A GENERALIZED NYQUIST-TYPE STABILITY CRITERION FOR MULTIVARIABLE FEEDBACK SYSTEMS

by

John F. Barman and Jacob Katzenelson

Memorandum No. ERL-383

15 June 1973

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

A Generalized Nyquist-type Stability Criterion for Multivariable Feedback Systems

John F. Barman and Jacob Katzenelson†

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory,
University of California, Berkeley, California 94720

Abstract

This article considers the stability of n-input n-output, linear time invariant convolution feedback systems. Stability theorems are expressed in terms of the Nyquist plots of the eigenvalues of $\hat{G}(s)$ where s varies along a Nyquist contour in the complex plane and $\hat{G}(s)$ is the transfer function of the open loop system which is allowed to have poles in the right half plane. Our objectives are to state clearly these theorems and to prove them. The paper investigates the geometry of the eigenvalues in the complex plane; in particular, the properties of the eigenvalues on and near the exceptional points, and the graph theoretic properties of the loci of the eigenvalues are studied. The stability theorems are proved using these geometric properties.

†On leave from the Technion, Israel Institute of Technology, Haifa, Israel.
This work has been supported in part by the National Science Foundation Grant GK-10656X2.

the Nyquist plots of the eigenvalues. Since the only practical way to obtain these plots is by the use of a digital computer, some remarks on the computational procedure are included.

Section 2 contains a precise definition of the linear feedback system whose stability is investigated here. Section 3 contains a discussion of the geometry of the eigenvalues of the transfer function $\hat{G}(s)$ in the complex plane. The lemmas proved in Section 2 are used to prove the theorems of Section 4. These theorems present necessary and sufficient conditions for stability in terms of the plots of the eigenvalues. Section 5 contains some concluding remarks.

Theorems 1 and 2 state stability conditions for open loop stable system. This result is extended in Theorem 3 to the open loop unstable case. It turns out that in this case to determine stability, one has to check the encirclements of the Nyquist plots of the eigenvalues and the encirclements of an additional plot, $\Gamma_D(R)$. Theorem 4 states conditions on the compensators so that the introduction of the compensator does not require recalculation of $\Gamma_D(R)$ for determining stability.

Appendix A contains some definitions of mathematical terms used in the paper. These terms are well known to the mathematician but are not commonly used by the engineer. The terms are mentioned here in order to help the reader and not to replace mathematical texts.

Appendix B contains some further stability results. Appendix C contains simple examples of the Nyquist plots of eigenvalues.

2. Definition of the System

Consider a continuous time, linear, time invariant, feedback system

1. Introduction

This article considers the stability of n-input n-output, linear time invariant convolution feedback systems. Necessary and sufficient stability conditions are expressed in terms of the Nyquist plots of the eigenvalues of $\hat{G}(s)$ where s varies along a Nyquist contour in the complex plane and $\hat{G}(s)$ is the transfer function of the open loop system. Our objectives are to clearly state these conditions and to prove the corresponding theorems.

The idea of expressing the stability conditions in terms of the Nyquist plots of the eigenvalues orgignated by A. G. J. MacFarlane [1, , 17]. MacFarlane used these loci for the design of compensators for n-dimensional systems. Generally speaking, MacFarlane uses these plots to compensate each eigenvalue "by itself," and thus by solving n one-dimensional compensation problems the stability requirement of the n-input n-output compensator problem is satisfied. The advantage of the eigenvalue approach is similar to the advantages of the use of the (single-input single-output) Nyquist criterion: it provides the designer with the insight which enables him to choose a compensator. Although not all the aspects of this design approach have been completely explored at this time, the provision of insight is of great importance and justifies further work on MacFarlane's design approach.

To the best of our knowledge, a clear statement of the conditions of stability via Nyquist plots of the eigenvalues and their proof does not appear in the literature. It is the paper's objective to provide a statement and a proof. Another contribution of this work is the investigation of the role played by the exceptional points in the description of complex variable s.

H and G are referred to as the closed loop and the open loop systems, respectively.

H is said to be <u>stable</u> if and only if $\hat{H}(s)$ is a proper matrix whose poles are in the open left half of the complex plane.

Note that since $\hat{H}(s)$ is rational the above definition is equivalent to the requirement that the closed loop system H be L^p stable for all $1 \leq p \leq \infty$.

3. The Geometry of the Eigenvalues in the Complex Plane

Consider a point s which is not a pole of $\hat{G}(s)$. Let $\lambda_i(s)$, $i = 1, \ldots, n$, denote the n (not necessarily distinct) eigenvalues of $\hat{G}(s)$. Generally speaking, our objective is to define n analytic functions, $\lambda_i(\cdot)$, such that at (almost) any point s, their values at the point are the set of eigenvalues of $\hat{G}(s)$. In this section we shall investigate the properties of the $\lambda_i(s)$. These properties will be used in Section 4 to express stability conditions of H.

Let $F(\lambda,s)$ be

$$F(\lambda,s) \triangleq \det[\lambda I - \hat{G}(s)] = \lambda^n + p_1(s)\lambda^{n-1} + \ldots + p_n(s). \tag{6}$$

$F(\lambda,s)$ is a polynomial in λ whose coefficients are proper rational functions of s. Let $p_0^1(s)$ be the least common denominator of $p_i(s)$, $i = 1, \ldots, n$. Then

$$F(\lambda,s) = \frac{1}{p_0^1(s)} \sum_{k=0}^{n} p_k^1(s)\lambda^{n-k} \triangleq \frac{1}{p_0^1(s)} F^1(\lambda,s) \; . \tag{7}$$

with n inputs and n outputs. The input u, output y and the error e are functions mapping $\mathbb{R}_+$ (defined as $[0,\infty)$) to $\mathbb{R}^n$ or corresponding distributions on $\mathbb{R}_+$. y, e and u are related by

$$y = G * e \tag{1}$$

$$e = u - y \tag{2}$$

where * denotes convolution and G is a convolution operator whose Laplace transform $\hat{G}(s)$ is given by

$$\hat{G}(s) = \hat{G}_a(s) + \sum_{\alpha=1}^{\ell} \sum_{k=1}^{m_\alpha} \frac{R_{\alpha k}}{(s - p_\alpha)^k} \tag{3}$$

where $\hat{G}_a(s)$ is a proper (bounded at $s = \infty$) matrix whose elements are rational functions of the complex variable s. Poles of $\hat{G}_a(s)$ are in the open left half plane only; ℓ and m_α, $\alpha = 1 \ldots \ell$ are finite integers; the matrices $R_{\alpha k}$ are elements of $\mathbb{C}^{n\times n}$ and the poles p_α are either real or occur in complex conjugate pairs and $\mathrm{Re}(p_\alpha) \geq 0$ for all $\alpha = 1, \ldots, \ell$.

Furthermore, assume that $\det [I + \hat{G}(s)] \not\equiv 0$. This is needed for defining the closed loop convolution operator H. Under the above assumptions there exists a convolution operator H such that

$$y = H * u \, . \tag{4}$$

Moreover $\hat{H}(s)$ exists and is given by

$$\hat{H}(s) = \hat{G}(s)[I + \hat{G}(s)]^{-1} \, . \tag{5}$$

Thus $\hat{H}(s)$ is also a matrix whose elements are rational functions of the

$i = 1, \ldots n$, approaches a constant as $|s| \to \infty$. Thus, at $s = \infty$, instead of considering $F^1(\lambda,s) = 0$ we consider $F(\lambda,s) = 0$. As $|s| \to \infty$, $F(\lambda,s) = 0$ reduces to a polynomial with constant coefficients denoted by $F(\lambda,\infty) = 0$. If this polynomial has multiple roots we call the point $s = \infty$ the exceptional point of the second kind at infinity.

Let us choose a point jd_0 on the $j\omega$ axis such that $F(\lambda,jd_0) = 0$ has n distinct roots and introduce branch cuts from all the (finite) exceptional points in a manner described by Figure 1.

Fact 1 [3, page 103].

There exist n functions $\lambda_1(\cdot), \ldots, \lambda_n(\cdot)$ defined on $\mathbb{C} - Q$ which are analytic everywhere except possibly on the branch cuts; and for any finite s, $s \in \mathbb{C} - Q, \{\lambda_1(s), \ldots, \lambda_n(s)\}$ is the set of roots of $F^1(\lambda,s) = 0$.

Remarks

(1) From Fact 1 and our previous discussion it follows that for any $s \in \mathbb{C} - Q$, $\{\lambda_1(s), \ldots, \lambda_n(s)\}$ is the set of eigenvalues of $\hat{G}(s)$. It will be shown that at infinity the values of functions are the eigenvalues of $G(\infty)$, the limit of $G(s)$ as $|s| \to \infty$.

(2) Fact 1 holds when $F(\lambda,s)$ is irreducible. It is now clear that if $F(\lambda,s) = \prod_{i=1}^{K} F_i(\lambda,s)$ where each $F_i(\lambda,s)$ is irreducible, for $i=1, \ldots, K$, then by Fact 1 we can define n_i functions $\lambda_1^i(\cdot) \ldots \lambda_{n_i}^i(\cdot)$ for each irreducible part $F_i(\lambda,s)$ with properties stated in Fact 1. Thus Fact 1 also holds for the case that $F(\lambda,s)$ is reducible and hence at this point we eliminate the irreducibility restriction on $F(\lambda,s)$.

$F^1(\lambda,s)$ is a polynomial in λ whose coefficients are polynomials in s.

At this point we assume that $F^1(\lambda,s)$ is an irreducible polynomial in (λ,s); i.e., $F^1(\lambda,s)$ can not be factored as a product of two polynomials in λ whose coefficients are polynomials in s. This restriction will be removed later.

Let Q and P be the set of all poles of $\hat{G}(s)$ and the set of poles of $\hat{G}(s)$ in the closed right half plane, respectively. It is clear that for any $s \notin Q$ and $|s| \neq \infty$ the roots and their corresponding multiplicates of $F(\lambda,s) = 0$ are exactly the same as those of $F^1(\lambda,s) = 0$. This is a key observation which allows us to make use of the extensive results obtained for roots of polynomials [3,4].

Except for a finite number of points, which we label the exceptional points, for any $s \notin Q$, $F^1(\lambda,s) = 0$ (and hence $F(s,\lambda) = 0$) has n-distinct roots. The set of exceptional points fall into three categories [3,page 93]:

(1) roots of $p_0^1(s) = 0$. These roots are some of the poles of $\hat{G}(s)$ and thus they belong to Q.

(2) Points $s \notin Q$ for which $F^1(\lambda,s) = 0$ (or $F(\lambda,s) = 0$) has multiple roots. These points are the roots of a polynomial in s (called the discriminant) [18, page 292] and thus are finite in number. It is also interesting to note that they occur in complex conjugate pairs and that they correspond to the solutions of the system of equations $F(\lambda,s) = 0$ and $\frac{\partial}{\partial\lambda} F(\lambda,s) = 0$. These points are called the (finite) exceptional points of the second kind.

(3) The point $s = \infty$. This point is a possible pole of some of the $p_i^1(s)$, $i = 1, \ldots n$; in which case $F^1(\lambda,s) = 0$ becomes meaningless at $s = \infty$. But since $\hat{G}(s)$ is bounded at infinity, $p_i(s)$,

$F(\lambda,\infty) = 0$. Given $\delta > 0$ there exists an $R_o > 0$ such that for any $R \geq R_o$ and any $s = Re^{j\phi}$, $\pi/2 \leq \phi \leq \pi/2$, and any i, $i=1, \ldots, q$, $F(\lambda,s)$ has exactly m_i roots, counting multiplicities, in $N(\tilde{\lambda}_i(\infty),\delta)$.

(iii)

$$\left\{\lim_{|s| \to \infty} \lambda_i(s),\ i = 1,\ldots, n\right\} = \left\{\tilde{\lambda}_i(\infty),\ i = 1,\ldots,q\right\}.$$

<u>Remark</u>:

Note that (ii) and (iii) are statements of Fact 2 when $s_o = \infty$.

<u>Proof</u>:

(i) follows directly from Fact 2 and (iii) follows from (ii). The difficulty with (ii) is that the $s = \infty$ point is not 'covered' by Fact 2. Consider the maping $z = \frac{1}{s}$, which maps the infinity point to zero. Let $F^2(\lambda,z)$ be defined by $F^2(\lambda,z) = F^1(\lambda,\frac{1}{z})$.

It is clear that $F^2(\lambda,z)$ is a polynomial in λ with rational coefficients in z which satisfies the conditions of Fact 2 with $s_o = 0$. However, the roots of $F^2(\lambda,0) = 0$ are the roots of $F(\lambda,\infty) = 0$ and all z such that $|z| < \varepsilon$ maps to all s such that $|s| > \frac{1}{\varepsilon}$ which completes the proof. ◻

Let us now define functions which are commonly called the Nyquist contours.

Since $\hat{G}(s)$ is rational the set consisting of the open L.H.P. poles of $\hat{G}(s)$ and the open left half plane zeros of $\det[I+G(s)]$ is bounded away from the $j\omega$ axis. Thus, there exists an $\varepsilon_o > 0$ such that $-\varepsilon_o$ is such a bound.

Let $N_q : [a,b] \to \mathbb{C}$ be a bijection whose image $(\tilde{N}_q)$ is plotted

The following fact is a restatement of Hurwitz Theorem [4, page 4] in terms of polynomials.

Let $s_o \in \mathbb{C}$. Let $N(s_o,\delta)$ denote an open ball, centered at s_o with radius δ.

Fact 2

Let $s_o \in \mathbb{C}$ and $s_o \notin Q$. Let $\tilde{\lambda}_i(s_o)$, $i = 1, \ldots, q$ be the distinct roots of $F(\lambda,s_o) = 0$. Let m_i, $i = 1, \ldots, q$, be the multiplicity of $\tilde{\lambda}_i(s_o)$ as a root of $F(\lambda,s_o) = 0$. Under these conditions, given any $\delta > 0$ there exists an $\varepsilon > 0$ such that for any s, $|s-s_o| < \varepsilon$, and for any i, $i = 1,\ldots,q$, $F(\lambda,s) = 0$ has exactly m_i roots, counting multiplicities, in $N(\tilde{\lambda}_i(s_o),\delta)$.

The following lemma is an immediate consequence of Fact 2. The lemma will be often used in the sequel.

Denote by $jd_1, jd_2, \ldots jd_m$, with $d_o < d_1 < \ldots d_m$, the exceptional points of the second kind on the $j\omega$ axis. Note that these points occur in complex conjugate pairs. Let $G(\infty)$ be defined by $G(\infty) = \lim_{|s|\to\infty} \hat{G}(s)$ (the limit exists since $\hat{G}(s)$ is rational. Let $\{\tilde{\lambda}_1(\infty), \ldots, \tilde{\lambda}_q(\infty)\}$ where $q \leq n$, be the distinct eigenvalues of $\hat{G}(\infty)$. Let $F(\lambda,\infty) \triangleq \det(\lambda I - G(\infty))$. Let ε be real and positive.

Lemma 1:

(i) For all k, $k = 1, \ldots, m$,

$$\left\{\lim_{\varepsilon \to 0+} \lambda_i(jd_k + j\varepsilon),\ i = 1, \ldots, n\right\} = \left\{\lim_{\varepsilon \to 0-} \lambda_i(jd_k + j\varepsilon),\ i = 1, \ldots, n\right\}$$

$$= \left\{\lambda_i(jd_k),\ i = 1, \ldots, n\right\};$$

(ii) Let m_i, $i = 1, \ldots, q$, be the multiplicity of $\tilde{\lambda}_i(\infty)$ as roots of families of paths. In the sequel we do not distinguish between the class and a representative member of it. No confusion arises and a simplification of discussion results from this convenience.

In the sequel the following notation is used: let γ be a continuous function defined on a compact interval I. $\tilde{\gamma}$ is another notation for $\gamma(I)$, the image of I under γ. When γ is a bijection a direction is associated with $\tilde{\gamma}$ in an obvious manner. If γ is an indexed family of functions, $\tilde{\gamma}$ denotes the indexed family of images.

Lemma 3:

The members of $\Gamma(R)$ (Γ and $\bar{\Gamma}(R)$) can be juxtaposed to form an indexed family of closed paths.

Proof:

In this proof we shall show that if the members of $\Gamma(R)$ are appropriately juxtaposed [5, page 217] then the result is an indexed family of closed paths. The proof for Γ and $\bar{\Gamma}(R)$ is similar and therefore we only prove Lemma 3 for $\Gamma(R)$.

The points $\{\lambda_i(jd_k), i = 1, \ldots, n, k = 0, \ldots, m\}$ play an important role in the geometry of the eigenvalues. We call these points λ-nodes. Each of these nodes is the image of both the beginning of one interval and the end of another. It follows from lemma 1 that (1) each $\tilde{\gamma}_{ik}$ leaves a λ-node and enters (possibly another) λ-node; (2) The number of $\tilde{\gamma}_{ik}$'s entering a λ-node is equal to the number of $\tilde{\gamma}_{ik}$'s leaving the same λ-node.

Construct a directed graph $\mathcal{G}$ whose nodes are in one to one correspondence with the λ-nodes and whose branches are in one to one correspondence in Fig. 2a: and where indentations to the L.H.P. of radius ε, $0 < \varepsilon < \varepsilon_o$, are taken around the poles of $\hat{G}(s)$ on the $j\omega$ axis. Similarly, let $N_q(R)$: $[a',b'] \to \mathbb{C}$ and $\tilde{N}_q(R)$: $[a'',b''] \to \mathbb{C}$ be bijections whose images $(\check{N}_q(R)$ and $\tilde{\tilde{N}}_q(R)$ are plotted in Fig. 2b and Fig. 2c respectively. Note that $N_q(R)$ is a closed path while, strictly speaking, N_q and $\tilde{N}_q(R)$ are not paths since their images are not compact.

For any ℓ, $\ell=0, \ldots, m+2$, let I_ℓ: $[c_\ell,d_\ell] \to \mathbb{C}$ be a bijection whose image $(\tilde{I}_\ell)$ is plotted in Fig. 2a; where ε-indentations to the L.H.P. of radius ε, $0 < \varepsilon < \varepsilon_o$, are taken around the poles of $\hat{G}(s)$ on the $j\omega$ axis. Let $\bar{I}_{m+2}$ be the path opposite to I_{m+2}[5, page 217].

Let $I_{-\infty}$ $:[c_{-\infty},d_{-\infty}] \to \mathbb{C}$ and $I_{+\infty}:[c_{+\infty},d_{+\infty}] \to \mathbb{C}$ be bijections whose images $(I_{-\infty}$ and $I_{+\infty})$ are plotted in Fig. 2a; where we assume that $I_{-\infty}(c_{-\infty}) = -j\infty$ and $I_{+\infty}(d_{+\infty}) = +j\infty$.

We emphasize that there is no need in exhibiting these functions explicitely.

It is thus clear that if the domains of the above defined bijections are appropriately chosen then:

(i) N_q is obtained by juxtaposing I_ℓ, $0 \leq \ell \leq m+1$, $I_{+\infty}$ and $I_{-\infty}$ (this involves a slight abuse of terminology [5, page 217] since, strictly speaking, $I_{+\infty}$, $I_{-\infty}$ are not paths).

(ii) $N_q(R)$ is obtained by juxtaposing I_ℓ, $0 \leq \ell \leq m+2$;

(iii) $\bar{N}_q(R)$ is obtained by juxtaposing $I_{+\infty}$, $I_{-\infty}$, $\bar{I}_{m+2}$.

For every $1 \leq i \leq n$ and $1 \leq k \leq m+2$, let γ_{ik} be functions defined by $\gamma_{ik} = \lambda_i \circ I_k$ where o denotes the composition of two functions. Let $\bar{\gamma}_{im+2} = \lambda_i \circ \bar{I}_{m+2}$ and let $\gamma_{i+\infty} = \lambda_i \circ I_{+\infty}$ and $\gamma_{i-\infty} = \lambda_i \circ I_{-\infty}$. Let us with the $\tilde{\gamma}_{ik}$ of $\Gamma(R)$. To each $\tilde{\gamma}_{ik}$ between λ-nodes there corresponds a branch between the corresponding nodes of $\mathcal{G}$ with a direction corresponding to the direction of the $\tilde{\gamma}_{ik}$.

Each node of $\mathcal{G}$ that has ℓ branches, $\ell > 1$, entering it is now split to ℓ nodes. The branches incident to the original node are now assigned <u>arbitrarily</u> to the new nodes such that each new node has exactly one branch entering it and one branch leaving it. It is clear that this new graph $\tilde{\mathcal{G}}$ now consists of subgraphs each consisting of one loop only.

Considering $\Gamma(R)$ again; from the construction above, it follows that to each loop of $\tilde{\mathcal{G}}$ there corresponds a collection of $\tilde{\gamma}_{ik}$ which forms a closed path. Since to each $\tilde{\gamma}_{ik}$ there corresponds a branch and since each branch of the graph is included in a loop, it follows that the members of $\Gamma(R)$ can be juxtaposed to form an indexed family of closed paths each path corresponding to a loop of $\tilde{\mathcal{G}}$.

Consider the members of $\Gamma(R)$. A number of these γ_{ik}'s 'go through' $\lambda_i(jd_k)$, i.e. $\lambda_i(jd_k)$ is either the origin or the extremity of γ_{ik} or both. The multiplicity of $\lambda_i(jd_k)$ as a root of $F(\lambda, jd_k) = 0$ is equal to the number of $\tilde{\gamma}_{ik}$'s which enter $\lambda_i(jd_k)$. An equal number of $\tilde{\gamma}_{ik}$ leave that λ-node.

To prove the lemma for Γ and $\bar{\Gamma}(R)$ we add the eigenvalues of $\hat{G}(\infty)$ to our collection of λ-nodes. $\gamma_{i+\infty}$ and $\gamma_{i-\infty}$ are paths entering and leaving λ-nodes. Using now Fact 2 and Lemma 1 and a directed graph the proof proceeds as in the case of $\Gamma(R)$. ◻

An indexed family of closed paths obtained by juxtaposition of members of $\Gamma(R)$ is said to <u>describe</u> $\Gamma(R)$. To simplify notations we shall use the notation $\Gamma(R)$ for this indexed family whenever the exact meaning is evident from the text.

families of paths. In the sequel we do not distinguish between the class and a representative member of it. No confusion arises and a simplification of discussion results from this convenience.

In the sequel the following notation is used: let γ be a continuous function defined on a compact interval I. $\tilde{\gamma}$ is another notation for $\overline{\gamma(I)}$, the image of I under γ. When γ is a bijection a direction is associated with $\tilde{\gamma}$ in an obvious manner. If γ is an indexed family of functions, $\tilde{\gamma}$ denotes the indexed family of images.

Lemma 3:

The members of $\Gamma(R)$ (Γ and $\bar{\Gamma}(R)$) can be juxtaposed to form an indexed family of closed paths.

Proof:

In this proof we shall show that if the members of $\Gamma(R)$ are appropriately juxtaposed [5, page 217] then the result is an indexed family of closed paths. The proof for Γ and $\bar{\Gamma}(R)$ is similar and therefore we only prove Lemma 3 for $\Gamma(R)$.

The points $\{\lambda_i(jd_k), i = 1, \ldots, n, k = 0, \ldots, m\}$ play an important role in the geometry of the eigenvalues. We call these points λ-nodes. Each of these nodes is the image of both the beginning of one interval and the end of another. It follows from lemma 1 that (1) each $\tilde{\gamma}_{ik}$ leaves a λ-node and enters (possibly another) λ-node; (2) The number of $\tilde{\gamma}_{ik}$'s entering a λ-node is equal to the number of $\tilde{\gamma}_{ik}$'s leaving the same λ-node.

Construct a directed graph $\mathcal{G}$ whose nodes are in one to one correspondence with the λ-nodes and whose branches are in one to one correspondence is well defined. The same holds for $\Gamma(R)$ and $\tilde{\Gamma}(R)$.

Proof:

We will prove this lemma for $\Gamma(R)$. The proofs for Γ and $\tilde{\Gamma}(R)$ are similar. Let $(\gamma_j)_{j=1}^{\ell}$ be any collection of closed paths describing $\Gamma(R)$. Then

$$2\pi j\, C(\Gamma(R), a) = 2\pi j \sum_{j=1}^{\ell} C(\gamma_j, a) = \sum_{j=1}^{\ell} \int_{\gamma_j} \frac{dz}{z-a}$$
$$= \sum_{i=1}^{n} \sum_{k=0}^{m+2} \int_{\gamma_{ik}} \frac{dz}{z-a} . \tag{8}$$

Note that the integrals above are well defined since $a \notin \tilde{\Gamma}(R)$. Each γ_j and each γ_{ik} can be covered by open sets such that $\frac{1}{z-a}$ is analytic on each set. Thus the integration along these paths is well defined [appendix A and page 251 in [5]].

The right hand side of (16) is independent of the way that the indexed family of closed paths $\Gamma(R)$ is constructed from $\left(\gamma_{ik}\right)_{\substack{1 \le i \le n \\ o \le k \le m+2}}$

Lemma 5:

Let $a \in \mathbb{C}$ and $a \notin \tilde{\Gamma}$.

For any $\delta > 0$ there exists an R_o such that for any $R \ge R_o$

(i) if $s \in \tilde{\Gamma}(R)$ then either $s \in \tilde{\Gamma}$ or

$s \in \bigcup_{i=1}^{q} N\,(\tilde{\lambda}_i(\infty), \delta)$ or both where $\{\tilde{\lambda}_i(\infty): i=1, \ldots q\}$ is the set

The <u>encirclement</u> of a point a, $a \in \mathbb{C}$, by a closed path γ, $a \notin \tilde{\gamma}$ is defined as the index of a with respect to the curve γ [see appendix A and note in particular the discussion of the relation between the argument function and the index]. The encirclement of a by γ is denoted by $C(\gamma,a)$.

Let γ be an indexed family of closed paths $\gamma = (\gamma_k)_{k=1}^{\ell}$. The encirclement of a point a by γ, $a \notin \gamma$, is defined as:

$$C(\gamma,a) \triangleq \sum_{k=1}^{\ell} C(\gamma_k,a).$$

Note the use of the term "indexed family" rather than "set" in the definition of γ. The reason is the following: if, say, γ_1, and γ_2 are identical we still want to sum their encirclements to define the encirclement of γ.

We have shown that Γ, ($\Gamma(R)$ and $\bar{\Gamma}(r)$), is an indexed family of closed paths. Thus, given this indexed family of closed paths $C(\Gamma,a)$, $a \notin \tilde{\Gamma}$, is well defined. Note, however, that the construction of the closed paths has some degree of arbitrariness to it: when a node of $\mathcal{G}$ is split the branches are arbitrarily connected to the new nodes; the only requirement being that each node has one and only one branch which enters it and one and only one branch which leaves it. Thus, it has to be proven that the actual construction of the closed paths does not change $C(\Gamma,a)$ (and the same for $\Gamma(R)$ and $\bar{\Gamma}(R)$).

<u>Lemma 4</u>:

For any a, $a \in C$, $a \notin \tilde{\Gamma}$, all indexed families of closed paths which describe Γ have the same encirclement with respect to a. Thus, $C(\Gamma,a)$

Proof:

(10) and the fact that $a \notin \tilde{\gamma}$ implies $0 \notin \tilde{\gamma}_o$.

For all i, i=1, ..., n, since $a \notin \tilde{\gamma}_i$ there exists an open set A_i such that $\tilde{\gamma}_i \subset A_i$, $a \notin A_i$ and $\frac{1}{s-a}$, is analytic in A_i; let γ_i' be loop-homotopic to γ_i (page 218, [5]) such that $\gamma_i' \subset A_i$. Let A_o be defined as $\{s: s = \prod_{i=1}^{n} (s_i - a),\ s_i \in A_i,\ i=1, \ldots, n\}$. It is clear that A_o is open; the image of γ_o and of $\gamma_o'(t) \triangleq \prod_{i=1}^{n} \gamma_i'(t-a)$, $t \in [0,1]$ A_o; $\frac{1}{s}$ is analytic on A_o. It is also clear that γ_o' is loop homotopic to γ_o. Thus we have [5, page 251].

$$2\pi j\ C(\gamma,a) = 2\gamma j \sum_{i=1}^{n} C(\gamma_i,a) = \sum_{i=1}^{n} \int_{\gamma_i'} \frac{dz}{z-a}$$

$$= \sum_{i=1}^{n} \int_0^1 \frac{\frac{d\gamma_i'(t)}{dt}}{\gamma_i'(t) - a}\, dt$$

$$= \int_0^1 \sum_{i=1}^{n} \frac{\frac{d(\gamma_i'(t) - a)}{dt}}{\gamma_i'(t) - a}\, dt = \int_0^1 \frac{\frac{d}{dt}\gamma_o'(t)}{\gamma_o'(t)}\, dt$$

$$= \int_{\gamma_o'} \frac{dz}{z} = 2\pi j\ C(\gamma_o,0).$$ ◻

4. Stability Theorems

In this section we present Nyquist type theorems to check the stability of the n-input, n-output feedback system. Unless a theorem explicitly states otherwise, the assumptions on G are stated in Section 2.

of distinct eigenvalues of $\hat{G}(\infty)$.

(ii) $a \notin \tilde{\Gamma}(R)$;

(iii) $C(\Gamma(R),a) = C(\Gamma,a)$.

Proof:

Part (i) is a direct result of Lemma 1. Since $a \notin \tilde{\Gamma}$ $a \notin \{\tilde{\lambda}_i(\infty),\ i=1,\ \ldots,\ q\}$. Let $d = \min_{1 \leq i \leq q} |\tilde{\lambda}_i(\infty) - a|$. Given $\delta = \frac{d}{2}$; it follows from part (i) of this lemma that those points of $\Gamma(R)$ which are not on Γ are in $\bigcup_{i=1}^{q} N(\tilde{\lambda}_i(\infty),\frac{d}{2})$ and thus cannot include the point a among them; which proves part (ii).

From $a \notin \tilde{\Gamma}$ and $a \notin \tilde{\Gamma}(R)$, it follows that $a \notin \bar{\tilde{\Gamma}}(R)$. Now, from the definition of the encirclement it follows that

$$C(\Gamma,a) = C(\Gamma(R),a) + C(\bar{\Gamma}(R),a). \tag{9}$$

But, if $R \geq R_o$ then $\bar{\tilde{\Gamma}}(R) \subset \bigcup_{\lambda=1}^{q} N(\tilde{\lambda}_i(\infty),\frac{d}{2})$. Thus a lies in the unbounded domain of C - image $\bar{\Gamma}(R)$ which means that if $R \geq R_o$ then $C(\bar{\Gamma}(R),a) = 0$ [9.8.3 and 9.86 in 5]. Thus from (9) we obtain that if $R \geq R_o$ then $C(\Gamma,a) = C(\Gamma(R),a)$. ∎

Lemma 6:

Let $\gamma_i : [0,1] \to \mathbb{C}$, $i=1,\ \ldots,\ n$, be closed paths. Let $\gamma = (\gamma_i)_{i=1}^{n}$ be the indexed family of the γ_i. Let $a \in C$, $a \notin \tilde{\gamma}$, and define

$$\gamma_o(t) = \prod_{i=1}^{n} (\gamma_i(t) - a) \text{ for all } t,\ 0 \leq t \leq 1. \tag{10}$$

Under these conditions, $0 \notin \gamma_o$ and $C(\gamma_o,0) = C(\gamma,a)$.

for any s, Re s $\geq$ 0 and any j, $1 \leq j \leq n$.

Hence,

$$|1 + \lambda_i(s)| \geq \frac{d}{(1 + k)^{n-1}} \text{ for any } i,\ 0 \leq i \leq n \text{ and for any } s,\ \text{Re } s \geq 0;$$

which completes the proof. □

At this point it is logical to discuss conditions for checking for each i, i=1, ..., n whether $|1 + \lambda_i(s)| > 0$ Res $\geq$ 0.

A theorem discussing this question and its difficulties is presented in Appendix B. We have moved this theorem to Appendix B since its presentation requires some additional notations and lemmas which are not used in theorems 2, 3 and 4.

Theorem 2:

Let G be stable. Under this condition

$$H \text{ is stable} \Longleftrightarrow \inf_{\text{Re } s \geq 0} |\det[I + \hat{G}(s)]| > 0$$

$\Longleftrightarrow$ (i) The point $-1 \notin \tilde{\Gamma}$;

and

(ii) $C(\Gamma, -1) = 0$.

Note: Since G is stable the N_q used is the one described in Figure 1a which has no ε indentation.

Proof:

For any radius R define $\Gamma_{det}(R)$ as

$$\Gamma_{det}(R):\quad t \rightarrow \det[I + \hat{G}(N_q(R)(t))], t \in I \tag{12}$$

where I is the interval on which $N_q(R)$ is defined. Since det[I + G(s)] is analytic on $\tilde{N}_q(R)$ and $N_q(R)$ is a closed path, so is $\Gamma_{det}(R)$.

Theorem 1:

Let G be stable. Under this condition

$$H \text{ is stable} \iff \inf_{\text{Re } s \geq 0} |\det[I + \hat{G}(s)]| > 0$$

$$\iff \inf_{\text{Re } s \geq 0} |1 + \lambda_i(s)| > 0 \text{ for all } i,\ i = 1, 2, \ldots, n.$$

Proof:

The first equivalence is shown in [6]. We shall prove the second equivalence only.

Since elements of $\hat{G}(s)$ are bounded on Re $s \geq 0$, there exist a number k such that $|\hat{G}(s)| < k$ on Re $s \geq 0$, where $|\hat{G}(s)|$ denote any induced matrix norm of $\hat{G}(s)$.

$$\max_{1 \leq i \leq n} |\lambda_i(s)| \leq |\hat{G}(s)| \leq k \text{ for any } s, \text{ Re } s \geq 0.$$

Thus, the fact that elements of $\hat{G}(s)$ are bounded on Re $s \geq 0$ implies that all the eigenvalues are bounded on Re $s \geq 0$.

($\Leftarrow$) The proof of ($\Leftarrow$) follows directly from the fact that

$$\det[I + \hat{G}(s)] = \prod_{i=1}^{n} (1 + \lambda_i(s)).$$

($\Rightarrow$) Let d be defined as

$$0 < d \triangleq \inf_{\text{Re } s \geq 0} |\det[I + \hat{G}(s)]|.$$

Thus, since each $\lambda_i(s)$ is bounded on Re $s \geq 0$ by k,

$$d \leq |\det[I + \hat{G}(s)]| = \prod_{i=1}^{n} |1 + \lambda_i(s)| \leq (1 + k)^{n-1} |1 + \lambda_j(s)|$$

Using lemma 6 we get

$$C(\Gamma_{det}(R),0) = C(\Gamma(R),-1) = 0. \tag{16}$$

Since $\det [(I + \hat{G}(s))] = \prod_{i=1}^{n} (1 + \lambda_i(s))$, $-1 \notin \tilde{\Gamma}(R)$ implies $0 \notin \tilde{\Gamma}_{det}(R)$. The principle of argument can be now applied to $\det[I + \hat{G}(s)]$ which is analytic on Re s $\geq$ 0 and non zero on $\tilde{N}_q(R)$ and together with (16) it implies that $\det [(I + \hat{G}(s))]$ does not have any zeros in any compact subset of Res $\geq$ 0. This together with (15) completes the proof. ¤

We shall now consider the general open loop unstable case.

<u>Fact 3</u>:

The proper (bounded at $s = \infty$) matrix $\hat{G}(s)$ can be factored as

$$\hat{G}(s) = N(s)D^{-1}(s)$$

where,

(a) N(s) and D(s) are n×n matrices whose elements are polynomials in s;

(b) N(s) and D(s) are right coprime;

(c) $\det D(s) \not\equiv 0$;

(d) p is a pole of $\hat{G}(s)$ if and only if it is a zero of det D(s).

This fact is due to several authors [8, 9, 10, 11]. For definitions and algorithms for this factorization see [12], [13].

Let $\Gamma_D(R): t \to \det D(N_q(R)(t))$, $t \in I$;

$\Gamma_{ND}(R): t \to \det[N(N_q(R)(t)) + D(N_q(R)(t))]$, $t \in I$, where I is the

($\Rightarrow$)

From the analiticity of $\det[I + \hat{G}(s)]$ on $\mathbb{C}$ and since $\det[I + \hat{G}(s)] > 0$ on $\tilde{N}_q(R)$ we can use the principle of argument to obtain

$$\frac{1}{2\pi j} \int_{N_q(R)} \frac{d \det[I + \hat{G}(s)]}{\det[I + \hat{G}(s)]} = C(\Gamma_{det}(R),0) = 0 \cdot \tag{13}$$

From Theorem 1 follows that under the conditions of Theorem 2 $1 + \lambda_i(s) \neq 0$, Re $s \geq 0$, $i = 0, \ldots, n$. Therefore, $-1 \notin \tilde{\Gamma}(R)$. Since $\Gamma(R)$ is an indexed family of closed paths (Lemma 3), $-1 \notin \tilde{\Gamma}(R)$ and $\det[I + \hat{G}(s)] = \prod_{i=1}^{n} (1 + \lambda_i(s))$, then, it follows from Lemma 6 that

$$C(\Gamma(R), -1) = C(\Gamma_{det}(R), 0) = 0. \tag{14}$$

Consider Lemma 5. For R sufficiently large points on $\tilde{\Gamma}(R)$ are on $\tilde{\Gamma}$ or within δ of $\{\tilde{\lambda}_i(\infty)\}_{i=1}^{q}$ which are on $\tilde{\Gamma}$. Therefore $-1 \notin \tilde{\Gamma}(R)$ for any R implies $-1 \notin \tilde{\Gamma}$ which proves (i). From (14) and Lemma 5 follows that for R large enough $C(\Gamma,-1) = C(\Gamma(R),-1) = 0$.

($\Leftarrow$) The proof of this part essentially requires retracing the steps of ($\Rightarrow$).

Since $s = \pm j\infty$ is on $\tilde{N}_q$ and $-1 \notin \tilde{\Gamma}$,

$$\det[I + \hat{G}(\infty)] = \prod_{i=1}^{q} (1 + \tilde{\lambda}_i(\infty)) \neq 0. \tag{15}$$

As above, using Lemma 5, R sufficiently large implies that $-1 \notin \tilde{\Gamma}(R)$ and $C(\Gamma(R),-1) = C(\Gamma,-1)$ which is equal to zero by (ii) of this theorem.

that

$$\det[I + \hat{G}(s)] \neq 0 \qquad \text{for all } s \in \tilde{N}_q(R). \tag{19}$$

(19) now implies that

$$1 + \lambda_i(s) \neq 0 \text{ for all } s \in \tilde{N}_q(R) \text{ and } i = 1, \ldots, n. \tag{20}$$

Now, (b) and the fact that $\det[\, I + \hat{G}(\infty)] = \prod_{i=1}^{q} [1 + \tilde{\lambda}_i(\infty)]$ implies that

$$1 + \tilde{\lambda}_i(\infty) \neq 0 \qquad \text{for all } i,\ i=1, \ldots, qr. \tag{21}$$

(21) and (20) now imply

$$0 \notin \tilde{\Gamma}_{det}(R); \tag{22}$$

$$-1 \notin \tilde{\Gamma}; \tag{23}$$

where $\Gamma_{det}(R)$ is defined in the proof of Theorem 2.

(23) proves (i).

To show (ii) we observe that (22), (a) and the fact that $\det D(s) \neq 0$ on $\tilde{N}_q(R)$, guarantee that all conditions of Lemma 5 are satisfied for equation (17) and hence

$$C(\Gamma_{ND}(R),0) = C(\Gamma_{det}(R),0) + C(\Gamma_D(R),0). \tag{24}$$

Moreover, since $\det[I + \hat{G}(s)] = \prod_{i=1}^{n} [1 + \lambda_i(s)]$, (20)guarantees that all conditions of Lemma 6 are satisfied and hence,

$$C(\Gamma_{ND}(R),0) = C(\Gamma(R),-1) + C(\Gamma_D(R),0). \tag{25}$$

(23) and Lemma 1 imply that there exists an R_4 such that if $R \geq R_4$,

interval on which $N_q(R)$ is defined and where $R > \max(d_m, |d_o|, R_3)$ where R_3 is such that all the roots of det D(s) = 0 lies in the inter or of $N_q(R)$.

Let $\varepsilon > 0$ be chosen sufficiently small such that $\det(N(s) + D(s)) = 0$ does not have any roots in the open left half plane with real part greater than or equal to $-\varepsilon$.

<u>Theorem 3</u>:

H is stable $\Longleftrightarrow$ (a) $|\det(N(s) + D(s))| \neq 0$ for all Re $s \geq 0$

and

(b) $\det[I + \hat{G}(\infty)] \neq 0$

$\Longleftrightarrow$ (i) $-1 \notin \tilde{\Gamma}$;

and

(ii) there exists an $R_4 > 0$ such that for all $R \geq R_4$

$C(\Gamma,-1) + C(\Gamma_D(R),0) = 0.$

<u>Proof</u>:

The first equivalence is shown in [14]. We shall prove the second equivalence.

($\Rightarrow$)

By construction there are no poles of $\hat{G}(s)$ on $\tilde{N}_q(R)$. Thus,

$$\det D(s) \neq 0, \text{ for all } s \in \tilde{N}_q(R); \text{ and}$$

$$\det[N(s) + D(s)] = \det[I + \hat{G}(s)]\det D(s) \text{ for all } s \in \tilde{N}_q(R). \qquad (17)$$

Moreover (a) implies that

$$\det(N(s) + D(s)) \neq 0 \qquad \text{for all } s \in \tilde{N}_q(R) \qquad (18)$$

and since $\det D(s) \neq 0$ for all $s \in \tilde{N}_q(R)$, we obtain from (17) and (18) tion of the $\Gamma_D(R)$. The following results are used in the sequel.

Callier and Desoer's [15] result concerning the open loop unstable case is as follows

$$H \text{ is stable} \Leftrightarrow C(\Gamma_{det}(R),0) = \sum_{\alpha=1}^{\ell} \Delta[R,p_\alpha] \tag{26}$$

where $R(s) = \sum_{\alpha=1}^{\ell} \sum_{k=1}^{m_\alpha} \frac{R_{\alpha k}}{(s-p_\alpha)^k}$ and $\Delta[R,p_\alpha]$ is the McMillan degree of R at p_α. It is also pointed out that

$$\Delta[R,p_\alpha] = \text{Rank}(H_\alpha) \tag{27}$$

where

$$H_\alpha = \begin{bmatrix} R_{\alpha 1} & \cdots & R_{\alpha m_{\alpha-1}} & R_{\alpha m_\alpha} \\ \vdots & ⋰ & ⋰ & \\ R_{\alpha m_{\alpha-1}} & ⋰ & & 0 \\ R_{\alpha m_\alpha} & & & \end{bmatrix} ; \tag{28}$$

and that

$$\det D(s) = c \prod_{\alpha=1}^{\ell} (s-p_\alpha)^{\Delta[R,p_\alpha]}, \tag{29}$$

where c is a nonzero constant. Thus (29) asserts that the multiplicity of p_k as a zero of det D(s) is equal to the McMillan degree of R at p_k which in turn equals to Rank (H_k). From (29) it thus follows that there exists an R_5 such that for any $R > R_5$,

$C(\Gamma(R),-1) = C(\Gamma,-1)$. Since $\det(D(s) + N(s))$ is analytic and different from zero on Re $s \geq 0$, $C(\Gamma_{ND}(R),0) = 0$.

Thus, $C(\Gamma,-1) + C(\Gamma_D(R),0) = 0$ for R sufficiently large; which completes the proof.

($\Leftarrow$)

The proof is similar to ($\Rightarrow$) and is thus omitted. ¤

Remarks:

(1) If is of paramount importance to note that the ε-indentations are taken to be in the left half plane rather than in the right half plane. The reason for doing this is that in the multiple-input, multiple output case it is possible to have both $\det D(s) = 0$ and $\det(N(s) + D(s)) = 0$ for some point s_o on the $j\omega$ axis. If an ε-indentation is taken to the right $C(\Gamma,-1) + C(\Gamma_D,0) = C(\Gamma_{ND},0) = 0$ and $\det(N(s) + D(s)) \neq 0$ on $\tilde{N}_q$ do not imply that $\det(N(s) + D(s)) \geq 0$ for Re $s \geq 0$, as the possible zero of $\det(N(s) + D(s)) = 0$ at s_o is not taken into account. Thus Theorem 3 does not hold with right indentations. Note, however, that this situation does not arise in the single input single output case since both det $D(s)$ and $\det(N(s) + D(s))$ cannot be zero at the same point $s = s_o$.

(2) Let $[\tilde{A},\tilde{B},\tilde{C},\tilde{D}]$ be a minimal realization of $\hat{G}(s)$. Then from [8,10] follows that $\det[sI-\tilde{A}] = c \det(D(s))$ for some $c \neq 0$. Thus instead of checking the encirclement of $\Gamma_D(R)$ one can check the encirclement of the mapping of $\tilde{N}_q(R)$ by det $[sI-\tilde{A}]$.

In the following theorem sufficient conditions are given under which the introduction of a compensating system does not require the recomputa-

<u>Remark</u>:

(1) Condition (a) and (3) of Section 2 imply that

$$\hat{G}_2(s) = \hat{G}_a^2(s) + \sum_{\alpha=1}^{\ell} \frac{R_{\alpha 1}^2}{(s-p_\alpha)},$$ where the superscript 2 in $R_{\alpha 1}^2$

denote the association with $G_2(s)$.

(2) Our interest in the above theorem and the way that conditions on $G_1(s)$ are stated have to do with the design of compensator using Mac Farlane's procedure. The procedure involves choosing $\{\lambda_{iG_1}(s),\ i=1, \ldots n\}$ to change the Γ_G, the Nyquist plots of the eigenvalues of the compensated system. When $G_2(s)$ is unstable such a procedure will change both $C(\Gamma_G,-1)$ and $C(\Gamma_D(R),0)$ which seems to require a recomputation of D(s) and $C(\Gamma_D(R),0)$; which is, to say the least, awkward. This theorem states that if the compensator is stable, and its $\lambda_{G_1 i}(s)$, which are the functions which used in compensation procedure, have no zeros in the closed right half plane, $C(\Gamma_D(R),0) = C(\Gamma_{D2}(R),0)$ and thus only $\Gamma_{D2}(R)$ has to be checked.

(3) From the proof it becomes clear that the $\lambda_{G_1 i}(s)$ have to be non-zero only at poles of $\hat{G}_2(s)$.

(4) Note that if $\lambda_{iG_1}(s)$ are rational, then conditions (b) and (c) imply that $\lambda_{iG_1}(s)$ is minimum phase for all i=1, ..., n.

<u>Proof</u>:

Since $\hat{G}_1(s)$ has no poles in Re s $\geq$ 0 the unstable part of $\hat{G}(s) = \hat{G}_1(s)\ \hat{G}_2(s)$ is given by $R(s) = \sum_{\alpha=1}^{\ell} \frac{\bar{R}_{\alpha 1}}{(s-p_\alpha)}$.

Thus

$$R_{\alpha 1} = [(s-p_\alpha)\hat{G}_1(s)\ \hat{G}_2(s)]\Big|_{s=p_\alpha} = \hat{G}_1(p_\alpha)R_{\alpha 1}^2 \tag{31}$$

$$C(\Gamma_D(R),0) = \sum_{k=1}^{\ell} \Delta[R,p_k] \tag{30}$$

Thus equation (30) gives a geometric way to calculate $\sum_{k=1}^{\ell} \Delta[R,p_k]$.

Let $\hat{G}(s) = \hat{G}_1(s)\hat{G}_2(s)$ where $\hat{G}_1(s)$ and $\hat{G}_2(s)$ are n×n matrices of proper rational functions satisfying the conditions imposed on $\hat{G}$ in section 2. Let the right coprime factorization of $\hat{G}(s)$, $\hat{G}_1(s)$ and $\hat{G}_2(s)$ be

$$\hat{G}(s) = N(s)D^{-1}(s),$$

$$\hat{G}_1(s) = N_1(s)D_1^{-1}(s),$$

$$\hat{G}_2(s) = N_2(s)D_2^{-1}(s).$$

Let λ_{iG}, λ_{iG_1}, λ_{iG_2} and $\Gamma_G, \Gamma_{G_1}, \Gamma_{G_2}, \Gamma_{D_1}$, Γ_{D_2} be appropriately defined using N_q of Figure 2.

Theorem 4:

Let G_1 and G_2 satisfy the following conditions:

(a) Poles of $\hat{G}_2(s)$ in Re $s \geq 0$ are simple.

(b) $\hat{G}_1(s)$ has no poles in Re $s \geq 0$ (G_1 is stable);

(c) $\lambda_{iG_1}(s) \neq 0$ for all s, Re $s \geq 0$, all i, i=1, ..., n.

Under these conditions,

H stable $\Longleftrightarrow$ (i) $-1 \notin \Gamma_G$,
and
(ii) there exists an R_0 such that for all R, $R \geq R_0$

$$C(\Gamma_G,-1) + C(\Gamma_{D_2}(R),0) = 0.$$

cases, the practical way to obtain such plots is by using a digital computer. It is therefore important to consider the numerical methods for obtaining these curves. The following are some remarks about the computation algorithms. We believe that this topic is far from being exhausted and it requires more theoretical and experimental work.

Consider first the size of n, the dimensions of $\hat{G}(s)$. The size of n gives one (among many) indication of the complexity of the compution. In many current applications of control theory this number is small; i.e. n = 5 is a fairly large problem. This implies that if other parameters are 'well behaved' we are not faced with a large computational problem.

To obtain Γ, the problem of finding $\lambda_i(j\omega)$, $j\omega \in I_k$, $k = 0, \ldots, m, m+2$; (or k stands for $+\infty$ or $-\infty$) is reduced to the solution of n ordinary differential equations. This is a method similar to the one commonly used for the calculation of the root-locus.

Given a point s_o where s_o is not a pole of $\hat{G}(s)$ the eigenvalue of $\hat{G}(s_o)$ can be calculated using, for example, the QR algorithm [16] (a subroutine which is commonly available at most computation centers). It is not necessary to use the QR algorithm for each point of N_q. If the eigenvalues at s_o are distinct the problem can be reduced to the solution of n differential equation where $\{\lambda_i(s_o): i = 1, \ldots n\}$ are the initial values:

$$F(\lambda_i(j\omega), j\omega) = 0 \quad \text{and}$$

$$\frac{d}{d(j\omega)} F(\lambda_i(j\omega), j\omega) = \frac{\partial F(\lambda_i(j\omega), j\omega)}{\partial \lambda_i} \frac{d\lambda_i}{d(j\omega)} + \frac{\partial F(\lambda_i(j\omega), j\omega)}{\partial (j\omega)} = 0.$$

From (26) it follows that if R is large enough

$$C(\Gamma_{D_2}(R),0) = \sum_{\alpha=1}^{\ell} \Delta[R^2,p_\alpha] = \sum_{\alpha=1}^{\ell} \text{Rank}(H_\alpha^2) \qquad (32)$$

where, since p_α is a simple pole of $\hat{G}_2(s)$, $H_\alpha^2 = R_{\alpha 1}^2$. Also from (26) it follows that if R is large enough

$$C(\Gamma_D(R),0) = \sum_{\alpha=1}^{\ell} \Delta[R,p_\alpha] = \sum_{\alpha=1}^{\ell} \text{Rank}(H_\alpha) \qquad (33)$$

where

$$H_\alpha = R_{\alpha 1}.$$

From (c) it follows that

$$\lambda_{i_{G_1}}(p_\alpha) \neq 0 \quad \text{for all } i = 1, \ldots, n,$$

and hence $\det \hat{G}_1(p_\alpha) \neq 0$. (34)

(31) and (34) imply that

$$\text{Rank}(R_{\alpha 1}^2) = \text{Rank}(R_{\alpha 1}) \text{ for all } \alpha = 1, \ldots, \ell. \qquad (35)$$

From (35), (32) and (33) it now follows that $C(\Gamma_D(R),0) = C(\Gamma_{D_2}(R),0)$ for all sufficiently large R. This result and Theorem 4 now establish Theorem 5. ∎

5. Some Concluding Remarks

This section contains some remarks concerning computational methods and future extensions of this work.

In order to use the above theorems for the design of compensators one needs $\tilde{\Gamma}$ and $\tilde{\Gamma}_D(R)$ in the complex plane. Except in trivial evaluate the derivatives and D(s) symbolically and check for the appearance of small differences between large numbers. If such numbers appear we have to return to point by point evaluation of the derivatives of $\det[I + \hat{G}(s)]$ and D(s); i.e. the value of s is substituted in $\hat{G}(s)$ and Gaus triangularization with pivots is performed to evaluate the determinant. A similar procedure is recommended for the factorization to $N(s)D^{-1}(s)$.

In conclusion further work has to be done on the numerical aspects of the problem. Other possible extensions are generalization to distributed systems along lines persued by Callier and Desoer, [15]. The generalization to sampled data systems is straightforward.

In the distributed case, however, the exceptional points of the second kind of $\det[\lambda I-\hat{G}(s)] = 0$ are isolated on Res > 0 but might be dense on Res = 0. The technique used in this paper is applicable when the exceptional points of the second are isolated on Res $\geq$ 0. Thus further work has to be done on the general distributed case.

Therefore,

$$\frac{d\lambda_i}{dj\omega} = - \frac{\frac{\partial F}{\partial j\omega}}{\frac{\partial F}{\partial \lambda}} \quad i = 1, \ldots, n \; ;$$

which is the differential equation to be solved numerically.

Having presented the main idea we have as usual to consider the questions of roundoff errors and sensitivity to changes in parameters (ill-conditioning). First, note that as we approach an exceptional point of the second kind $\frac{\partial F(\lambda, j\omega)}{\partial \lambda}$ approaches zero [3]. While we still can determine $\arg \frac{d\lambda_i}{d(j\omega)}$. Low accuracy is expected in the evaluation of $\left|\frac{d\lambda_i}{d(j\omega)}\right|$ whenever $\frac{\partial F}{\partial \lambda_i}$ remains finite (note that the analyticity of the $\lambda_i(s)$ imply differentiability but the $\lambda_i(s)$ might not be analytic and differentiable at the ends of the interval). At this point one can proceed in one of several alternatives: one alternative is to use $\arg \frac{d\lambda_i}{d(j\omega)}$, where i ranges over those $\lambda_i(j\omega)$ which are close in value to each other, to determine an estimation of the location of the singular point.

If the -1 point is not near the image of an exceptional point the above possible inaccuracy does not effect the stability result. We have just to record that an exceptional point has been met and repeat the above procedure (applying the QR algorithm, etc.) to the next interval of the $j\omega$ axis. On the other hand, it seems that if the -1 point is near one of the exceptional points of the second kind we have to evaluate Γ accurately and may expect trouble.

The actual evaluation of $\det[I + \hat{G}(s)]$ is needed for the evaluation of the partial derivatives; also needed is the factorization of $\hat{G}(s)$ to $N(s)D^{-1}(s)$ to find Γ_D. For small systems of the order of $n = 5$ one can the juxtaposition of γ and γ_1.

We shall extend this term to the case where $I_1 = [a',b']$, a' not necessarily equal to b but $\gamma(b) = \gamma_1(a')$.

Let γ_2 be a path defined on $[a,b+b'-a']$, γ_2 is equal to γ in I and to $\gamma_3 : t \to \gamma_1(t-b'+b)$ in $[b,b+b'-a']$ then γ_2 is denoted by $\gamma \, v\gamma_1$ and is called the juxtaposition of γ and γ_1.

Let γ be a road defined in $I = [a,b]$, and let f be a continuous mapping of the compact set $\gamma(I)$ into $\mathbb{C}$. $t \to f(\gamma(t)) \, \gamma'(t)$ is a regulated function in I; the integral $\int_a^b f(\gamma(t)) \, \gamma'(t) df$ is called the integral of f along the road γ and is denoted by $\int_\gamma f(z)dz$. If γ is a road equivalent to γ_1 then $\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz$ and; if $\gamma_1 v\gamma_2$ is defined then,

$$\int_{\gamma_1 v\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz \quad .$$

Let γ_0, γ_1 be two paths defined on the same interval I, and let A be an open set in $\mathbb{C}$ such that $\gamma_0(I) \subset A$ and $\gamma_1(I) \subset A$. A homotopy of γ_0 into γ_1 in A is a continuous mapping ρ of $I \times [\alpha,\beta]$ ($\alpha<\beta$ in R) into A such that $\rho(t,\alpha) = \gamma_0(t)$ and $\rho(t,\beta) = \gamma_1(t)$ in I; γ_1 is said to be homotopic to γ_0 in A if such a ρ exists. When both γ_0 and γ_1 are closed paths, ρ is a closed path homotopy if $t \to \rho(t,\xi)$ is a closed path for any $\xi \in [a,b]$; when we say that two loops γ_0 and γ_1 are homotopic in A we mean that there is a closed path homotopy.

Fact A.1 (9.6.4 in [5]): Let γ_1, γ_2 be two roads in an open set $A \subset \mathbb{C}$, having the same origin u and the same extremity v, such that there is a

Appendix A: Mathematical Terms

The appendix contains a short exposition of several mathematical terms which are used in the paper and which are not often used by engineers. The definitions and statement of theorems follow Dieudonné [5] and the reader is referred to that text for more detail and examples. We have introduced some remarks related to the usage of the mathematical concepts in this work, and to avoid conflict with terms common in electrical engineering; we have introduced some changes in terminology.

A path in $\mathbb{C}$ is a continuous mapping γ of a compact interval $I = [a,b] \subset R$, not reduced to a point, into $\mathbb{C}$. If $\gamma(I) \subset A \subset \mathbb{C}$ we say that γ is a path in A; $\gamma(a)$ and $\gamma(b)$ are called the origin and the extremity of γ. If $\gamma(a) = \gamma(b)$, γ is called a closed path.

A mapping γ^o of I into $\mathbb{C}$ such that $\gamma^o(t) = \gamma(a+b-t)$ is a path which is said to be opposite to γ.

A path γ is called a road if γ is a primitive of a regulated function (i.e. there exists a regulated function whose integral is γ). If $\gamma(a) = \gamma(b)$, γ is called a closed road.

Let γ, γ_1 be two roads defined in the intervals I, I_1, respectively. γ and γ_1 are called equivalent if there exists a bijection ρ of I into I_1, such that ρ and ρ^{-1} are primitives of regulated functions and $\gamma = \gamma_1 o \rho (\gamma_1 = \gamma o \rho^{-1})$, where o denotes composition).

Let $I_1 = [b,c]$ be a compact inberval in R, and let $I_2 = I \cup I_1 = [a,c]$. γ_1 be a path defined in I_1, such that $\gamma_1(b) = \gamma(b)$; if we define γ_2 to equal to γ in I and to γ_1 in I_1, γ_2 is a path denoted by $\gamma v \gamma_1$ and called

$$C(\Gamma,0) = \int_{\gamma} \frac{f'(z)}{f(z)}\, dz = \sum_{a \in T} C(\gamma,a) - \sum_{a \in S} C(\gamma,a).$$

The above definition and facts consider roads. Our objective is to state the same properties with respect to paths. This extension is done in [5] by the use of the concept of homotopy.

Fact A.6 (Ap.1.1 in [5]) If $t \to \gamma(t)$ ($a \le t \le b$) is a path in an open subset A of $\mathbb{C}$, there is in A a homotopy ρ of γ into a road γ_1, such that ρ is defined in $[a,b] \times [0,1]$ and $\rho(a,\xi) = \gamma(a)$ and $\rho(b,\xi) = \gamma(b)$ for every $\xi \in [0,1]$.

The line integral along a path is defined in the following way: Let A be a simply connected open domain in $\mathbb{C}$, f a complex valued function analytic in A, γ a path such that $\gamma(I) \subset A$, γ_1 a road homotopic to γ such that $\gamma_1(I) \subset A$; then

$$\int_{\gamma} f(s)\, ds \triangleq \int_{\gamma_1} f(s)\, ds.$$

Note that from Cauchy theorem (9.6.3 in [5], Fact A.1) follows that the definition is independent of the particular road γ_1 which is chosen. The method of actually finding a γ_1 which is homotopic to γ is illustrated in the proof of Ap. 1.1 in [5]: A partition $\{t_o=a, t_1, \ldots, t_k = b\}$ is chosen and a piecewise linear function γ_k is constructed with $f_k(t_i) = \gamma(t_i)$ and $\gamma_k(t) = \gamma(t_i) + \frac{t - t_i}{t_{i+1} - t_i}(\gamma(t_{i+1}) - \gamma(t_i))$ for $t_i \le t \le t_{i+1}$, $0 \le i \le k-1$. The partition is now chosen fine enough to that $\gamma_k(t)$ is included in A. This γ_k is the desired γ_1.

homotopy of γ_1 into γ_2 in A which leaves u and v fixed (i.e. $\rho(a,\xi) = u$ and $\rho(b,\xi) = v$ for any $\xi \in [\alpha,\beta]$). Then, for every analytic function f in A

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz \quad .$$

The index of a closed road γ with respect to a point $a \in \mathbb{C}$, $a \notin \gamma(I)$, is defined by $C(\gamma,a) = \frac{1}{2\pi j} \int_\gamma \frac{dz}{z-a}$.

Fact A.2 (9.8.1 of [5]). For any γ and any γ satisfying the above condition the index $C(\gamma,a)$ is an integer.

Fact A.3 (9.8.5 of [5]) If a closed road is contained in a closed ball D: $|z-a| \leq r$, then $C(\gamma,z) = 0$ for any point z exterior to D.

The following fact is called the principle of the argument.

Fact A.4 (9.17.1 in [5]): Let A be a simply connected domain in $\mathbb{C}$, f a complex valued meromorphic function in A, S(resp. T) the set of its poles (resp. zeros). Then, for any closed road in A $-(S \cup T)$

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{a \in T} C(\gamma,a) - \sum_{a \in S} C(\gamma,a).$$

Fact A.5 (9.17.2 in [5]) With the assumption of A.4, let $t \to \gamma(t)$ be a closed road in A$-(S \cup T)$. If Γ is the closed road $t \to f(\gamma(t))$, then

$$\frac{\frac{dz(t)}{dt}}{z(t)-a} = \frac{d}{dt} \log|z(t)-a| + j \frac{d}{dt} \arg(z(t)-a) \tag{A.1}$$

and when the cut is crossed we choose the branch of arg(z(t)-a) which maintains this equality. To make the integral of both sides of (A.1) the initial value of arg(z(t)-a) is taken to be equal to $\text{Im}\left(\frac{z'(t)}{z(t)-a}\right)$ at the initial value of t.

The index $C(\gamma,a)$, γ is a closed path and $a \notin \gamma(I)$, is defined as $C(\gamma_1,a)$ where γ_1 is any closed path homotopic to γ. Thus facts A.1 through A.5 hold with the word 'road' replaced by 'path'. Since usually we use paths we shall in an obvious way extend also the definition of equivalence and juxtaposition to include paths.

Remark: The argument function.

(1) The index of the curve with respect to a point is commonly thought as the net increase in the argument function, $\frac{1}{2\pi}\arg(z-a)$ as z travels around γ. This point of view is supported by

$$\int_\gamma \frac{dz}{z-a} = \int_\gamma d\Big(\log(z-a)\Big) = \int_\gamma d\Big(\log|z-a|\Big) + j\int_\gamma d\Big(\arg(z-a)\Big).$$

The first integral is zero on a closed curve. The trouble is([18], pg. 115) that the function arg(z-a) whose value has to be taken into account. above equation is meaningless without specifying at each point of the integration the branch of arg(z-a) whose value has to be taken into account.

One proper way for the choice of the branches (i.e. the choice which guarantees $c(\gamma,a) = \frac{1}{2\pi j}\int_\gamma d\,(\arg(z-a))$ is the following. Choose a cut

$\{s:\ \operatorname{Re} s \geq \operatorname{Re} a,\ \operatorname{Im} s = \operatorname{Im} a\}$. The branches of arg(z-a) are ..., $[-2\pi,0]$, $[0,2\pi]$, $[2\pi,4\pi]$, ... One starts, say at the branch $[0,2\pi]$. When γ crosses the cut from "up" to "down" one moves to a lower branch, say from $[0,2\pi]$ to $[-2\pi,0]$. When γ crosses the cut from "down" to "up" one moves to a higher branch, say, from $[0,2\pi]$ to $[2\pi,4\pi]$. The correctness of this procedure follows from the fact that inside each branch we have

Figure 3b and 3c.

In Figure 3c it is understood that $\tilde{N}_2$ and $\tilde{N}_3$ contain a portion of the branch cut (between jd_o and B) and that two lines have been drawn along the cut in the way of illustration only. Similarly, a portion of the circle $|s-s_o| = \delta$ is common to $\tilde{N}_1$ and $\tilde{N}_2$($\tilde{N}_1$ and $\tilde{N}_3$).

Let $I_2 \subset I$ and $I_3 \subset I$ be the two intervals on which $\tilde{N}_2 = \tilde{N}_3$. The path $t \to N_2(t)$, $t \in I_2$, is equivalent to the opposite of the path $t \to N_3(t)$, $t \in I_3$. A similar statement can be made with regard to N_1 and N_2, etc.

We shall define $N_{q\lambda}(R)$ in detail. Let us define the following:

$$I_c \triangleq \{s: s \in \mathbb{C},\ \mathrm{Re}\, s \geq 0,\ s \text{ on the branch cut}\};$$

$t \to \beta_1(t)$: a continuous one-to-one mapping from $[-1,0]$ onto I_c

such that $\beta_1(-1) = jd_o$, $\beta_1(o) - s_o$;

$t \to \beta_2(t)$: a continuous one-to-one mapping from $[0,1]$ onto I_c

such that $\beta_2(o) = s_o$, $\beta_2(1) = jd_o$;

The path $t \to N_{q\lambda}(R,t)$ is defined as the juxtaposition

$$I_o \vee \beta_2 \vee \beta_1 \vee I_1 \vee I_2 \vee \ldots \vee I_{m+2}.$$

$t \to N_{q\gamma}(t)$ is defined similarly with $I_{+\infty}$ and $I_{-\infty}$ replacing I_{m+2} (with a slight abuse of the juxtaposition notation since $\tilde{I}_{+\infty}$, for example, is not compact).

The path $\Gamma_{\lambda i}(R,\delta)$ is defined as $t \to \lambda_i(N_{q\lambda}(R,\delta,t)), t \in I$. Similarly

$\Gamma_{iN1}: t \to \lambda_i(N_1(t))$, $t \in I$. The definition of $\Gamma_{\lambda_i}(R)$ requires more work: Let λ_i^- be a mapping defined on I_c in the following way: for any $s \in I_c$

$$\lambda_i^-(s) \triangleq \lim_{\substack{s_1 \to s \\ Re(s_1 - s) = 0 \\ I_m(s_1 - s) < 0}} \lambda_i(s_1)$$

Since $\lambda_i(s)$, i=1, ..., n, are continuous and bounded, the limit exists. Since the $\lambda_i(s)$, i=1, ..., n are the function elements of an algebraic function then for $s \neq s_o$ the result of the limit operation is a value of another function elements at s. From this, or alternatively by using analytic continuation argument, it follows that $\lambda_i^-(s)$ is continuous on I_c.

Let the paths I_o be defined on [-2,-1]; I_ℓ, $1 \leq \ell \leq m+1$ be defined on $[\ell-1,\ell]$; β_1 be defined on [m+1,m+2]; β_2 be defined on [-1,0]. The mapping $t \to \Gamma_{\lambda i}(R,t)$ is defined as

$$\Gamma_{\lambda i}(R,t) \triangleq \lambda_i^-(N_q(R,t)) \text{ for } t \in [-2,-1] \text{ or } t \in [0,m+2]$$

$$\Gamma_{\lambda i}(R,t) \triangleq \lambda_i(N_q(R,t)) \text{ for } t \in [-1,0].$$

$\Gamma_{\lambda i}$ is defined similarly to $\Gamma_{\lambda i}(R)$. The definition of Γ_{iN_1} and Γ_{iN_3} can be done in the same detail. For simplicity we shall use a general description only:

Consider Γ_{iN_1}. Let s_o be the origin of N_I. As t is increased and $N_1(t)$ is on the branch cut, $\Gamma_{iN_1}(t) = \lambda_i^-(N_1(t))$. As the point B is reached and $N_1(t)$ is on the circle, $\lambda_i(s)$ is used. $\lambda_i(s)$ is used again when the branch cut is transversed again.

Consider Γ_{iN_3}. When $N_3(t) \in I_c$,

$t \to \Gamma_{iN_3}(t) = \bar{\lambda}_i(N_3(t))$; otherwise, $\lambda_i(s)$ is used.

Lemma 1.B

Let G be stable (i.e. $R_{\alpha k} = 0$ for all $1 \leq \alpha \leq \ell$, $1 \leq k < m_\alpha$), then for all i, i=1, ..., n,

(i) $\Gamma_{\lambda i}$, $\Gamma_{\lambda i}(R)$, $\Gamma_{\lambda i}(R,\delta)$ are closed paths.

(ii) Let $a \in \mathbb{C}$ be a point such that $a \notin \tilde{\Gamma}_{\lambda i}$. There exists an $R_o > 0$ and a $\delta_o > 0$ such that for all $R \geq R_o$ and all $0 < \delta \leq \delta_o$,

(ii.1) $a \notin \tilde{\Gamma}_{\lambda i}(R)$, $a \notin \tilde{\Gamma}_{\lambda i}(R,\delta)$;

(ii.2) $C(\Gamma_{\lambda i},a) = C(\Gamma_{\lambda i}(R),a) = C(\Gamma_{\lambda i}(R,\delta),a)$.

Proof:

(i) follows from the continuity of the eigenvalues (Fact 1 and Lemma 1), the properness of $\hat{G}(s)$, and the construction of the paths. (ii.1) is proven similarly to Lemma 5 and, therefore, details are omitted.

Consider (ii.2). Let $a \notin \tilde{\Gamma}_{\lambda i}$ and let R and δ be such that (ii.1) holds.

We claim that

$$C(\Gamma_{\lambda i}(R,\delta),a) = C(\Gamma_{\lambda i}(R),a) + C(\Gamma_{N1},a) + C(\Gamma_{N2},d) + C(\Gamma_{N3},a). \quad (B.1)$$

This follows from the definitions and the fact that each of Γ_{N1} and Γ_{N2} and Γ_{N3} is a juxtaposition of paths which are either equivalent to portions of $\Gamma_{\lambda i}(R,\delta)$ or opposite to portions of $\Gamma_{\lambda i}(R)$.

For δ small enough, (ii.1) and A.3(9.8.5 of [5]) imply that $C(\Gamma_{N1},a) = 0$.

Let $a \in \mathbb{C}$ be given and let δ_o and R be such that (ii.1) holds; under these conditions $a \notin \tilde{\lambda}_i(A)$ where A denotes the interior of $N_2(N_3)$.

(Otherwise, by choosing a smaller δ we shall get $a \in \tilde{\Gamma}_{\lambda i}(R,\delta)$.)
This and 9.8.7 of [5] imply that $C(\Gamma_{iN_2},a) = 0$ ($C(\Gamma_{iN_3},a) = 0$), which together with (B.1) imply that $C(\Gamma_{\lambda i}(R,\delta),a) = C(\Gamma_{\lambda i}(R),a)$. Using the same procedure as in Lemma 5 it can be shown that for R sufficiently large $C(\Gamma_{\lambda i}(R),a) = C(\Gamma_{\lambda i},a)$ which completes the proof. □

Theorem B:

Let G be stable (i.e. $R_{\alpha k} = 0$ for all $1 \leq \alpha \leq \ell$, $1 \leq k \leq m_\alpha$).
Under this condition

H is stable $\Leftrightarrow$ for all i, i=1, ..., n

(i) $-1 \notin \tilde{\Gamma}_{\lambda i}$;

and

(ii) $C(\tilde{\Gamma}_{\lambda i},-1) = 0$.

Proof:

($\Rightarrow$) Theorem I implies that

$$\inf_{\text{Res} \geq 0} |1+\lambda_i(s)| > 0 \text{ for all } i=1, \ldots, n. \tag{B.2}$$

(i) now follows from (B.2) (note that the definition of $\lambda_i^-(s)$ as a limit of a sequence of values of $\lambda_i(s)$, with Res > 0).

Since, for all i, $1 + \lambda_i(s)$ is analytic in the interior and on the closed path $N_{q\lambda}(R,\delta)$ and since (i) now implies that $1 + \lambda_i(s)$ is different from zero on $\tilde{N}_{q\lambda}(R,\delta)$, the principle of argument implies that, for all i, $C(\Gamma_{\lambda i}(R,\delta), -1) = 0$. (ii) now follows from (ii.2) of Lemma 1.B.

($\Leftarrow$)

(i) implies that $1 + \lambda_i(s)$, i=1, ..., n, is bounded away from zero on the $j\omega$ axis, at ∞, and on the branch cuts. Since $\prod_{i=1}^{n} [1 + \lambda_i(s)] = \det[I + \hat{G}(s)]$, and the $\lambda_i(s)$ are bounded, zeros of $1 + \lambda_i(s)$, any i, are zeros of $\det[I + \hat{G}(s)]$. Since the zeros of the determinant are isolated and since $1 + \lambda_i(s) \neq 0$ on the $j\omega$ and the branch cuts, we can find a δ sufficiently small and R sufficiently large such that inside the interiors of $\tilde{N}_1$, $\tilde{N}_2$ and $\tilde{N}_3$ there are not zeros of $1 + \lambda_i(s)$, $i=1, \ldots, n$.

From Lemma 1.B, part (ii) and condition (ii) of this theorem follows that

$$C(\Gamma_{\lambda i}(R,\delta), -1) = C(\Gamma_{\lambda i}, -1) = 0;$$

Since $1 + \lambda_i(s)$ is analytic on and in the interior of $N_{q\lambda}(R,\delta)$ and different from zero on $\tilde{N}_{q\lambda}(R,\delta)$ the principle of argument can be used to imply that $1 + \lambda_i(s)$, $i=1, \ldots, n$ has no zeros in any bounded subset of $\operatorname{Re} s \geq 0$. Since, for $i=1, \ldots, n$, $1 + \lambda_i(\infty) \neq 0$ we get that

$\inf_{\operatorname{Re} s \geq 0} |1 + \lambda_i(s)| > 0$ and by Theorem 1, H is stable. ⌑

Remark:

Note that in order to apply the theorem the exceptional point of the second kind have to be found which is an obvious practical limitation.

Appendix C: Some Simple Examples.

The appendix contains some simple examples of the Nyquist plots of the eigenvalues. These examples illustrate the role that the exceptional points of the second kind play in the Nyquist plots of the eigenvalues. Note that if there are no finite exceptional points of the second kind in Re $s > 0$ then the image of N_q under each eigenvalue forms a closed curve.

References

[1] A.G.J. MacFarlane, "A Survey of Some Recent Results in Linear Multivariable Feedback Theory," Automatica, Vol. 8, no. 4, July 1972, pp. 455-492.

[2] ________________, "Notes on the Vector Frequency Response Approach to the Analysis and Design of Multivariable Feedback Systems," Department of Electrical Engineering and Electronics, University of Manchester Institute of Science and Technology, Manchester, England, August 1972.

[3] E. Hille, Analytic Function Theory, Volumes I, II, Ginn and Company, 1959 (esp. Volume II, Chapter 12).

[4] M. Marden, "Geometry of Polynomials," Mathematical Survey No. 3, American Mathematical Society, Providence, R.I., 1966.

[5] J. Dieudonné, "Foundations of Modern Analysis," Vol. I, Academic Press, 1969.

[6] C.T. Chen, Introduction to Linear System Theory," Holt Reinhart & Winston Inc. 1970, theorem 9-10, page 376.

[7] B. Nobel, "Applied Linear Algebra," Prentice Hall Inc., 1969, pages 303 and 430.

[8] V.M. Popov, "Some properties of the control systems with irreducible matrix-transfer functions," Lecture Notes in Mathematics, vol. 144, Seminar on Differential Equations and Dynamical Systems, Springer Verlag, Berlin-Heidelberg-New York, 1970, pp. 169-180.

[9] W.A. Wolovich, "The determination of state-space representations for linear multivariable systems," Second IFAC Symposium on Multivariable Technical Control Systems, Düsseldorf, Germany, October 1971.

[10] S.H. Wang, "Design of Linear multivariable systems," Memorandum No. ERL-M309, Electronics Research Laboratory, University of California, Berkeley, October 1971.

[11] N. Bourbaki, Algèbre. Paris: Hermann et Cie, 1964 (esp. Chap. VI, §1, no. 11, Prop. 9 (Div)).

[12] C.C. MacDuffee, The Theory of Matrices. New York; Chelsea, 1956 (esp. p. 35).

[13] H.H. Rosenbrock, State-Space and Multivariable Theory. New York: J. Wiley and Sons, 1970.

[14] C.A. Desoer and J.D. Schulman, "Cancellations in Multivariable Continuous-Time and Discrete-Time Feedback Systems," Memorandum No. ERL-M346, College of Engineering, University of California, Berkeley, California, July 1972.

[15] F.M. Callier and C.A. Desoer, "Necessary and Sufficient Conditions for Stability for n-Input n-Output Convolution Feedback Systems with a Finite Number of Unstable Poles," IEEE Trans. on Automatic Control, June 1973.

[16] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford University Press, 1965.

[17] J. J. Belletrutti, and A. G. J. Mac Farlane," Characteristic Loci Techniques in Multivariable Control Systems," Proc. IEE, Vol. 118, No. 9, September 1971, pp. 1291-1297.

[18] L. V. Ahlfors, "Complex Analysis," 2nd edition, McGraw-Hill Book Co., 1966.

Captions

Figure 1: Cuts for the definitions of the λ_i.

Figure 2: The images of the paths N_q, N_q(R(and $\bar{N}(R)$.

Figure 3: Images of the Nyquist paths (a) $N_{q\lambda}(R)$ (b) $N_{q\lambda}(R,\delta)$ and (c) the images of the paths N_1, N_2 and N_3.

Figure 4: (a) The plots of the eigenvalues on N_q which corresponds to a 3 x 3 matrix $\hat{G}(s)$ such that $\lambda^3 = \frac{1}{s+1}$. No finite exceptional point of the second kind appears on Re $s \geq 0$. $s = \pm j\infty$ is an exceptional point of the second kind. The image of N_q under each eigenvalue is a closed curve.

(b) The plot eigenvalues on N_q for a 2 x 2 matrix $\hat{G}(s)$ such that $\lambda^2 = \frac{1}{s}$; note the ε indentation in N_q. A pole is present on the $j\omega$ axis.

(c) The plot of the eigenvalues on N_q for a 3 x 3 matrix $\hat{G}(s)$ such that $\lambda^3 = \frac{s-1}{s+1}$; there is an exceptional point of the second kind at $s = + 1$. The image of N_q under each eigenvalue does not form a closed curve.

(d) The plot of λ_1 and λ_1^- on $N_{\lambda q}$ for example c (the branch cut is shown in (c)). Note that the image of $N_{\lambda q}$ is a closed curve.

(e) The plot of the eigenvalues on N_q for a 2 x 2 matrix $\hat{G}(s)$, $\lambda^2 = \frac{s}{s+1}$. A singular point on the $j\omega$ axis, $s = 0$. No singular points on Re $s \geq 0$. The image of N_q under each eigenvalue forms a close curve.

Acknowledgements

We gratefully acknowledge stimulating discussions with Professors F. M. Callier and C.A. Desoer of the Department of Electrical Engineering and Computer Science, and Professor A.J. Chorin, Department of Mathematics, University of California at Berkeley.

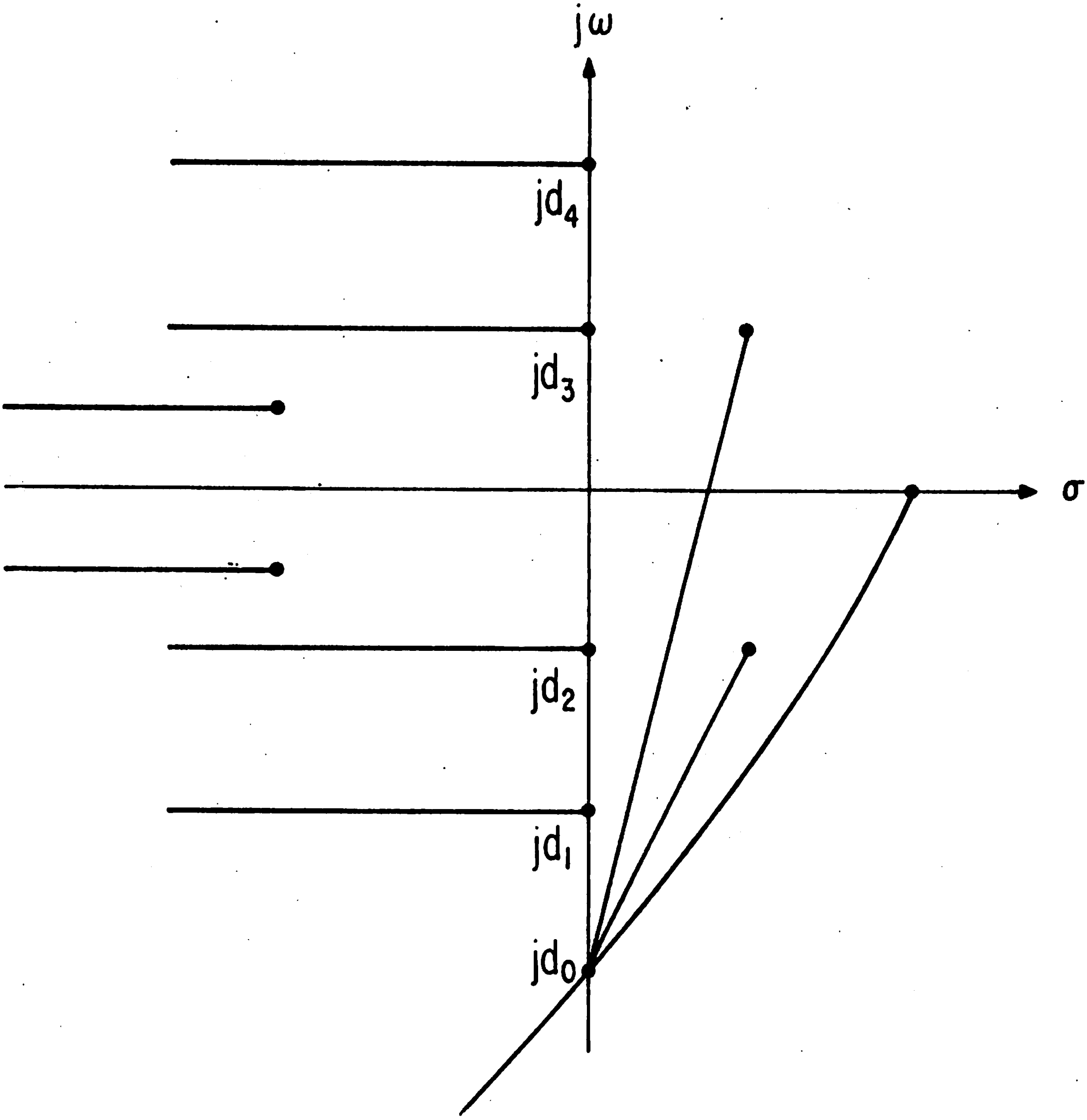
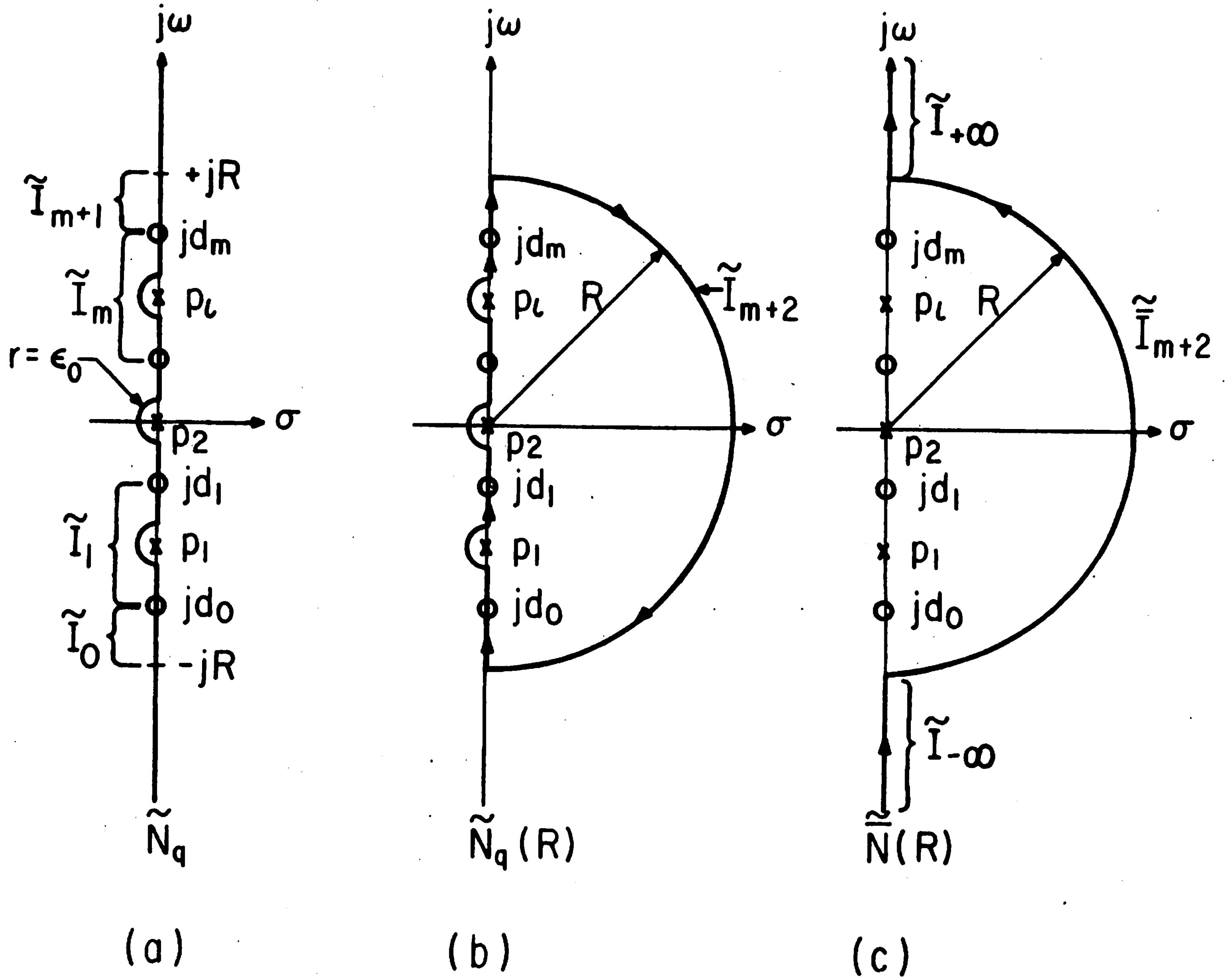


Figure 1



jω
+jR
$\tilde{I}_{m+1}$
jdm
$\tilde{I}_m$
pℓ
r = ε0
σ
p2
jd1
$\tilde{I}_1$
p1
jd0
$\tilde{I}_0$
-jR
$\tilde{N}_q$
jω
jdm
pℓ
R
$\tilde{I}_{m+2}$
σ
p2
jd1
p1
jd0
$\tilde{N}_q(R)$
jω
$\tilde{I}_{+\infty}$
jdm
pℓ
R
$\tilde{\tilde{I}}_{m+2}$
σ
p2
jd1
p1
jd0
$\tilde{I}_{-\infty}$
$\tilde{\tilde{N}}(R)$
(a)
(b)
(c)

Figure 2

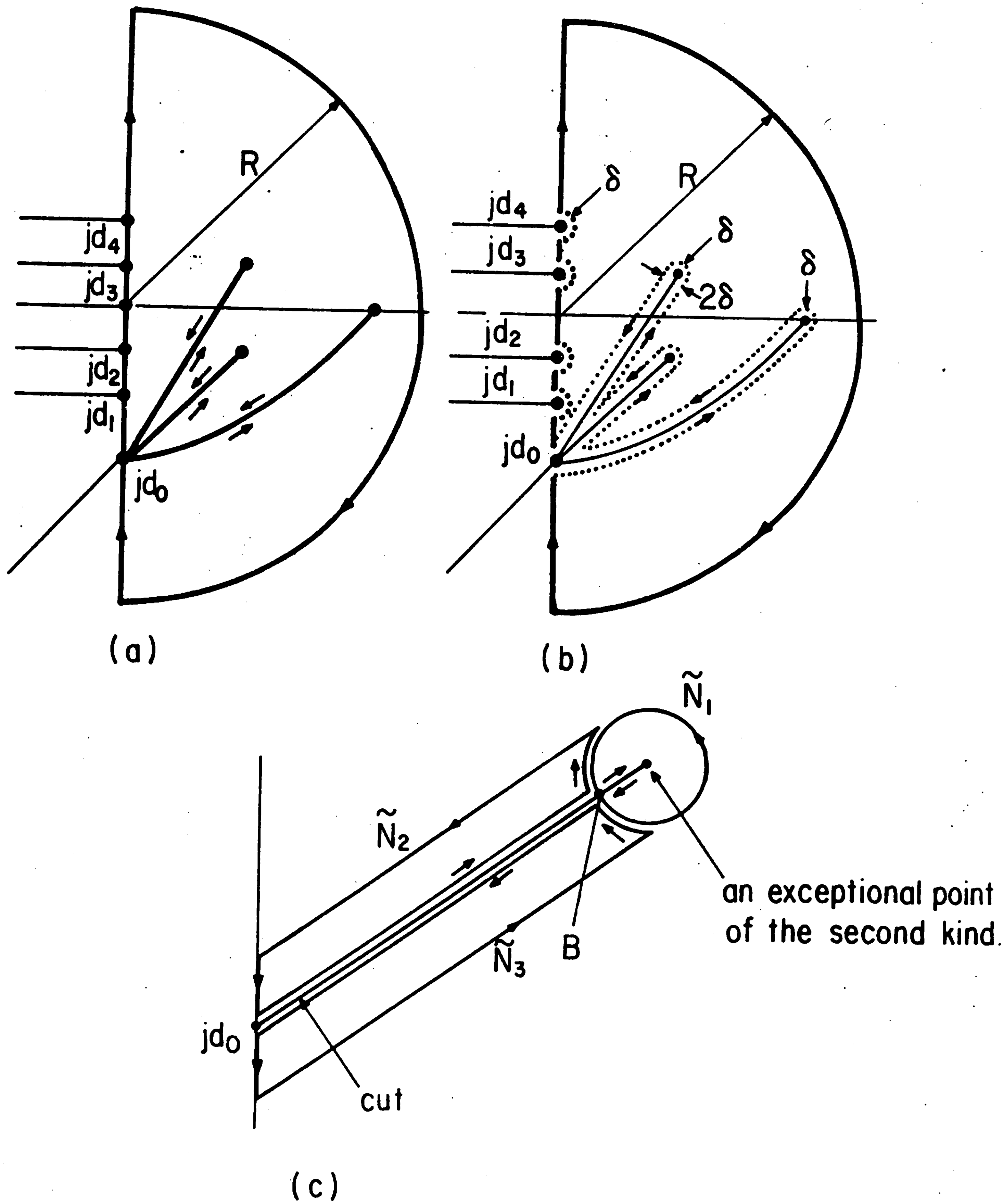

R
jd4
jd3
jd2
jd1
jd0
(a)
δ
2δ
(b)
Ñ1
Ñ2
Ñ3
B
cut
an exceptional point
of the second kind.
jd0
(c)

Figure 3

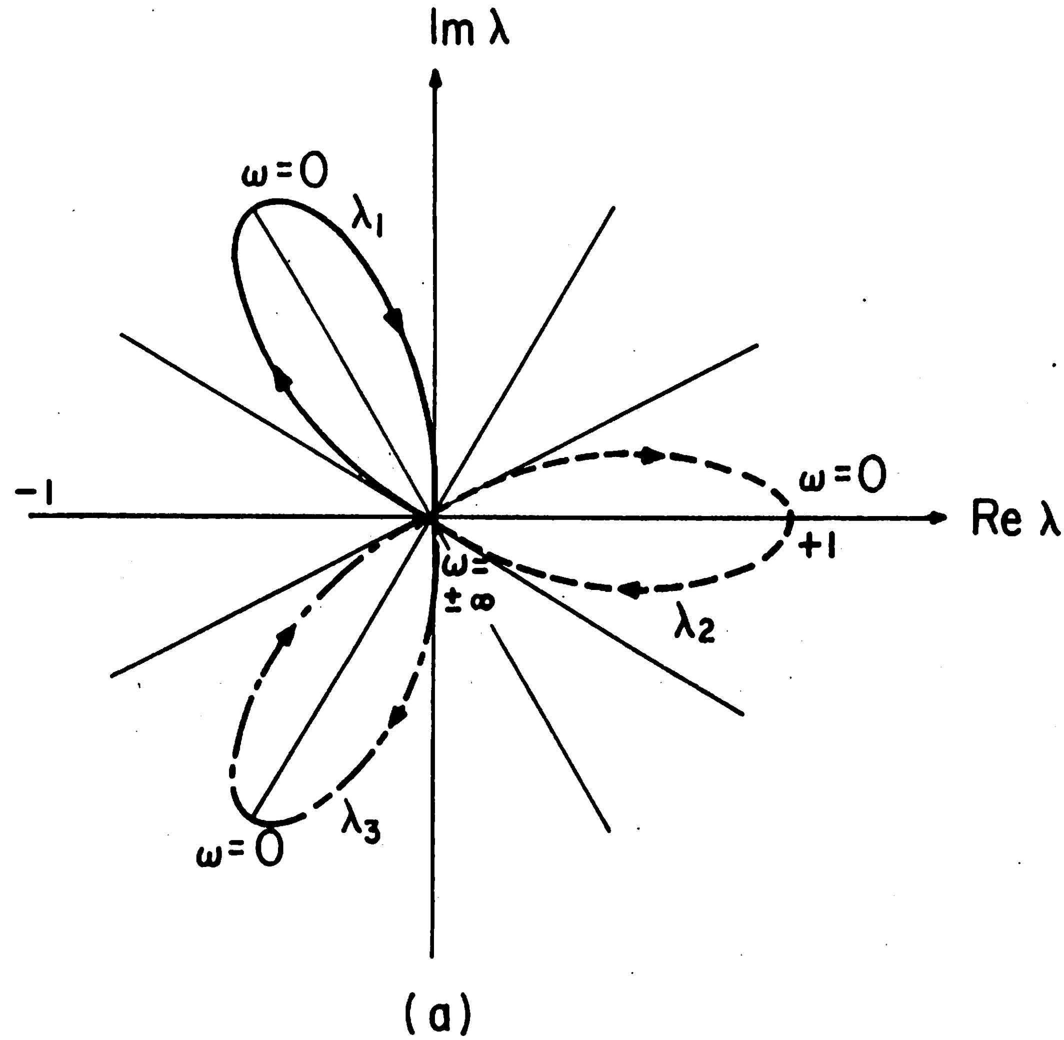
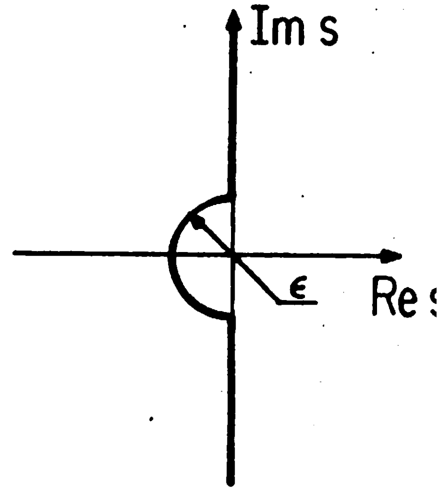


Figure 4a



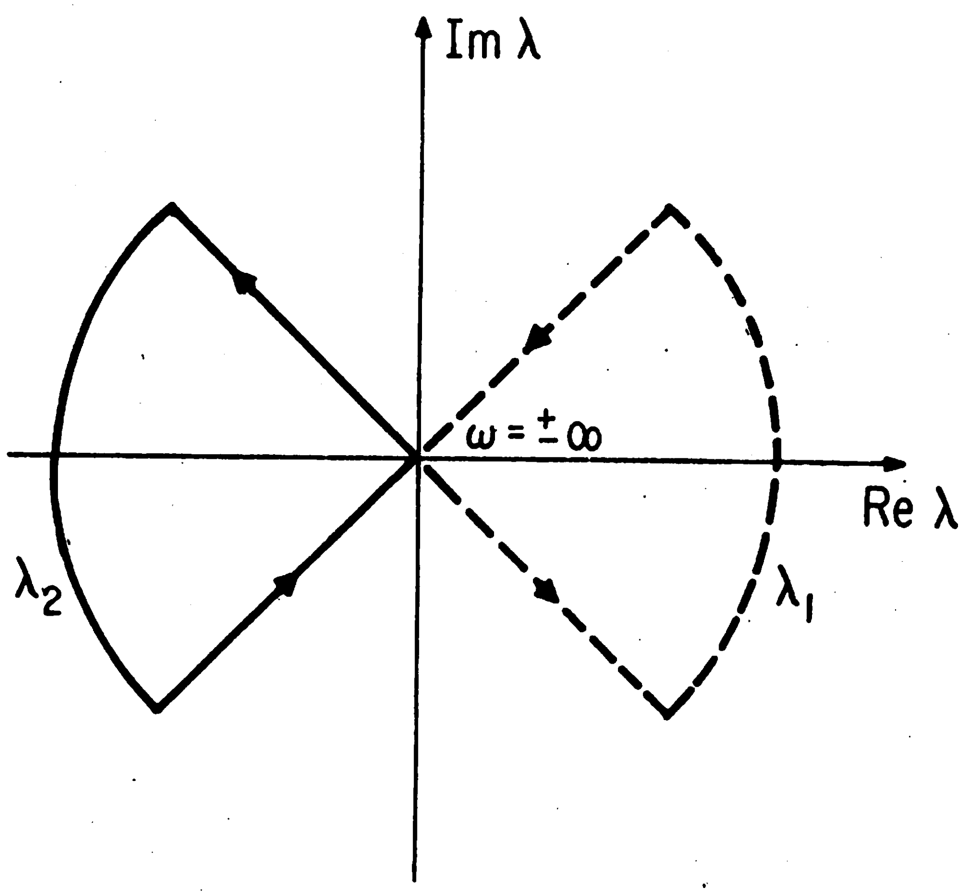


Figure 4b

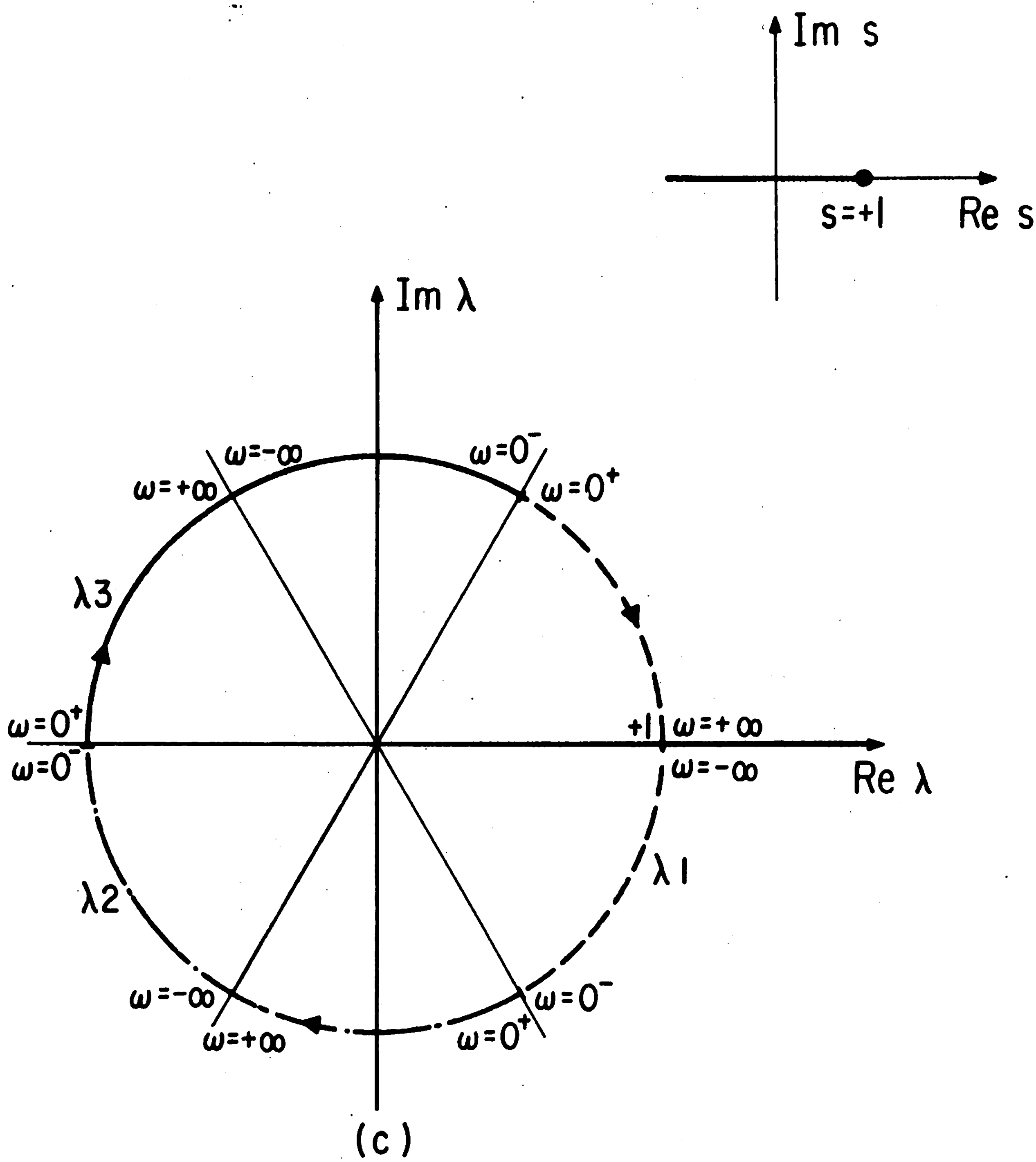


Figure 4c

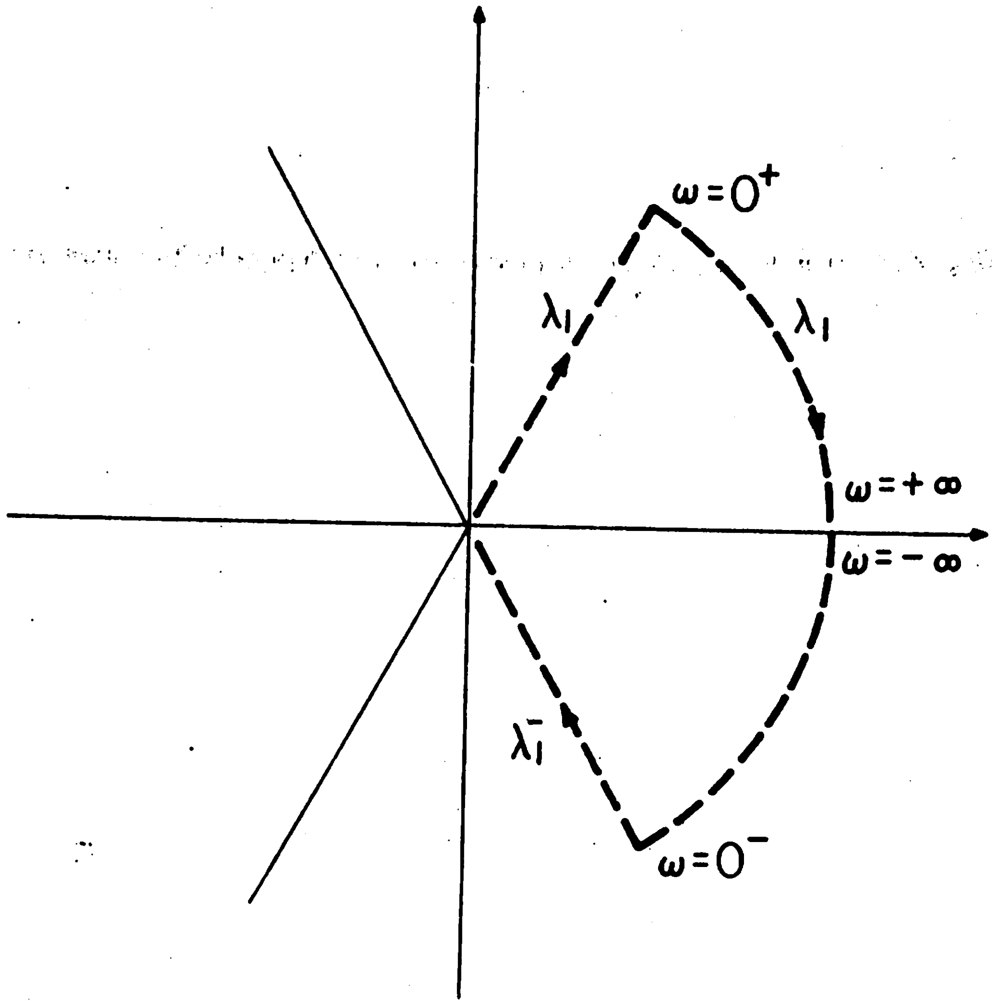
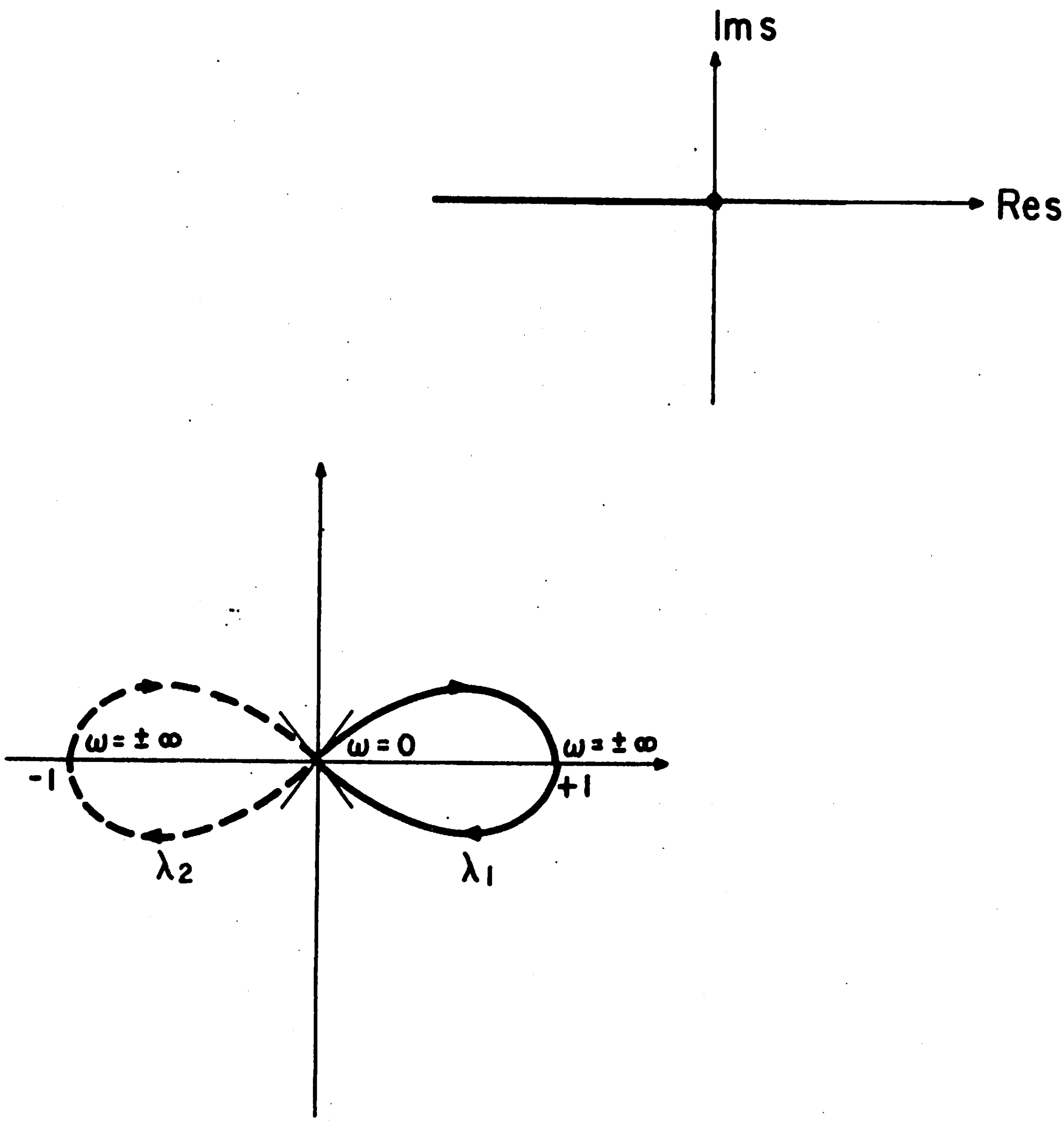


Figure 4d



(e)

Figure 4e