ON THE USE OF OPTIMIZATION ALGORITHMS IN THE DESIGN OF LINEAR SYSTEMS

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Abstract

This paper discusses how optimization techniques can be used to deal with various constraints occurring in the design of a feedback system.

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Abstract

This paper discusses how optimization techniques can be used to deal with various constraints occurring in the design of a feedback system.

1. Introduction

In recent years there has been a considerable resurgence of activity in the area of linear systems. As a result, it has been shown [1, 2, 3] that, by using state feedback, poles of a linear system can be assigned to desired values, and that by using both state feedback and pre-compensation it is possible to obtain desired transfer functions [4, 5]. Furthermore, the decoupling problem has been largely resolved [6, 7, 8, 9] and classical Bode-Nyquist type design techniques have been extended to multi-input - multi-output systems [10, 11, 12]. The two main virtues of this work are that it has revealed very powerful compensation configurations, and that it enables us to utilize the accumulated experience with single input - single output systems,while its two main shortcomings are that it provides only very indirect guidance for the selection of compensators to meet design specifications and that it does not permit us to take into account hard constraints (e.g., energy and amplitude limitations) and saturation effects.

In circuit and filter design, difficulties arising from hard constraints and sophisticated specifications are frequently resolved by using nonlinear programming techniques. Control system designers, however, seem to make only very elementary use of these techniques. The reason for this may be that the popular penalty function algorithms do not perform well in the context of control system design, and that most control designers are not familiar with the more complex methods of feasible directions, which, though slow, do not succumb to ridge paralysis. The purpose of this note is purely tutorial: to exhibit the flexibility of problem formulation which results from the use of optimization techniques and to point out to designers a few of the more suitable feasible directions algorithms. Almost all of these nonlinear programming algorithms can be found in [13] or in any other modern books on nonlinear programming. In addition, [13] shows how these algorithms can be extended to control applications. References [14-19] present a few optimal control algorithms in which the author has some confidence because of personal experience. Of course, these do not represent all the available algorithms for optimal control. For an exhaustive survey of optimal control algorithms, the reader is referred to [20].

2. Canonical Forms

For a design problem to be solvable by existing optimization algorithms, it must be cast in the form of either a nonlinear programming problem or of an optimal control problem. The following two canonical problems are solvable by methods of feasible directions, such as those described in [13, 14, 15, 16, 17].

The first problem is:

$$\min \{f^0(z) \mid f^j(z) \leq 0,\ j = 1, 2, .., s;\ Wz = \xi\} \tag{1}$$

where $f^j : \mathbb{R}^\ell \to \mathbb{R}^1$ for $j = 0, 1, 2, .., s$, are continuously differentiable functions, W is a $q \times \ell$ matrix and $\xi \in \mathbb{R}^q$.

The second problem is:

$$\min \{\int_{t_0}^{t_1} h^0\ (x(t),z(t),t)\ dt \mid$$

$$\frac{d}{dt} x(t) = h(x(t),z(t),t),\ t \in [t_0,t_1];\ q_0(x(t_0)) \leqq 0;$$

$$q_1(x(t_1)) \leqq 0;\quad q_c(z(t)) \leqq 0\} \tag{2}$$

where $h^0 : \mathbb{R}^n \times \mathbb{R}^\mu \times \mathbb{R}^1 \to \mathbb{R}^1$ and $h: \mathbb{R}^n \times \mathbb{R}^\mu \times \mathbb{R}^1 \to \mathbb{R}^n$ are piecewise continuously differentiable, while the vector valued functions q_0, q_1 and q_c are assumed to be continuously differentiable. Computation and convergence are facilitated by choosing constraint functions with convex components.

Thus, to solve a design problem by optimization algorithms, we must write it in the form (1) or (2), show that the functions satisfy the hypothesis stated, and, most important, we must make sure that all the quantities these algorithms utilize can be readily computed. These quantities always include values of the functions and, usually, also values of gradients. We shall now give a simple example to illustrate the power of the suggested approach.

3. Design of a State Feedback Compensator.

Suppose we are given a dynamical system described for small signals by the equations

$$\frac{d}{dt}\, x(t) = Ax(t) + Bu(t) \tag{3}$$

$$y(t) = Cx(t) \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ and A, B, C are constant matrices of appropriate dimensions. Suppose that we are also aware of the fact that when signals become large, saturation effects set in, as well as other nonlinearities, so that the large signal representation of the system is of the form

$$\frac{d}{dt}\, x(t) = h(x(t),u(t)) \tag{5}$$

$$y(t) = g(x(t)) \tag{6}$$

where, in the simplest case, h and g are piecewise linear functions, but, in any event, h and g must be assumed to be piecewise continuously differentiable. Now suppose that we wish to approximate the desired output $v(t)$, $t \in [0,T]$ (where T may be infinite) and that we wish to use a linear state feedback law. As is customary in dealing with linearizations, we assume that $x(0) = 0$. However, we may set $x(0) \neq 0$ if we wish. To define this problem as an optimization problem, we must introduce a performance index and define our constraints, if any. Thus, we set

$$u(t) = v(t) - Kx(t) \tag{7}$$

where K is an $m \times n$ matrix, to be computed so that (3) becomes

$$\frac{d}{dx} x(t) = (A - BK)\, x(t) + Bv(t) \tag{8}$$

and (5) becomes

$$\frac{d}{dt} x(t) = h(x(t), v(t) - Kx(t)) \tag{9}$$

Let k_i denote the i^{th} column of K, i=1, 2, .., n, then it is sometimes convenient to rewrite (9) in the expanded form

$$\frac{d}{dt} x(t) = h(x(t), v(t) - \sum_{i=1}^{n} x_i(t)\, k_i) \tag{10}$$

where $x_i(t)$ is the i^{th} component of x(t). Treating K as a vector $(k_1^T, k_2^T, .., k_n^T)$, we introduce the cost function

$$f^0(K) = \frac{1}{2}\int_0^T \| Q(t)[y(t) - v(t)] \|^2 \, dt \tag{11}$$

where Q(t) is a continuous, symmetric, positive definite matrix whose choice can be considered as a design parameter. The function $f^0(\cdot)$ is differentiable,and one way of calculating the gradient of $f^0(\cdot)$ at K, taking nonlinearities into account, is as follows.

(i) Solve (5) for $u(t) \equiv v(t) - Kx(t)$ and compute y(t) for $t \in [0,T]$.

(ii) Solve the adjoint equation

$$\frac{d}{dt} p(t) = - \frac{\partial}{\partial x} [h(x(t), v-Kx(t)]^T p(t)$$

$$+ [\frac{\partial g}{\partial x} (x(t))]^T Q(t)^2 [y(t)-v(t)], \quad t \in [0,T]$$

$$p(T) = 0 \tag{12}$$

(iii) Compute $\nabla_{k_i} f(K)$ according to the formula

$$\nabla_{k_i} f^0(K) = - \int_0^T \left[\frac{\partial h}{\partial k_i} (x(t), v(t) - \sum_{i=1}^{n} x_i(t) k_i)\right]^T p(t)\, dt. \tag{13}$$

We can now examine what constraints can be introduced. First, suppose that there are limitations on the gains k_{ij}. These can be expressed as

$$|k_{ij}| \leq \gamma_{ij},\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n, \tag{14}$$

i.e., $k_{ij} - \gamma_{ij} \leq 0$, $-k_{ij} - \gamma_{ij} \leq 0$.

Then we may have an energy constraint of the form

$$f^1(K) = \int_0^T \frac{1}{2} \|Q_e(t)\ u(t,K)\|^2\, dt \leq E \tag{15}$$

where $Q_e(t)$ is a continuous, symmetric, positive definite matrix and $u(t,K) = v(t) - Kx(t)$.

The gradients of $\nabla f^1_{k_i}(K)$ can be calculated in a manner similar to the one used for $\nabla_{k_i} f^0(K)$.

Next, we have the requirement that the resulting small signal model is stable, which can be expressed by the inequality

$$f^2(K) = \int_0^\infty \frac{1}{2} \|e^{t(A-BK)}\|_F^2 \, dt \leq S^\dagger \tag{16}$$

where $\|\cdot\|_F$ is the Frobenius norm of a matrix (i.e., $\|(d_{ij})\|_F^2 = \sum_{i,j} d_{ij}^2$)

and S is a large number which can be used to control the stability margin of the linearized system (8), (4). Let I_{ij} be an $m \times n$ matrix such that its ij^{th} element is 1 and all other elements are zero. Then

$$\frac{d}{dk_{ij}} e^{t(A-BK)} = - te^{t(A-BK)} BI_{ij} \tag{17}$$

and hence

$$\frac{\partial}{\partial k_{ij}} f^2(K) = \sum_{\alpha,\beta} \int_0^\infty - t[e^{t(A-BK)}]_{\alpha\beta} \, [e^{t(A-BK)} BI_{ij}]_{\alpha\beta} \, dt \tag{18}$$

where $[\ ]_{ij}$ is the ij^{th} element of the bracketed matrix. Thus, $f^2(K)$ is differentiable and hence can be included as a constraint function.#

Finally, we may wish to impose a model matching type constraint on the transfer function of the linearized system (8), (4). Thus, let

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†In practice, the upper limit of integration is not infinite but an adequately large number.
Alternatively, set $f^2(K) = \frac{1}{2} \|P(K)\|_F^2$, with $P(K) = \int_0^\infty e^{t(A-BK)} e^{t(A-BK)^T} dt$. Then, (see p. 65 of [21]) P(K) satisfies $P(K)(A-BK)^T + (A-BK)P(K) + I = 0$. This relationship can be used to calculate both P(K) and its partial derivatives.

$\hat{H}_d(s)$ be an $m \times n$ desired transfer function, then we can require that

$$f^3(K) = \int_0^{\bar{\Omega}} \|C(j\omega I - (A-BK))^{-1} B - \hat{H}_d(j\omega)\|_F^2 d\omega \le \tau \tag{19}$$

where τ is a tolerance factor and $\|\cdot\|_F$ is extended to complex valued matrices according to $\|d_{ij}\|_F^2 = \sum_{i,j} d_{ij} d_{ij}^*$, where $*$ denotes the complex conjugate. To calculate the partial derivatives of $f^3(K)$ we need the fact that

$$\frac{\partial}{\partial k_{ij}} C(j\omega I-(A-BK))^{-1}B$$

$$= - C(j\omega I-(A-BK))^{-1} BI_{ij}(j\omega I-(A-BK))^{-1}B \tag{20}$$

The rest of the calculation is obvious.

Assuming that the transfer function $\hat{H}_d(s)$ can be obtained for some feedback law K_0, i.e., $\hat{H}_d(s) = C(sI-(A-BK_0))^{-1}B$, an initial value for S in (16) can be $S = 100f^2(K_0)$.

At this stage, it should be pointed out that the criterion (11) is by far not the only one that can be used in the design. For example, we may set

$$f^0(K) = \alpha \max_{t\in I} \|y(t)-v(t)\| + \beta t_r + \gamma \int_0^T \|Q(t)\ (y(t)-v(t)\|^2\, dt \tag{21}$$

where $\alpha \ge 0$, $\beta \ge 0$, $\gamma \ge 0$, $I = [0,T]$ and t_r is a rise time criterion, e.g., it is the first time t such that $\|y(t) - v(t)\| \le \frac{1}{10}\|v(t)\|$.

In the case where $f^0(K)$ is defined as in (21), it is no longer possible to obtain analytic expressions for its partial derivatives. However, these can be approximated numerically and algorithms such as the ones described in [18, 19] can be applied. These algorithms automatically set the precision for the numerical differentiation. Obviously, in this case computation will be more expensive than with the simpler cost (11). Note that since y(t) is calculated as a numerical sequence as part of the integration process, the calculation of $f^0(K)$ is not particularly encumbered by the two seemingly difficult terms in (21).

To apply methods of feasible directions to a problem such as (1), it is necessary to calculate a feasible starting point first. Although these algorithms include a subprocedure for finding such a point, it may be preferable in our case to start instead with a compensator K_0 calculated by techniques such as those described in [1, 2, 3].

4. Conclusion

The example in the preceeding section demonstrates the flexibility in design which results from the use of optimization techniques. It should be clear by now that if we had wished, we could have allowed K to be a function of time. This would result in an optimal control problem of the form (2). In this case, however, constraints such as (19) would be meaningless. Also, it is possible to approach in the same manner, time varying systems. The gradient formulas for this case can be developed by following the pattern set in Section 3. Finally, it should be clear that state feedback is not the only configuration which can be optimized. Any fixed configuration, dynamic linear compensator can be optimized provided that it is specified by a reasonable number of parameters (say, up to 50).

In conclusion, to apply the optimization approach to system design, (i) one selects a finite parameter configuration to optimize, (ii) one selects a cost function f^0 and constraint functions, $f^1, f^2, \ldots, f^m$ and (iii) one applies an optimization algorithm to calculate the required parameters. If upon simulation one discovers that the behavior of the system does not correspond to one's intuitive idea of good performance, then one modifies the cost function or constraints and one starts over again. In the end, as in any other approach, this empirical part simply cannot be avoided.

It is hoped that this note will help clarify the extent to which optimization methods apply to feedback system design.

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