AN ASYMPTOTIC EXPANSION FOR THE QUANTIZATION ERROR OF CLOSELY SPACED UNIFORM QUANTIZERS WITH GAUSSIAN INPUT

by

E. R. Berlekamp

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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We begin by computing a sequence of constants, C_n, which are nonzero only for even n and which satisfy the expression

$$\int_{a-\Delta}^{a+\Delta} (x-a)^2 \; p(x) \; dx \approx \Delta^2 \sum_{\substack{n \text{ even} \\ = 0}}^{\infty} C_n \, \Delta^n \int_{a-\Delta}^{a+\Delta} p^{(n)}(x) \; dx \tag{1}$$

where $p^{(n)}(x) = \frac{d^n p(x)}{dx^n}$. To find the constants C_n, we expand $p^{(n)}(x)$ in a Taylor series about a,

$$p^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k p^{(n+k)}(a)}{k!}$$

Substituting this into Eq. (1) gives

$$\int_{a-\Delta}^{a+\Delta} (x-a)^2 \sum_{k=0}^{\infty} \frac{(x-a)^k p^{(k)}(a)}{k!} \; dx = \Delta^2 \sum_{\substack{n \text{ even} \\ = 0}}^{\infty} C_n \, \Delta^n \int_{a-\Delta}^{a+\Delta} \sum_{k=0}^{\infty} \frac{(x-a)^k p^{(n+k)}(a)}{k!} \; dx.$$

Exchanging the summation and integration signs and replacing x - a by y gives

$$\sum_{k=0}^{\infty} \frac{p^{(k)}(a)}{k!} \int_{-\Delta}^{\Delta} y^{k+2} \; dy = \Delta^2 \sum_{\substack{\text{even } n \\ = 0}}^{\infty} C_n \, \Delta^n \sum_{k=0}^{\infty} \frac{p^{(n+k)}(a)}{k!} \int_{-\Delta}^{\Delta} y^k \; dy$$

$$2 \sum_{\substack{\text{even} \\ m=0}}^{\infty} \frac{p^{(n)}(a) \;\; \Delta^{m+3}}{m! \;\; (m+3)} = \Delta^2 \sum_{\substack{\text{even } n \\ = 0}}^{\infty} C_n \, \Delta^n \, 2 \sum_{\substack{\text{even } k \\ = 0}}^{\infty} \frac{p^{(n+k)}(a) \;\; \Delta^{k+1}}{k! \;\; (k+1)}$$

$$= 2 \sum_{\substack{\text{even } m \\ = 0}}^{\infty} \sum_{\substack{\text{even } n \\ = 0}}^{m} \frac{C_n}{(m+1-n)!} \, p^{(m)}(a) \Delta^{m+3}$$

$$\sum_{\substack{\text{even } n \\ = 0}}^{m} \frac{C_n}{(m+1-n)!} = \frac{1}{m!(m+3)} \, ,$$

$$C_m = \frac{1}{m!(m+3)} - \sum_{\substack{\text{even } n \\ = 0}}^{m-2} \frac{C_m}{(m+1-n)!}$$

$$c_0 = \frac{1}{3} = \frac{2^3}{4!} = \frac{1}{2} \times \frac{2}{3!}$$

$$c_2 = \frac{2}{45} = \frac{2^5}{6!} = \frac{1 \times 2^4}{3 \times 5!}$$

$$c_4 = \frac{-4}{3\times5\times7\times9} = -\frac{4}{3}\frac{\times 2^7}{8!} = -\frac{1\times 2^6}{3\times 7!}$$

$$c_6 = \frac{2}{3\times5^2\times7\times9} = \frac{3\times 2^9}{10!} = \frac{3\times 2^8}{5\times 9!}$$

$$c^8 = \frac{-2^9}{3\times 9!\times 11} = -\frac{10\times 2^{11}}{12!} = -\frac{5}{3}\frac{2^{10}}{11!}$$

Using Eq. (1), we now compute the errors associated with the quantizer outputs at $V_{min} + \Delta$, $V_{min} + 3\Delta$, $V_{min} + 5\Delta, \ldots,$ $V_{max} - \Delta$ as

$$\sum_{\text{even } j = 0}^{\frac{(V_{max}-V_{min}-2)}{\Delta}} \int_{V_{min}+j\Delta}^{V_{min}+(j+2)\Delta} (x-(V_{min}+(j+1)\Delta))^2 \, p(x) \, dx$$

$$= \Delta^2 \sum_{\substack{\text{even } n \\ = 0}}^{\infty} c_n \, \Delta^n \int_{V_{min}}^{V_{max}} p^{(n)}(x) \, dx \qquad (2)$$

In the case when $V_{min} = -\infty$, $V_{max} = +\infty$, and $p^{(0)}(x) = \frac{\exp - \frac{\beta_x{}^2}{2}}{\sqrt{2\pi}}$,

we may differentiate the equation

$\int p^{(0)}(x) \, dx = \beta^{-\frac{1}{2}}$ with respect to β n times to obtain

$$\int x^{2n} \, p^{(0)}(x) \, dx = (-\tfrac{1}{2}) \, (-\tfrac{3}{2}) \, (-\tfrac{5}{2}) \cdots \left(-\frac{(2n+1)}{2}\right) \beta^{-(\frac{2n+1}{2})}$$

Defining the $(n+1)^{st}$ Hermite polymonial as

$$H_n(x) = \sqrt{2\pi}\, p^{(n+1)}(x) \, \exp\frac{x^2}{2} = \left(-x + \frac{d}{dx}\right)\left(\sqrt{2\pi}\, p^{(n)}(x) \, \exp\frac{x^2}{2}\right).$$

we have $H(x) = 1, -x, x^2 - 1, -x^3 + 3x, x^4 - 6x^2 + 3, -x^5 + 10x^3 - 15x,$
$x^6 - 15x^4 + 45x^2 - 15, -x^7 + 21x^5 - 105x^3 + 105x, x^8 - 28x^6 + 210x^4 - 420x^2 + 105$

from which

$$\int p^{(0)}(x)\ dx = 1$$
$$\int p^{(2)}(x)\ dx = -\frac{1}{2} - 1 = -\frac{3}{2}$$
$$\int p^{(4)}(x)\ dx = \frac{3}{4} - 6(-\frac{1}{2}) + 3 = 6\ \frac{3}{4} = \frac{27}{4}$$
$$\int p^{(6)}(x)\ dx = -\frac{15}{8} - 15\ \frac{3}{4} + 45(-\frac{1}{2}) - 15 = -\frac{5x81}{8} = -50\ \frac{5}{8}$$

so that Eq. (2) becomes

$$\frac{\overline{\ell_\infty^2}}{2} = \frac{\Delta^2}{3} - \frac{\Delta^4}{15} - \frac{\Delta^6}{35} - \frac{3\Delta^8}{140} - \frac{\Delta^{10}}{44} - \cdots \tag{3}$$

We now consider the case in which there are a finite even number, M, of quantizer outputs, spaced 2Δ apart, and located symmetrically about 0 so that $V_{max} = -V_{min} = V$. The mean-square error can now be expressed as the sum in Eq. (2), which gives the contribution to the error when the input is between $-V$ and V, and the "overflow" error when the input lies outside of this range. The latter expression is

$$2\int_V^\infty (x-V+\Delta)^2\ p^{(0)}(x)\ dx$$

which in the Gaussian case can be rewritten as

$$2\int_V^\infty \frac{(x+\Delta)^2}{\sqrt{2\pi}} \exp - \frac{(x+V)^2}{2}\ dx$$
$$= V^{-3} \exp - \frac{V^2}{2}\int_0^\infty (y+V\Delta)^2 \exp \frac{-y^2}{2V^2} \exp - y\ dy$$
$$= V^{-3} \exp - \frac{V^2}{2}\int_0^\infty (y+V\Delta)^2\left(\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{-y^2}{2V^2}\right)^k\right)\exp - y\ dy$$

Since V is large, the integral may be tightly bounded between the first two terms of the following divergent series

$$\sum_{k=0}^{\infty} \frac{(-2)^{-k} V^{-2k}}{k!} \int_0^{\infty} \left(y^{2k+2} + 2V\Delta y^{2k+1} + V^2 \Delta^2 y^{2k} \right) \exp - y \, dy$$

$$\doteq \sum_{k=0}^{\infty} \frac{(-2)^{-k} V^{-2k}}{k!} \left\{ (2k+2)! + 2V\Delta \; (2k+1)! + V^2 \Delta^2 (2k)! \right\} .$$

or

$$\frac{\overline{\ell_{0v}^{2}}}{2} \doteq V^{-3} \exp \frac{-V^2}{2} \left\{ \sum_{k=0}^{\infty} \frac{(-2)^{-k} V^{-2k}}{k!} \left[(2k+2)! + 2V\Delta (2k+1)! + V^2 \Delta^2 (k)! \right] \right\}$$

$$\frac{\overline{\ell^2_{trunc}}}{2} \doteq V^{-3} \exp \frac{-V^2}{2} \left\{ - \frac{(V\Delta)^2}{3} + \frac{2}{45} (V\Delta)^4 + \frac{2^6}{3x7!} (V\Delta)^6 + \dots \right\} \left\{ 1 + 0 \left(\frac{1}{V^2} \right) \right\}$$

or $$\overline{\ell^2} = \frac{\Delta^2}{3} - \frac{\Delta^4}{15} - \frac{\Delta^6}{35} - \dots + V^{-3} \exp \frac{-V^2}{2} \left\{ 2 + 0(V\Delta) \right\}$$