Proof of Theorem V: $\Rightarrow$. By assumption, $p \in \mathbb{C}$ is a pole of $H(\cdot)$; hence, by (33), $\det D(p) = 0$. So there is a nonzero vector, $r \in \mathbb{C}^{n_o}$, such that

$D(p)r = \theta_{n_o}$. Hence the polynomial vector function $s \mapsto D(s)r$ has a zero at $s = p$ and can be written as

$$D(s)r = k(s)\cdot(s-p), \tag{58}$$

where $k(s)$ is a polynomial vector, i.e., $k(\cdot) \in \mathbb{R}[s]^{n_o \times 1}$.

Now, with (2) in mind, choose the input to be given by

$$\hat{u}(s) \triangleq P(s)k(s). \tag{59}$$

The zero-state response to this input is

$$\hat{y}(s) = H(s)P(s)k(s) = D(s)^{-1}N(s)P(s)k(s). \tag{60}$$

Using (2) in (60) we obtain

$$\hat{y}(s) = D(s)^{-1}[-D(s)Q(s)k(s) + k(s)] \tag{61}$$

$$= -Q(s)k(s) + r/(s-p), \tag{62}$$

where we used (58). Since $Q(s)k(s)$ is a polynomial vector in s, the conclusion follows.

$\Leftarrow$. Suppose that, starting from the zero-state at $t = 0^-$, some polynomial vector input $\hat{u}(s)$ would produce as a zero-state response re^{pt}, for $t > 0$, where r is a nonzero vector in $\mathbb{C}^{n_o}$ and $p \in \mathbb{C}$, then

$$\hat{y}(s) = H(s)\hat{u}(s) = D(s)^{-1}N(s)\hat{u}(s) \tag{63}$$

$$= q(s) + r/(s-p), \tag{64}$$

where $q(s)$ is a polynomial vector in s contributed by the linear combination of $\delta(t)$, $\delta'(t)$, etc. which occur at $t = 0$. By (64), $\hat{y}(s)$ has a pole at p. By (63), this implies that $\det D(p) = 0$; hence, by (33), $p \in \mathbb{C}$ is a pole of $H(\cdot)$. Q.E.D.

Conclusion

This paper is based on the factorizations of the rational matrix transfer function $H(\cdot)$ given in (23). The zeros of $H(\cdot)$ are defined in terms of the local rank of the polynomial matrices $N(\cdot)$ or $\tilde{N}(\cdot)$. The dynamic properties associated with the zeros are given in Theorems II, III, IV.1 and IV.2. The poles of $H(\cdot)$ are defined by classical analysis and are characterized in Theorem V. If the complex variable s is changed into z and if the resulting elements of $H(\cdot)$ are interpreted as z-transfer functions, then the algebraic techniques used above can be applied to the sampled-data case and, except for a few modifications in interpretations, the results still hold in this case.

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Footnotes

1 The local rank of $N(\cdot)$, $(\tilde{N}(\cdot))$, at z is simply the rank of the matrix $N(z)$, $(\tilde{N}(z))$, where the elements are complex numbers. The normal rank of $N(\cdot)$, $(\tilde{N}(\cdot))$, is in fact the rank of the matrix $N(\cdot)$, $(\tilde{N}(\cdot)$, resp.), provided that when we calculate minors we view their elements as members of $\mathbb{C}[s]$. Equivalently, in determining linear independence, we view the rows, (columns), as elements of the module $(\mathbb{C}[s])^{n_i}$, $((\mathbb{C}[s])^{n_o}$, resp.), over $\mathbb{C}[s]$.

2 Any left-inverse, $H^L(\cdot) \in \mathbb{R}(s)^{n_i \times n_o}$, of $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$ has the property

$$H^L(s)\,H(s) = I_{n_i} \quad \forall s \in \mathbb{C}. \tag{37}$$

One candidate for $H^L(\cdot)$ is $H^L(s) = [H(s)'H(s)]^{-1}H(s)'$

3 Any right-inverse, $H^R(\cdot) \in \mathbb{R}(s)^{n_i \times n_o}$, of $H(\cdot) \in \mathbb{R}(s)^{n_o \times n_i}$ has the property

$$H(s)H^R(s) = I_{n_o} \quad \forall s \in \mathbb{C}. \tag{46}$$

A candidate for $H^R(\cdot)$ is $H(s)'[H(s)H(s)']^{-1}$.