NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY FOR n-INPUT n-OUTPUT CONVOLUTION FEEDBACK SYSTEMS WITH A FINITE NUMBER OF UNSTABLE POLES

by

Frank M. Callier and Charles A. Desoer

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY FOR n-INPUT n-OUTPUT CONVOLUTION FEEDBACK SYSTEMS WITH A FINITE NUMBER OF UNSTABLE POLES

Frank M. Callier,* Member IEEE
Charles A. Desoer, Fellow IEEE

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

This paper considers n-input n-output convolution feedback systems characterized by $y = G * e$ and $e = u - Fy$, where the open-loop transfer function $\hat{G}$ contains a finite number of unstable multiple poles and F is a constant nonsingular matrix. Theorem 1 gives necessary and sufficient conditions for stability. A basic device is the following: the principal part of the Laurent expansion of $\hat{G}$ at the unstable poles is factored as a ratio of two right-coprime polynomial matrices. There are two necessary and sufficient conditions, the first is the usual infimum one and the second is required to prevent the closed-loop transfer function from being unbounded in some small neighborhood of each open-loop unstable pole. The latter condition is given an interpretation in concepts of McMillan degree theory. The modification of the theorem for the discrete-time case is immediate.

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I. INTRODUCTION

A number of increasingly sophisticated sufficient conditions for the input-output stability of distributed linear time-invariant feedback systems have been published over the last few years. Desoer treated the single-input, single-output system where the open-loop system includes an integrator [1]. Desoer and Wu allowed an integrator and considered the n-input n-output case both for continuous-time and discrete-time systems [2,3,4]. Further refinements were obtained by Baker and Vakharia [5], Desoer and Lam [6], Nasburg and Baker [7], and M. Vidyasagar [8]. General necessary conditions were published by Desoer and Vidyasagar [9]. Necessary and Sufficient conditions were obtained by Willems for the open-loop-stable, single-input, single-output case [10], by Vidyasagar [8], and by Desoer and Callier for open-loop-unstable, n-input, n-output systems both continuous-time [11] and discrete-time [12]. Graphical tests were developed by Willems [10], Davis [13], Callier and Desoer [14], and Callier [15].

The main result of this paper is Theorem 1 which gives necessary and sufficient conditions for the stability of a multivariable, distributed, linear, time-invariant, feedback system whose open-loop matrix transfer function includes a finite number of unstable poles. The conditions obtained are better than those of [11] because (i) they are conceptually and computationally simpler, (ii) they allow poles on the boundary and (iii) the derivation is much simpler. Corollary 1 relates the necessary and sufficient conditions to the McMillan Degree Theory, in fact, for an interior pole, it expresses condition (9) in terms of the rank of a

Hankel matrix.

An important difference between the single variable case, (n=1), and the multivariable case, (n>1), is the following: if a scalar transfer function, say, $1 + f\hat{g}(s)$, has a pole at p, then its inverse, $(1 + f\hat{g}(s))^{-1}$, has necessarily a zero at p; on the other hand, in the multivariable case, if a matrix transfer function, say, $(I + F\hat{G}(s))$, has a pole at p, then its inverse, $[I + F\hat{G}(s)]^{-1}$, may very well have a pole at p. To see this consider

$$I + F\hat{G}(s) = \mathrm{diag}[(s-1)/(s-p),\ (s-p)/(s-1)].$$

In view of the essentially algebraic nature of the proofs, it will be clear to the reader that the same results apply to the discrete-time case. To carry out the simple required modifications, see [4, 12].

II. System Description and Useful Facts.

We consider a distributed, linear, time-invariant, feedback system with n inputs and n outputs. The input u, output y and error e are functions from $\mathbb{R}_+$, (defined as $[0,\infty)$), to $\mathbb{C}^n$ or corresponding distributions on $\mathbb{R}_+$. The open-loop system is of the convolution type so that we have

$$y = G*e \tag{1}$$

$$e = u - Fy \tag{2}$$

where $F \in \mathbb{C}^{n\times n}$ with $\det F \neq 0$ and G is an n×n matrix whose elements are complex-valued distributions on $\mathbb{R}_+$. We shall repeatedly use the convolution algebra $\mathcal{A}$ [2,3]; recall that $f \in \mathcal{A}$ iff, for $t < 0$, $f(t) = 0$, and, for $t \geq 0$, $f(t) = f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i)$ where $f_a(\cdot) \in L^1[0,\infty)$, $f_i \in \mathbb{C}$ for all i, $\sum_{i=0}^{\infty} |f_i| < \infty$ and $0 = t_0 < t_i$ for all integers $i > 0$. Thus f is a distribution of order 0 with support on $\mathbb{R}_+$. An n-vector v, (n×n matrix A), is said to be in $\mathcal{A}^n$, ($\mathcal{A}^{n\times n}$, resp.) iff all its elements are in $\mathcal{A}$. Let $\hat{f}$ denote the Laplace transform of f: f belongs to the convolution algebra $\mathcal{A}$ if and only if $\hat{f}$ belongs to the algebra $\hat{\mathcal{A}}$ (with pointwise product). Similarly we write $\hat{v} \in \hat{\mathcal{A}}^n$, $\hat{A} \in \hat{\mathcal{A}}^{n\times n}$. We denote by $\mathbb{C}_+$ the closed right half-plane i.e. $\mathbb{C}_+ \triangleq \{s | \text{Re } s \geq 0\}$.

Throughout the paper we consider a system defined by (1) and (2) where the open-loop transfer function $\hat{G}(s)$ has a finite number of "unstable" poles in $\mathbb{C}_+$, i.e.

$$\hat{G}(s) = \sum_{k=1}^{\ell} \sum_{\alpha=0}^{m_k-1} R_{k\alpha}(s-p_k)^{-m_k+\alpha} + \hat{G}_\rho(s) \quad \text{for Re } s \geq 0, \tag{3}$$

where Re $p_k \geq 0$ for $k = 1, 2, \ldots, \ell$; the poles p_k and the coefficient matrices $R_{k\alpha}$ are either real or occur in complex conjugate pairs; $\hat{G}_\rho(s) \in \hat{\mathcal{A}}^{n\times n}$. We define the system given by (1), (2) and (3) to be stable iff its closed-loop impulse response H belongs to $\mathcal{A}^{n\times n}$. This definition of stability has a very important interpretation: for closed-loop impulse responses having such a form, the closed-loop system is stable in the sense above if and only if it is $L^P[0,\infty]$ stable for all $p \in [1,\infty]$, [10].

Useful Facts and Definitions. $\mathbb{C}^{n\times n}(s)$, ($\mathbb{C}^{n\times n}[s]$), denotes the noncommutative ring of matrices of rational functions (polynomials, resp.) with complex coefficients. An element A of $\mathbb{C}^{n\times n}(s)$ is said to be proper iff A is bounded at infinity. An element A of $\mathbb{C}^{n\times n}[s]$ is said to be unimodular iff det[A(s)] is a nonzero constant. Let A and B be two elements of $\mathbb{C}^{n\times n}[s]$; a matrix M is said to be a common right divisor of A and B iff there exist matrices $\tilde{A}$ and $\tilde{B}$ such that $A = \tilde{A}M$ and $B = \tilde{B}M$ where M, $\tilde{A}$ and $\tilde{B}$ are elements of $\mathbb{C}^{n\times n}[s]$; both A and B are said to be left multiples of M; an element L of $\mathbb{C}^{n\times n}[s]$ is said to be a greatest common right divisor (g.c.r.d.) of A and B iff (i) it is a common right divisor of A and B, and (ii) it is a left multiple of every common right divisor of A and B. Two elements A and B of $\mathbb{C}^{n\times n}[s]$ are said to be right coprime iff they have a unimodular g.r.c.d.. An element A of $\mathbb{C}^{n\times n}[s]$ for which det $A(s) \not\equiv 0$ is said to be column proper, [16], iff degree (det A(s)) = $\sum_{j=1}^{n} \delta_j$ where δ_j is the highest degree of the polynomials in the j-th column of A.

Fact 1, [17]. Two elements A and B of $\mathbb{C}^{n\times n}[s]$ are right coprime if and only if there exist elements P and Q in $\mathbb{C}^{n\times n}[s]$ such that

$$P(s)\ A(s) + Q(s)\ B(s) = I \quad \text{for all } s \text{ in } \mathbb{C}.$$

Fact 2. Let R be a proper element of $\mathbb{C}^{n\times n}(s)$, then there exist two elements N and D in $\mathbb{C}^{n\times n}[s]$ such that

$$R = N\,D^{-1}$$

where (a) N and D are right-coprime; (b) det $D(s) \not\equiv 0$; (c) p is a pole of R(s) if and only if it is a zero of det D(s). Furthermore, if D is column proper with δ_j denoting the highest degree of the polynomials in the j-th column of D and with n_{ij} denoting the (i,j)-th element of N, then degree $(n_{ij}) \le \delta_j$ for all (i,j).

This fact is due to several authors, [18], [22], [19], [20], and is closely connected to a proposition of [21].

III. Necessary and Sufficient Conditions for Stability.

With $\hat{G}$ defined by (3), let $I + F\hat{G}(s) = F \sum_{k=1}^{\ell} R_k(s) + I + F\hat{G}_\rho(s)$ for all s in $\mathbb{C}_+$, thus

$$R_k(s) = \sum_{\alpha=0}^{m_k-1} R_{k\alpha}(s-p_k)^{-m_k+\alpha} \quad \text{for all s in } \mathbb{C}. \tag{4}$$

At each pole p_k, $k = 1, 2, \ldots, \ell$, consider the principal part of the Laurent expansion of $I + F\hat{G}(s)$ up to and including the constant term. This proper rational function can be represented as the product $N_k(s)\, D_k(s)^{-1}$ where N_k and D_k are right-coprime elements of $\mathbb{C}^{n\times n}[s]$ and $\det D_k(s) \not\equiv 0$. In other words, for all $s \in \mathbb{C}$ and for $k = 1, 2, \ldots, \ell$,

$$N_k(s)\, D_k(s)^{-1} = FR_k(s) + I + \sum_{\substack{\beta=1 \\ \beta\neq k}}^{\ell} FR_\beta(p_k) + F\hat{G}_\rho(p_k) \tag{4a}$$

Hence we have for $k = 1, 2, \ldots, \ell$ and for all $s \in \mathbb{C}_+$

$$I + F\hat{G}(s) = N_k(s)\, D_k(s)^{-1} + \sum_{\substack{\beta=1 \\ \beta\neq k}}^{\ell} F[R_\beta(s) - R_\beta(p_k)]$$

$$+ F[\hat{G}_\rho(s) - \hat{G}_\rho(p_k)]. \tag{5}$$

Note that (5) has the following form

$$I + F\hat{G}(s) = [N_k(s) + \Phi_k(s)]\, D_k(s)^{-1} \tag{6}$$

where the thus defined function $\Phi_k(\cdot)$ has the property that $\Phi_k(s) \to 0$ as $s \to p_k$ and if $\mathcal{N}(p_k)$ denotes a sufficiently small neighborhood of p_k so that $p_\beta \notin \mathcal{N}(p_k)$ for $\beta \neq k$, then Φ_k is continuous in $\mathcal{N}(p_k) \cap \mathbb{C}_+$.

Theorem 1. Consider an n-input n-output convolution feedback system described by (1) to (3). Under these conditions,

the closed-loop impulse response $H \in \mathcal{A}^{n\times n}$ (7)

if and only if

(i)

$$\inf_{\text{Re } s \geq 0} |\det [I + F\hat{G}(s)]| > 0, \tag{8}$$

and

(ii) for $k = 1, 2, \ldots, \ell$

$$\det N_k(p_k) \neq 0. \tag{9}$$

Comment. If the system were open-loop stable, i.e. $G \in \mathcal{A}^{n\times n}$, (hence $L^p[0,\infty)$-stable for all $p \in [1,\infty]$), then (8) would be the necessary and sufficient condition for $H \in \mathcal{A}^{n\times n}$. The additional ℓ conditions (9) can be viewed as additional conditions required to prevent $\hat{H}(s)$ from mis-behaving at the unstable poles of $\hat{G}(s)$. This idea is made more precise by the following.

Lemma 1. Let $\mathcal{N}(p_k)$ be a sufficiently small neighborhood of p_k. Under these conditions

$$[I + F\hat{G}]^{-1} \text{ is bounded in } \mathcal{N}(p_k) \cap \mathbb{C}_+ \tag{10}$$

if and only if

$$\det N_k(p_k) \neq 0. \tag{9}$$

Proof. By construction N_k and D_k are right-coprime elements of $C^{n\times n}[s]$, hence by fact 1 there exist elements P_k and Q_k of $\mathbb{C}^{n\times n}[s]$ such that

$$P_k(s)\ N_k(s) + Q_k(s)\ D_k(s) = I \quad \text{for all } s \text{ in } \mathbb{C}.$$

We obtain successively

$$P_k(s)[N_k(s) + \Phi_k(s)] + Q_k(s)\ D_k(s) = I + P_k(s)\Phi_k(s)$$

or, for all s in $\mathcal{N}(p_k) \cap \mathbb{C}_+$

$$P_k(s) + Q_k(s)\ D_k(s)\ [N_k(s) + \Phi_k(s)]^{-1}$$
$$= [I + P_k(s)\Phi_k(s)]\ [N_k(s) + \Phi_k(s)]^{-1}.$$

Finally, by (6), we have for all $s \in \mathcal{N}(p_k) \in \mathbb{C}_+$,

$$P_k(s) + Q_k(s)\ [I + F\hat{G}(s)]^{-1} = [I + P_k(s)\Phi_k(s)]\ [N_k(s) + \Phi_k(s)]^{-1}. \tag{11}$$

Observe that $I + P_k \Phi_k$ is continuous and tends to I as $s \to p_k$, hence from (11), we obtain the following successive implications:

$(I + F\hat{G})^{-1}$ is bounded in $\mathcal{N}(p_k) \cap \mathbb{C}_+$

$\Rightarrow$ the right hand side of (11) is bounded in $\mathcal{N}(p_k) \cap \mathbb{C}_+$

$\Rightarrow$ $(N_k + \Phi_k)^{-1}$ is bounded in $\mathcal{N}(p_k) \cap \mathbb{C}_+$

$\Rightarrow$ $\det N_k(p_k) \neq 0.$ (12)

The last implication is established as follows: Suppose $\det N_k(p_k) = 0$. By assumption $s \mapsto [N_k(s) + \Phi_k(s)]^{-1}$ is bounded in $\mathcal{N}(p_k) \cap \mathbb{C}_+$, hence so is

$$s \mapsto \det\{[N_k(s) + \Phi_k(s)]^{-1}\} = \{\det[N_k(s) + \Phi_k(s)]\}^{-1}$$

This however is contradictory since, as $s \to p_k$, the last determinant tends to $\det N_k(p_k)$ which has been assumed to be zero. So we must have $\det N_k(p_k) \neq 0$. Hence by (12) necessity has been established. To prove sufficiency, observe that from (6), for all s in $\mathcal{N}(p_k) \cap \mathbb{C}_+$

$$[I + F\hat{G}(s)]^{-1} = D_k(s)\ [N_k(s) + \Phi_k(s)]^{-1}. \qquad (13)$$

Now $[N_k + \Phi_k]^{-1} = \mathrm{Adj}[N_k + \Phi_k]/\det[N_k + \Phi_k]$, where $\mathrm{Adj}[N_k + \Phi_k]$ and $\det[N_k + \Phi_k]$ are continuous functions, and $\det[N_k + \Phi_k]$ is nonzero in a sufficiently small neighborhood $\mathcal{N}(p_k) \cap \mathbb{C}_+$. Since by assumption, $\det N(p_k) \neq 0$, $[N_k + \Phi_k]^{-1}$ is bounded in $\mathcal{N}(p_k) \cap \mathbb{C}_+$. Finally, D_k, as a polynomial matrix, is bounded in $\mathcal{N}(p_k) \cap \mathbb{C}_+$. Therefore sufficiency follows from (13). ◻

We now generate conditions which are equivalent to the conditions (9). With $\hat{G}$ defined by (3), let $I + F\hat{G}(s) = R(s) + I + F\hat{G}_\rho(s)$ for all s in $\mathbb{C}_+$, thus

$$R(s) = \sum_{k=1}^{\ell} F\, R_k(s) \quad \text{for all } s \text{ in } \mathbb{C} \tag{15}$$

where $R_k(s)$ is given by (4). Then there exist two right-coprime elements N and D of $\mathbb{C}^{n\times n}[s]$ such that

$$R(s) = N(s)\, D(s)^{-1} \quad \text{for all } s \text{ in } \mathbb{C}, \tag{16}$$

Since R has poles only at $p_1, p_2, \dots p_\ell$

$$\det D(s) = c \prod_{k=1}^{\ell} (s - p_k)^{\gamma_k} \tag{17}$$

where c is a nonzero constant. Now by appropriate column operations, D can be made to be column proper, thus for that case

$$\sum_{k=1}^{\ell} \gamma_k = \sum_{j=1}^{n} \delta_j \tag{18}$$

with the δ_j defined in Fact 2.

Finally if

$$\tilde{N}_k(s) \triangleq N(s) + [I + F\hat{G}_\rho(p_k)]\, D(s) \tag{19}$$

then for all $k = 1, 2, \ldots, \ell$ and for all $s \in \mathbb{C}_+$

$$I + \hat{F}G(s) = \tilde{N}_k(s)\, D(s)^{-1} + F[\hat{G}_\rho(s) - \hat{G}_\rho(p_k)]. \tag{20}$$

As before, (20) can be written in the form

$$I + F\hat{G}(s) = [\tilde{N}_k(s) + \Psi_k(s)]D(s)^{-1} \tag{21}$$

where $\Psi_k(s)$ is a continuous function in $\mathcal{N}(p_k) \cap \mathbb{C}_+$ and $\Psi_k(s) \to 0$ as $s \to p_k$.

<u>Lemma 2</u>. Let N_k and $\tilde{N}_k$ be the elements of $\mathbb{C}^{n\times n}[s]$ defined by (4a) and (19), then

$$\det N_k(p_k) \neq 0 \tag{9}$$

if and only if

$$\det \tilde{N}_k(p_k) \neq 0 \tag{22}$$

<u>Proof</u>. By (19), since N and D are right-coprime elements of $\mathbb{C}^{n\times n}[s]$, so are $\tilde{N}_k$ and D. A reasoning analogous to that of Lemma 1 establishes the following equivalence:

$$[I + F\hat{G}]^{-1} \text{ is bounded in } \mathcal{N}(p_k) \cap \mathbb{C}_+ \tag{10}$$

if and only if

$$\det \tilde{N}_k(p_k) \neq 0. \tag{22}$$

Hence Lemma 2 follows by transitivity.

Proof of Theorem 1.

$\Leftarrow$. By assumption, condition (8) holds and, for k = 1, ..., ℓ, condition (9) and its equivalent (22) hold. Observe that F is nonsingular and that $I - F\hat{H} = [I + F\hat{G}]^{-1}$ hence

$$\hat{H} \in \hat{\mathcal{A}}^{n\times n} \text{ if and only if } [I + F\hat{G}]^{-1} \in \hat{\mathcal{A}}^{n\times n}. \tag{23}$$

From (3), (4), (15) and (16)

$$\begin{aligned} I + F\hat{G} &= N D^{-1} + I + F\hat{G}_\rho = [N + (I + F\hat{G}_\rho)D]D^{-1} \\ &= L D^{-1} \end{aligned} \tag{24}$$

with obvious notation. The matrix-valued function L is continuous in Re s $\geq$ 0 and analytic in Re s > 0. In the analysis below we may assume that D is column proper: indeed if it were not, it could be made so by elementary column operations. (Of course the same elementary column operations have to be applied to N so that (16) be still valid). Consequently condition (18) holds. Define the matrix multiplier M by

$$M(s) \triangleq \text{diag}\{(s+1)^{\delta_1}, (s+1)^{\delta_2}, \ldots, (s+1)^{\delta_n}\} \tag{25}$$

where the δ_j are defined in Fact 2. Observe that because D is column proper and because of Fact 2, DM^{-1} and $NM^{-1} \in \hat{\mathcal{A}}^{n\times n} \cap \mathbb{C}^{n\times n}(s)$. With (23) in mind, we write (24) as follows

$$[I + F\hat{G}]^{-1} = [DM^{-1}]\,[LM^{-1}]^{-1}. \tag{26}$$

Since $\hat{G}_\rho \in \hat{\mathcal{A}}^{n\times n}$ we have

$$LM^{-1} = NM^{-1} + (I + F\hat{G}_\rho)\, D\, M^{-1} \in \hat{\mathcal{A}}^{n\times n}.$$

From [3], $[LM^{-1}]^{-1} \in \hat{\mathcal{A}}^{\,n\times n}$ if and only if

$$\inf_{\text{Re } s \geq 0} |\det[L(s)M(s)^{-1}]| > 0. \tag{27}$$

Thus sufficiency will be established if we show that (8) and (22) imply (27). By (26) we have

$$\det[L(s)M(s)^{-1}] = \det[I + F\hat{G}(s)].\;\; \det D(s) \cdot [\det M(s)]^{-1}$$

and, by (17), (18) and (25), we obtain for all s in $\mathbb{C}_+$

$$\det[L(s)\; M(s)^{-1}] = \{c \prod_{k=1}^{\ell} \left(\frac{s-p_k}{s+1}\right)^{\gamma_k}\} \det[I + F\hat{G}(s)].$$

By assumption (8), the right hand side is bounded away from zero everywhere in $\mathbb{C}_+$, except possibly in some neighborhoods of the p_k's. To investigate the behavior in these neighborhoods, we use (24) to write

$$\det[L(s)\; M(s)^{-1}] = \det\{[N(s) + (I + F\hat{G}_\rho(s))D(s)]\; M(s)^{-1}\}$$

and we use (19) and (25) to obtain as $s \to p_k$

$$\det[L(s)\; M(s)^{-1}] \to \det[\tilde{N}_k(p_k)] \cdot \prod_{j=1}^{n} (p_k+1)^{-\delta_j}.$$

Assumption (9) and its equivalent (22) guarantee that the right hand side is nonzero. Therefore the continuous function $s \mapsto \det[L(s)\ M(s)^{-1}]$ is bounded away from zero in sufficiently small neighborhoods of the p_k's. Therefore (27) is established.

$\Rightarrow$. By assumption, $\hat{H} \in \hat{\mathcal{A}}^{n\times n}$, hence, by (23), $[I + F\hat{G}]^{-1} \in \hat{\mathcal{A}}^{n\times n}$ and is bounded in $\mathbb{C}_+$. Hence (9) follows for $k = 1, 2, \ldots, \ell$ in view of Lemma 1. Finally $\det[I + F\hat{G}(s)]^{-1} = \{\det[I + F\hat{G}(s)]\}^{-1} \in \hat{\mathcal{A}}$, hence is bounded in $\mathbb{C}_+$, so (8) follows, [9]. ¤

<u>Example</u>. Let $\mathrm{Re}\ p_1 > 0$ and $F = I$; assume that

$$[I + \hat{G}(s)] = \begin{bmatrix} \frac{s-p_1}{s+1} & .5e^{-s} \\ .2e^{-s} & \frac{s+1}{s-p_1} \end{bmatrix}.$$

Immediately

$$\det[I + \hat{G}(s)] = 1 - .1\, e^{-2s},$$

hence (8) holds by the graphical test [10, 14]. However we see that

$$N_1(p_1) = \left[\begin{array}{c|c} 0 & 0 \\ \hline .2e^{-p_1} & 1 + p_1 \end{array}\right],$$ hence $\det N_1(p_1) = 0$ and (9) does not hold;

hence $\hat{H}$ is unstable. This can be checked by direct calculation: indeed

$$[I + \hat{G}(s)]^{-1} = \frac{1}{1 - .1e^{-2s}} \left[\begin{array}{c|c} \frac{s+1}{s-p_1} & -\ .5e^{-s} \\ \hline -\ .2e^{-s} & \frac{s-p_1}{s+1} \end{array}\right]$$

is unstable since Re $p_1 > 0$. For this case $I + \hat{G}(s)$, $\hat{H}(s)$ and $[I + \hat{G}(s)]^{-1}$ have a pole at p_1. We emphasize that the condition $\det N_1(p_1) \neq 0$ is required because $\det[I + \hat{G}(p_1)] \neq 0$ is not sufficient to prevent $\hat{H}(s)$ and $[I + \hat{G}(s)]^{-1}$ from having a pole at p_1.

IV. Interpretation of Condition (9) in Terms of McMillan Degree Theory.

Let Re $p_k > 0$. Let $\Omega\,[\det[I + F\hat{G}];\, p_k]$ denote the order of the pole p_k of $\det[I + F\hat{G}]$. Let R denote the strictly proper element of $\mathbb{C}^{n\times n}(s)$ defined by (15). Let $\psi[R](s)$ be the least monic common denominator of <u>all</u> minors of R, let $\Delta[R]$ denote the McMillan degree of R and $\Delta[R;\, p_k]$ denote the maximal order of the pole p_k in <u>all</u> minors of R; $\Delta[R;\, p_k]$ is the McMillan degree of the pole p_k of R and $\Delta[R] = \sum_{k=1}^{\ell} \Delta[R;\, p_k]$, [24].

<u>Corollary 1</u>. Let $\hat{G}$ be defined by (3) and let N_k be defined by Fact 3. Under these conditions, for any $k \in \{1, 2, \ldots, \ell\}$ for which Re $p_k > 0$,

$$\det N_k(p_k) \neq 0 \tag{9}$$

if and only if

$$\Omega[\det[I + F\hat{G}];\, p_k] = \Delta[R;\, p_k] = \text{rank} \begin{bmatrix} R_{k(m_k-1)} & \cdots & R_{k1} & R_{k0} \\ \vdots & ⋰ & ⋰ & 0 \\ \vdots & ⋰ & ⋰ & \vdots \\ R_{k1} & ⋰ & & 0 \\ R_{k0} & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

where in the triangular Hankel matrix, the $R_{k\alpha}$'s are the coefficient matrices of R_k defined by (4).

<u>Proof</u>. From (17) and (19) - (20):

$$c \prod_{\beta=1}^{\ell} (s-p_\beta)^{\gamma_\beta} \det[I + F\hat{G}(s)] = \det[\tilde{N}_k(s) + F(\hat{G}_\rho(s) - \hat{G}_\rho(p_k))D(s)],$$

hence

$$\lim_{s \to p_k} (s - p_k)^{\gamma_k} \det[I + \hat{G}(s)] = \{c \prod_{\substack{\beta=1 \\ \beta\neq k}}^{\ell} (p_k - p_\beta)^{\gamma_\beta}\}^{-1} \det \tilde{N}_k(p_k)$$

and for any integer $\varepsilon_k > \gamma_k$

$$\lim_{s \to p_k} (s - p_k)^{\varepsilon_k} \det[I + F\hat{G}(s)] = 0.$$

Hence by Lemma 2, (9) is true if and only if $\Omega[\det[I + F\hat{G}]; p_k] = \gamma_k$. Let [A, B, C, 0] be any minimal realization of the strictly proper element R of $C^{n\times n}(s)$ defined by (15) then, because of the coprime factorization (16), det D(s) = c det(sI - A), [18, 20]. Now det(sI - A) = $\psi[R](s) = \prod_{k=1}^{\ell} (s - p_k)^{\Delta[R;\, p_k]}$, [25, 23, 24]. Comparing with (17) it follows that $\gamma_k = \Delta[R; p_k]$. The last equality of (29) is established by observing that in (15) F is nonsingular and by using a result of [26].

<u>Remarks</u>. (a) The conditions of Theorem 9-10 of [25], are a special case of those of Theorem 1 above: indeed for the lumped case, i.e. $G_\rho(s) \in \hat{\mathcal{A}}^{n\times n} \cap \mathbb{C}^{n\times n}(s)$, we have

$$\Delta[\hat{G}; p_k] = \Delta[R; p_k] \text{ for } k = 1, 2, \ldots \ell$$

and Chen's conditions are equivalent to (8) and

$$\Omega[\det[I + F\hat{G}]; p_k] = \Delta[\hat{G}; p_k] \quad \text{for } k = 1, 2, \ldots \ell.$$

(b) The conditions of Theorem 3 of [11] can be shown to be equivalent to (8) and (29) (for details see [15]).

(c) The equivalence of Corollary 1 is stated only for interior poles (i.e. Re $p_k > 0$). It is still valid for poles on the boundary (i.e. Re $p_k = 0$) whenever $\hat{G}_\rho(s)$ is meromorphic in a neighborhood of such p_k. This will always be the case for differential delay systems.

(d) Graphical interpretation. Consider the case where Re $p_k > 0$, for $k = 1, 2, \ldots, \ell$, and $G_\rho(\cdot) \in L^1_{n\times n}[0,\infty) \subset \mathcal{A}^{n\times n}$; then the necessary and sufficient conditions for stability, i.e. (8) and (9) for $k = 1, 2, \ldots \ell$, will hold if and only if the Nyquist diagram of $\omega \mapsto \det[I + F\hat{G}(j\omega)]$, (from $\omega = -\infty$ to $\omega = \infty$), encircles the origin of complex plane $\sum_{k=1}^{\ell} \Delta[R; p_k]$ times in counterclockwise sense. This graphical interpretation can be extended to cases where $G(\cdot) \in \mathcal{A}^{n\times n}$.

V. Conclusions

For the continuous-time, multivariable, linear, time-invariant, feedback system described by (1), (2) and (3), the conditions (8) and (9) are necessary and sufficient for stability. The conditions (9) can be replaced by either (22) or (29) and prevent the closed-loop transfer function to be unbounded in small neighborhoods of the unstable open-loop poles. The nature of the proofs makes it clear that the same type of conditions apply to the discrete-time case.

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