THE MULTIPLICITY OF AN INCREASING FAMILY OF σ-FIELDS

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ABSTRACT

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_t$, $t \in R_+$, be an increasing family of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \bigvee_t \mathcal{F}_t$. Let $\mathcal{M}^2$ be the family of all square-integrable martingales m_t with $m_0 = 0$. Suppose that $L^2(\Omega, \mathcal{F}, P)$ is separable. Then there exists a finite or countable sequence in $\mathcal{M}^2$, m_t^1, m_t^2, ..., such that i) the stable subspaces generated by m_t^i, m_t^j are orthogonal for $i \neq j$; ii) $\langle m^1 \rangle \succ \langle m^2 \rangle \succ \ldots$ where $\langle m^i \rangle$ is the nonnegative measure on the predictable σ-field on $\Omega \times R_+$ induced by the quadratic variation process $\langle m^i \rangle$ of m^i, and iii) every m in $\mathcal{M}^2$ has a representation $m_t = \sum_i \int_0^t \phi_i(s) dm_s^i$ a.s. for some predictable integrands ϕ_i. Furthermore, if n_t^1, n_t^2, ... is another such sequence, then $\langle n^i \rangle \sim \langle m^i \rangle$ for all i.

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The Multiplicity of an Increasing Family of σ-fields*

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I. Introduction. Let x_t, $t \in R_+$, be a zero-mean, real-valued, right-continuous, separable Gaussian process. Cramer (1964, 1967) has shown that there exists a finite or countable sequence of Gaussian processes $z_t^1, z_t^2, \ldots$ with independent increments such that i) z_t^i, z_t^j are orthogonal for $i \neq j$, ii) $\langle z^1 \rangle \succ \langle z^2 \rangle \succ \ldots$ where $\langle z^i \rangle$ is the nonnegative Stieltjes measure on R_+ defined by the increasing function $\langle z^i \rangle_t = E(z_t^i)^2$, and iii) $x_t = \sum_i \int_0^t K_i(t,s)dz_s^i$ for some real-valued kernels K_i; the integral is a Wiener integral. Furthermore if $y_t^1, y_t^2, \ldots$ is any other sequence of processes with independent increments satisfying i), ii) and iii) then $\langle z^i \rangle \sim \langle y^i \rangle$ for all i. The result depends crucially upon the Hellinger-Hahn Theorem.

This paper extends the result above in the following manner. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_t$, $t \in R_+$, be an increasing family of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \bigvee_t \mathcal{F}_t$. Let $\mathcal{M}^2$ be the family of all square-integrable martingales m_t with $m_0 = 0$. Suppose that $L^2(\Omega, \mathcal{F}, P)$ is separable. Then there exists a finite or countable sequence in $\mathcal{M}^2$, $m_t^1, m_t^2, \ldots$, such that i) the stable subspaces

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generated by m_t^i, m_t^j are orthogonal for $i \neq j$ i.e., the quadratic co-variation of m^i, m^j, $\langle m^i, m^j \rangle_t = 0$ a.s., $t \in R_+$; ii) $\langle m^1 \rangle \succ \langle m^2 \rangle \succ \ldots$ where $\langle m^i \rangle$ is the nonnegative measure on the predictable σ-field on $\Omega \times R_+$ induced by the quadratic variation process $\langle m^i \rangle$ of m^i, and iii) every m in $\mathcal{M}^2$ has a representation $m_t = \sum_i \int_0^t \phi_i(s) dm_s^i$ a.s. for some predictable integrands ϕ_i; the integral is a stochastic integral. Furthermore, if $n_t^1, n_t^2, \ldots$ is another sequence in $\mathcal{M}^2$ satisfying i), ii) and iii) then $\langle n^i \rangle \sim \langle m^i \rangle$ for $i=1,2,3,\ldots$. The proof of this result is an adaptation of that of the Hellinger-Hahn theorem as presented in Stone (1932), and uses some important properties of the space $\mathcal{M}^2$. The length of the 'base' sequence $m^1, m^2, \ldots$ is called the multiplicity of the family $(\Omega, \mathcal{F}_t, P)$ and as a corollary of the result it follows that the multiplicity is the minimum number of martingales necessary to generate (via stochastic integration) the entire family $\mathcal{M}^2$.

One example illustrating the main theorem is presented in Section 4 while the main result is proved in Section 3. The next section is merely a collection, from the literature, of definitions and results which are preliminary to the main theorem.

II. <u>Preliminaries</u>. Throughout $(\Omega, \mathcal{F}, P)$ is a fixed probability space and $\mathcal{F}_t$, $t \in R_+$, is a fixed, increasing family of sub-σ-fields of $\mathcal{F}$ with $\mathcal{F}_0$ trivial and $\mathcal{F} = \bigvee_t \mathcal{F}_t$. $\mathcal{M}^2$ denotes the family of all martingales m_t (with respect to $(\Omega, \mathcal{F}_t, P)$, $t \in R_+$) such that $m_0 = 0$ and $\sup_{t \in R_+} Em_t^2 < \infty$. $\mathcal{M}^2$ is a Hilbert space under the inner product $(m,n) = E\, m_\infty n_\infty$ (see [4, Thm. 1]). The <u>predictable</u> σ-field $\mathcal{P}$ is the σ-field of subsets of $\Omega \times R_+$ generated by all the adapted process y_t on $(\Omega, \mathcal{F}_t, P)$ which have left-continuous sample paths. A process x_t on $(\Omega, \mathcal{F}_t, P)$ is said to be <u>predictable</u> if the function $(\omega, t) \mapsto x_t(\omega)$ is $\mathcal{P}$-measurable. The set of all predictable processes is also denoted by $\mathcal{P}$ and there will not be any confusion between the two interpretations.

To each $m \in \mathcal{M}^2$ there is associated a unique[1] predictable process denoted $\langle m \rangle$ with nondecreasing sample paths and with $\langle m \rangle_0 = 0$, such that the process $m_t - \langle m \rangle_t$ is a martingale [4, Thm. 2]. To each pair m, n in $\mathcal{M}^2$ is associated a unique process $\langle m,n \rangle \in \mathcal{P}$ with $\langle m,n \rangle_0 = 0$ such that $\langle m,n \rangle$ is the difference of two processes in $\mathcal{P}$ with nondecreasing sample paths and such that $m_t n_t - \langle m,n \rangle_t$ is a martingale. It turns out that $\langle m,n \rangle \equiv \frac{1}{2} (\langle m+n \rangle - \langle m \rangle - \langle n \rangle)$ and $\langle m \rangle \equiv \langle m,m \rangle$ (see [4, p. 80]). The processes $\langle m \rangle$, respectively $\langle m,n \rangle$, are called the predictable quadratic variation of m, respectively predictable quadratic covariation of m and n; the adjective "predictable" will be dropped in the sequel but should be kept in mind when referring to Meyer (1970).

Let $m, n \in \mathcal{M}^2$. Then $\langle m,n \rangle$ induces a finite (signed) measure, also denoted by $\langle m,n \rangle$, on the σ-field $\mathcal{P}$ via the linear functional $\phi \in \mathcal{P} \mapsto E \int_0^\infty \phi_s \, d\langle m,n \rangle_s$. The measure $\langle m \rangle$ is nonnegative. Let $L^p(\langle m,n \rangle) = \{\phi \in \mathcal{P} | E \int_0^\infty |\phi_s|^p \, |d\langle m,n \rangle|_\infty < \infty\}$, $p = 1, 2$. Let $\phi \in L^2(\langle m \rangle)$, then $\phi \in L^1(\langle m,n \rangle)$ for all $n \in \mathcal{M}^2$ and there is a unique martingale $\phi \circ m \in \mathcal{M}^2$ called the stochastic integral of ϕ with respect to m such that for all $n \in \mathcal{M}^2$

$$\langle \phi \circ m, n \rangle_t = \int_0^t \phi_s d\langle m,n \rangle_s \text{ a.s.}$$

[1] Throughout "unique" means unique modulo modification; x_t is a modification of y_t if $x_t = y_t$ a.s. for $t \in R_+$, and this will sometimes be written $x_t \equiv y_t$ or $x \equiv y$.

(see [4, Thm. 3.])

A closed subspace $\mathcal{L}$ of the Hilbert space $\mathcal{M}^2$ is <u>stable</u> if $m \in \mathcal{L}$, $\phi \in L^2(\langle m \rangle)$ implies $\phi \circ m \in \mathcal{L}$. If $\mathcal{N} \subset \mathcal{M}$ then $\mathcal{L}(\mathcal{N})$ is the smallest stable subspace containing $\mathcal{N}$. The fact that for $m \in \mathcal{M}^2$ the mapping $\phi \mapsto \phi \circ m$ is an isometric injection from $L^2(\langle m \rangle)$ into $\mathcal{M}^2$ implies that $\mathcal{L}(m) = \{\phi \circ m | \phi \in L^2(\langle m \rangle)\}$. If $\mathcal{L}$ is a stable subspace $\mathcal{L}^{\perp} = \{m \in \mathcal{M}^2 | (m,n) = Em_\infty n_\infty = 0 \text{ for all } n \in \mathcal{L}\}$. If $\mathcal{L}$, $\mathcal{L}'$ are two stable subspaces $\mathcal{L} \perp \mathcal{L}'$ means $\mathcal{L}' \subset \mathcal{L}^{\perp}$.

<u>Lemma 1</u>. If $\mathcal{L}$ is a stable subspace and $n \in \mathcal{L}^{\perp}$ then $\langle m,n \rangle \equiv 0$ for $m \in \mathcal{L}$.

<u>Proof</u>. $En_\infty(\phi \circ m)_\infty = 0$ for all $\phi \in L^2(\langle m \rangle)$. By the differential rule (see [4])

$$n_\infty(\phi \circ m)_\infty = \int_0^\infty n_{t-}\phi_t dm_t + \int_0^\infty (\phi \circ m)_{t-} dn_t + \int_0^\infty \phi_t d[n,m]_t$$

Thus

$$0 = En_\infty(\phi \circ n)_\infty = E\int_0^\infty \phi_t \, d[n,m]_t$$

However, from [4] again,

$$E\int_0^\infty \phi_t \, d[n,m]_t = E\int_0^\infty \phi_t \, d\langle n,m \rangle_t$$

Since these relations must hold for all $\phi \in L^2(\langle m \rangle)$ it follows that $\langle n,m \rangle \equiv 0$. ¤

The following result follows easily from the Hilbert space structure of $\mathcal{M}^2$ and Lemma 1 above.

<u>Theorem 1</u>. Let $\mathcal{L} \subset \mathcal{M}^2$ be a stable subspace. Then

(i) $\mathcal{L}^{\perp}$ is stable,

(ii) for each $m \in \mathcal{M}^2$ there is a unique decomposition of m, $m = n + n'$, such that $n \in \mathcal{L}$ and $\langle n', y \rangle \equiv 0$ for all $y \in \mathcal{L}$.

<u>Remark</u>. If $\mathcal{L} = \mathcal{L}(y)$ it is easily shown that $\frac{d\langle m,y\rangle}{d\langle y\rangle} \in L^2(\langle y \rangle)$ and in the above decomposition, $n = \frac{d\langle m,y\rangle}{d\langle y\rangle} \circ y$.

III. <u>The Main Result</u>. For m, n in $\mathcal{M}^2$, $\langle m \rangle \succ \langle n \rangle$ or $\langle n \rangle \prec \langle m \rangle$ means that the measure $\langle n \rangle$ on $\mathcal{P}$ is absolutely continuous with respect to $\langle m \rangle$ i.e., $E\int_0^\infty \phi_s d\langle m \rangle_s = 0$ implies $E\int_0^\infty \phi_s d\langle n \rangle_s = 0$ for $\phi \in \mathcal{P}$. $\langle m \rangle \sim \langle n \rangle$ means $\langle m \rangle \succ \langle n \rangle$ and $\langle m \rangle \prec \langle n \rangle$, whereas $\langle m \rangle \perp \langle n \rangle$ means that the two measures are mutually singular i.e., for some set $A \in \mathcal{P}$ $\langle m \rangle (A) = 0$ and $\langle n \rangle (A^c) = 0$.

<u>Proposition 1</u>. Let $\mathcal{L} = \mathcal{L}(m^1, \ldots, m^K)$, $K \leq \infty$[2]. Then there exists a sequence $n^1, \ldots, n^K$ in $\mathcal{L}$ such that $n^1 = m^1$ and

(i) $\mathcal{L}(n^1, \ldots, n^K) = \mathcal{L}$,

(ii) $\mathcal{L}(n^i) \perp \mathcal{L}(n^j)$, $i \neq j$

<u>Proof</u>. Let $n^1 \equiv m^1$. For $i > 1$ let $m^i = n^i + n'^i$ be the unique decomposition of m^i such that $n^i \in (\mathcal{L}(m^1, \ldots, m^{i-1}))^{\perp}$ and $n'^i \in \mathcal{L}(m^1, \ldots, m^{i-1})$. It is easy to check that (i) and (ii) are satisfied.

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[2] $K = \infty$ simply means that $m^1, m^2 \ldots$ is a countable sequence.

Theorem 2. Let $\mathcal{L} = \mathcal{L}(m^1, \ldots, m^K)$, $K \leq \infty$. Then there exists a sequence $n^1, \ldots, n^R$, in $\mathcal{L}$ with $R \leq K$ and $n^i \not\equiv 0$ for all i such that

(i) $\mathcal{L}(n^1, \ldots, n^R) = \mathcal{L}$

(ii) $\mathcal{L}(n^i) \perp \mathcal{L}(n^j)$, $i \neq j$

(iii) $\langle n^1 \rangle \succ \langle n^2 \rangle \succ \langle n^3 \rangle \succ \ldots$

Proof. By Proposition 1 it can be assumed that $\mathcal{L}(m^i) \perp \mathcal{L}(m^j)$ $i \neq j$. Applying the Lebesgue decomposition theorem to the measures $\langle m^i \rangle$ for $i > 1$, there exist measures μ_1^i, μ_2^i on $\mathcal{P}$ such that

$$\langle m^i \rangle = \mu_1^i + \mu_2^i$$

$$\mu_1^i \prec \sum_{j=1}^{i-1} \langle m^j \rangle$$

$$\mu_2^i \perp \sum_{j=1}^{i-1} \langle m^j \rangle,$$

and sets $A_i \in \mathcal{P}$ such that

$$\mu_2^i(A_i^c) = 0, \quad \langle m^j \rangle (A_i) = 0 \quad j = 1, \ldots, i-1.$$

Now set $A_1 = \Omega \times R_+$ and define

$$n^1 = \sum_{i=1}^{K} \frac{1}{2^i E \langle m^i \rangle_\infty} I_{A_i} \circ m^i. \tag{1}$$

Clearly $n^1 \in \mathcal{L}$. It will be shown that it has the following properties:

(a) $m^1 \in \mathcal{L}(n^1)$

(b) $\langle n^1 \rangle \succ \langle m^i \rangle$ for all i

To prove (a) let $B = (\bigcup_{i=2}^{K} A_i)^c$. Then

$$2E\langle m_\infty^1 \rangle\ I_B \circ n^1 = \sum_{i=1}^{K} \frac{2E\langle m^1 \rangle_\infty}{2^i E\langle m^i \rangle_\infty} I_{B \cap A_i} \circ m^i = I_B \circ m^1$$

Now $\langle m^1 \rangle (A_i) = 0$ for $i \geq 2$ so $\langle m^1 \rangle\ (B^c) = 0$. Thus $I_{B^c} \circ m^1 = 0$ and hence

$$m_1 = 2E\langle m_\infty^1 \rangle\ I_B \circ n^1 \in \mathcal{L}(n^1)$$

To prove (b) take $E \in \mathcal{P}$ such that $\langle n^1 \rangle (E) = 0$. Since the m^j are orthogonal, this implies, from (1), that

$$\langle m^i \rangle\ (E \cap A_i) = 0 \quad i = 1, 2, \ldots \tag{2}$$

For $i = 1$ this says $\langle m^1 \rangle\ (E) = 0$ so that $\langle n^1 \rangle \succ \langle m^1 \rangle$ and the result is established by induction, if

$$\langle m^j \rangle\ (E) = 0 \text{ for } j = 1, \ldots, i-1 \tag{3}$$

implies $\langle m^i \rangle\ (E) = 0$. Suppose (3) is true. Now

$$\langle m^i \rangle\ (E) = \langle m^i \rangle\ (E \cap A_i) + \langle m^i \rangle\ (E \cap A_i^c)$$

$$= \langle m^i \rangle\ (E \cap A_i^c) \text{ from (2)}$$

$$= \mu_1^i\ (E \cap A_i^c) \text{ since } \mu_2^i(A_i^c) = 0$$

But $\mu_1^i < \sum_{j=1}^{i-1} \langle m^j \rangle$ so that, using (3), $\langle m^i \rangle (E) = 0$ as required.

Thus n^1 satisfies (a) and (b) above. Now let q^i, $i \geq 2$, be the projection of m^i on $(\mathcal{L}(n^1))^{\perp}$ i.e.,

$$q^i = m^i - \frac{d \langle m^i, n^1 \rangle}{d \langle n^1 \rangle} \circ n^1, = m^i - \alpha_i \circ n^1 \text{ say.} \tag{4}$$

Apply Proposition 1 to $\mathcal{L}(q^2, q^3, \ldots)$ to obtain a sequence $p^2, p^3, \ldots$ with $p^2 = q^2$ such that $\mathcal{L}(q^2, q^3, \ldots) = \mathcal{L}(p^2, p^3, \ldots)$ and $\mathcal{L}(p^i) \perp \mathcal{L}(p^j)$, $i \neq j$. Then, using (4), it follows that

$$\mathcal{L} = \mathcal{L}(n^1, m^2, m^3, \ldots) = \mathcal{L}(n^1, p^2, p^3, \ldots).$$

To obtain n^2, start with the sequence $p^2, p^3, \ldots$ and construct n^2 in the same way that n^1 was constructed from $m^1, m^2, \ldots$. Then n^2 will have the properties

(a') $p^2 \in \mathcal{L}(n^2)$

(b') $\langle n^2 \rangle \succ \langle p^i \rangle$ for all $i \geq 2$.

Now $m^2 \in \mathcal{L}(n^1, n^2)$ since $m^2 - p^2 = m^2 - q^2 \in \mathcal{L}(n^1)$ by (4) and $p^2 \in \mathcal{L}(n^2)$ by (a'). Also from (4)

$$\langle q^i \rangle = \langle m^i \rangle + \langle \alpha_i \circ n^1 \rangle - 2 \langle m^i, \alpha_i \circ n^1 \rangle.$$

From (b) above, $\langle n^1 \rangle \succ \langle m^i \rangle$ for all i so that this equation implies $\langle n^1 \rangle \succ \langle q^i \rangle$ and consequently $\langle n^1 \rangle \succ \langle n^2 \rangle$.

Having constructed n^2 one projects $p^3, p^4, \ldots$ on $(\mathcal{L}(n^1, n^2))^{\perp}$ and then n^3 is constructed. This procedure continues for n^4, n^5, ... and it is clear that the sequence thus obtained satisfies the assertion. ⊠

<u>Theorem 3</u>. Let $\mathcal{L} = \mathcal{L}(m^1, \ldots, m^K) = \mathcal{L}(n^1, \ldots, n^R)$ K, R ≤ ∞ and suppose that

(i) $\mathcal{L}(m^i) \perp \mathcal{L}(m^j)$; $\mathcal{L}(n^i) \perp \mathcal{L}(n^j)$, $i \neq j$

(ii) $\langle m^1 \rangle \succ \langle m^2 \rangle \succ \ldots$; $\langle n^1 \rangle \succ \langle n^2 \rangle \succ \ldots$.

Then $\langle m^i \rangle \sim \langle n^i \rangle$ for all i, in particular K = R.

<u>Proof</u>. Because of (ii) there exist predictable processes $\phi_i \in L^2(\langle n^1 \rangle)$, $\phi_i \equiv \dfrac{d\langle n^i \rangle}{d\langle n^1 \rangle}$, such that for $A \in \mathcal{P}$

$$\langle n^i \rangle (A) = E \int_0^\infty I_A \, \phi_i \, d\langle n^1 \rangle \quad i = 1, 2, \ldots \tag{5}$$

Also for each i $m^i \in \mathcal{L}(n^1, \ldots, n^R)$ so that there exist predictable processes $f_{ij} \in L^2(\langle n^j \rangle)$ such that

$$m^i \equiv \sum_j f_{ij} \circ n^j$$

Because of (i) it follows from this representation that

$$\langle m^i, m^j \rangle_t = \sum_k \int_0^t f_{ik} f_{jk} \, d\langle n^k \rangle$$

$$= \sum_k \int_0^t f_{ik} f_{jk} \phi_k \, d\langle n^1 \rangle \quad \text{from (5)}$$

In particular, putting i = j gives

$$\frac{d\langle m^i\rangle}{d\langle n^1\rangle} = \sum_k f_{ik}^2 \phi_k \, , \tag{6}$$

and since $\mathcal{L}(m^i) \perp \mathcal{L}(m^j)$ for $i \neq j$

$$\sum_k f_{ik} f_{jk} \phi_k = 0 \quad \text{a.s. } \langle n^1\rangle \text{ for } i \neq j \tag{7}$$

It is immediate from (6) that $\langle m^1\rangle \prec \langle n^1\rangle$ and by symmetry $\langle n^1\rangle \prec \langle m^1\rangle$, thus $\langle m^1\rangle \sim \langle n^1\rangle$.

Now assume that $\langle m^i\rangle \sim \langle n^i\rangle$ for $i = 1, \ldots, r$. It will be shown that $\langle m^{r+1}\rangle \sim \langle n^{r+1}\rangle$. By the Lebesgue decomposition theorem there are measures μ^1, μ^2 on $\mathcal{P}$ with $\langle m^{r+1}\rangle = \mu^1 + \mu^2$, such that

$$\mu^1 \prec \langle n^{r+1}\rangle, \; \mu^2 \perp \langle n^{r+1}\rangle$$

i.e., there exists $B \in \mathcal{P}$ such that $\mu^2(B^c) = 0$, $\langle n^{r+1}\rangle(B) = 0$. Suppose $\mu^2(B) > 0$. Then $\langle m^{r+1}\rangle(B) > 0$ and there exists $B_0 \subset B$, $B_0 \in \mathcal{P}$ such that

$$\int_0^\infty I_{B_0} \, d\langle n^{r+1}\rangle = 0 \quad \text{a.s.} \tag{8}$$

$$E\int_0^\infty I_{B_0} \, d\langle m^{r+1}\rangle > 0 \text{ and } \frac{d\langle m^{r+1}\rangle}{d\langle n^1\rangle} > 0 \text{ a.s. } \langle n^1\rangle \text{ on } B_0 \tag{9}$$

Now $\langle n^1\rangle \succ \langle n^{r+1}\rangle \succ \langle n^k\rangle$ for $k \geq r+1$, so, using (5), (8) implies

$$I_{B_0} \phi_k = 0 \quad \text{a.s. } \langle n^1\rangle \text{ for } k \geq r+1. \tag{10}$$

Also $\langle n^1 \rangle \sim \langle m^1 \rangle \succ \langle m^k \rangle \succ \langle m^{r+1} \rangle$ for $1 \le k \le r+1$ so that from (9)

$$\frac{d\langle m^k \rangle}{d\langle n^1 \rangle} > 0 \quad \text{as.} \quad \langle n^1 \rangle \text{ on } B_0 \text{ for } 1 \le k \le r+1 \tag{11}$$

Combining (7) with (10) it follows that

$$\sum_{k=1}^{r} f_{ij} f_{jk}\, \phi_k = 0 \quad \text{a.s.} \quad \langle n^1 \rangle \text{ on } B_0 \text{ for } i \ne j \tag{12}$$

whereas from (6), (10) and (11)

$$\sum_{k=1}^{r} f_{ik}^2\, \phi_k > 0 \quad \text{a.s.} \quad \langle n^1 \rangle \text{ on } B_0 \text{ for } 1 \le i \le r+1 \tag{13}$$

Now fix $(\omega,t) \in B_0$ such that (12), (13) hold, and let $x^i \in R^r$ be the vector with jth component equal to $\sqrt{\phi_j(\omega,t)}\, f_{ij}(\omega,t)$. Then (12) implies that x^i, x^j are orthogonal in R^r for $1 \le i \ne j \le r+1$ whereas (13) implies that these vectors are non-zero which is a contradiction so it must be the case that $\langle m^{r+1} \rangle \prec \langle n^{r+1} \rangle$ and by symmetry $\langle m^{r+1} \rangle \succ \langle n^{r+1} \rangle$. The result follows by induction. ⊠

__Definition.__ The __multiplicity__ of the family $(\Omega, \mathcal{F}_t, P)_{t \in R_+}$ is the length K of (any) sequence $m^1, m^2, \ldots, m^K$ of non-zero martingales in $\mathcal{M}^2$ such that

(i) $\mathcal{M}^2 = \mathcal{L}(m^1, \ldots, m^K)$

(ii) $\mathcal{L}(m^i) \perp \mathcal{L}(m^j)$ $i \ne j$

(iii) $\langle m^1 \rangle \succ \langle m^2 \rangle \succ \ldots$

Denote the multiplicity by $M(\mathcal{M}^2)$.

From Theorems 2 and 3 follows immediately the following result.

Corollary. If $\mathcal{L}(n^1, \ldots, n^R) = \mathcal{M}^2$ then $R \geq M(\mathcal{M}^2)$.

Remarks. 1. Kunita and Watanabe [6, p. 227] have shown that every martingale which is measurable with respect to the family of σ-fields generated by a vector Brownian motion can be represented as a Stochastic integral with respect to the Brownian motion. This result implies that if $(\Omega, \mathcal{F}_t, P)_{t \in R_+}$ is generated by a Gaussian process then the definition of multiplicity given here coincides with the one due to Cramer.

2. The definition of multiplicity and the main results similarly extend the corresponding ones given in Motoo and Watanabe (1965) for the case of martingales of a Hunt process.

3. Let x_t, $t \in R_+$ be a process defined on the probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t$ be the sub-σ-algebra of $\mathcal{F}$ generated by x_s, $0 \leq s \leq t$. Suppose that $\mathcal{F}_0$ is trivial and $\mathcal{F}_1 = \bigvee_t \mathcal{F}_t$ and suppose that $L^2(\Omega, \mathcal{F}, P)$ is separable. Then one could define the multiplicity of the process x, $M(x)$, as equal to $M(\mathcal{M}^2)$ where $\mathcal{M}^2$ is the family of square-integrable martingales on $(\Omega, \mathcal{F}_t, P)$. It should be kept in mind however that $M(x)$ is not invariant under modifications i.e. $y_t = x_t$ a.s., $t \in R_+$, does not necessarily imply that $M(x) = M(y)$. Of course if x_t and y_t are separable modifications of each other then $M(x) = M(y)$.

IV. An example. Let x_t, $t \in R_+$, be a process on $(\Omega, \mathcal{F}, P)$ taking values in a countable state space Z with discrete topology. The sample paths of the process are assummed right-continuous, and x_0 is fixed. Let $\mathcal{F}_t$ be the σ-field generated by x_s, $s \leq t$, and $\mathcal{F} = \bigvee_t \mathcal{F}_t$. Let $\mathcal{M}^2$ be defined as usual. Suppose that from each state $z \in Z$ the process can jump to at most n different values. This is illustrated in the state-transition diagram, Figure 1, where the different transitions are labeled $\sigma_1, \ldots, \sigma_n$. Define the "counting" processes P_t^i, $1 \leq i \leq n$, by $P_t^i(\omega)$ = the number of transitions of type i taken by $x_s(\omega)$ in the interval [0,t]. The integer-valued processes P^i are locally square-integrable and so there exist nondecreasing predictable processes $\tilde{P}^i$ such that

$$m_t^i = P_t^i - \tilde{P}_t^i, \quad 1 \leq i \leq n$$

is a locally square-integrable martingale. Furthermore $\mathcal{L}(m^i) \perp \mathcal{L}(m^j)$ if $i \neq j$.

Assume that the processes $\tilde{P}_t^i$ have continuous sample paths. Then it is shown in [1] that $\langle m^i \rangle = \tilde{P}^i$, and $\mathcal{M}^2 = \mathcal{L}(m^1, \ldots, m^n)$. Assume further that the $\tilde{P}_t^i$ have a representation of the form

$$\tilde{P}_t^i = \int_0^t \lambda_s^i ds$$

for some predictable processes λ^i, $1 \leq i \leq n$, and that $\lambda_t^i > 0$ a.s. for all i and all t. (Note that $\lambda_t^i \geq 0$ a.s. always since $\tilde{P}^i$ is nondecreasing). Then $\langle m^i \rangle \sim \langle m^j \rangle$ for all i, j so that $M(\mathcal{M}^2) = n$.

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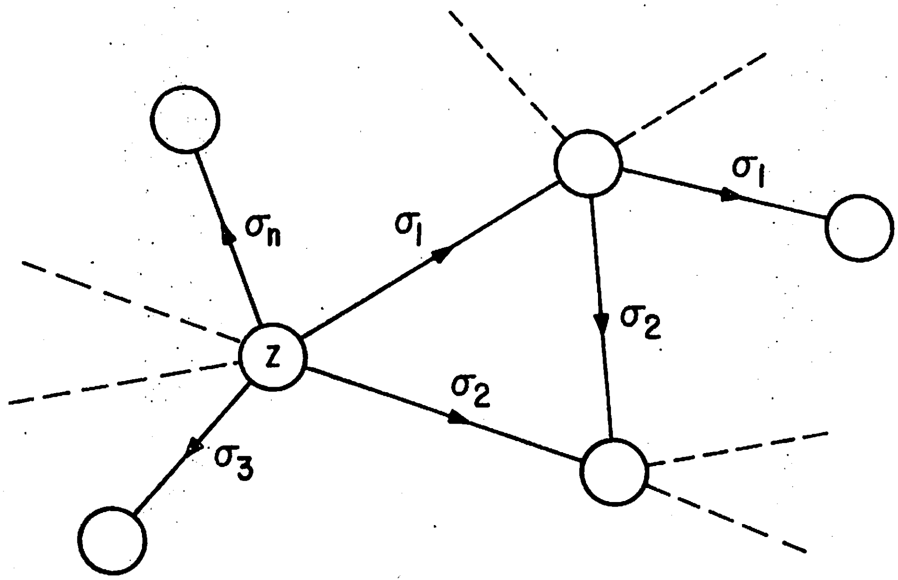

Fig. 1. State-transition diagram for the example.