INTERCONNECTION, DECOMPOSITION, AND SYNTHESIS OF NONLINEAR N-PORTS

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INTERCONNECTION, DECOMPOSITION, AND SYNTHESIS OF NONLINEAR N-PORTS

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ABSTRACT

This paper represents a sequel to the recent work on algebraic n-ports [1]. It relates the external representation of nonlinear n-ports in terms of the constitutive relations of the internal elements composing the n-ports, and the topological matrices defining the elements' interconnection. Various closure properties associated with interconnection of nonlinear 1-ports are presented. The problem of synthesis leads naturally to a consideration of canonic decomposition of nonlinear n-ports into basic building blocks. In particular, every voltage-controlled (current-controlled) resistive 2-port is shown to be realizable in a canonic form consisting of a series (parallel) connection between a reciprocal nonlinear 2-port, and a new class of nonlinear 2-ports called quasi-antireciprocal 2-ports. This basic result is then generalized to allow the synthesis of a very large class of nonlinear n-ports in terms of only two building blocks; namely, reciprocal n-ports and quasi-antireciprocal n-ports. Moreover, the class of quasi-antireciprocal n-ports is shown to be realizable in terms of only nonlinear resistive 1-ports, reciprocal 2-ports, and gyrators.

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I. INTRODUCTION

A theory of nonlinear n-ports has recently been presented strictly from a "black box" approach [1]. In this paper, we "open" the box and express the external black box characterizations of nonlinear algebraic n-ports in terms of their internal constituents. This study is motivated by the observation that most n-ports with n>2 do not represent intrinsic devices but are often created conceptually or physically through an interconnection of 1-ports and 2-ports.[1] Such "composite" n-ports have served as invaluable tools for both analysis [2,3] and synthesis [4] purposes. Since any (n+1)-terminal element can be considered as a grounded n-port and since any n-port can be considered as a system of "n" controlled 1-ports -- i.e., n 1-ports with mutual couplings among the port variables -- there is no loss of generality in assuming that our n-ports consist of an interconnection of only 1-ports and controlled 1-ports. We adopt this point of view in this paper because each n-port can then be represented topologically by n separate branches, thereby allowing standard graph-theoretic techniques to be brought to bear.

In Section II, we consider the problem of expressing the external representation of an n-port in terms of the constitutive relations of the internal elements and the topological matrices defining the interconnection. To understand why appropriate conditions must be imposed not only on the nature of the element's nonlinearity, but also on the network topology,

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[1] To economize on symbols, the index "n" is used in a generic sense. Hence two n-ports need not have the same number of ports. We will assume that whenever necessary, our n-ports are provided with internal isolation transformers so that arbitrary interconnection among n-ports will not introduce circulation currents.

consider the following examples:

Example 1. This example illustrates the types of composite v-i characteristics of a 1-port that could arise as a result of a simple series connection of two 1-ports R_1 and R_2 as shown in Fig. 1. Observe that with only two segments per v_j-i_j curve, $j = 1, 2$, it is possible to obtain a composite curve having self-intersections, branching segments, and a finite perimeter as in (a), (b), and (c), respectively. More complicated v-i curves can be obtained as in (d)-(f) with only three segments per v_j-i_j curves, $j = 1,2$.

When interconnections of multiports are involved, the problem is even more acute as indicated in Examples 2, 3, 4 and 5.

Example 2. The circuit in Fig. 2(a) consists of a 2-port N (in fact, a current-controlled current source) and two 1-ports R_A and R_B characterized by the v_j-i_j curves shown in Figs. 2(b) and (c). The composite v-i curve shown in Fig. 2(d) consists of the union of a closed line segment and two isolated points. It is easy to see that if we replace R_B by a short circuit, the composite v-i curve would reduce to 3 isolated points; namely, (0,-1), (0,0), and (0,1). If we also replace R_A by an open circuit, the composite v-i curve degenerates into one point at the origin and becomes a nullator [4].

Example 3. The circuit in Fig. 3(a) consists of a 2-port N and three 1-ports. With the v_j-i_j curves shown in Figs. 3(b) and (c) for R_A and R_B, the composite v-i relationship covers an entire area, as shown in Fig. 3(d). In fact, if we replace R_A and R_C by short circuits, and R_B by an open circuit, the composite v-i relationship would cover the entire v-i plane and become a norator [4].

Example 4. To show that any unicursal v-i curve [5] $v = v(\rho)$, $i = i(\rho)$,

$\rho \in R^1$, including those with self-intersections and cusps, could be synthesized, we introduce a new linear 3-port N_ρ in Fig. 4(a)--called a <u>unicursal</u> 3-port--characterized by the hybrid matrix:

$$\begin{bmatrix} i_1 \\ v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \\ v_3 \end{bmatrix}$$

If we connect two voltage-controlled resistors R_A and R_B (characterized by $i_A = g_A(v_A)$ and $i_B = g_B(v_B)$ respectively) across ports 2 and 3 of N_ρ, as shown in Fig. 4(b), we obtain a 1-port with the composite v_1-i_1 curve $i_1 = g_A(\rho)$, $v_1 = g_B(\rho)$. To show that the unicursal 3-port is nothing exotic, we offer a simple realization using only a voltage-controlled voltage source N_a and a current-controlled voltage source N_b, as shown in Fig. 4(c).

<u>Example 5</u>. To show that v-i curves more complicated than unicursal curves could be realized, we introduce yet another linear 3-port in Fig. 5(a)--called an <u>intersection</u> 3-port--characterized by: $v_1 = v_2 = v_3$ and $i_1 = -i_2 = -i_3$. If we connect two 1-ports R_A and R_B across ports 2 and 3 of an intersection 3-port as in Fig. 5(b), then the composite v_1-i_1 relationship of the resulting 1-port is simply the <u>point set intersection</u> of the v-i curves of R_A and R_B. Some examples of v-i curves that can be realized with the intersection 3-port are shown in Figs. 5(c)-(e) where R_A and R_B are connected as in Fig. 5(b). Notice that in all cases, R_A and R_B are unicursal resistors which can be realized by a unicursal 3-port, a v-controlled resistor and an i-controlled resistor.[2] To show that even an intersection 3-port is not too

[2]It has been shown in [6] that every voltage-controlled or current-current 1-port resistor can be realized by a linear 2-port called an LTC (linear transformation converter) and an <u>increasing</u> 1-port resistor.

exotic, we offer a simple realization in Fig. 5(f) using only three common 2-ports, namely: two voltage-controlled voltage sources N_a and N_c, and a current-controlled current source N_b.

Observe that there exist an infinitely many distinct pairs of R_A and R_B for realizing a prescribed v_1-i_1 relationship. What is required is merely that the point set <u>intersection</u> of the v-i characteristics of R_A and R_B consists of only the points of the prescribed v_1-i_1 relationship, no more and no less.

In Section III, we will be mainly concerned with the closure properties of 1-port elements. That is, we are trying to answer the basic question: "Does a composite 1-port resulting from an interconnection[3] of 1-ports all having property P also possess property P?" This question is of great importance in the <u>qualitative analysis</u> of nonlinear networks.

In Section IV, we introduce an important class of nonlinear n-ports--called <u>quasi-antireciprocal</u> n-ports--which represents a generalization of anti-reciprocal n-ports. It is shown that every i-controlled or v-controlled nonlinear 2-port can be realized using only a reciprocal 2-port and a quasi-antireciprocal 2-port. Properties of quasi-antireciprocal n-ports are investigated. In addition, certain classes of nonlinear n-ports are shown to be realizable by an appropriate interconnection of a reciprocal n-port and a quasi-antireciprocal n-port.

Throughout this paper, we let R^k denote the Euclidean k-space and $\|\cdot\|$ the usual Euclidean norm. Vectors are denoted by lower case letters and matrices by upper case letters. A column vector $x = [x_1, x_2, \cdots, x_n] \in R^n$

[3] All networks considered in this paper are assumed to be <u>connected</u> and <u>non-separable</u>.

is partitioned into $x = [x_a, x_b]$ if $x_a = [x_1, x_2, \cdots, x_k] \in R^k$ and $x_b = [x_{k+1}, x_{k+2}, \cdots, x_n] \in R^{n-k}$. In addition, we let $\dot{x}$ denote the time-derivative of x, $J_f(x)$ denote the Jacobian matrix of a function f: $R^n \to R^n$ at the point x (when n=1, we use $f'(x)$ instead of $J_f(x)$) and $\langle \cdot , \cdot \rangle$ denote the Euclidean inner product.

In this paper, we will use the symbols v and i instead of ξ and η as used in [1] for general discussion even though most results are applicable to algebraic n-ports. To distinguish results that are applicable to algebraic n-ports from those that are applicable to n-port resistors only, we will use the word "elements" for results associated with the former case[4] and the word "resistors" for results associated with the latter case.

II. EXPLICIT REPRESENTATION OF NONLINEAR N-PORTS VIA TOPOLOGICAL MATRICES

In this section, we will derive conditions which guarantee a "composite" n-port resulting from an arbitrary interconnection of 1-ports and <u>controlled</u> 1-ports to possess a <u>hybrid representation</u>. Additional conditions will be imposed to guarantee that the n-port is either <u>increasing</u> or <u>non-decreasing</u> [1].[5] The hybrid representations will be derived in <u>explicit</u> topological forms. Unlike the results presented in [7] which are valid only for <u>reciprocal</u> networks,

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[4] Throughout this paper, we assume that all charge variables and flux-linkage variables at the initial time are zero. Under this assumption, any topological equations which apply for resistive n-ports would also apply for algebraic n-ports. Otherwise, appropriate constants of integration will have to be introduced.

[5] The class of <u>increasing</u> n-ports is the appropriate generalization of 1-ports characterized by a <u>strictly</u> monotonically increasing v-i curve. Similarly, the class of <u>non-decreasing</u> n-ports is the generalization of 1-ports with a monotone increasing v-i curve. It is important to observe that for n>1, the class of increasing n-ports is a <u>proper</u> subset of <u>homeomorphic</u> n-ports [1]. The study of <u>increasing</u> n-ports is of basic importance because most results involving strictly monotone increasing v-i curves have a natural generalization <u>only</u> for this class of n-ports but not for homeomorphic n-ports.

our results in this section allow coupled elements, such as controlled sources, and are therefore much more general.

For a given (connected and nonseparable) network $\mathcal{N}$ with m nodes and b branches, let T be a spanning tree and L be the corresponding co-tree. Let T1 be any subset of T; L1 be the subset of L that form loops with elements in T1; T2 = T - T1 and L2 = L - L1, where the symbol "-" denotes set subtraction. Let t, t_1, t_2, ℓ, ℓ_1, and ℓ_2 be correspondingly the number of elements in T, T1, T2, L, L1, and L2. Then $t = t_1 + t_2 = m-1$ and $\ell = \ell_1 + \ell_2 = b - m+1$. By numbering the branches of $\mathcal{N}$ in the order of L1, L2, T1 and T2, we obtain the following topological equations [7-9]:[6]

$$v_L = \begin{bmatrix} v_{L1} \\ v_{L2} \end{bmatrix} = - \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} v_{T1} \\ v_{T2} \end{bmatrix} \triangleq - B_T v_T \tag{1}$$

$$i_T = \begin{bmatrix} i_{T1} \\ i_{T2} \end{bmatrix} = \begin{bmatrix} B_{11}^t & B_{21}^t \\ 0 & B_{22}^t \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix} \triangleq B_T^t i_L \tag{2}$$

where

(i) B_{11}, B_{21} and B_{22} are topological submatrices of dimensions $\ell_1 \times t_1$, $\ell_2 \times t_1$ and $\ell_2 \times t_2$ respectively;

(ii) The superscript "t" of a matrix A indicates the transpose matrix of A;

(iii) $z_{L1} = [z_1, z_2, \cdots, z_{\ell_1}]$, $z_{L2} = [z_{\ell_1+1}, z_{\ell_1+2}, \cdots, z_\ell]$, $z_{T1} = [z_{\ell+1}, z_{\ell+2}, \cdots, z_{\ell+t_1}]$, $z_{T2} = [z_{\ell+t_1+1}, z_{\ell+t_1+2}, \cdots, z_b]$, $z_L = [z_{L1}, z_{L2}]$, $z_T = [z_{T1}, z_{T2}]$ and $z = [z_L, z_T]$ where the generic symbol z stands for either v or i.

Let $\hat{\mathcal{N}}$ be an n-port resulting from an interconnection of 1-ports, i.e. two-terminal elements. Since each port of $\hat{\mathcal{N}}$ can be considered as a branch as far as network topology is concerned, we will call these branches port

[6] For convenience, we simply use "0" to denote a zero matrix of appropriate dimension.

branches. Let $\mathcal{N}$ be the network consisting of the 1-ports inside $\hat{\mathcal{N}}$ and the port branches of $\hat{\mathcal{N}}$, and let $\mathcal{G}$ be the graph representing $\mathcal{N}$. Then $\mathcal{G} = \mathcal{G}_p \cup \mathcal{G}_q$, where $\mathcal{G}_p$ represents the set of n "external" port branches of $\hat{\mathcal{N}}$ and $\mathcal{G}_q$ represents the set of "internal" 1-ports contained in $\hat{\mathcal{N}}$. We will write, for example, $\mathcal{G}_p$ = L2 if the port branches consist of the set of L2 elements associated with a particular tree T under consideration.[7]

Due to the sign convention that we have adopted for ports and port-branches as shown in Fig. 6, the port currents of $\hat{\mathcal{N}}$ are the negative of the port-branch currents of $\mathcal{G}_p$ of $\mathcal{N}$, while the port voltages of $\hat{\mathcal{N}}$ are equal to the port-branch voltages of $\mathcal{G}_p$ of $\mathcal{N}$.

Let Z be a set of elements in $\mathcal{N}$ numbered from α to β inclusively. Let $x_Z = [x_\alpha, x_{\alpha+1}, \cdots, x_\beta]$, $y_Z = [y_\alpha, y_{\alpha+1}, \cdots, y_\beta]$, $v_Z = [v_\alpha, v_{\alpha+1}, \cdots, v_\beta]$ and $i_Z = [i_\alpha, i_{\alpha+1}, \cdots, i_\beta]$; where $[x_Z, y_Z] = \sum[v_Z, i_Z]$ for some permutation matrix $\sum$ described in [1], v_k and i_k are the port variables associated with the k^{th} element in Z. We will say that the elements in Z can be represented by $y_Z = \hat{y}_Z(x_Z)$ if each element in Z, say the k^{th} element, can be represented by

$$y_k = \hat{y}_k(x_\alpha, x_{\alpha+1}, \cdots, x_\beta) = \hat{y}_k(x_Z) \text{ for } k = \alpha, \alpha+1, \cdots, \beta$$

and $\hat{y}_Z(\cdot) = [\hat{y}_\alpha(\cdot), \hat{y}_{\alpha+1}(\cdot), \cdots, \hat{y}_\beta(\cdot)]$. It is clear that if element k in Z is not a controlled 1-port, then $y_k = \hat{y}_k(x_k)$.

Since the theorems to be presented in this section share a number of common hypotheses, we have collected these conditions in Table 1 in order to conserve space. Observe that each hypothesis is identified by a literal-numeric code. Hence, rather than stating the entire hypothesis in a par-

[7]Throughout Section II, we will use $\hat{\mathcal{N}}$ to denote an n-port, $\mathcal{N}$ the corresponding network, $\mathcal{G}$ the graph of $\mathcal{N}$, $\mathcal{G}_p$ the port branches of $\mathcal{N}$, and $\mathcal{G}_q$ the graph representing the elements inside $\hat{\mathcal{N}}$.

ticular theorem, we simply state the corresponding identification code. The symbols in Table 1 which have not been previously defined carry the following meanings:

(i) k is a nonnegative integer;

(ii) $L2a \cup L2b = L2$, $L1a \cup L1b = L1$, $T2a \cup T2b = T2$ and $T1a \cup T1b = T1$;

(iii) ℓ_{2a}, ℓ_{2b}, ℓ_{1a}, ℓ_{1b}, t_{2a}, t_{2b}, t_{1a} and t_{1b} denote the number of elements in L2a, L2b, L1a, L1b, T2a, T2b, T1a, T1b, respectively;

(iv) pd, psd, and upd stand for positive definite, positive semidefinite and uniformly positive definite respectively.[8]

Theorem 1a.

Suppose there exists a tree T such that $\mathcal{G}_P = T1 \cup L2$.

(H_1^a) If $L1_{A(k)}$ and $T2_{A(k)}$ hold, then $\hat{\mathcal{N}}$ admits a C^k-hybrid representation.

(H_2^a) If $L1_{A(k)}$, $T2_{A(k)}$, $T2_B$ and $L1_B$ hold, then $\hat{\mathcal{N}}$ admits a C^k-hybrid representation and $\hat{\mathcal{N}}$ is nondecreasing.

(H_3^a) If $L1_{A(k)}$, $T2_{A(k)}$, $L1_C$, $T2_C$, M_A and M_E are satisfied, then $\hat{\mathcal{N}}$ admits a hybrid representation and $\hat{\mathcal{N}}$ is an increasing n port, where $n = t_1 + \ell_2$.

Proof. Let $i_P = [i_{P1}, i_{P2}] = [-i_{T1}, -i_{L2}]$ and $v_P = [v_{P1}, v_{P2}] = [v_{T1}, v_{L2}]$, where i_P and v_P are the port-current and port-voltage vectors of $\hat{\mathcal{N}}$, respectively. Then (1) and (2) become:

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[8] Let S be a subset of R^m and A(x) be an n×n matrix. A(x) is positive definite {positive semidefinite; uniformly positive definite} at x if $z^t A(x) z > 0$ $\forall\, z \neq 0 \in R^n$ {$z^t A(x) z \geq 0\ \forall\, z \in R^n$; there exists a constant $c > 0$ such that $z^t A(x) z \geq c\|z\|^2\ \forall\, z \in R^n$}. $A(\cdot)$ is said to be positive definite {positive semidefinite; uniformly positive definite} on S if A(x) is positive definite {positive semidefinite; uniformly positive definite} for every $x \in S$.
In addition, a function f: $R^n \to R^n$ is said to be non-decreasing {increasing, uniformly increasing} if $\alpha(x_a, x_b) \triangleq \langle f(x_a) - f(x_b), x_a - x_b \rangle \geq 0\ \forall\, x_a, x_b \in R^n$ {$> 0\ \forall\, x_a \neq x_b \in R^n$; there exists a positive constant c such that $\alpha(x_a, x_b) \leq c\|x_a - x_b\|^2$}.

$$v_{L1} = -B_{11}v_{P1} \tag{3a}$$

$$v_{P2} = -B_{21}v_{P1} - B_{22}v_{T2} \tag{3b}$$

$$i_{P1} = -B_{11}^t i_{L1} + B_{21}^t i_{P2} \tag{3c}$$

$$i_{T2} = -B_{22}^t i_{P2} \tag{3d}$$

Substituting $L1_{A(k)}$ and (3a) into (3c); and $T2_{A(k)}$ and (3d) into (3b), we obtain:

$$\boxed{\begin{aligned} i_{P1} &= -B_{11}^t \hat{i}_{L1}\circ(-B_{11}v_{P1}) + B_{21}^t i_{P2} \triangleq \hat{i}^a_{P_1}(v_{P1},i_{P2}) \\ v_{P2} &= -B_{21}v_{P1} - B_{22}\hat{v}_{T2}\circ(-B_{22}^t i_{P2}) \triangleq \hat{v}^a_{P2}(v_{P1},i_{P2}) \end{aligned}} \begin{matrix} (4a) \\ (4b) \end{matrix}$$

That is, $\hat{\mathcal{N}}$ admits a <u>hybrid</u> representation, given by (4).

To prove (H_2^a), let $[v_{P1},i_{P2}]$ and $[v'_{P1},i'_{P2}]$ be any two distinct points in R^n where $n = t_1 + \ell_2$. Then, we have:

$$\alpha(v_{P1},i_{P2};v'_{P1},i'_{P2}) \triangleq \left\langle \begin{bmatrix} \hat{i}^a_{P1}(v_{P1},i_{P2})-\hat{i}^a_{P1}(v'_{P1},i'_{P2}) \\ \hat{v}^a_{P2}(v_{P1},i_{P2})-\hat{v}^a_{P2}(v'_{P1},i'_{P2}) \end{bmatrix}, \begin{bmatrix} v_{P1}-v'_{P1} \\ i_{P2}-i'_{P2} \end{bmatrix} \right\rangle$$

$$= \langle -B_{11}^t[\hat{i}_{L1}\circ(-B_{11}v_{P1})-\hat{i}_{L1}\circ(-B_{11}v'_{P1})], v_{P1}-v'_{P1} \rangle$$

$$+ \langle B_{21}^t[i_{P2}-i'_{P2}], v_{P1}-v'_{P1} \rangle - \langle B_{21}[v_{P1}-v'_{P1}], i_{P2}-i'_{P2} \rangle$$

$$+ \langle -B_{22}[\hat{v}_{T2}\circ(-B_{22}^t i_{P2})-\hat{v}_{T2}\circ(-B_{22}^t i'_{P2})], i_{P2}-i'_{P2} \rangle$$

$$= \langle [\hat{i}_{L1}\circ(-B_{11}v_{P1})-\hat{i}_{L1}\circ(-B_{11}v'_{P1})], [-B_{11}v_{P1}-(-B_{11}v'_{P1})] \rangle$$

$$+ \langle \hat{v}_{T2}\circ(-B_{22}^t i_{P2})-\hat{v}_{T2}\circ(-B_{22}^t i'_{P2})], [-B_{22}^t i_{P2}-(-B_{22}^t i'_{P2})] \rangle \tag{5}$$

By assumptions $T2_B$ and $L1_C$, $\alpha(v_{P1},i_{P2};v'_{P1},i'_{P2}) \geq 0 \;\forall\; [v_{P1},i_{P2}]$ and $[v'_{P1},i'_{P2}]$ in R^n. Hence $\hat{\mathcal{N}}$ is a <u>nondecreasing</u> n-port.

To prove (H_3^a), assumptions M_A and M_E imply that it is not possible to have

$$B_{11}v_{P1} = B_{11}v'_{P1} \text{ and } B_{22}^t i_{P2} = B_{22}^t i'_{P2}$$

whenever $[i_{P1}, v_{P2}] \neq [i'_{P1}, v'_{P2}]$ due to the maximal column rank requirements of the matrices B_{11} and B_{22}^t. That is

$$\begin{bmatrix} B_{11}v_{P1} \\ B_{22}^t i_{P2} \end{bmatrix} \neq \begin{bmatrix} B_{11}v'_{P1} \\ B_{22}^t i'_{P2} \end{bmatrix} \quad \text{whenever} \quad \begin{bmatrix} v_{P1} \\ i_{P2} \end{bmatrix} \neq \begin{bmatrix} v'_{P1} \\ i'_{P2} \end{bmatrix}, \tag{6}$$

Together, assumptions $T2_C$, $L1_C$ and (6) imply that $\alpha(v_{P1}, i_{P2}; v'_{P1}, i'_{P2}) > 0$ $\forall\ [v_{P1}, i_{P2}] \neq [v'_{P1}, i'_{P2}]$ in R^n. Hence, $\hat{\mathcal{N}}$ is an increasing n-port. Q.E.D.

Theorem 1b.

Suppose there exists a tree T such that $\mathcal{G}_P = T1 \cup L2b$.

(H^b) If $T2_{A(k+1)}$, $T2_B$, $L1_{A(k+1)}$ and $L2a_{A(k+1)}$ hold, then $\hat{\mathcal{N}}$ admits a hybrid representation.

Proof. With $i_{P1} = -i_{T1}$, $i_{P2} = -i_{L2b}$, $v_{P1} = v_{T1}$ and $v_{P2} = v_{L2b}$, (1) and (2) become:

$$v_{L1} = -B_{11}v_{P1} \tag{7a}$$

$$v_{L2a} = -B_{21a}v_{P1} - B_{22a}v_{T2} \tag{7b}$$

$$v_{P2} = -B_{21b}v_{P1} - B_{22b}v_{T2} \tag{7c}$$

$$i_{P1} = -B_{11}^t i_{L1} - B_{21a}^t i_{L2a} + B_{21b}^t i_{P2} \tag{7d}$$

$$i_{T2} = B_{22a}^t i_{L2a} - B_{22b}^t i_{P2} \tag{7e}$$

where

$$B_{21} \triangleq \begin{bmatrix} B_{21a} \\ B_{21b} \end{bmatrix} \quad \text{and} \quad B_{22} \triangleq \begin{bmatrix} B_{22a} \\ B_{22b} \end{bmatrix}$$

and the partitioned matrices are of appropriate dimensions. Substituting $L1_{A(k+1)}$ and (7a) into (7d); $T2_{A(k+1)}$ and (7e) into (7c); and $L2a_{A(k+1)}$, $T2_{A(k+1)}$ and (7e) into (7b); we obtain:

$$i_{P1} = -B_{11}^t \hat{i}_{L1} \circ (-B_{11} v_{P1}) - B_{21a}^t i_{L2a} + B_{21b}^t i_{P2} \tag{8a}$$

$$v_{P2} = -B_{21b} v_{P1} - B_{22b} \hat{v}_{T2} \circ (B_{22a}^t i_{L2a} - B_{22b}^t i_{P2}) \tag{8b}$$

$$\hat{v}_{L2a}(i_{L2a}) = -B_{21a} v_{P1} - B_{22a} \hat{v}_{T2} \circ (B_{22a}^t i_{L2a} - B_{22b}^t i_{P2}) \tag{8c}$$

Let $f\colon R^{\ell_{2a}} \times R^{t_2} \to R^{\ell_{2a}}$ be the C^{k+1} function defined by

$$f(i_{L2a}, -B_{22b}^t i_{P2}) \triangleq \hat{v}_{L2a}(i_{L2a}) + B_{22a} \hat{v}_{T2} \circ (B_{22a}^t i_{L2a} - B_{22b}^t i_{P2})$$

$$= -B_{21a} v_{P1}. \tag{9a}$$

By assumptions $T2_B$ and $L2a_{A(k+1)}$, $\partial f(i_{L2a}, -B_{22b}^t i_{P2})/\partial i_{L2a}$ is upd on $R^{\ell_{2a}} \times R^{t_2}$. By a lemma in [10], there is a C^{k+1} function $g\colon R^{\ell_{2a}} \times R^{t_2} \to R^{\ell_{2a}}$ defined by

$$i_{L2a} = g(-B_{21a} v_{P1}, -B_{22b}^t i_{P2}) \tag{9b}$$

and $g(\cdot,\cdot)$ satisfies the following:

$$g(f(i_{L2a}, -B_{22b}^t i_{P2}), -B_{22b}^t i_{P2}) \triangleq i_{L2a}$$

$$f(g(-B_{21a} v_{P1}, -B_{22b}^t i_{P2}), -B_{22b}^t i_{P2}) \triangleq -B_{21a} v_{P1}.$$

Substituting (9b) into (8a) and (8b), we obtain:

$$\boxed{\begin{aligned} i_{P1} &= -B_{11}^t \hat{i}_{L1} \circ (-B_{11} v_{P1}) - B_{21a}^t g(-B_{21a} v_{P1}, -B_{22b}^t i_{P2}) + B_{21b}^t i_{P2} \triangleq \hat{i}_{P1}^b(v_{P1}, i_{P2}) && (10a) \\ v_{P2} &= -B_{22b} \hat{v}_{T2} \circ (B_{22a}^t g(-B_{21a} v_{P1}, -B_{22b}^t i_{P2}) - B_{22b}^t i_{P2}) - B_{21b} v_{P1} \triangleq \hat{v}_{P2}^b(v_{P1}, i_{P2}) && (10b) \end{aligned}}$$

Thus, $\hat{\mathcal{N}}$ admits a hybrid representation given by (10). Q.E.D.

Theorem 1c.

Suppose there exists a tree T such that $\mathcal{G}_p = L2$.

(R_1^a) If $T1_{A(k+1)}$, $T2_{A(k+1)}$, $L1_{A(k+1)}$, $L1_D$ and N_C are satisfied, then $\hat{\mathcal{N}}$ has a C^{k+1} i-controlled representation.

(R_2^a) If, in addition to the assumptions in (R_1^a), $T2_B$ holds, then $\hat{\mathcal{N}}$ admits a C^{k+1} i-controlled representation and $\hat{\mathcal{N}}$ is nondecreasing.

(R_3^a) If, in addition to the assumptions in (R_1^a), one of the following two conditions holds:

(i) $T2_B$ and M_D are satisfied

(ii) $T2_C$ and M_E are satisfied

Then $\hat{\mathcal{N}}$ is an increasing i-controlled n-port with a C^{k+1} representation function.

Proof. In this case, we have $i_P = -i_{L2}$ and $v_P = v_{L2}$. Thus, (1) and (2) become:

$$v_{L1} = -B_{11}v_{T1} \tag{11a}$$

$$v_P = -B_{21}v_{T1}-B_{22}v_{T2} \tag{11b}$$

$$i_{T1} = B_{11}^t i_{L1}-B_{21}^t i_P \tag{11c}$$

$$i_{T2} = -B_{22}^t i_P \tag{11d}$$

Substituting $T1_{A(k+1)}$, $T2_{A(k+1)}$ and (11d) into (11b); and $L1_{A(k+1)}$ and (11a) into (11c), we obtain:

$$v_P = -B_{21}\hat{v}_{T1}(i_{T1})-B_{22}\hat{v}_{T2}\circ(-B_{22}^t i_P) \tag{12a}$$

$$i_{T1} = B_{11}^t \hat{i}_{L1}\circ(-B_{11}\hat{v}_{T1}(i_{T1}))-B_{21}^t i_P \tag{12b}$$

Rearranging (12b), we obtain:

$$\tilde{f}(i_{T1}) \triangleq i_{T1}-B_{11}^t \hat{i}_{L1}\circ(-B_{11}\hat{v}_{T1}(i_{T1})) = -B_{21}^t i_P \tag{13a}$$

where $\tilde{f}\colon R^{t_1} \to R^{t_1}$ is clearly a C^{k+1} map and the Jacobian matrix of $\tilde{f}(\cdot)$ is[9]

[9]We let 1_k denote the identity matrix of order k.

$$J_{\tilde{f}}(i_{T1}) = 1_{t_1} + B_{11}^t J_{\hat{i}_{L1}} \circ (-B_{11}\hat{v}_{T1}(i_{T1}))B_{11}J_{\hat{v}_{T1}}(i_{T1}) \quad (13b)$$

Since $J_{\hat{v}_{T1}}(\cdot)$ is C^k and pd on R^{t_1}, $[J_{\hat{v}_{T1}}(\cdot)]^{-1}$ exists and is C^k and pd on R^{t_1} [10]. Postmultiplying (13b) by $[J_{\hat{v}_{T1}}(\cdot)]^{-1}$, we obtain:

$$A(i_{T1}) \triangleq J_{\tilde{f}}(i_{T1})[J_{\hat{v}_{T1}}(i_{T1})]^{-1}$$

$$= [J_{\hat{v}_{T1}}(i_{T1})]^{-1} + B_{11}^t J_{\hat{i}_{L1}} \circ (-B_{11}\hat{v}_{T1}(i_{T1}))B_{11} \quad (14a)$$

It follows from $T1_{A(k+1)}$ and $L1_D$ that $A(i_{T1})$ is pd and symmetric $\forall\ i_{T1} \in R^{t_1}$. Since both $A(i_{T1})$ and $J_{\hat{v}_{T1}}(i_{T1})$ are pd and symmetric, it follows from Lemma 4.2.5 of [11] that all eigenvalues of

$$A(i_{T1})J_{\hat{v}_{T1}}(i_{T1}) = J_{\tilde{f}}(i_{T1}) \quad (14b)$$

are positive and real $\forall\ i_{T1} \in R^{t_1}$. Hence,

$$\det J_{\tilde{f}}(i_{T1}) > 0 \qquad \forall\ i_{T1} \in R^{t_1} \quad (15)$$

Together, (15) and N_A imply that $\tilde{f}$: $R^{t_1} \to R^{t_1}$ is a C^{k+1}-diffeomorphic onto mapping[10] [12,13]. That is, there exists a unique C^{k+1} function $\tilde{g}$: $R^{t_1} \to R^{t_1}$ such that

$$i_{T1} = \tilde{g} \circ (-B_{21}^t i_P) \quad (16)$$

where $g(\cdot)$ is the inverse function of $f(\cdot)$ on R^{t_1}. Substituting (1b) into (12a), we obtain:

$$\boxed{v_P = -B_{21}\hat{v}_{T1} \circ \tilde{g} \circ (-B_{21}^t i_P) - B_{22}\hat{v}_{T2} \circ (-B_{22}^t i_P) \triangleq \hat{v}_P^a(i_P)} \quad (17)$$

[10] A function f: $R^n \to R^n$ is said to be a C^k-diffeomorphic onto mapping on R^n if its inverse function g: $R^n \to R^n$ exists and both $f(\cdot)$ and $g(\cdot)$ are C^k onto functions on R^n.

This proves (R_1^a).

Let $h_1(\cdot)$ and $h_2(\cdot)$ be two mappings from R^n into R^n, where $n = \ell_2$, defined by

$$h_1(i_P) \triangleq -B_{21}\hat{v}_{T1}\circ\tilde{g}\circ(-B_{21}^t i_P) \tag{18a}$$

$$h_2(i_P) \triangleq -B_{22}\hat{v}_{T2}\circ(-B_{22}^t i_P) \tag{18b}$$

Clearly, $h_1(\cdot)$ and $h_2(\cdot)$ are C^{k+1} functions on R^n. Taking derivatives, we obtain:

$$J_{h_1}(i_P) = B_{21}J_{\hat{v}_{T1}}\circ(\tilde{g}(-B_{21}^t i_P))J_{\tilde{g}}\circ(-B_{21}^t i_P)B_{21}^t \tag{19a}$$

$$J_{h_2}(i_P) = B_{22}J_{\hat{v}_{T2}}\circ(-B_{22}^t i_P)B_{22}^t \tag{19b}$$

It follows from assumption $T2_B$ that $\hat{v}_{T2}(\cdot)$ is nondecreasing on R^{t_2} and $J_{\hat{v}_{T2}}(\cdot)$ is psd on R^{t_2} [14,15]. Hence $J_{h_2}(\cdot)$ is psd on R^n. That is, $h_2(\cdot)$ is nondecreasing on R^n [14,15].

To discuss $h_1(\cdot)$, let us first note that $\tilde{g}\circ\tilde{f}(i_{T1}) = i_{T1}$ and hence $J_{\tilde{g}}(\tilde{f}(i_{T1}))J_{\tilde{f}}(i_{T1}) = 1_{t_1}$. Therefore, $J_{\tilde{g}}(-B_{21}^t i_P) = [J_{\tilde{f}}(i_{T1})]^{-1}$, with the relationship of i_{T1} and $-B_{21}^t i_P$ constrained by (13a) and (16). With the help of (14b), further manipulations give:

$$J_{\hat{v}_{T1}}(i_{T1})J_{\tilde{g}}(-B_{21}^t i_P) = J_{\hat{v}_{T1}}(i_{T1})]J_{\tilde{f}}(i_{T1})]^{-1}$$

$$= J_{\hat{v}_{T1}}(i_{T1})\ [A(i_{T1})J_{\hat{v}_{T1}}(i_{T1})]^{-1}$$

$$= A(i_{T1}) = A\circ(\tilde{g}(-B_{21}^t i_P)).$$

Since $A(i_{T1})$ has been shown to be pd $\forall\ i_{T1}$, $J_{v_{T1}}(\tilde{g}(-B_{21}^t i_P))J_{\tilde{g}}(-B_{21}^t i_P)$ is pd $\forall\ i_P \in R^n$. Hence, $J_{h_1}(\cdot)$ is psd on R^n. Thus, $h_1(\cdot)$ is nondecreasing on R^n. Consequently, $\hat{v}_P^a(\cdot)$ as defined in (11) is nondecreasing on R^n, and this proves (R_2^a).

Suppose now the assumptions of (R_1^a) and condition (i) hold, then B_{21}^t is of maximal column rank. By Lemma 4.2.7 of [11] and the fact that $J_{\hat{v}_{T1}}\circ(\breve{g}(-B_{21}^t i_P))J_{\breve{g}}\circ(-B_{21}^t i_P)$ is pd $\forall\ i_P \in R^n$, $J_{h_1}(\cdot)$ is pd on R^n. Hence, $h_1(\cdot)$ is increasing on R^n [14,15]. It has been shown that $h_2(\cdot)$ is nondecreasing on R^n. Hence, $\hat{v}_P^a(\cdot)$ is increasing on R^n.

On the other hand, if the assumptions of (R_1^a) and condition (ii) hold, then $h_1(\cdot)$ is nondecreasing by previous result and $h_2(\cdot)$ is increasing on R^n since, for any two distinct points i_P^a and i_P^b in R^n, we have:

$$
\begin{aligned}
&\langle h_2(i_P^a)-h_2(i_P^b),\ i_P^a-i_P^b \rangle \\
&= \langle -B_{22}[\hat{v}_{T2}\circ(-B_{22}^t i_P^a) - \hat{v}_{T2}\circ(-B_{22}^t i_P^b)],\ i_P^a-i_P^b \rangle \\
&= \langle \hat{v}_{T2}\circ(-B_{22}^t i_P^a)-\hat{v}_{T2}\circ(-B_{22}^t i_P^b),\ -B_{22}^t i_P^a - (-B_{22}^t i_P^b) \rangle \\
&= \langle \hat{v}_{T2}(i_{T2}^a) - \hat{v}_{T2}(i_{T2}^b),\ i_{T2}^a-i_{T2}^b \rangle > 0
\end{aligned}
$$

The last inequality was due to the fact that a). $\hat{v}_{T2}(\cdot)$ is increasing on R^{t_2} and b). the rank of B_{22}^t equals its number of columns by assumption M_E, and hence $i_{T2}^a \triangleq B_{22}^t i_P^a \neq i_{T2}^b \triangleq B_{22}^t i_P^b$ whenever $i_P^a \neq i_P^b$. Consequently, $\hat{v}_P^a(\cdot)$ is increasing on R^n. This proves (R_3^a). Q.E.D.

<u>Remarks 1.</u>

Several additional theorems analogous to Theorems 1a, b, and c can be formulated to account for other combinations and interconnections of 1-ports and controlled 1-ports. To conserve space, these results, including Theorems 1a, b, and c, are summarized in Table 2.[11] The interpretation of this table

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[11] For ease of reference, each assertion in Theorems 1a, b, and c, as well as those listed in Table 2, are identified with a literal code $H_i^{(j)}$, $R_i^{(j)}$, and $G_i^{(j)}$, j = a, b, c, d; i = 1, 2, 3, where the letters are chosen to correspond to the <u>hybrid</u> (H), <u>resistance</u> (R), or conductance (G) representation of the n-port under consideration.

is as follows: If there exists a tree T of $\mathcal{N}$ such that the assumptions of a row in the left column are satisfied, then the conclusions of the corresponding row in the center column are true. The symbols used under the "assumptions" column are explained in Table 1. Those under the "conclusions" column are explained in the rightmost column "Notations and Definitions" of Table 2. Examples of this interpretation can be made by referring to Theorem 1. Note that we have further partitioned the submatrices of the fundamental loop matrix B_T as follows:

$$B_{11} = \begin{bmatrix} B_{11a} \\ B_{11b} \end{bmatrix} = \begin{bmatrix} B_{11c} & B_{11d} \end{bmatrix}, \quad B_{21} = \begin{bmatrix} B_{21a} \\ B_{21b} \end{bmatrix} = \begin{bmatrix} B_{21c} & B_{21d} \end{bmatrix}, \quad B_{22} = \begin{bmatrix} B_{22a} \\ B_{22b} \end{bmatrix} = \begin{bmatrix} B_{22c} & B_{22d} \end{bmatrix}$$

where the partitioned matrices are of appropriate dimensions.

(ii) In Table 2, the proofs for cases (H^a), (H^b) and (R^a) are shown in Theorem 1. The proof for (R^b) is similar to that of (R^a). Case (R^c) is a special case of (R^b). Cases (G^a), (G^b) and (G^c) are dual cases of (R^a), (R^b) and (R^c), respectively. The proofs for cases (H^c) and (H^d) are similar to the proofs for cases (H^b) and (H^a), respectively.

(iii) Concerning the norm conditions,

(a) N_A is true if $B_{22}\hat{v}_{T2}(\cdot)$ is a globally bounded mapping[12] on R^{t_2}.

(b) N_B is true if either $\hat{v}_{L2}(\cdot)$ satisfies the norm condition[12] and $B_{22}\hat{v}_{T2}(\cdot)$ is globally bounded; or $J_{\hat{v}_{L2}}(\cdot)$ is upd on R^{ℓ_2} and $\hat{v}_{T2}(\cdot)$ is nondecreasing on R^{t_2}.

---

[12] A function $f\colon R^n \to R^n$ is said to be globally bounded if there exists a constant $c > 0$ such that

$$\sup_{x \in R^n} \|f(x)\| \leq c$$

The function f is said to satisfy the norm condition if

$$\lim_{\|x\| \to \infty} \|f(x)\| = \infty.$$

(c) N_C is true if $B_{11}^t \hat{i}_{L1}(\cdot)$ is globally bounded on R^{ℓ_1}.

(d) N_D is true if either $\hat{i}_{T1}(\cdot)$ satisfies the norm condition and $B_{11}^t \hat{i}_{L1}(\cdot)$ is globally bounded; or $J_{\hat{i}_{T1}}(\cdot)$ is upd on R^{t_1} and $\hat{i}_{L1}(\cdot)$ is non-decreasing on R^{ℓ_1}.

(iv) It should be noted that the <u>maximal</u> column rank requirements in those cases involving the <u>increasing property</u> of n-ports are necessary. These conditions guarantee that the port-branches do not form loops or cutsets. It is clear that if the port-branches form a loop or a cutset, then the Jacobian matrix of the function describing the n-port in question will always be singular. That is, the representation function of the n-port can never be increasing on R^n. A case in point is as follows:

<u>Example 6</u>. Consider the 3-port $\hat{\mathcal{N}}$ shown in Fig. 7(a). The corresponding graph $\mathcal{G}$ is shown in Fig. 7(b). Let the tree T = {branches 4, 5 and 6}. Hence the corresponding co-tree L = {branches 1, 2 and 3} = $\mathcal{G}_P$. Obviously $T_{A(k)}$ and T_B are satisfied. Hence case (R_2^c) implies that $\hat{\mathcal{N}}$ can be represented by an i-controlled representation as

$$v_P \triangleq \begin{bmatrix} v_{P_1} \\ v_{P_2} \\ v_{P_3} \end{bmatrix} = \begin{bmatrix} r_4+r_5 & -r_5 & r_4 \\ -r_5 & r_5+r_6 & r_6 \\ r_4 & r_6 & r_4+r_6 \end{bmatrix} \begin{bmatrix} i_{P_1} \\ i_{P_2} \\ i_{P_3} \end{bmatrix} \triangleq Ri_P$$

Since the matrix R defined above is singular, $\hat{\mathcal{N}}$ can not be an increasing 3-port. An examination of case (R_3^c) shows that all conditions are satisfied except the condition M_F since in this case

$$B_T^t = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

is of rank 2 and is not equal to the number of columns (which is 3 for this example).

III. CLOSURE PROPERTIES

In [1], we have classified algebraic n-ports in terms of four distinguishing properties: (1) constitutive relation (resistor, inductor, capacitor, memristor, etc. (2) global mathematical property (parametrizable, v-controlled, i-controlled, non-decreasing, increasing, uniformly increasing, strongly uniformly increasing, proper, bijective, etc. (3) local mathematical property (reciprocal and anti-reciprocal) and (4) circuit-theoretic property (passive, active, lossless, and non-energic). Now suppose all internal elements of a composite n-port $\hat{\mathcal{N}}$ are known to possess a given property, say all of them are passive. A fundamental question to raise is whether the composite n-port $\hat{\mathcal{N}}$ also possesses the passivity property. The answer in this case is of course yes. However, not all properties are preserved under arbitrary interconnections. Before we present some examples to illustrate this point, let us define the notion of "closure" more precisely.

Definition 1. Closure Property

A property P is said to be closed relative to some prescribed internal constraint K if any composite n-port $\hat{\mathcal{N}}$ containing elements having property P and satisfying the prescribed internal constraint K also possesses property P. If no internal constraint is necessary, then the class of n-ports having property P is said to be closed.

It can be shown that passive n-ports are closed since arbitrary interconnection of passive n-ports always result in another passive n-port. However, the following examples show that most other classes of n-ports are not closed.

Example 7. Examples illustrating non-closure properties

(a) Let $\hat{\mathcal{N}}$ be a 1-port resulting from a series connection of two algebrais 1-ports; namely; a resistor and a capacitor as shown in Fig. 8(a). The composite 1-port $\hat{\mathcal{N}}$ is clearly not an algebraic n-port. Hence, algebraic n-ports are not closed.

(b) Case (d) of Example 1 shows that v-controlled n-ports are not closed.

(c) By taking the dual [16] of Example (b), we conclude that i-controlled n-ports are not closed.

(d) Let $\hat{\mathcal{N}}$ be the 1-port shown in Fig. 8(b). Let $R_1 = 1\Omega$ and $R_2 = -1\Omega$. Then both R_1 and R_2 are proper 1-ports. However, $\hat{\mathcal{N}}$ is not. Hence, proper n-ports are not closed.

(e) Example (d) shows that homeomorphic n-ports are not closed.

(f) Example (d) shows that bijective n-ports are not closed.

(g) Let $\hat{\mathcal{N}}$ be the 1-port as shown in Fig. 8(b), where R_1 and R_2 are defined as follows:

$$R_1: \quad i_1 = 3 + \exp(v_1) \qquad \text{for } v_1 < 0$$
$$\qquad = 5 - \exp(-v_1) \qquad \text{for } v_1 \geq 0$$

$$R_2: \quad i_2 = \exp(v_2) \qquad \text{for } v_2 < 0$$
$$\qquad = 2 - \exp(-v_2) \qquad \text{for } v_2 \geq 0$$

It can be shown that both R_1 and R_2 are C^1-increasing 1-ports. However the v-i relationship of $\hat{\mathcal{N}}$ is an empty set in the v-i plane.[13] Hence, increasing n-ports are not closed.

(h) Example (g) shows that non-decreasing n-ports are not closed.

(i) Example (g) shows that regular n-ports are not closed.[14]

(j) Let $\hat{\mathcal{N}}$ be the 1-port shown in Fig. 8(b), with the characteristics of R_1 and R_2 shown in Figs. 8(c) and (d). It is clear that the resulting v-i relationship of $\hat{\mathcal{N}}$ is 1). An empty set (i.e. $\hat{\mathcal{N}}$ is singular) if $a < -1$ and 2). A horizontal line (i.e. $\hat{\mathcal{N}}$ is regular) if $a = -1$ as shown in Fig. 8(e). Hence, dense n-ports are not closed. To see that both R_1 and R_2 are dense 1-ports, the following is a C^1-parametrizable description of R_1:

$$
\begin{aligned}
v(\rho) &= \rho_1 && \forall\ \rho_1 \in R^1 \\
i(\rho) &= 1-(1+\rho_2)^{-1} && \text{for } \rho_2 \geq 0 \\
&= (1-\rho_2)^{-1}-1 && \text{for } \rho_2 \leq 0
\end{aligned}
$$

where $\rho = [\rho_1,\rho_2] \in R^2$. It can be shown easily that $J_{[v,i]}(\rho)$ is of rank 2 $\forall\ \rho \in R^2$.

(k) Let $\hat{\mathcal{N}}$ be the 1-port shown in Fig. 8(b), where the characteristics of R_1 and R_2 are shown in Figs. 8(f) and (g). Then the v-i relationship of $\hat{\mathcal{N}}$ is shown in Fig. 8(h). Clearly both R_1 and R_2 are active but $\hat{\mathcal{N}}$ is

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[13] A 1-port element characterized by an empty subset of $V \times I$, where $V = R^1$ and $I = R^1$ is called an empty resistor. Analytically, an empty resistor is characterized by a constitutive relation $f(i,v) = 0$ where f: $R^1 \times R^1 \to R^1$ is always positive or always negative. It has been shown in [11] that the concept of an empty resistor is required to prove that 1-port resistors are closed under an arbitrary interconnection, i.e., a 1-port resulting from an arbitrary interconnection of 1-port resistors is a resistor. An empty resistor is said to have a dimension [1] equal to -1, in accordance with the dimension theory [17]. A similar statement can be made for an empty inductor, capacitor, and memistor [18].

[14] An n-port is said to be regular {singular, dense} if it has dimension [1,19] equal to {less than, greater than} n.

passive. Hence, active n-ports are not closed.

The preceding examples demonstrate that additional constraints must be imposed in order for a composite n-port to preserve the common properties shared by its internal elements. Some of these constraints pertaining to increasing and non-decreasing n-ports are already formulated in Table 2. In the remainder of this section, we will restrict our attention to the formulation of closure property of composite 1-ports. Except for properties 1, 6, and 7, the generalization to composite n-ports where n > 1 is a much more difficult problem.

Property 1. Closure Property for Similar-Kind 1-Ports[15]

Let $\hat{\mathcal{N}}$ be a 1-port resulting from an arbitrary interconnection of 1-port resistors {inductors, capacitors, memristors}. Then $\hat{\mathcal{N}}$ is also a 1-port resistor {inductor, capacitor, memristor}.

Proof. The proof of this property is rather straightforward and can be found in [11].

Property 2. Closure Property for Non-decreasing 1-ports.

Let $\mathcal{N}$ be a network containing 1-port elements only. Suppose there exists a tree T such that all its tree branch elements are i-controlled and all its link elements are v-controlled. If the representation function of each 1-port in $\mathcal{N}$ is continuous and non-decreasing, then the driving-point characteristic of a 1-port $\hat{\mathcal{N}}_1$ {$\hat{\mathcal{N}}_2$}, created by a soldering-iron entry across any two nodes {plier-type entry through any wire} in $\mathcal{N}$, is i-controlled {v-controlled} and the representation function is continuous and

[15] By a straightforward generalization of the proof of Property 1 [11], one can show that the following is true: Let $\hat{\mathcal{N}}$ be an n-port ($n \geq 1$) resulting from an arbitrary interconnections of 1-port resistors {inductors, capacitors, memristors}. Then $\hat{\mathcal{N}}$ is an n-port resistor {inductor, capacitor, memristor}.

non-decreasing.

Proof. The proof of this property can be found in [20]. Some extensions of this property are given in a recent paper by Desoer and Wu [21].

Property 3. Closure Property for Increasing 1-Ports.

Let $\hat{\mathcal{N}}$ be a 1-port resulting from a series-parallel connection [16] of increasing 1-ports. Then $\hat{\mathcal{N}}$ is either an increasing 1-port with a v-controlled or i-controlled representation, or is an empty resistor.

Proof. Follows from the graphical construction techniques presented in [16].

Property 4. Closure Property for Increasing and Proper 1-Ports.[16]

Let $\mathcal{N}$ be a network containing increasing and proper 1-ports only. Let $\hat{\mathcal{N}}$ be a 1-port created either by a soldering-iron entry across any two nodes in $\mathcal{N}$, or by a plier-type entry through any wire in $\mathcal{N}$. Then $\hat{\mathcal{N}}$ is increasing and proper.

Proof. The proof of this property follows from results in [20] and [22] and is given in Appendix A.

Before we proceed to the discussion of uniformly increasing and strongly uniformly increasing 1-ports[17] [1], we would like to point out that strongly uniformly increasing 1-ports are uniformly increasing while uniformly increasing 1-ports are increasing and proper. However, the converse statements are

[16] A proper 1-port is represented by a function mapping from R^1 onto R^1. An increasing and proper 1-port is represented by an increasing function mapping from R^1 onto R^1.

It can be shown easily that an increasing and proper 1-port can be described both by $i = g(v)$ and $v = f(i)$, where $f(\cdot)$ and $g(\cdot)$ are inverse functions of each other on R^1.

Conversely, if a 1-port admits a continuous v-controlled {i-controlled} representation $i = g(v)$ $\{v = f(i)\}$ where $g(\cdot)$ $\{f(\cdot)\}$ is nondecreasing and if the 1-port is also i-controlled {v-controlled}, then the 1-port is increasing and proper.

[17] A C^1-uniformly increasing 1-port is represented by a function whose derivative is always greater than a positive constant. A C^1-strongly uniformly increasing 1-port is represented by a function whose derivative is always bounded by two finite positive constants.

not true as shown by the following examples:

Example 8.

(a) An example of an increasing proper 1-port which is not uniformly increasing.

Let $\hat{\mathcal{N}}$ be a 1-port represented by

$$v = f_1(i) = i(1 + 2i^2)^{-1/4}$$

Then $f_1'(i) = (1 + i^2)/(1 + 2i^2)^{-5/4}$. It can be shown that $|f_1(i)| \to \infty$ as $|i| \to \infty$ and $f'(i) > 0 \;\forall\; i \in R^1$. This implies that $\hat{\mathcal{N}}$ is increasing and proper. However, $\hat{\mathcal{N}}$ is not uniformly increasing since

$$f_1'(i) \to 0 \text{ as } |i| \to \infty.$$

(b) An example of a uniformly increasing 1-port which is not strongly uniformly increasing.

Let $\hat{\mathcal{N}}$ be a 1-port represented by

$$\begin{aligned} v = f_2(i) &= i + 1 \qquad \text{for } i < 0 \\ &= \exp(i) \qquad \text{for } i \geq 0 \end{aligned}$$

It can be shown that $f_2(\cdot)$ is C^1 on R^1, and $f_2'(i) \geq 1 \;\forall\; i \in R^1$. Hence $\hat{\mathcal{N}}$ is uniformly increasing. However $\hat{\mathcal{N}}$ is not strongly uniformly increasing since $f'(i)$ is not bounded on R^1.

Property 5. Closure Property for Strongly Uniformly Increasing 1-Ports.

Let $\hat{\mathcal{N}}$ be a 1-port containing C^{k+1} strongly uniformly increasing 1-ports. Then $\hat{\mathcal{N}}$ is C^{k+1} strongly uniformly increasing, where k is a nonnegative integer.

Proof. Since $\hat{\mathcal{N}}$ contains only strongly uniformly increasing 1-ports, each element inside $\hat{\mathcal{N}}$ can have both a C^{k+1} v-controlled and a C^{k+1} i-controlled representation with uniformly positive definite and bounded derivatives [1]. Clearly, the assumptions of case (G_3^b) of Table 2 are satisfied with T1 being the only port-branch. Hence $\hat{\mathcal{N}}$ has a C^{k+1} v-controlled representation:

$$i_P = \hat{i}_P(v_P) \triangleq -B_{11}^t \hat{i}_{L1} \circ (-B_{11}v_P) - B_{21}^t \hat{g} \circ (-B_{21}v_P) \tag{20}$$

where $g(\cdot)$ is the inverse function of the C^{k+1}-diffeomorphic onto mapping of

$$\hat{f}(i_{L2}) \triangleq \hat{v}_{L2}(i_{L2}) + B_{22}\hat{v}_{T2} \circ (B_{22}^t i_{L2}) = -B_{21}v_P \tag{21a}$$

on R^{ℓ_2}. Since $\hat{v}_{L2}(\cdot)$ and $\hat{v}_{T2}(\cdot)$ are uniformly increasing with bounded Jacobian matrices,

$$J_{\hat{f}}(i_{L2}) = J_{\hat{v}_{L2}}(i_{L2}) + B_{22}J_{\hat{v}_{T2}} \circ (B_{22}^t i_{L2})B_{22}^t \tag{21b}$$

is upd with bounded Jacobian matrix. This implies that $J_{\hat{g}}(w)$ is upd with bounded Jacobian matrix [23,24], where $w \triangleq -B_{21}v_P$.

Since B_{11} is $\ell_1 \times 1$ and B_{21} is $\ell_2 \times 1$ and the only port-branch T1 must belong to at least one fundamental loop, we know that at least one of these two matrices; namely, B_{11} and B_{21}, has rank 1. Taking derivative of (20), we obtain:

$$\hat{i}_P'(v_P) = B_{11}^t J_{\hat{i}_{L1}} \circ (-B_{11}v_P)B_{11} + B_{21}^t J_{\hat{g}} \circ (-B_{21}v_P)B_{21}$$

Since both $J_{\hat{i}_{L1}}(\cdot)$ and $J_{\hat{g}}(\cdot)$ are upd matrices with bounded entries and either B_{11} or B_{21} or both has a maximal column rank, i.e. one, $\hat{i}_P'(\cdot)$ is upd and bounded. That is, $\hat{\mathcal{N}}$ is strongly uniformly increasing. Note that, by a theorem in [23,24], we can also write $v_P = \hat{v}_P(i_P)$ where $\hat{v}_P(\cdot)$ is the inverse function of $\hat{i}\ (\cdot)$ defined in (20), and $\hat{v}\ (\cdot)$ is also a C^{k+1} uniformly increasing function with bounded derivative on R^1. Q.E.D.

Due to some technical difficulties, we are unable to present a closure property on uniformly increasing 1-ports. However, based on the works of [21] and the preceding two properties, we conjecture that uniformly increasing 1-ports are closed.

<u>Property 6</u>. Closure Property for Singular 1-Ports.[14]

Arbitrary interconnection of singular 1-ports is also singular.

<u>Proof</u>. Let $\hat{\mathcal{N}}$ be a 1-port containing singular 1-ports only. Since each element in $\hat{\mathcal{N}}$ is characterized either by an empty set (dimension = -1) or by a set of <u>isolated</u> points (dimension = 0) on the v-i plane, the driving point characteristic of $\hat{\mathcal{N}}$ contains at most a set of isolated points and hence is singular. Q.E.D.

<u>Property 7</u>. Closure Properties for Passive, Lossless and Nonenergic 1-Ports.

Let $\hat{\mathcal{N}}$ be an n-port containing passive {lossless, nonenergic} 1-ports. Then $\hat{\mathcal{N}}$ is passive {lossless, nonenergic}.

<u>Proof</u>. Let $i_p = [i_{p_1}, i_{p_2}, \cdots, i_{p_n}]$ and $v_p = [v_{p_1}, v_{p_2}, \cdots, v_{p_n}]$ be the port-current and port-voltage vectors, i_k and v_k be the k^{th} element's current and voltage, k = 1, 2, $\cdots$, b, where b is the total number of elements inside $\hat{\mathcal{N}}$. Then Tellegen's Theorem [25] states

$$\langle i_p, v_p \rangle = \sum_{j=1}^{b} v_j i_j \tag{22a}$$

$$\frac{1}{T}\int_0^T \langle i_p(t), v_p(t) \rangle \, dt = \frac{1}{T}\int_0^T \sum_{j=1}^{b} v_j(t) i_j(t) dt \tag{22b}$$

Equation (22a) implies the closure of lossless and nonenergic 1-ports while (22b) implies the closure of passive 1-ports. Q.E.D.

IV. DECOMPOSITION AND SYNTHESIS OF NONLINEAR N-PORTS

In [1], we have shown that every antireciprocal n-port with a hybrid representation (including i-controlled and v-controlled representations) can be described by $y = Hx + c$, where H is an n×n constant matrix, c an n-vector and $[x,y] = \Sigma[v,i]$ for some 2n × 2n permutation matrix Σ. Hence,

[14] Op. Cit.

it is clear that the well-known result "Every linear n-port can be decomposed into a reciprocal and an antireciprocal n-port" does not admit a nonlinear generalization. In this section, we introduce a new class of n-ports which permits a partial generalization of this result to nonlinear n-ports.

Definition 1. An n-port $\mathcal{N}$ with a C^1 i-controlled representation $v = f(i)$ {a C^1 v-controlled representation $i = g(v)$} is said to be quasi-antireciprocal if $J_f(i)$ {$J_g(v)$} can be written as:

$$J_f(i) = J_1(i) + J_2(i) \tag{23a}$$

$$\{J_g(v) = \hat{J}_1(v) + \hat{J}_2(v)\} \tag{23b}$$

where $J_1(\cdot)$ {$\hat{J}_1(\cdot)$} is a diagonal matrix and $J_2(\cdot)$ {$\hat{J}_2(\cdot)$} is a skew symmetric matrix. Notice that the decomposition in (23) is unique. We will henceforth call a matrix that admits the above decomposition a quasi-skew symmetric matrix.

The motivation for introducing quasi-antireciprocal n-ports is partly given by the next theorem.

Theorem 2.

Every 2-port $\mathcal{N}$ with a C^2 i-controlled {v-controlled} representation can be realized by a series {parallel} connection of a reciprocal 2-port $\hat{\mathcal{N}}$ and a quasi-antireciprocal 2-port $\tilde{\mathcal{N}}$.

Proof. Since the proofs for both cases are similar, we will present the i-controlled case only.

Let the desired 2-port $\mathcal{N}$ be represented by

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} f_1(i_1,i_2) \\ f_2(i_1,i_2) \end{bmatrix} = f(i) \tag{24}$$

where $i = [i_1, i_2]$. Let the quasi-antireciprocal 2-port $\tilde{\mathcal{N}}$ be represented by

$$\tilde{v} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} g_1(\tilde{i}_1, \tilde{i}_2) \\ g_2(\tilde{i}_1, \tilde{i}_2) \end{bmatrix} = g(\tilde{i}) \tag{25a}$$

and the reciprocal 2-port $\hat{\mathcal{N}}$ be represented by

$$\hat{v} = \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix} = \begin{bmatrix} h_1(\hat{i}_1, \hat{i}_2) \\ h_2(\hat{i}_1, \hat{i}_2) \end{bmatrix} = h(\hat{i}) \tag{25b}$$

The problem is to find $g(\cdot)$ and $h(\cdot)$ of (25) such that when $\hat{\mathcal{N}}$ and $\tilde{\mathcal{N}}$ are connected in series, as shown in Fig. 9, i.e., $i_1 = \hat{i}_1 = \tilde{i}_1$ and $i_2 = \hat{i}_2 = \tilde{i}_2$, the resulting 2-port is represented by (24).

Taking the Jacobian matrix of (24), we obtain:

$$J_f(i) = \begin{bmatrix} \dfrac{\partial f_1(i)}{\partial i_1} & \dfrac{\partial f_1(i)}{\partial i_2} \\ \dfrac{\partial f_2(i)}{\partial i_1} & \dfrac{\partial f_2(i)}{\partial i_2} \end{bmatrix}$$

Let

$$\frac{\partial g_1(\tilde{i})}{\partial \tilde{i}_2} \triangleq \frac{1}{2}\left[\frac{\partial f_1(i)}{\partial i_2} - \frac{\partial f_2(i)}{\partial i_1}\right]_{i=\tilde{i}} = \frac{1}{2}\left[\frac{\partial f_1(\tilde{i})}{\partial \tilde{i}_2} - \frac{\partial f_2(\tilde{i})}{\partial \tilde{i}_1}\right] \tag{26a}$$

$$\frac{\partial g_2(\tilde{i})}{\partial \tilde{i}_1} \triangleq - \frac{\partial g_1(\tilde{i})}{\partial \tilde{i}_2} = -\frac{1}{2}\left[\frac{\partial f_1(\tilde{i})}{\partial \tilde{i}_2} - \frac{\partial f_2(\tilde{i})}{\partial \tilde{i}_1}\right] \tag{26b}$$

$$\boxed{\begin{aligned} g_1(\tilde{i}_1, \tilde{i}_2) &= \frac{1}{2} f_1(\tilde{i}_1, \tilde{i}_2) - \frac{1}{2}\int_{a_2}^{\tilde{i}_2} [\partial f_2(\tilde{i}_1, z_2)\, \partial \tilde{i}_1] dz_2 + k_1(\tilde{i}_1) & \text{(27a)} \\ g_2(\tilde{i}_1, \tilde{i}_2) &= -\frac{1}{2}\int_{a_1}^{\tilde{i}_1} [\partial f_1(z_1, \tilde{i}_2)/\partial i_2] dz_1 + \frac{1}{2} f_2(\tilde{i}_1, \tilde{i}_2) + k_2(\tilde{i}_2) & \text{(27b)} \end{aligned}}$$

where a_1 and a_2 are arbitrary constants, $k_1(\cdot)$ and $k_2(\cdot)$ are arbitrary functions of $\tilde{i}_1$ and $\tilde{i}_2$ respectively. Let the quasi-antireciprocal 2-port $\tilde{\mathcal{N}}$ be represented by (27). It remains to find the representation $h(\cdot)$ of the reciprocal 2-port $\hat{\mathcal{N}}$. Let

$$\frac{\partial h_1(\hat{i})}{\partial \hat{i}_1} \triangleq \left.\frac{\partial f_1(i)}{\partial i_1}\right|_{i=\hat{i}} - \left.\frac{\partial g_1(\tilde{i})}{\partial \tilde{i}_1}\right|_{\tilde{i}=\hat{i}} = \frac{\partial f_1(\hat{i})}{\partial \hat{i}_1} - \frac{\partial g_1(\hat{i})}{\partial \hat{i}_1} \tag{28a}$$

$$\frac{\partial h_2(\hat{i})}{\partial \hat{i}_2} \triangleq \left.\frac{\partial f_2(i)}{\partial i_2}\right|_{i=\hat{i}} - \left.\frac{\partial g_2(\tilde{i})}{\partial \tilde{i}_2}\right|_{\tilde{i}=\hat{i}} = \frac{\partial f_2(\hat{i})}{\partial \hat{i}_2} - \frac{\partial g_2(\hat{i})}{\partial \hat{i}_2} \tag{28b}$$

From (28), we can certainly find $h(\cdot)$ (by integration). If the computed $h(\cdot)$ satisfies the following two equations

$$\frac{\partial h_1(\hat{i})}{\partial \hat{i}_2} = \frac{1}{2}\left[\frac{\partial f_1(i)}{\partial i_2} + \frac{\partial f_2(i)}{\partial i_1}\right]_{i=\hat{i}} = \frac{1}{2}\left[\frac{\partial f_1(\hat{i})}{\partial \hat{i}_2} + \frac{\partial f_2(\hat{i})}{\partial \hat{i}_1}\right] \tag{29a}$$

$$\frac{\partial h_2(\hat{i})}{\partial \hat{i}_1} = \frac{\partial h_1(\hat{i})}{\partial \hat{i}_2} \tag{29b}$$

then $\hat{\mathcal{N}}$ is clearly reciprocal and it can be shown easily that

$$J_f(i) = \left.J_g(\tilde{i})\right|_{\tilde{i}=i} + \left.J_h(\hat{i})\right|_{\hat{i}=i}$$

By an appropriate choice of the constants of integration in (28), we obtain

$$f(i) = \left.g(\tilde{i})\right|_{\tilde{i}=i} + \left.h(\hat{i})\right|_{\hat{i}=i}$$

Hence, it suffices to compute $h(\cdot)$ from (28) and show that the resulting function satisfies (29). Integrating (28), we obtain:

$$\boxed{\begin{aligned} h_1(\hat{i}_1,\hat{i}_2) &= \tfrac{1}{2}\, f_1(\hat{i}_1,\hat{i}_2) + \tfrac{1}{2}\int_{a_2}^{\hat{i}_2} [\partial f_2(\hat{i}_1,z_2)/\partial \hat{i}_1]dz_2 - k_1(\hat{i}_1) && \text{(30a)} \\ h_2(\hat{i}_1,\hat{i}_2) &= \tfrac{1}{2}\int_{a_1}^{\hat{i}_1} [\partial f_1(z_1,\hat{i}_2)/\partial \hat{i}_2]dz_1 + \tfrac{1}{2}\, f_2(\hat{i}_1,\hat{i}_2) - k_2(\hat{i}_2) && \text{(30b)} \end{aligned}}$$

By taking appropriate partial derivatives of the functions in (30), we obtain (29) easily. This concludes the proof of Theorem 2. Q.E.D.

Example 8. An Example to Illustrate Theorem 2.

Suppose we want to realize a 2-port $\mathcal{N}$ defined by the following Ebers-Moll equations:

$$i_1 = A_1[\exp(Kv_1)-1]-B_1[\exp(Kv_2)-1] \triangleq f_1(v_1,v_2)$$

$$i_2 = -A_2[\exp(Kv_1)-1] + B_2[\exp(Kv_2)-1] \triangleq f_2(v_1,v_2)$$

By Theorem 2, $\mathcal{N}$ can be realized by a parallel connection of a quasi-antireciprocal 2-port $\tilde{\mathcal{N}}$ defined by (from (27)):

$$\begin{aligned}\tilde{i}_1 = g_1(\tilde{v}_1,\tilde{v}_2) &= \frac{1}{2} A_1[\exp(K\tilde{v}_1)-1] - \frac{1}{2} B_1[\exp(K\tilde{v}_2)-1] \\ &\quad + \frac{1}{2} A_2\tilde{v}_2[\exp(K\tilde{v}_1)-1] - \frac{1}{2} \frac{B_2}{K}[\exp(K\tilde{v}_2)-1] \\ &= \frac{1}{2} [A_1+A_2\tilde{v}_2][\exp(K\tilde{v}_1)-1] - \frac{1}{2} [B_1+ \frac{B_2}{K}][\exp(K\tilde{v}_2)-1]\end{aligned}$$

$$\tilde{i}_2 = g_2(\tilde{v}_1,\tilde{v}_2) = - \frac{1}{2} [\frac{A_1}{K} + A_2][\exp(k\tilde{v}_1)-1] + \frac{1}{2} [B_1\tilde{v}_1+ B_2][\exp(K\tilde{v}_2)-1]$$

and a reciprocal 2-port $\hat{\mathcal{N}}$ defined by (from (30)):

$$\hat{i}_1 = h_1(\hat{v}_1,\hat{v}_2) = \frac{1}{2} [A_1-A_2\hat{v}_2][\exp(K\hat{v}_1)-1] - \frac{1}{2} [B_1- \frac{B_2}{K}][\exp(K\hat{v}_2)-1]$$

$$\hat{i}_2 = h_2(\hat{v}_1,\hat{v}_2) = - \frac{1}{2} [- \frac{A_1}{K} + A_2][\exp(k\hat{v}_1)-1] + \frac{1}{2} [-B_1\hat{v}_1+B_2][\exp(K\hat{v}_2)-1]$$

Here, we have let $k_1(\cdot) \triangleq 0 \triangleq k_2(\cdot)$. Notice that each 2-port $\mathcal{N}$ can be decomposed into many distinct pairs of $\tilde{\mathcal{N}}$ and $\hat{\mathcal{N}}$.

Definition 2. A C^1 v-controlled {i-controlled} n-port $\mathcal{N}$ is said to be

(i) in the class $\mathcal{N}_{R+Q}$ if $\mathcal{N}$ can be decomposed into a reciprocal i-controlled {v-controlled} n-port and a quasi-antireciprocal i-controlled {v-controlled} n-port.

(ii) <u>separable</u> if $\mathcal{N}$ can be represented by

$$v_j = \sum_{k=1}^{n} f_{jk}(i_k) \quad \{i_j = \sum_{k=1}^{n} g_{jk}(v_k)\} \; j = 1, 2, \cdots, n$$

(iii) <u>simply coupled</u> if $\mathcal{N}$ can be represented by

$$v_j = \sum_{k=1}^{n} f_{jk}(i_j,i_k) \qquad \{i_j = \sum_{k=1}^{n} g_{jk}(v_j,v_k)\} \; j = 1, 2, \cdots, n$$

where $i = [i_1,i_2, \cdots, i_n]$ and $v = [v_1,v_2, \cdots, v_n]$.

<u>Lemma 1</u>.

(i) Every separable n-port is simply coupled.

(ii) Every simply coupled n-port is in the class $\mathcal{N}_{R+Q}$.

<u>Proof</u>. Statement (i) is obvious. We shall prove the i-controlled case of Statement (ii) only.

Let $\mathcal{N}$ be represented by

$$v_j = \sum_{k=1}^{n} f_{jk}(i_j,i_k) \quad j = 1, 2, \cdots, n. \tag{31}$$

For each $j > k = 1, 2, \cdots, n$, let us define the following:[18]

$$g_{jk}(i_j,i_k) \triangleq \frac{1}{2} f_{jk}(i_j,i_k) - \frac{1}{2} \int_{a_k}^{i_k} [\partial f_{kj}(z_k,i_j)/\partial i_j]dz_k \tag{32a}$$

$$g_{kj}(i_k,i_j) \triangleq - \frac{1}{2} \int_{a_j}^{i_j} [\partial f_{jk}(z_j,i_k)/\partial i_k]dz_j + \frac{1}{2} f_{kj}(i_k,i_j) \tag{32b}$$

$$h_{jk}(i_j,i_k) \triangleq \frac{1}{2} f_{jk}(i_j,i_k) + \frac{1}{2} \int_{a_k}^{i_k} [\partial f_{kj}(z_k,i_j)/\partial i_j]dz_k \tag{33a}$$

$$h_{kj}(i_j,i_k) \triangleq \frac{1}{2} \int_{a_j}^{i_j} [\partial f_{jk}(z_j,i_k)/\partial i_k]dz_j + \frac{1}{2} f_{kj}(i_k,i_j) \tag{33b}$$

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[18] It doesn't matter whether one chooses $j>k$, or $k>j$. The resulting decompositions may not be identical, but <u>lemma</u> 1 remains valid in either case. Observe that (32) and (33) are modeled after (27) and (30) with $k_1(\cdot) \triangleq k_2(\cdot) = 0$.

$$\tilde{v}_j = \sum_{\substack{k=1 \\ k\neq j}}^{n} g_{jk}(\tilde{i}_j, \tilde{i}_k) \quad \text{for } j = 1, 2, \cdots, n \tag{34}$$

$$\hat{v}_j = \sum_{\substack{k=1 \\ k\neq j}}^{n} h_{jk}(\hat{i}_j, \hat{i}_k) + f_{jj}(\hat{i}_j, \hat{i}_j) \quad \text{for } j = 1, 2, \cdots, n \tag{35}$$

where $a \triangleq [a_1, a_2, \ldots, a_n]^t$ is a constant n-vector.

Let $\tilde{\mathcal{N}}$ be an n-port represented by (34) and let $\hat{\mathcal{N}}$ be an n-port represented by (35). It is clear that $\tilde{\mathcal{N}}$ is quasi-antireciprocal and $\hat{\mathcal{N}}$ is reciprocal. Furthermore, if we connect $\tilde{\mathcal{N}}$ and $\hat{\mathcal{N}}$ in series, the result is the desired n-port $\mathcal{N}$ prescribed by (31). Q.E.D.

Remarks 2.

It has been shown in [26] that nonlinear systems with prescribed singularities can be realized by the following canonical form:

$$\dot{x}_1 = g_1(x_2)$$

$$\dot{x}_2 = f_2(x_2)g_2(x_3) + h_2(x_2)$$

$$\vdots$$

$$\dot{x}_{n-1} = f_{n-1}(x_{n-1})g_{n-1}(x_n) + h_{n-1}(x_{n-1})$$

$$\dot{x}_n = f_n(x_n)g_n(x_1) + h_n(x_n)$$

This system can be synthesized by connecting n linear capacitors across a simply-coupled resistive n-port characterized by the above equations (with x_j and $\dot{x}_j$ replaced by v_j and i_j, respectively). It follows that a nonlinear network with prescribed singularities can always be synthesized with a reciprocal and a quasi-anti-reciprocal n-port and n linear capacitors.

Since a very large class of nonlinear n-ports can be decomposed into a reciprocal and a quasi-antireciprocal n-port, the problem of synthesizing n-ports belonging to the class $\mathcal{N}_{R+Q}$ reduces to that of synthesizing a reciprocal and a quasi-antireciprocal n-port. This observation motivates the following study on the structure and realization of quasi-antireciprocal n-ports.

<u>Lemma 2</u>. Properties of Representation Functions of Quasi-antireciprocal n-Ports.

Let $f = [f_1, f_2, \cdots, f_n]$: $R^n \to R^n$ be a C^2 map such that $J_f(x) = J_1(x) + J_2(x)$, where $x = [x_1, x_2, \cdots, x_n]$, $J_1(x)$ is diagonal and $J_2(x)$ is an n×n skew-symmetric matrix. Then

(i) $\partial f_j(x)/\partial x_k$ is a function of x_j and x_k only for all

$$j \neq k = 1, 2, \cdots, n \tag{36a}$$

(ii) $\partial f_j(x)/\partial x_j$ can be written as:

$$\partial f_j(x)/\partial x_j = \sum_{k=1}^{n} h_{jk}(x_j, x_k) \qquad j = 1, 2, \cdots, n \tag{36b}$$

(iii) $f_j(x)$ can be written as:

$$f_j(x) = \sum_{k=1}^{n} f_{jk}(x_j, x_k) \qquad j = 1, 2, \cdots, n \tag{37}$$

<u>Proof</u>. Let $A(x) = [a_{ij}(x)] \triangleq J_f(x)$, i.e., $a_{ij}(x) = \partial f_i(x)/\partial x_j$ $\forall$ $i, j = 1, 2, \cdots, n$. Since $f(\cdot)$ is C^2 on R^n, we have:

$$\frac{\partial}{\partial x_j}[a_{ik}(x)] = \frac{\partial}{\partial x_k}[a_{ij}(x)] \qquad \forall\ i, j, k = 1, 2, \cdots, n \tag{38}$$

By hypothesis, we have:

$$a_{ij}(x) = -a_{ji}(x) \qquad \forall\ i \neq j = 1, 2, \cdots, n \tag{39}$$

Assume $i \neq j \neq k$. By repeated use of (38) and (39), we obtain:

$$\frac{\partial}{\partial x_j}[a_{ik}(x)] = -\frac{\partial}{\partial x_j}[a_{ki}(x)] = -\frac{\partial}{\partial x_i}[a_{kj}(x)] = \frac{\partial}{\partial x_i}[a_{jk}(x)] \quad (40a)$$

$$\frac{\partial}{\partial x_j}[a_{ik}(x)] = \frac{\partial}{\partial x_k}[a_{ij}(x)] = -\frac{\partial}{\partial x_k}[a_{ji}(x)] = -\frac{\partial}{\partial x_i}[a_{jk}(x)] \quad (40b)$$

Equation (40) implies that $a_{jk}(x)$ is independent of x_i for $i \neq j \neq k = 1, 2, \cdots, n$. Consequently $a_{jk}(x)$ depends (at most) on x_j and x_k only $\forall\ j \neq k = 1, 2, \cdots, n$. Hence, (i) is true.

Let $k = i$ in (38), we obtain:

$$\frac{\partial}{\partial x_j}[a_{ii}(x)] = \frac{\partial}{\partial x_i}[a_{ij}(x)] \quad (41)$$

Equation (41) holds for all $i, j = 1, 2, \cdots, n$. Assume $i \neq j$. Since $a_{ij}(x)$ is a function of x_i and x_j only, we conclude that $\partial[a_{ii}(x)]/\partial x_j$ is also a function of x_i and x_j only; namely,

$$a_{ii}(x) = \sum_{k=1}^{n} h_{ik}(x_i, x_k) \qquad \forall\ i = 1, 2, \cdots, n.$$

This proves (ii).

Since $\nabla f_j(x)$ is always a state function[19] [27,1] on R^n, we have [28]:

$$f_j(x) = \sum_{k=1}^{n} \int_0^1 \frac{\partial f_j(tx)}{\partial x_k} x_k dt \quad (42)$$

Substituting (36) into (42), we obtain:

$$f_j(x) = \sum_{k=1, k\neq j}^{n} \int_0^1 a_{jk}(tx_j, tx_k) x_k dt$$

$$+ \int_0^1 \sum_{i=1}^{n} h_{ji}(tx_j, tx_i) x_j dt \quad (43)$$

[19] Let S be a convex subset of R^n. A C^1 function h: $S \to R^n$ is said to be a <u>state</u> function on S if $J_h(x)$ is symmetric for all $x \in S$.

For each $j \neq k = 1, 2, \cdots, n$, let

$$f_{jk}(x_j,x_k) \triangleq \int_0^1 [a_{jk}(tx_j,tx_k)x_k + h_{jk}(tx_j,tx_k)x_j]dt \tag{44a}$$

and for each $j = 1, 2, \cdots, n$, let

$$f_{jj}(x_j,x_j) \triangleq \int_0^1 h_{jj}(tx_j,tx_j)x_j dt \tag{44b}$$

Substituting (44) into (43), we obtain (37). This completes the proof of Lemma 2. Q.E.D.

In view of Lemma 2, we have the following theorem on the structure of quasi-antireciprocal n-ports.

Theorem 3.

Let $\mathcal{N}$ be a quasi-antireciprocal n-port with a C^2 v-controlled {i-controlled} representation $i = g(v)$ $\{v=f(i)\}$. Then $\mathcal{N}$ can be realized by an appropriate parallel {series} connection of $n(n-1)/2$ quasi-antireciprocal C^2 v-controlled {i-controlled} 2-ports and n C^2 v-controlled $\{C^2$ i-controlled} 1-ports.

Proof. Since the proofs for both cases are similar, we will consider the v-controlled case only.

In view of Lemma 2, we can write $g = [g_1,g_2, \cdots, g_n]$ as follows:

$$g_j(v) = \sum_{k=1}^{n} g_{jk}(v_j,v_k) \qquad j = 1, 2, \cdots, n \tag{45}$$

where $g_{jk}(x_j,x_k)$ are C^2 functions on R^2. Let $\mathcal{N}_{jk}$ be a 2-port described by:

$$i_{j(k)} = g_{jk}(v_j,v_k) \tag{46a}$$

$$i_{k(j)} = g_{kj}(v_k,v_j) \tag{46b}$$

where $j > k = 1, 2, \cdots, n$. Since $\partial g_j(v)/\partial v_k = - \partial g_k(v)/\partial v_j$ $\forall$ $k \neq j = 1,$

2, ⋯, n, (45) implies

$$\frac{\partial g_{jk}(v_j,v_k)}{\partial v_k} = -\frac{\partial g_{kj}(v_k,v_j)}{\partial v_j} \tag{47}$$

Hence, $\mathcal{N}_{jk}$ is a quasi-antireciprocal 2-port with a C^2 conductance representation as given by (46). This is true for all $j > k = 1, 2, \cdots, n$. That is, there are $n(n-1)/2$ of them. For convenience, let us label the two ports of $\mathcal{N}_{jk}$ as follows: the port defined by (46a) is called the $j(k)$ port and the port defined by (46b) is called the $k(j)$ port. In addition, let $\mathcal{N}_j$ denote the 1-port described by

$$i_{j(j)} = g_{jj}(v_j,v_j) \qquad j = 1, 2, \cdots, n \tag{48}$$

Then $\mathcal{N}_j$ is a v-controlled 1-port. Let $\tilde{\mathcal{N}}$ be the n-port obtained by connecting in parallel across each port, say port k, the following:

(i) port $k(j)$ of $\mathcal{N}_{jk}$ for $j = k+1, k+2, \cdots, n$

(ii) port $k(j)$ of $\mathcal{N}_{kj}$ for $j = 1, 2, \cdots, k-1$

(iii) the 1-port $\mathcal{N}_j$

where $j = 1, 2, \cdots, n$. Fig. 10 shows how these connections are made in the case when $n = 4$. Hence, we have

$$i_k = \sum_{j=1}^{n} i_{k(j)} = \sum_{j=1}^{n} g_{kj}(v_k,v_j) = g_k(v)$$

where i_k, v_k are the port current and port voltage at port k of $\tilde{\mathcal{N}}$. That is, $\tilde{\mathcal{N}}$ is represented by $i = g(v)$. Hence $\tilde{\mathcal{N}}$ and $\mathcal{N}$ are equivalent. This completes the proof of Theorem 3. Q.E.D.

<u>Remarks 3</u>.

In view of Theorem 3, the problem of realizing a quasi-antireciprocal <u>n-port</u> is reduced to that of realizing quasi-antireciprocal <u>2-ports</u>. It can be shown that a quasi-antireciprocal 2-port can be synthesized by

(a) a cascade connection of a reciprocal 2-port and a gyrator.

(b) a cascade connection of a reciprocal 2-port and an NIC.

(c) an appropriate connection of a reciprocal 2-port and a controlled source.

We will establish the validity of remarks 3 for the case of i-controlled 2-ports only. The case of v-controlled 2-ports can be established by duality.

Suppose we wish to realize a 2-port $\mathcal{N}$ represented by

$$v_1 = g_1(i_1,i_2) \tag{49a}$$

$$v_2 = g_2(i_1,i_2) \tag{49b}$$

where $\partial g_1(i_1,i_2)/\partial i_2 = -\partial g_2(i_1,i_2)/\partial i_1 \quad \forall\ [i_1,i_2] \in R^2$, i.e., $\mathcal{N}$ is quasi-antireciprocal.

(a) Consider the circuit in Fig. 11(a). The problem is 1). To find the representation of $\mathcal{N}_R$ so that the composite 2-port has the characteristics represented by (49), and 2). To show that $\mathcal{N}_R$ is reciprocal.

With the references shown in Fig. 11(a), we find that if $v_3 = g_1(i_3,-v_4)$ and $i_4 = -g_2(i_3,-v_4)$, then the composite 2-port $\mathcal{N}$ is represented by (49). Hence, $\mathcal{N}_R$ is represented by

$$\begin{bmatrix} v_3 \\ i_4 \end{bmatrix} = h(i_3,v_4) = \begin{bmatrix} h_3(i_3,v_4) \\ h_4(i_3,v_4) \end{bmatrix} \triangleq \begin{bmatrix} g_1(i_3,v_4) \\ -g_2(i_3,-v_4) \end{bmatrix}$$

Note that $\partial h_4(i_3,v_4)/\partial i_3 = -\partial h_3(i_3,v_4)/\partial v_4 \quad \forall\ [i_3,v_4] \in R^2$. Hence, $\mathcal{N}_R$ is reciprocal [1].

(b) Consider the circuit in Fig. 11(b). With the references defined in Fig. 11(b), we can show that if $\mathcal{N}_R$ is represented by

$$\begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} h_3(i_3,i_4) \\ h_4(i_3,i_4) \end{bmatrix} \triangleq \begin{bmatrix} g_1(i_3,-i_4) \\ g_2(i_3,-i_4) \end{bmatrix} \triangleq h(i_3,i_4) \tag{50}$$

then the overall 2-port $\mathcal{N}$ is represented by (49). In addition, the Jacobian matrix $J_h(i_3,i_4)$ is symmetric. Hence $\mathcal{N}_R$ is reciprocal [1].

(c) Consider the circuit in Fig. 11(c). It is clear that if $\mathcal{N}_R$ is represented by (50), then $\mathcal{N}$ is represented by (49). From (b), $\mathcal{N}_R$ is reciprocal.

Remarks 4.

It follows from Remarks 3 that every v-controlled or i-controlled 2-port can be realized by two reciprocal nonlinear 2-ports and a non-reciprocal linear 2-port. Moreover, it follows from case (b) of Remarks 3 that the two reciprocal nonlinear 2-ports can be combined into one equivalent reciprocal nonlinear 2-port. Similarly, every nonlinear n-port belonging to the class $\mathcal{N}_{R+Q}$ can be realized by two reciprocal nonlinear n-ports and a non-reciprocal linear n-port.

V. CONCLUDING REMARKS

The representation of composite nonlinear n-ports in explicit topological form as presented in Section II constitutes only the first step toward the formulation of a unified theory of nonlinear n-ports. The numerous criteria summarized in Table 2 could serve as the vital link toward a systematic study of dynamic nonlinear networks where the dynamic elements are extracted as "loads" across a resistive n-port. The closure properties presented in Section III and the decomposition theorems derived in Section IV could serve as a foundation for the synthesis of nonlinear n-ports. Since not all n-ports ($n > 2$) can be decomposed into a reciprocal and a quasi-antireciprocal n-ports, we close this paper by posing the following unsolved fundamental problem: Characterize the class $\mathcal{N}(?)$ of nonlinear n-ports such that every resistive n-port ($n > 2$) can be decomposed into a reciprocal n-port and a member of $\mathcal{N}(?)$.

APPENDIX A

Proof of Closure Property 4. Let v and i be the port voltage and port current of the created 1-port $\hat{\mathcal{N}}$, respectively.

Suppose $\hat{\mathcal{N}}$ is created by a plier-type entry through a wire in $\mathcal{N}$ as shown in Figs. 12(a) and (b). Since $\mathcal{N}$ contains only increasing and proper 1-ports, any tree T of $\mathcal{N}$ will satisfy the assumptions of Property 2. Hence $\hat{\mathcal{N}}$ is v-controlled and the representation function $i = g(v)$ is continuous and non-decreasing.

Let $\mathcal{N}_I$ be the network obtained from $\hat{\mathcal{N}}$ by terminating the only port of $\hat{\mathcal{N}}$ with a current source as shown in Fig. 12(c). Since $\hat{\mathcal{N}}$ is created by a plier type entry, there is a loop in $\mathcal{N}_I$ containing the port current source. By i-shift theorem [16,20], the port current source is shifted through that loop of $\mathcal{N}_I$ containing the port current source. The composite elements (containing the original 1-port in parallel with the shifted current source) in the loop is still increasing and proper, and the port voltage v is equal to the algebraic sum of the voltages of the elements in the loop. Let $\mathcal{N}_{IS}$ be the network resulting from the i-shift transformation. Then all elements in $\mathcal{N}_{IS}$ are increasing and proper. Hence, $\mathcal{N}_{IS}$ has a unique solution [20,22]. That is, for each value of the port current source, there corresponds uniquely a value of the port voltage. Hence, $\hat{\mathcal{N}}$ is i-controlled. This means that $\hat{\mathcal{N}}$ is both i-controlled and v-controlled. Hence $\hat{\mathcal{N}}$ is proper and increasing.

Suppose now the 1-port $\hat{\mathcal{N}}$ is created by soldering-iron entry across two nodes in $\mathcal{N}$ as shown in Figs. 12(d) and (e). Since $\mathcal{N}$ contains only increasing and proper 1-ports, by Property 2, $\hat{\mathcal{N}}$ is i-controlled and the representation function is continuous and non-decreasing. Let $\mathcal{N}_E$ be the network obtained by terminating the only port of $\hat{\mathcal{N}}$ with a voltage source as shown in Fig. 12(f). Since we can always choose a tree T which contains the port-voltage source for $\mathcal{N}_E$ and that all other elements in $\mathcal{N}_E$ are proper and increasing, it follows from a theorem in [20] that $\mathcal{N}_E$ has a unique solution. That is, for each value of the port voltage source, there corresponds a unique port current for $\hat{\mathcal{N}}$. Hence $\hat{\mathcal{N}}$ is also v-controlled. This implies that $\hat{\mathcal{N}}$ is increasing and proper and our conclusion follows. Q.E.D.

LIST OF FIGURE CAPTIONS

Fig. 1. Examples illustrating the complex nature of the composite v-i curve corresponding to two resistors with rather simple v_j-i_j curves in series.

Fig. 2. An example showing simple interconnection of nonlinear resistors with a current-controlled current source could lead to a composite v-i relationship consisting of line segments and isolated points.

Fig. 3. An example showing simple interconnection of nonlinear resistors with a current-controlled current source could lead to a composite v-i relationship consisting of subsets of points having a finite area.

Fig. 4. An example of a unicursal 3-port and its circuit realization in terms of two controlled sources.

Fig. 5. An example of an intersection 3-port and the arbitrary composite v-i relationships that could be realized in terms of two nonlinear resistors and three controlled sources.

Fig. 6. An illustration of an n-port $\hat{\mathcal{N}}$ and its associated network $\mathcal{N}$.

Fig. 7. A 3-port $\hat{\mathcal{N}}$ and its associated network graph $\mathcal{G}$ with T = {4,5,6} and L = {1,2,3} = $\mathcal{G}_p$.

Fig. 8. Examples for illustrating the non-closure property of various n-ports.

Fig. 9. A canonic current-controlled 2-port synthesis method consisting of a series connection of a reciprocal 2-port and a quasi-antireciprocal 2-port.

Fig. 10. Example illustrating the synthesis of a quasi-antireciprocal voltage-controlled 4-port in terms of nonlinear 1-ports and quasi-anti-

reciprocal 2-ports.

Fig. 11. Synthesis of a quasi-antireciprocal 2-port in terms of a reciprocal 2-port and a gyrator, an NIC, or a controlled source.

Fig. 12. An illustration of various 1-ports created via soldering iron or plier entry.

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Table 1. Collection of Hypotheses for the "Assumption Column" in Table 2.

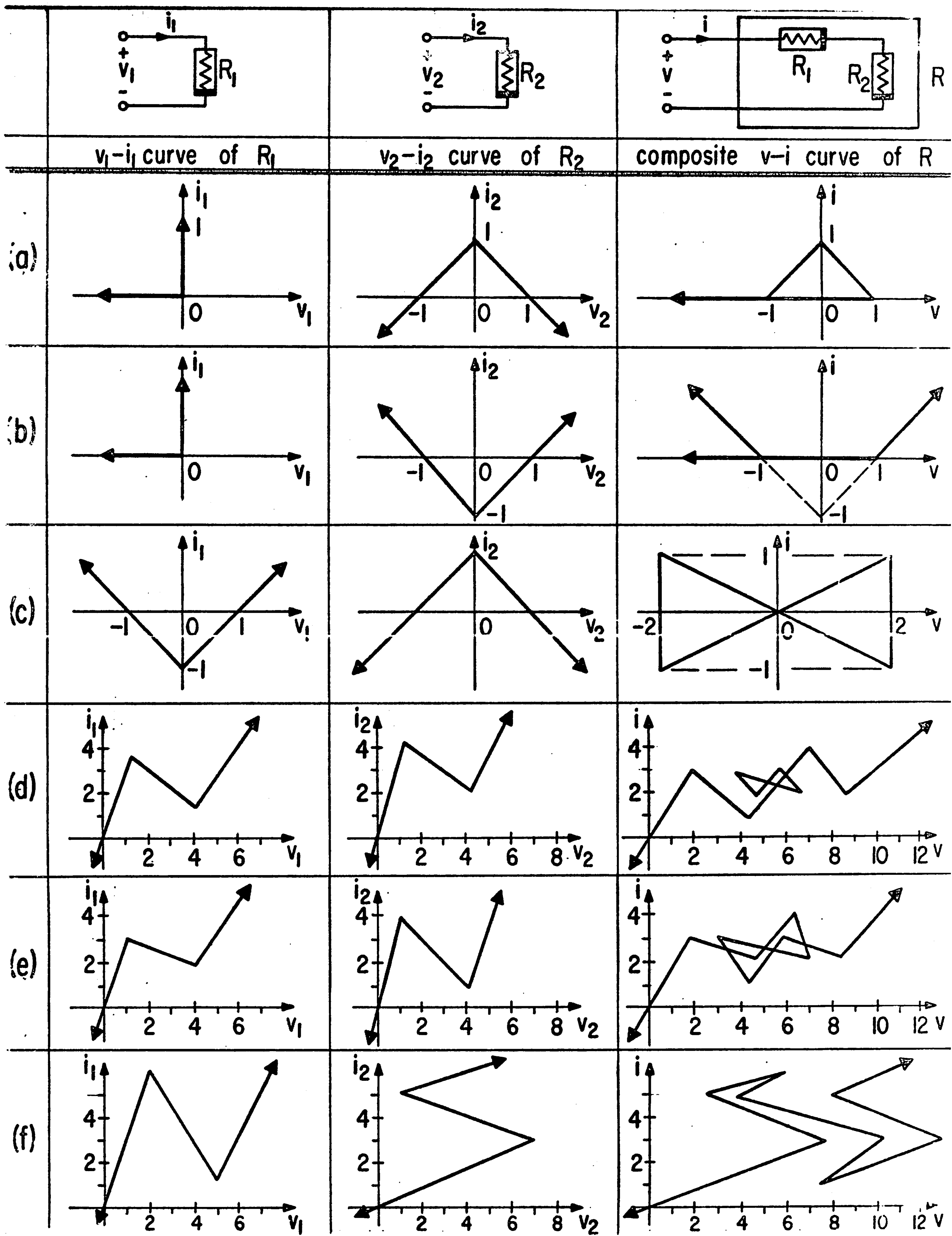
Hypothesis	Statement of the Hypothesis	Hypothesis	Statement of the Hypothesis
$T2_{A(k)}$	T2: $v_{T2} = \hat{v}_{T2}(i_{T2})$ where $\hat{v}_{T2}: R^{t_2} \to R^{t_2}$ is C^k.	$L1_{A(k)}$	L1: $i_{L1} = \hat{i}_{L1}(v_{L1})$ where $\hat{i}_{L1}: R^{\ell_1} \to R^{\ell_1}$ is C^k.
$T2_B$	$\hat{v}_{T2}(\cdot)$ is nondecreasing on R^{t_2}.	$L1_B$	$\hat{i}_{L1}(\cdot)$ is nondecreasing on R^{ℓ_1}
$T2_C$	$\hat{v}_{T2}(\cdot)$ is increasing on R^{t_2}	$L1_C$	$\hat{i}_{L1}(\cdot)$ is increasing on R^{ℓ_1}
$T2_D$	$J_{\hat{v}_{T2}}(\cdot)$ is psd and symmetric on R^{t_2}	$L1_D$	$J_{\hat{i}_{L1}}(\cdot)$ is psd and symmetric on R^{ℓ_1}.
$L2_{A(k+1)}$	L2: $i_{L2} = \hat{i}_{L2}(v_{L2})$ where $\hat{i}_{L2}: R^{\ell_2} \to R^{\ell_2}$. and $J_{\hat{i}_{L2}}(\cdot)$ is pd and symmetric on R^{ℓ_2}.	$T1_{A(k+1)}$	T1: $v_{T1} = v_{T1}(i_{T1})$ where $v_{T1}: R^{t_1} \to R^{t_1}$ is C^{k+1} and $J_{\hat{v}_{T1}}(\cdot)$ is pd and symmetric on R^{t_1}.
$L2_{B(k+1)}$	L2: $v_{L2} = \hat{v}_{L2}(i_{L2})$ where $\hat{v}_{L2}: R^{\ell_2} \to R^{\ell_2}$ is C^{k+1} and $J_{\hat{v}_{L2}}(\cdot)$ is pd on R^{ℓ_2}.	$T1_{B(k+1)}$	T1: $i_{T1} = \hat{i}_{T1}(v_{T1})$ where $\hat{i}_{T1}: R^{t_1} \to R^{t_1}$ is C^{k+1} and $J_{\hat{i}_{T1}}(\cdot)$ is pd on R^{t_1}.
$L2a_{A(k+1)}$	L2a: $v_{L2a} = \hat{v}_{L2a}(i_{L2a})$ where $\hat{v}_{L2a}: R^{\ell_{2a}} \to R^{\ell_{2a}}$ is C^{k+1} and $J_{\hat{v}_{L2a}}(\cdot)$ is upd on $R^{\ell_{2a}}$.	$T1b_{A(k+1)}$	T1b: $i_{T1b} = \hat{i}_{T1b}(v_{T1b})$ where $\hat{i}_{T1b}: R^{t_{1b}} \to R^{t_{1b}}$ is C^{k+1} and $J_{\hat{i}_{T1b}}(\cdot)$ is upd on $R^{t_{1b}}$.
$T2b_{A(k)}$	T2b: $v_{T2b} = \hat{v}_{T2b}(i_{T2b})$ where $\hat{v}_{T2b}: R^{t_{2b}} \to R^{t_{2b}}$ is C^k.	$L1a_{A(k)}$	L1a: $i_{L1a} = \hat{i}_{L1a}(v_{L1a})$ where $\hat{i}_{L1a}: R^{\ell_{1a}} \to R^{\ell_{1a}}$ is C^k.
$T2b_B$	$\hat{v}_{T2b}(\cdot)$ is nondecreasing on $R^{t_{2b}}$.	$L1a_B$	$\hat{i}_{L1a}(\cdot)$ is nondecreasing on $R^{\ell_{1a}}$.
$T_{A(k)}$	T: $v_T = \hat{v}_T(i_T)$ where $\hat{v}_T: R^t \to R^t$ is C^k.	$L_{A(k)}$	L: $i_L = \hat{i}_L(v_L)$ where $\hat{i}_L: R^{\ell} \to R^{\ell}$ is C^k
T_B	$\hat{v}_T(\cdot)$ is nondecreasing on R^t.	L_B	$\hat{i}_L(\cdot)$ is nondecreasing on R^{ℓ}.
T_C	$\hat{v}_T(\cdot)$ is increasing on R^t.	L_C	$\hat{i}_L(\cdot)$ is increasing on R^{ℓ}.
M_A	B_{11} has rank t_1.	M_D	B_{21}^t has rank ℓ_2.
M_B	B_{21} has rank t_1.	M_E	B_{22}^t has rank ℓ_2.
M_C	B_T has rank t.	M_F	B_T^t has rank ℓ.
N_A	$\lim_{\Vert v_{L2}\Vert \to \infty} \Vert v_{L2} + B_{22}\hat{v}_{T2} \circ (B_{22}^t \hat{i}_{L2}(v_{L2}))\Vert = \infty$	N_C	$\lim_{\Vert i_{T1}\Vert \to \infty} \Vert i_{T1} - B_{11}^t \hat{i}_{L1} \circ (-B_{11}\hat{v}_{T1}(i_{T1}))\Vert = \infty$
N_B	$\lim_{\Vert i_{L2}\Vert \to \infty} \Vert \hat{v}_{L2}(i_{L2}) + B_{22}\hat{v}_{T2} \circ (B_{22}^t i_{L2})\Vert = \infty$	N_D	$\lim_{\Vert v_{T1}\Vert \to \infty} \Vert \hat{i}_{T1}(v_{T1}) - B_{11}^t \hat{i}_{L1} \circ (-B_{11} v_{T1})\Vert = \infty$

Table 2. Conditions for the Existence of Various Nonlinear n-Port Representations Via Topological Matrices

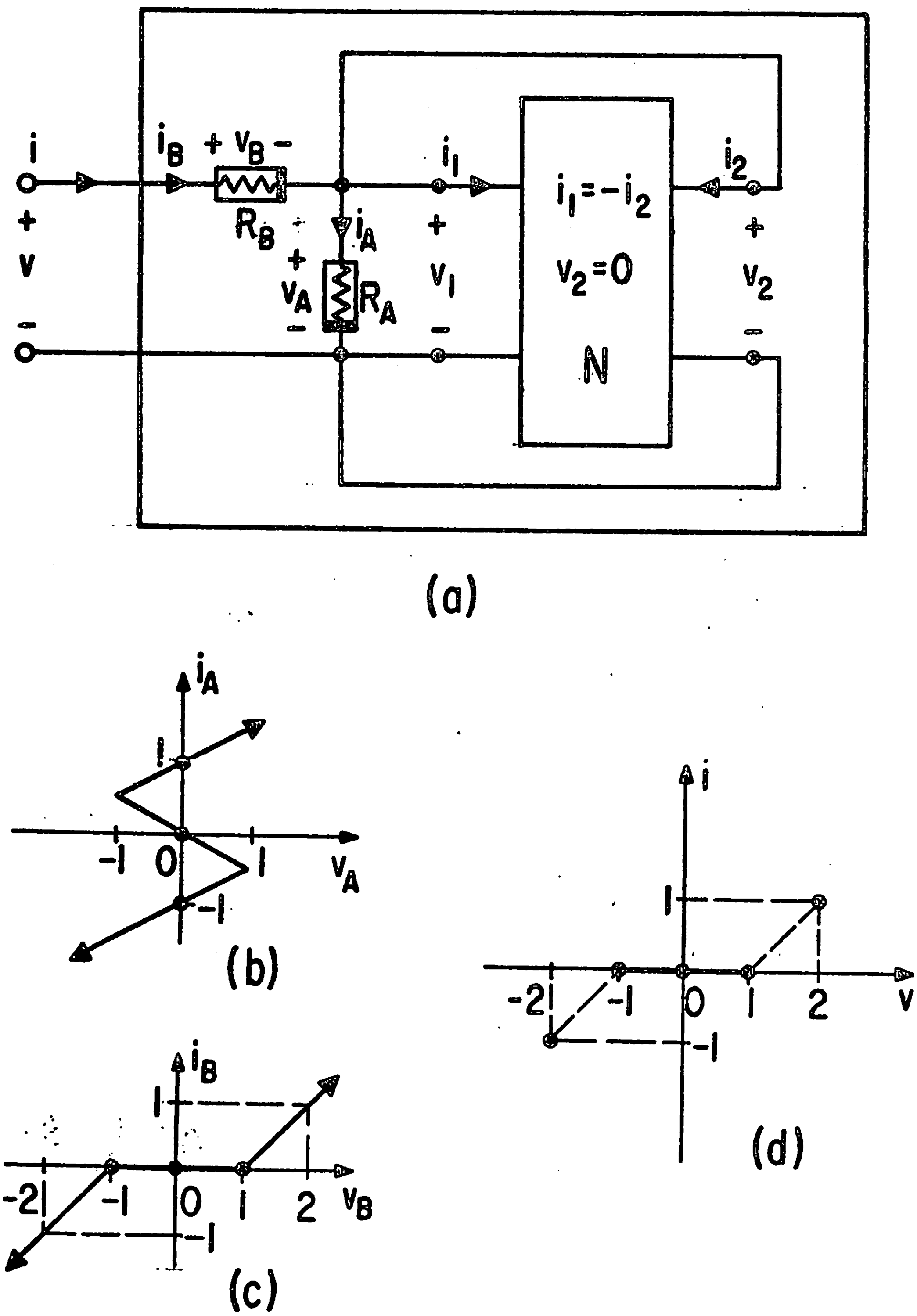
Cases	Assumptions	Conclusions	
(H_1^a)	$\mathcal{G}_P = T1 \cup L2$ and $T2_{A(k)} + L1_{A(k)}$	$\hat{\mathcal{N}}$ has a C^k hybrid representation $i_{P1} = \hat{i}_{P1}^a(v_{P1}, i_{P2})$ (α^a) $v_{P2} = \hat{v}_{P2}^a(v_{P1}, i_{P2})$	1. $i_P = [i_{P1}, i_{P2}]$, $v_P = [v_{P1}, v_{P2}]$, $i_{P1} = -i_{T1}$, $i_{P2} = -i_{L2}$, $v_{P1} = v_{T1}$, $v_{P2} = v_{L2}$ 2. $n = t_1 + \ell_2$ 3. $\boxed{\begin{array}{l} \hat{i}_{P1}^a(v_{P1}, i_{P2}) \triangleq -B_{11}^t \hat{i}_{L1} \circ (-B_{11} v_{P1} + B_{21}^t i_{P2}) \\ \hat{v}_{P2}^a(v_{P1}, i_{P2}) \triangleq -B_{21} v_{P1} - B_{22} \hat{v}_{T2} \circ (-B_{22}^t i_{P2}) \end{array}}$ (α^a)
(H_2^a)	$\mathcal{G}_P = T1 \cup L2$ and $T2_{A(k)} + L1_{A(k)} + T2_B + L1_B$	$\hat{\mathcal{N}}$ has a C^k hybrid representation (α^a) and $\hat{\mathcal{N}}$ is nondecreasing.	
(H_3^a)	$\mathcal{G}_P = T1 \cup L2$ and $T2_{A(k)} + L1_{A(k)} + T2_C + L1_C + M_A + M_E$	$\hat{\mathcal{N}}$ has a C^k hybrid representation (α^a) and $\hat{\mathcal{N}}$ is an increasing n-port.	
(H^b)	$\mathcal{G}_P = T1 \cup L2b$ and $T2_{A(k+1)} + L1_{A(k+1)} + L2a_{A(k+1)} + T2_B$	$\hat{\mathcal{N}}$ has a C^{k+1} hybrid representation: $i_{P1} = \hat{i}_{P1}^b(v_{P1}, i_{P2})$ (α^b) $v_{P2} = \hat{v}_{P2}^b(v_{P1}, i_{P2})$	1. $i_P = [i_{P1}, i_{P2}]$, $v_P = [v_{P1}, v_{P2}]$, $i_{P1} = -i_{T1}$, $i_P = -i_{L2b}$, $v_{P1} = v_{T1}$, $v_{P2} = v_{L2b}$ 2. $n = t_1 + \ell_{2b}$ 3. $\boxed{\begin{array}{l} \hat{i}_{P1}^b(v_{P1}, i_{P2}) \triangleq -B_{11}^t \hat{i}_{L1} \circ (-B_{11} v_{P1}) - B_{21a}^t g(-B_{21a} v_{P1}, B_{22b}^t i_{P2}) \\ \hat{v}_{P2}^b(v_{P1}, i_{P2}) \triangleq -B_{21b} v_{P1} - B_{22b} \hat{v}_{T2} \circ (B_{22a}^t g(-B_{21a} v_{P1}, B_{22b}^t i_{P2})) \end{array}}$ (α^b) where g: $R^{\ell_{2a}} \times R^{t_2} \to R^{\ell_{2a}}$ is the C^{k+1} function such that $g(f(i_{L2a}, B_{22b}^t i_{P2}), B_{22b}^t i_{P2}) \triangleq i_{L2a}$, $f(g(-B_{21a} v_{P1}, B_{22b}^t i_{P2}), B_{22b}^t i_{P2}) \triangleq -B_{21a} v_{P1}$ and f: $R^{\ell_{2a}} \times R^{t_2} \to R^{\ell_{2a}}$ is a C^{k+1} function defined by $f(i_{L2a}, B_{22b}^t i_{P2}) \triangleq v_{L2a}(i_{L2a}) + B_{22a} \hat{v}_{T2} \circ (-B_{22a}^t i_{L2a} - B_{22b}^t i_{P2})$ $= -B_{21a} v_{P1}$
(H^c)	$\mathcal{G}_P = T1a \cup L2$ and $L1_{A(k+1)} + T2_{A(k+1)} + T1b_{A(k+1)} + L1B$	$\hat{\mathcal{N}}$ has a C^{k+1} hybrid representation $i_{P1} = \hat{i}_{P1}^c(v_{P1}, i_{P2})$ (α^c) $v_{P2} = \hat{v}_{P2}^c(v_{P1}, i_{P2})$	1. $i_P = [i_{P1}, i_{P2}]$, $v_P = [v_{P1}, v_{P2}]$, $i_{P1} = -i_{T1a}$, $i_{P2} = i_{L2}$, $v_{P1} = v_{T1a}$, $v_{P2} = v_{L2}$ 2. $n = t_{1a} + \ell_2$ 3. $\boxed{\begin{array}{l} \hat{i}_{P1}^c(v_{P1}, i_{P2}) \triangleq -B_{11c}^t \hat{i}_{L1} \circ (-B_{11c} v_{P1} - B_{11d} g(-B_{21d}^t i_{P2}, -B_{11c} v_{P1})) - B_{21c}^t i_{P2} \\ \hat{v}_{P2}^c(v_{P1}, i_{P2}) \triangleq -B_{21c} v_{P1} - B_{21d} g(-B_{21d}^t i_{P2}, -B_{11c} v_{P1}) \quad (\alpha^c) \\ \qquad -B_{22} \hat{v}_{T2} \circ (B_{22}^t i_{P2}) \end{array}}$ where g: $R^{t_{1b}} \times R^{\ell_1} \to R^{t_{1b}}$ is the C^{k+1} function such that $g(f(v_{T1b}, -B_{11c} v_{P1}), -B_{11c} v_{P1}) \triangleq v_{T1b}$ $f(g(-B_{21d}^t i_{P2}, -B_{11c} v_{P1}), -B_{11c} v_{P1}) \triangleq -B_{21d}^t i_{P2}$ and f: $R^{t_{1b}} \times R^{\ell_1} \to R^{t_{1b}}$ is a C^{k+1} function defined by $f(v_{T1b}, -B_{11c} v_{P1}) \triangleq \hat{i}_{T1b}(v_{T1b}) - B_{11c}^t \hat{i}_{L1} \circ (-B_{11c} v_{P1} - B_{11d} v_{T1b}) = -B_{21d}^t i_{P2}$

(H_1^d)	$\mathcal{G}_P = T1 \cup T2a \cup L1b \cup L2$ and $L1a_{A(k)} + T2b_{A(k)}$	$\hat{\mathcal{N}}$ has a C^k <u>hybrid</u> representation $i_{P1} = \hat{i}^d_{P1}(v_{P1}, v_{P2}, i_{P3}, i_{P4})$ $i_{P2} = \hat{i}^d_{P2}(v_{T1}, v_{P2}, i_{P3}, i_{P4})$ (α^d) $v_{P3} = \hat{v}^d_{P3}(v_{P1}, v_{P2}, i_{P3}, i_{P4})$ $v_{P4} = \hat{v}^d_{P4}(v_{P1}, v_{P2}, i_{P3}, i_{P4})$	1. $i_P = [i_{P1}, i_{P2}, i_{P3}, i_{P4}]$, $v_P = [v_{P1}, v_{P2}, v_{P3}, v_{P4}]$, $i_{P1} = -i_{T1}$, $i_{P2} = -i_{T2a}$, $i_{P3} = -i_{L1b}$, $i_{P4} = -i_{L2}$, $v_{P1} = v_{T1}$, $v_{P2} = v_{T2a}$, $v_{P3} = v_{L1b}$, $v_{P4} = v_{L2}$ 2. $n = t_1 + t_{2a} + \ell_{1b} + \ell_2$ 3. $\boxed{\begin{aligned} \hat{i}^d_{P1}(v_{P1}, v_{P2}, i_{P3}, i_{P4}) &\triangleq -B^t_{11a}\hat{i}_{L1a}\circ(B_{11a}v_{P1}) + B^t_{11b}i_{P3} + B^t_{21}i_{P4} \\ \hat{i}^d_{P2}(v_{P1}, v_{P2}, i_{P3}, i_{P4}) &\triangleq -B^t_{22c}i_{P4} \\ \hat{v}^d_{P3}(v_{P1}, v_{P2}, i_{P3}, i_{P4}) &\triangleq -B_{11b}v_{P1} \\ \hat{v}^d_{P4}(v_{P1}, v_{P2}, i_{P3}, i_{P4}) &\triangleq -B_{21}v_{P1} - B_{22c}v_{P2} - B_{22d}\hat{v}_{T2b}\circ(-B^t_{22d}i_{P4}) \end{aligned}}$ (α^d)
(h_2^d)	$\mathcal{G}_P = T1 \cup T2a \cup L1b \cup L2$ and $L1a_{A(k)} + T2b_{A(k)}$ $+ L1a_B + T2b_B$	$\hat{\mathcal{N}}$ has a C^k <u>hybrid</u> representation (α^d) and $\hat{\mathcal{N}}$ is a <u>nondecreasing</u> n-port.	
(G_1^a)	$\mathcal{G}_P = T1$ and $T2_{A(k+1)} + T2_D + L2_{A(k+1)}$ $+ L1_{A(k+1)} + N_A$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>v-controlled</u> representation $i_P = \hat{i}^a_P(v_P)$ (β^a)	1. $i_P = -i_{T1}$, $v_P = v_{T1}$ 2. $n = t_1$ 3. $\boxed{\hat{i}^a_P(v_P) \triangleq -B^t_{11}\hat{i}_{L1}\circ(-B_{11}v_P) - B^t_{21}\hat{i}_{L2}\circ g(-B_{21}v_P)}$ (β^a) where $g\colon R^{\ell_2} \to R^{\ell_2}$ is the inverse function of the C^{k+1}-diffeomorphic onto mapping $f\colon R^{\ell_2} \to R^{\ell_2}$ by $f(v_{L2}) \triangleq v_{L2} + B_{22}\hat{v}_{T2}\circ(B^t_{22}\hat{i}_L(v_{L2})) = -B_{21}v_P$.
(G_2^a)	$\mathcal{G}_P = T1$ and $T2_{A(k+1)} + T2_D + L2_{A(k+1)}$ $+ L1_{A(k+1)} + L1_B + N_A$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>v-controlled</u> representation (β^a) and $\hat{\mathcal{N}}$ is <u>nondecreasing</u>.	
(G_3^a)	$\mathcal{G}_P = T1$ and $T2_{A(k+1)} + T2_D + L2_{A(k+1)}$ $+ L1_{A(k+1)} + N_A$ + either $(L1_B + M_B)$ or $(L1_C + M_A)$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>v-controlled</u> representation (β^a) and $\hat{\mathcal{N}}$ is an <u>increasing</u> n-port.	
(G_1^b)	$\mathcal{G}_P = T1$ and $T2_{A(k+1)} + T2_B + L2_{B(k+1)}$ $+ L1_{A(k+1)} + N_B$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>v-controlled</u> representation $i_P = \hat{i}^b_P(v_P)$ (β^b)	1. $i_P = -i_{T1}$, $v_P = v_{T1}$ 2. $n = t_1$ 3. $\boxed{\hat{i}^b_P(v_P) \triangleq -B^t_{11}\hat{i}_{L1}\circ(-B_{11}v_P) - B^t_{21}\hat{g}\circ(-B_{21}v_P)}$ (β^b) where $\hat{g}\colon R^{\ell_2} \to R^{\ell_2}$ is the inverse function of a C^{k+1}-diffeomorphic onto mapping $\hat{f}\colon R^{\ell_2} \to R^{\ell_2}$ defined by $\hat{f}(i_{L2}) \triangleq \hat{v}_{L2}(i_{L2}) + B_{22}\hat{v}_{T2}\circ(B^t_{22}i_{L2})$
(G_2^b)	$\mathcal{G}_P = T1$ and $T2_{A(k+1)} + T2_B + L2_{B(k+1)}$ $+ L1_{A(k+1)} + L1_B + N_B$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>v-controlled</u> representation (β^b) and $\hat{\mathcal{N}}$ is <u>nondecreasing</u>.	
(G_3^b)	$\mathcal{G}_P = T1$ and $T2_{A(k+1)} + T2_B + L2_{B(k+1)}$ $+ L1_{A(k+1)} + N_B$ + either $(L1_B + M_B)$ or $(L1_C + M_A)$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>v-controlled</u> representation (β^b) and $\hat{\mathcal{N}}$ is an <u>increasing</u> n-port.	

(G_1^c)	$\mathcal{G}_P$ = T and $L_{A(k)}$	$\hat{\mathcal{N}}$ has a C^k <u>v-controlled</u> representation $i_P = \hat{i}_P^c(v_P)$ (β^c)	1. $i_P = -i_T$, $v_P = v_T$ 2. $n = t$
(G_2^c)	$\mathcal{G}_P$ = T and $L_{A(k)} + L_B$	$\hat{\mathcal{N}}$ has a C^k <u>v-controlled</u> representation (β^c) and $\hat{\mathcal{N}}$ is <u>nondecreasing</u>.	3. $\boxed{\hat{i}_P^c(v_P) \triangleq -B_T^t \hat{i}_L \circ (-B_T v_P)}$ (β^c)
(G_3^c)	$\mathcal{G}_P$ = T and $L_{A(k)} + L_B + M_C$	$\hat{\mathcal{N}}$ has a C^k <u>v-controlled</u> representation (β^c) and $\hat{\mathcal{N}}$ is an <u>increasing</u> n-port.	
(R_1^a)	$\mathcal{G}_P$ = L2 and $L1_{A(k+1)} + L1_D + T1_{A(k+1)}$ $+ T2_{A(k+1)} + N_C$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>i-controlled</u> representation $v_P = \hat{v}_P^a(i_P)$ (γ^a)	1. $i_P = -i_{L2}$, $v_P = v_{L2}$ 2. $n = \ell_2$
(R_2^a)	$\mathcal{G}_P$ = L2 and $L1_{A(k+1)} + L1_D + T1_{A(k+1)}$ $+ T2_{A(k+1)} + T2_B + N_C$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>i-controlled</u> representation (γ^a) and $\hat{\mathcal{N}}$ is <u>nondecreasing</u>.	3. $\boxed{\hat{v}_P^a(i_P) = -B_{21}\hat{v}_{T1}\circ\tilde{g}\circ(-B_{21}^t i_P) - B_{22}\hat{v}_{T2}\circ(-B_{22}^t i_P)}$ (γ^a) where $\tilde{g}: R^{t_1} \to R^{t_1}$ is the inverse function of the C^{k+1} - diffeomorphic onto mapping $\tilde{f}: R^{t_1} \to R^{t_1}$ defined by $\tilde{f}(i_{T1}) \triangleq i_{T1} - B_{11}^t \hat{i}_{L1}\circ(-B_{11}\hat{v}_{T1}(i_{T1}))$
(R_3^a)	$\mathcal{G}_P$ = L2 and $L1_{A(k+1)} + L1_D + T1_{A(k+1)}$ $+ T2_{A(k+1)} + N_C$ + either $(T2_B + M_D)$ or $(T2_C + M_E)$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>i-controlled</u> representation (γ^a) and $\hat{\mathcal{N}}$ is an <u>increasing</u> n-port.	
(R_1^b)	$\mathcal{G}_P$ = L2 and $L1_{A(k+1)} + L1_B + T1_{B(k+1)}$ $+ T2_{A(k+1)} + N_D$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>i-controlled</u> representation $v_P = \hat{v}_P^b(i_P)$ (γ^b)	1. $i_P = -i_{L2}$, $v_P = v_{L2}$ 2. $n = \ell_2$
(R_2^b)	$\mathcal{G}_P$ = L2 and $L1_{A(k+1)} + L1_B + T1_{B(k+1)}$ $+ T2_{A(k+1)} + T2_B + N_D$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>i-controlled</u> representation (γ^b) and $\hat{\mathcal{N}}$ is <u>nondecreasing</u>.	3. $\boxed{\hat{v}_P^b(i_P) \triangleq -B_{21}\bar{g}\circ(-B_{21}^t i_P) - B_{22}\hat{v}_{T2}\circ(-B_{22}^t i_P)}$ (γ^b) where $\bar{g}: R^{t_1} \to R^{t_1}$ is the inverse function of the C^{k+1}-diffeomorphic onto mapping $\bar{f}: R^{t_1} \to R^{t_1}$ defined by $\bar{f}(v_{T1}) \triangleq \hat{i}_{T1}(v_{T1}) - B_{11}^t \hat{i}_{L1}\circ(-B_{11}(v_{T1}))$.
(R_3^b)	$\mathcal{G}_P$ = L2 and $L1_{A(k+1)} + L1_B + T1_{B(k+1)}$ $+ T2_{A(k+1)} + N_D$ + either $(T2_B + M_D)$ or $(T2_C + M_E)$	$\hat{\mathcal{N}}$ has a C^{k+1} <u>i-controlled</u> representation (γ^b) and $\hat{\mathcal{N}}$ is an <u>increasing</u> n-port.	
(R_1^c)	$\mathcal{G}_P$ = L and $T_{A(k)}$	$\hat{\mathcal{N}}$ has a C^k <u>i-controlled</u> representation $v_P = \hat{v}_P^c(i_P)$ (γ^c)	1. $i_P = -i_L$, $v_P = v_L$ 2. $n = \ell$
(R_2^c)	$\mathcal{G}_P$ = L and $T_{A(k)} + T_B$	$\hat{\mathcal{N}}$ has a C^k <u>i-controlled</u> representation (γ^c) and $\hat{\mathcal{N}}$ is <u>nondecreasing</u>.	3. $\boxed{\hat{v}_P^c(i_P) \triangleq -B_T\hat{v}_T\circ(-B_T^t i_P)}$ (γ^c)
(R_3^c)	$\mathcal{G}_P$ = L and $T_{A(k)} + T_B + M_F$	$\hat{\mathcal{N}}$ has a C^k <u>i-controlled</u> representation (γ^c) and $\hat{\mathcal{N}}$ is an <u>increasing</u> n-port.	

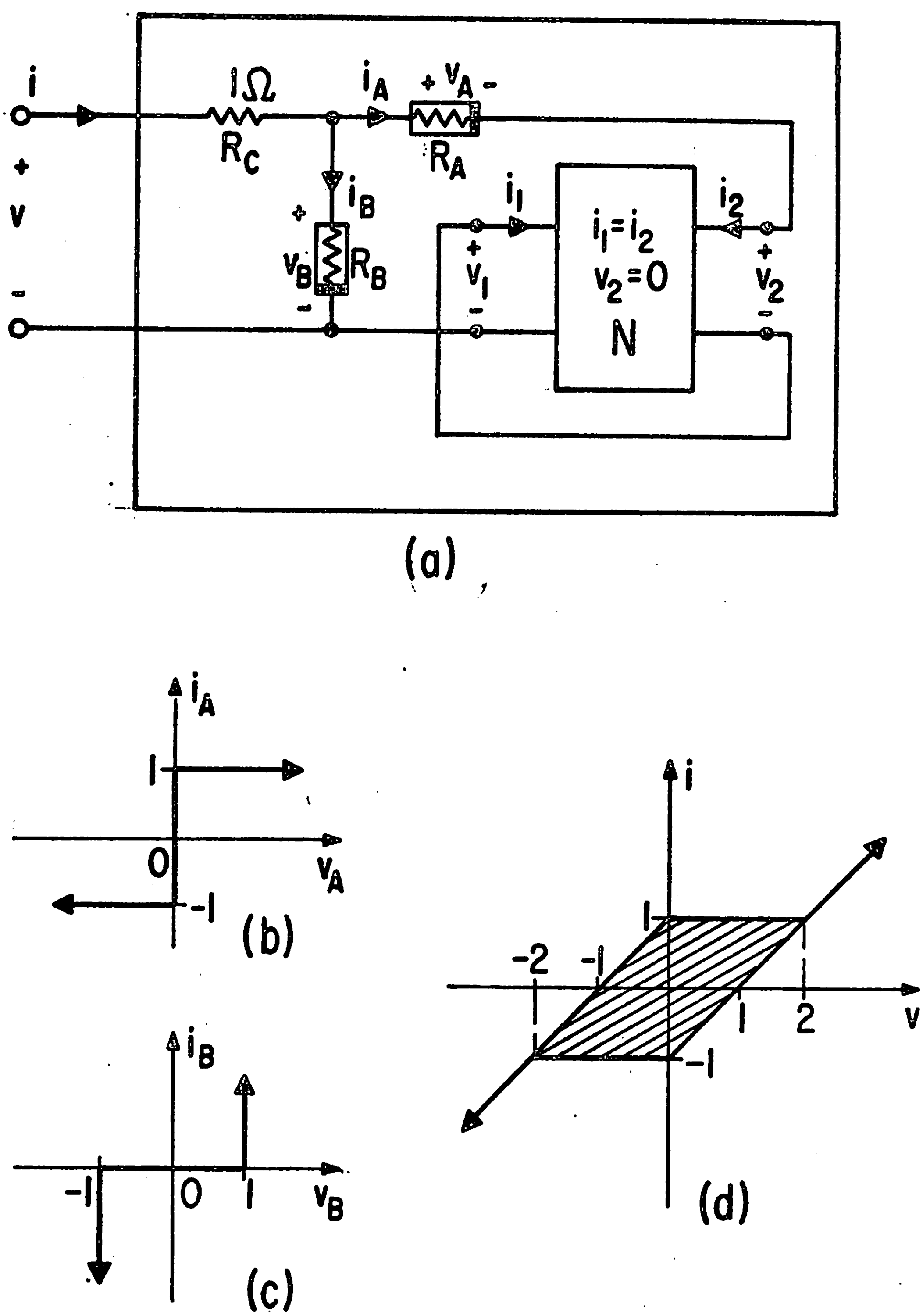


v1-i1 curve of R1
v2-i2 curve of R2
composite v-i curve of R
(a)
(b)
(c)
(d)
(e)
(f)



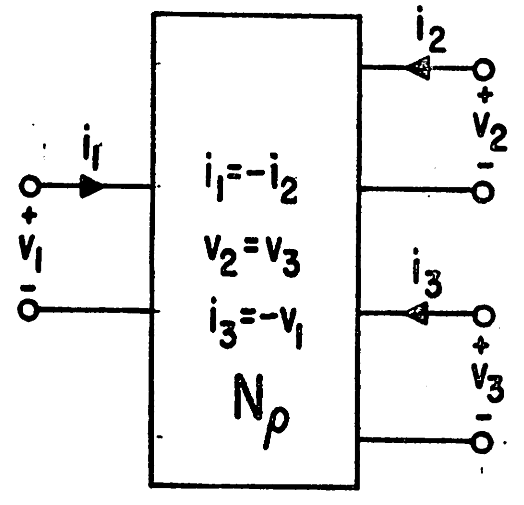
i
+
v
−
i_B
$+ v_B -$
R_B
i_A
+
v_A
−
R_A
i_1
+
v_1
−
$i_1 = -i_2$
$v_2 = 0$
N
i_2
+
v_2
−
(a)
i_A
1
-1
0
1
v_A
-1
(b)
i_B
1
-2
-1
0
1
2
v_B
-1
(c)
i
1
-2
-1
0
1
2
v
-1
(d)

FIGURE 2

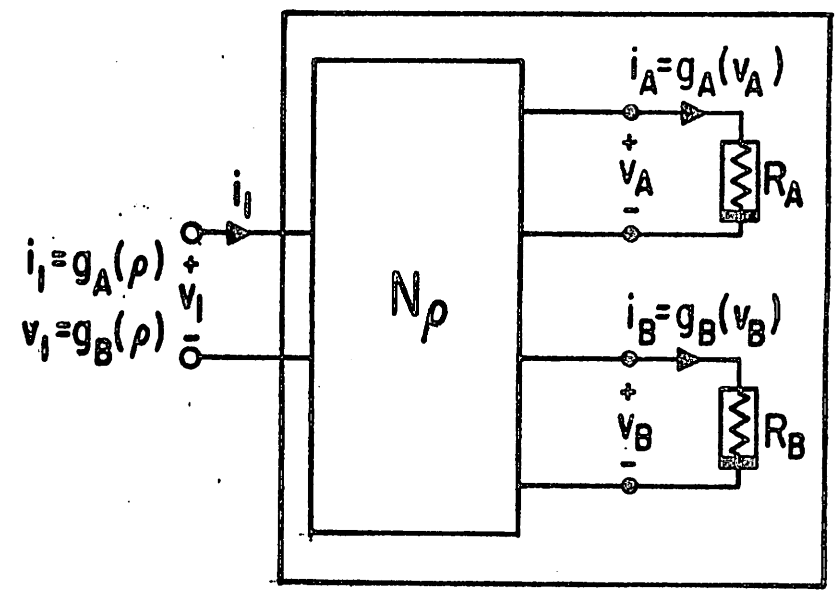


i
+
v
-
1Ω
R_C
i_A
+ v_A -
R_A
i_B
+
v_B
R_B
-
i_1
+
v_1
-
i_1 = i_2
v_2 = 0
N
i_2
+
v_2
-
(a)
i_A
1
0
v_A
-1
(b)
i_B
-1
0
1
v_B
(c)
i
1
-2
-1
1
2
v
-1
(d)

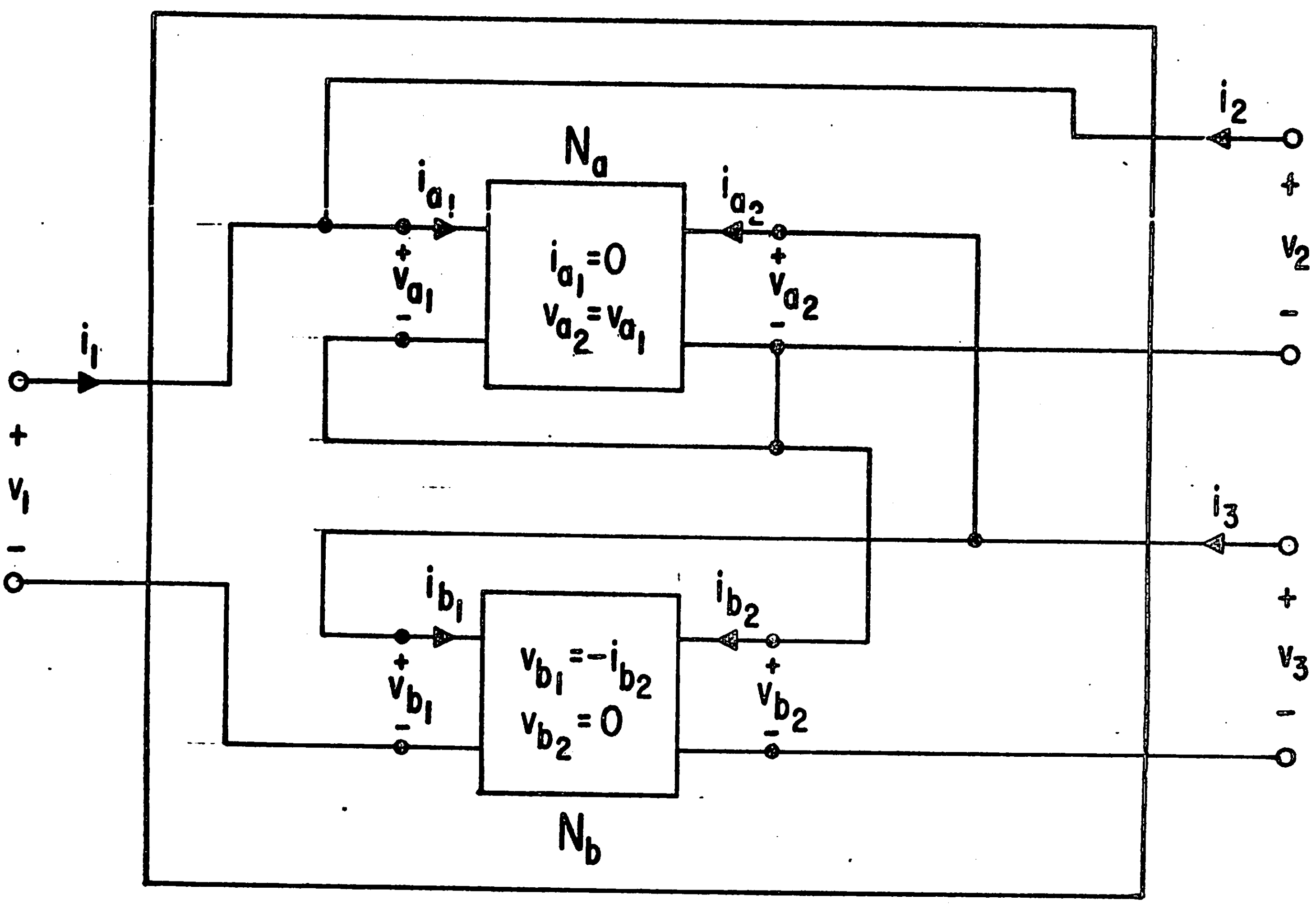
FIGURE 3



(a) A unicursal 3-port

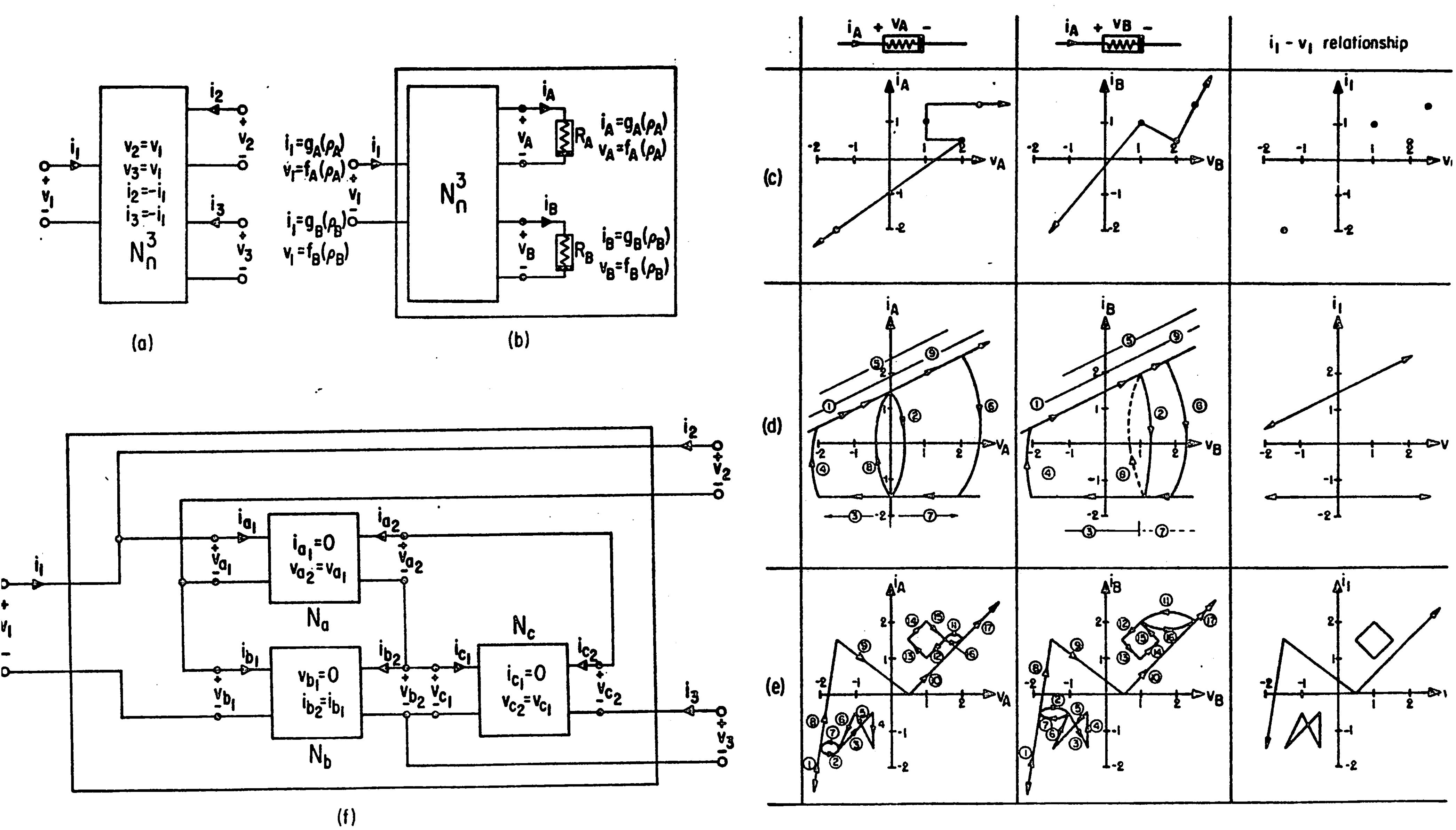


(b)



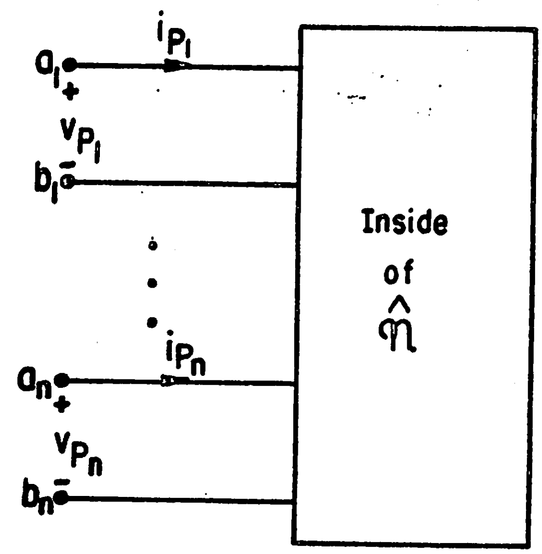
(c) Circuit realization of a unicursal 3-port N_ρ

FIGURE 4

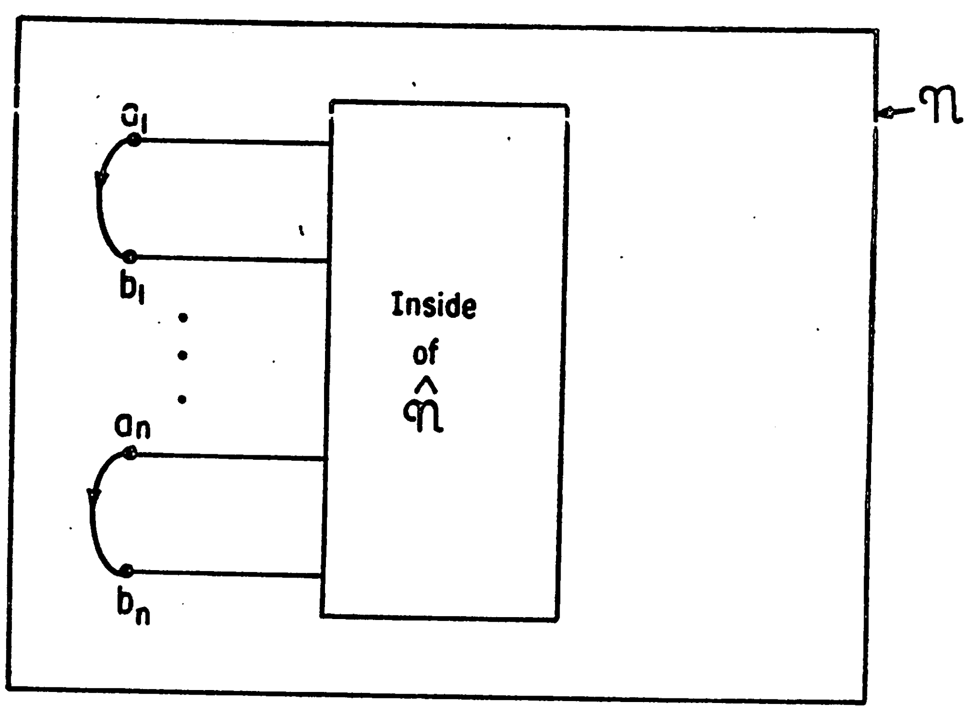


$v_2=v_1$
$v_3=v_1$
$i_2=-i_1$
$i_3=-i_1$
N_n^3
(a)
$i_1=g_A(\rho_A)$
$v_1=f_A(\rho_A)$
$i_1=g_B(\rho_B)$
$v_1=f_B(\rho_B)$
$i_A=g_A(\rho_A)$
$v_A=f_A(\rho_A)$
$i_B=g_B(\rho_B)$
$v_B=f_B(\rho_B)$
R_A
R_B
(b)
$i_{a_1}=0$
$v_{a_2}=v_{a_1}$
N_a
$v_{b_1}=0$
$i_{b_2}=i_{b_1}$
N_b
$i_{c_1}=0$
$v_{c_2}=v_{c_1}$
N_c
(f)
i_1 - v_1 relationship
(c)
(d)
(e)

FIGURE 5


i_{P_1}
a_1
v_{P_1}
b_1
i_{P_n}
a_n
v_{P_n}
b_n
Inside of $\hat{\eta}$

(a)


a_1
b_1
a_n
b_n
Inside of $\hat{\eta}$
η

(b)

FIGURE 6

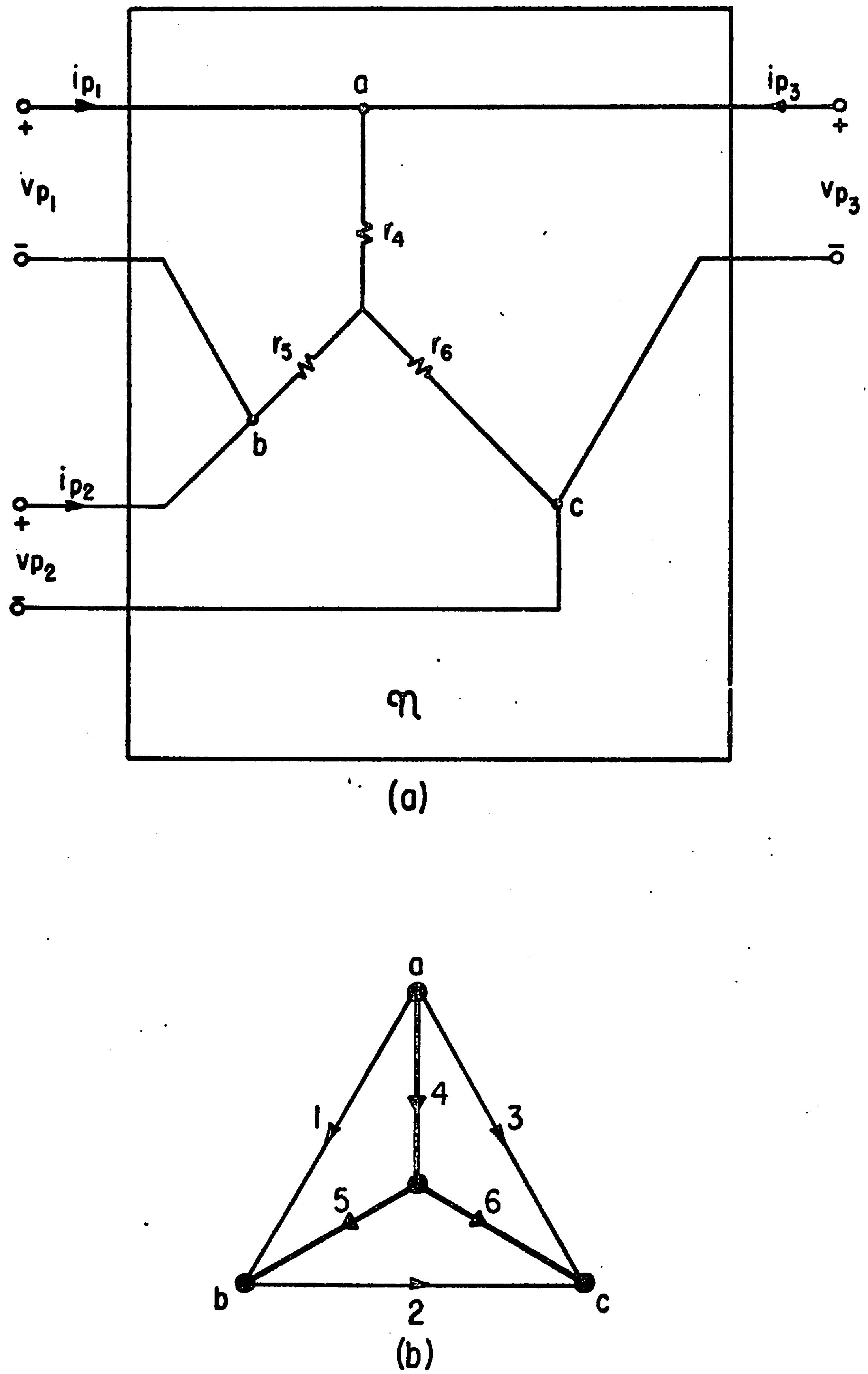


FIGURE 7

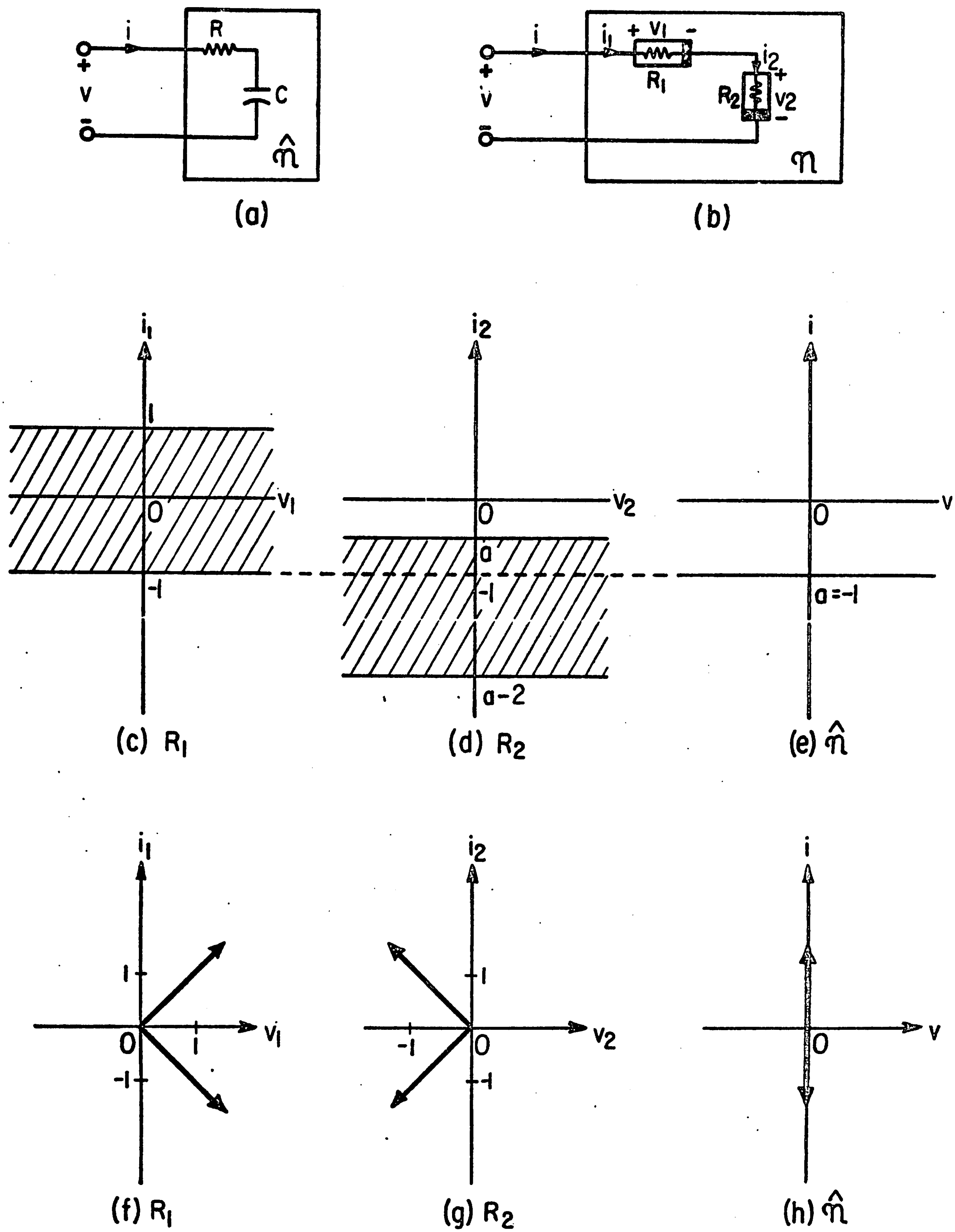
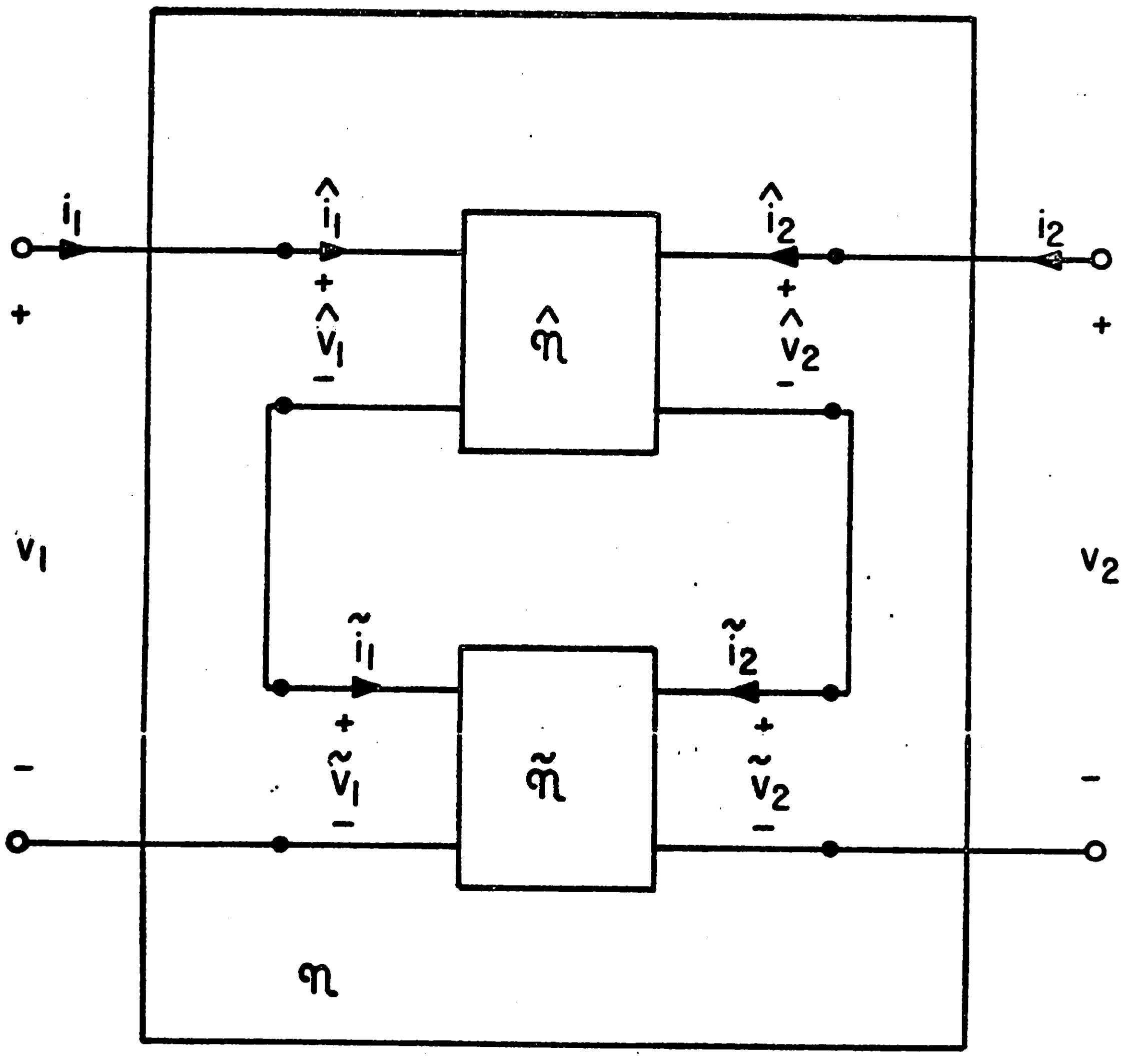

i
R
+
v
C
-
$\hat{n}$
(a)
i
i_1
+ v_1 -
R_1
i_2
+
R_2
v_2
-
+
v
-
n
(b)
i_1
1
v_1
0
-1
(c) R_1
i_2
v_2
0
a
-1
a-2
(d) R_2
i
v
0
a=-1
(e) $\hat{n}$
i_1
1
v_1
0
1
-1
(f) R_1
i_2
1
v_2
-1
0
-1
(g) R_2
i
v
0
(h) $\hat{n}$

FIGURE 8



i_1
$\hat{i}_1$
$\hat{i}_2$
i_2
+
$\hat{v}_1$
$\hat{\eta}$
$\hat{v}_2$
v_1
v_2
$\tilde{i}_1$
$\tilde{i}_2$
$\tilde{v}_1$
$\tilde{\eta}$
$\tilde{v}_2$
η

FIGURE 9

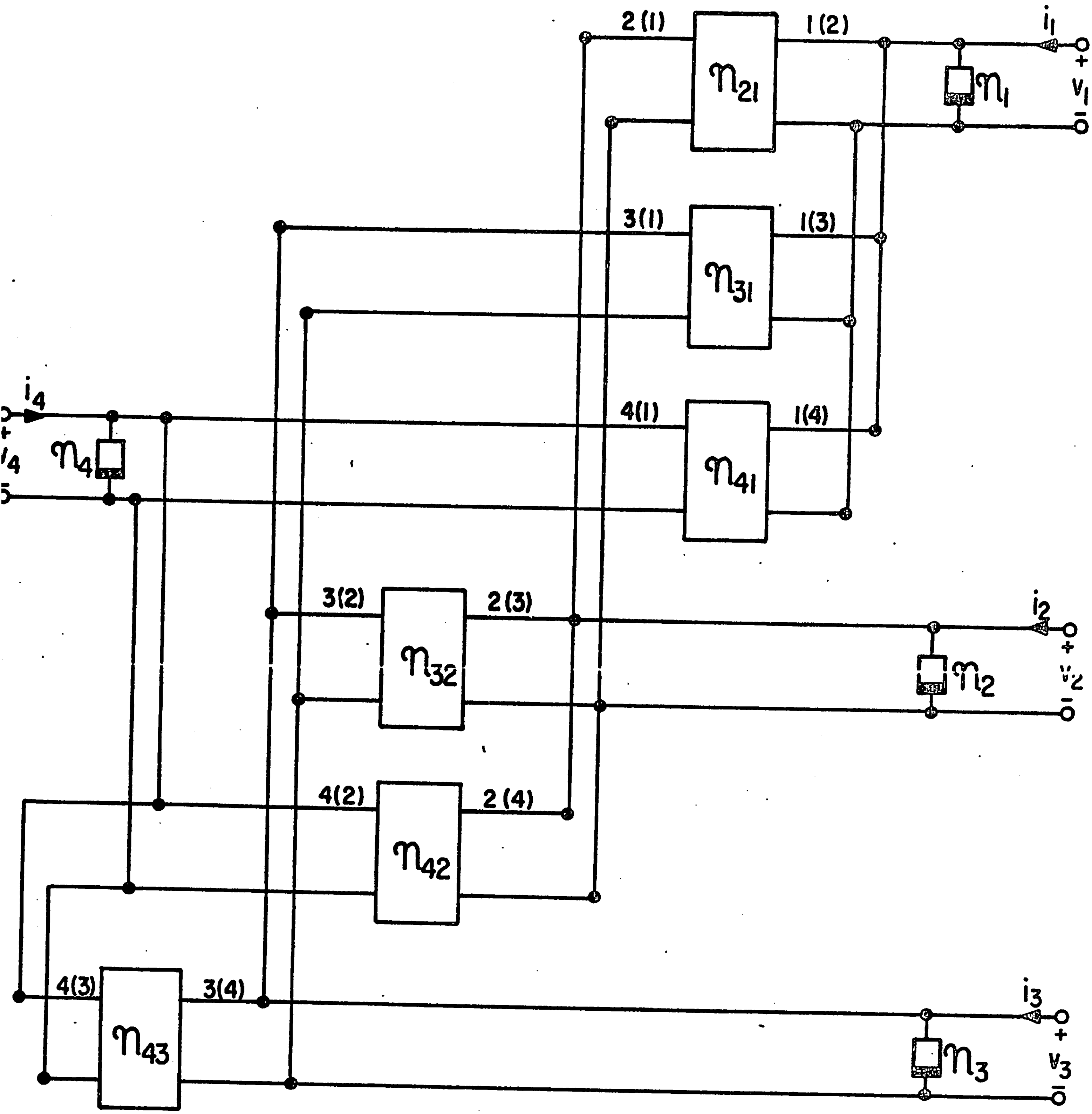
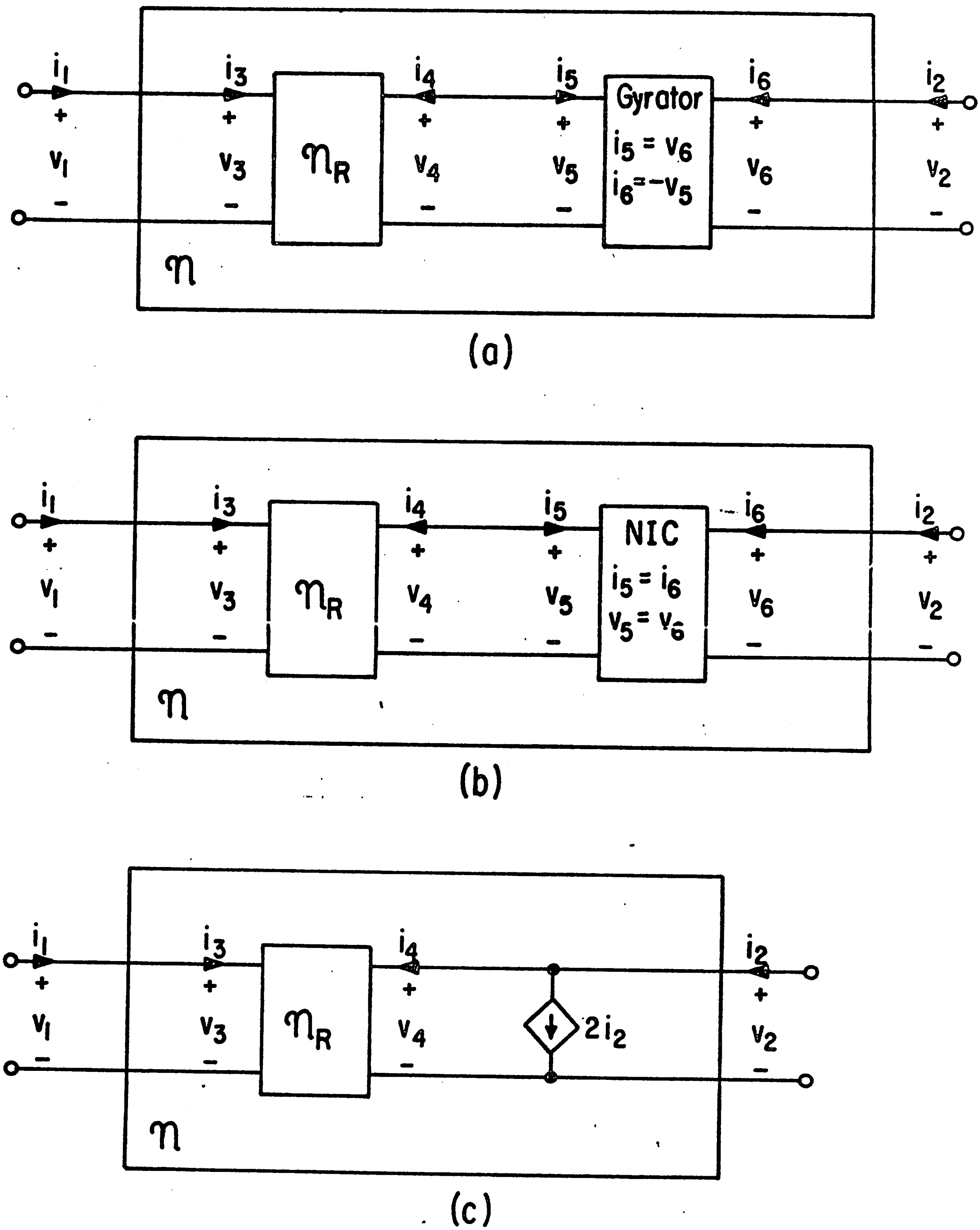

2 (1)
1 (2)
i1
+
v1
-
n21
n1
3 (1)
1 (3)
n31
i4
+
v4
-
n4
4 (1)
1 (4)
n41
3 (2)
2 (3)
n32
i2
+
v2
-
n2
4 (2)
2 (4)
n42
4 (3)
3 (4)
n43
i3
+
v3
-
n3

FIGURE 10



i_1
v_1
i_3
v_3
η_R
i_4
v_4
i_5
v_5
Gyrator
$i_5 = v_6$
$i_6 = -v_5$
i_6
v_6
i_2
v_2
η
(a)
NIC
$i_5 = i_6$
$v_5 = v_6$
(b)
$2i_2$
(c)

FIGURE 11

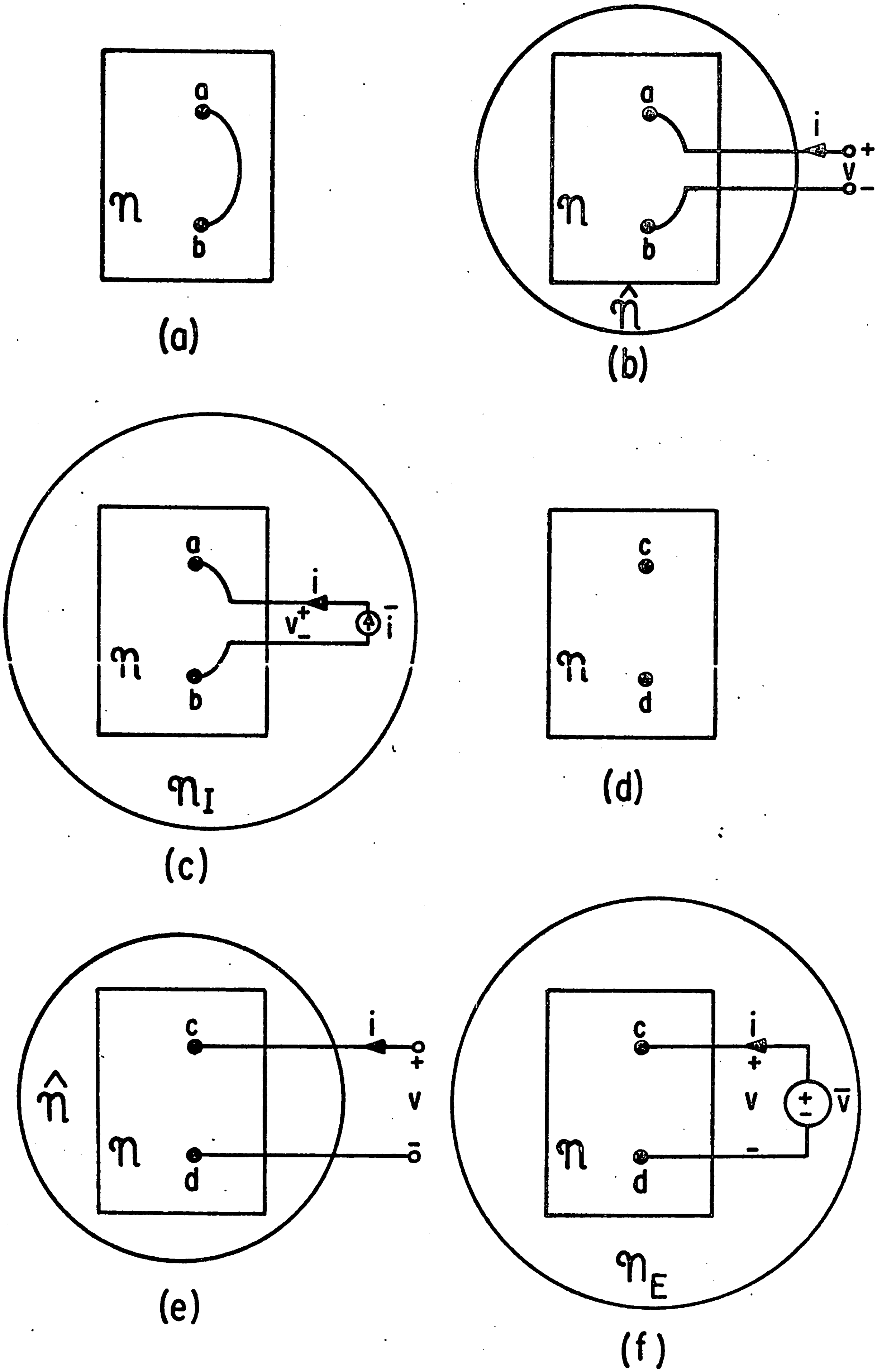


FIGURE 12