ON THE SOLUTIONS OF A STOCHASTIC CONTROL SYSTEM II

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ABSTRACT. This paper presents generalizations of the work in [1,2] to
include controlled stochastic processes which take values in a certain
class of Fréchet spaces. The crucial result is an extension of the
technique for defining solutions of stochastic differential equations
by an absolutely continuous transformation of measures. The result is
used to prove existence results for stochastic control problems and for
a class of two-person zero sum games.

1. Introduction and Summary.

Consider a controlled stochastic process represented by the stochastic differential equation

$$dX(t) = f(t,X,u(t,X))dt + dB(t), \quad t \in [0,1],$$

where $B(t)$ is a Fréchet-valued Brownian motion, $X(t)$ is the state and $u(t,X)$ is the control applied at time t. For the case where the Fréchet space is R^n a satisfactory theory dealing with the problem of existence of solutions of the differential equation and existence of optimal control laws is now available [1,2]. A crucial building block in this theory consists in defining a solution of the differential equation via an absolutely continuous transformation of measures. Each control thereby defines a solution characterized by its (unique) probability law which is absolutely continuous with respect to Wiener measure. Thus the influence of a control law upon the system is captured in the Radon-Nikodym derivative of the resulting probability law with respect to Wiener measure. Questions dealing with the existence of an optimal control can then be converted into questions about the compactness (in an appropriate sense) of the set of Radom-Nikodym derivatives. The measure transformation technique mentioned above is due originally to Girsanov [16].

This paper deals with these same questions for the case where the state space is infinite-dimensional. The problem of characterizing Brownian motion in infinite-dimensional spaces is a difficult one and has been resolved for certain Fréchet spaces. This is described in the next section, where some additional properties of Brownian motion such as sample continuity and stochastic integration are established also.

In Section 3 the result of Girsanov alluded to above is extended to cover the differential equation under consideration. Once this has been achieved the techniques of [1,2] apply without change and the existence of optimal control and saddle points follows easily. This is sketched in Section 4.

2. Preliminaries for defining Brownian motion.

Measures on Fréchet spaces are defined using the results of Dudley-Feldman-LeCam [3] which generalize the work of Gross [4]. Some notation and definitions will be introduced before introducing the main topics. While the generality of the discussion is not used subsequently it serves to indicate some directions along which the results reported here can be further pursued.

For a locally convex Hausdorff topological (real) vector space X its topological dual is denoted by X^*. A dual system or duality over the reals consists of two vector spaces X, Y and a bilinear form $\langle \cdot, \cdot \rangle : X \times Y \to R$ that separates points for both X and Y. For a pair X, Y in duality FD(X) denotes the collection of all finite-dimensional subspaces of X. For $G \subset X$, $\mathcal{M}(Y,G)$ is the smallest σ-algebra on Y for which every $x \in G$, regarded as functions on Y, is measurable. Finally, let $\mathcal{C}(Y,X) = \cup \{\mathcal{M}(Y,G) \mid G \in FD(x)\}$. A cylinder set measure on Y is any non-negative, finitely additive set function m on $\mathcal{C}(Y,X)$ with m(Y) = 1 such that m is countably additive on $\mathcal{M}(Y,G)$ for each $G \in FD(X)$.

The following notion of a measurable seminorm given in [3] generalizes the one described in [4].

Definition 1. Given a duality X, Y, $\langle \cdot, \cdot \rangle$, and a cylinder set measure m on Y, a seminorm $|\cdot|$ on Y is said to be m-measurable if for each $\epsilon > 0$

there is $G \in FD(Y)$ such that if $F \in FD(X)$ and $F \perp G$ (i.e., $\langle x,y \rangle = 0$ for $x \in F$, $y \in G$) then

$$m\{y: \ |y-F^\perp| \leq \epsilon\} \geq 1-\epsilon$$

or equivalently

$$m^*\{y: \ |y-G| \leq \epsilon\} \geq 1-\epsilon$$

where m^* is the outer measure on $\mathscr{B}(Y,X)$ induced by m.

The next result follows from [3, Theorem 2].

<u>Fundamental Theorem.</u> Let $|\cdot|$ be a Mackey-continuous seminorm on Y and suppose that it is m-measurable. Then the cylinder set measure induced by m on the Banach space $Y/|\cdot|$ obtained from Y via the seminorm $|\cdot|$ extends to a regular Borel measure.

From here on the discussion is specialized to a fixed, separable Hilbert space H. It is assumed that there is given for each $t \in R_+$ a cylinder set measure p_t on H such that p_t is a canonical normal distribution on H with variance parameter t (see [4] or [5])[1]. It is also assumed that there is given an increasing family of Mackey-continuous seminorms $|\cdot|_j$, $j = 1, 2, \ldots$ on H. Let F be the Fréchet space obtained from H with respect to the topology defined by the seminorms $|\cdot|_j$ by completion modulo the intersection of their null spaces (which without loss of generality

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[1]This means that if $C \in \mathscr{C}(H,H)$ is of the form $C = P^{-1}(E)$ where P is the orthogonal projection of H onto a subspace $L \in FD(H)$ and E is a Borel subset of L then

$$p_t(C) = (2\Pi t)^{-n/2} \int_E \exp\left(- \frac{1}{2t} |x|_H^2 \right) dx$$

where n is the dimension of L, and $|\cdot|_H$ denotes the norm on H induced by its inner product.

is assumed to equal $\{0\}$). As a corollary to the Fundamental Theorem it is proved in [3, Corollary 2.1] that each p_t, $t \in R_+$ extends to a regular Borel measure, denoted μ_t, on F. For future reference note that $H \subset F$, $F^* \subset H^* = H$, and define the maps i: $H \to F$, j: $F^* \to H^*$, as the canonical injections. Finally let $\mathcal{B}(F)$ denote the Borel sets of F. Throughout H and F denote the spaces introduced here.

For each $t \in R_+$ let $F^t = F$. For each finite collection $t_1 < t_2 < \ldots < t_n$ in R_+ the joint distribution $\mu_{t_1 \ldots t_n}$ on $(F^{t_1} \times \ldots \times F^{t_n}$, $\mathcal{B}(F^{t_1}) \times \ldots \times \mathcal{B}(F^{t_n}))$ is obtained from the marginal distributions μ_{t_i} on $(F^{t_i}, \mathcal{B}(F^{t_i}))$ by using the fact that Brownian motion has independent increments. Since each of the measures μ_t is a regular Borel measure the projective system of measures admits a projective limit (see [6], p. 49). The projective limit thus obtained is denoted $(\Omega, \mathcal{F}, P)$. Evidently $(\Omega, \mathcal{F}) = (\prod_{t \in R_+} F^t, \prod_{t \in R_+} \mathcal{B}(F^t))$. It will be assumed that $(\Omega, \mathcal{F}, P)$ is complete. Let X_t, $t \in R_+$, denote the F-valued evaluation map. Let $\mathcal{F}_t$ be the smallest σ-algebra with respect to which X_s is measurable for $s \leq t$. It is assumed that $\mathcal{F}_t$ is complete for each t. Then $(X_t, \mathcal{F}_t, P)_{t \in R_+}$ is a Brownian motion as defined below.

<u>Definition 2</u>. A stochastic process $(X_t, \mathcal{F}_t, P)_{t \in R_+}$ (or simply $(X_t, P)_{t \in R_+}$ or X_t if there is no ambiguity) is said to be a <u>Brownian motion with values in the Fréchet space F</u> induced from a family of canonical normal distributions on a Hilbert space H, if for each $\ell \in F^*$ (the topological dual of F) the real-valued process $(\langle \ell, X_t \rangle, \mathcal{F}_t, P)_{t \in R_+}$ is a real-valued Brownian motion with $E\langle \ell, X_t \rangle^2 = t|j\ell|_H^2$. (Here and throughout $|\cdot|_H$ denotes the norm on H and j: $F^* \to H^*$ is the canonical injection mentioned before.)

The process $(Y_t, P)_{t \in R_+}$ is said to be a <u>modification</u> of the process

$(X_t, P)_{t \in R_+}$, if for each $t \in R_+$ $X_t(\omega) = Y_t(\omega)$ a.s. (the null set $\{X_t \neq Y_t\}$ may depend on t).

For Fréchet-valued Brownian motion there is a modification which has continuous sample paths as shown by the following lemma.

Lemma 1. Let $(X_t, \mathcal{F}_t, P)_{t \in R_+}$ be a Fréchet-valued Brownian motion. There is a modification of X_t with continuous sample paths.

Proof. Since a countable number of seminorms determine the topology of a Fréchet space it suffices to verify the continuity of the sample function of Brownian motion with respect to each of these countable number of seminorms and therefore it is enough to prove the lemma for a Banach-valued Brownian motion.

Fernique [7] has shown that for a Gaussian random variable X on a topological vector space with a measurable seminorm $|\cdot|$ there is an $\alpha > 0$ such that

$$E \exp \alpha |X|^2 < \infty \qquad (1)$$

Combining (1) with a result of Nelson ([8], Theorem 2) as used by Gross ([9] p. 134) it follows that the Banach valued Brownian motion has a modification with continuous sample paths.   ¤

Since stochastic integrals will be used subsequently, a family of processes have to be described that will be the integrands for the stochastic integrals. The following definition gives such a family.

Definition 3. Let $\tilde{H}$ be a separable Hilbert space. An $\tilde{H}$-valued stochastic process $(\psi_t)_{t \in R_+}$ on $(\Omega, \mathcal{F}, P)$ that is adapted to $(\mathcal{F}_t)_{t \in R_+}$ is said to be

<u>predictable</u> if the map $\psi_t(\omega)$: $R_+ \times \Omega \to \tilde{H}$ is measurable with respect to the σ-algebra on $R_+ \times \Omega$ generated by the left-continuous $\tilde{H}$-valued processes adapted to $(\mathcal{F}_t)_{t \in R_+}$.

Real-valued stochastic integrals will now be defined from F-valued Brownian motion and predictable H-valued processes.

<u>Lemma 2.</u> Let $(\psi_t)_{t \in R_+}$ be a predictable H-valued process with

$$E \int_0^\infty |\psi_t|_H^2 dt < \infty, \tag{2}$$

and let $(\mathcal{B}_t, \mathcal{F}_t, P)_{t \in R_+}$ be a F-valued Brownian motion. Then the real-valued process $(Y_t, \mathcal{F}_t, P)_{t \in R_+}$,

$$Y_t = \int_0^t \langle \psi_s, dB_s \rangle, \tag{3}$$

is a square-integrable martingale which has a modification with continuous sample paths.

<u>Proof.</u> Let $\varepsilon > 0$. By the monotone convergence theorem there exists a finite-dimensional orthogonal projection P_ε on H with $P_\varepsilon H \subset jF^*$ such that

$$E \int_0^\infty |(I-P_\varepsilon)\psi_t|_H^2 dt < \varepsilon \tag{4}$$

where I is the identity operator on H and j: $F^* \to H^* = H$ is the natural injection. Let $(Y_t(P_\varepsilon))_{t \in R_+}$ denote a modification with continuous sample paths of the real-valued martingale

$$Y_t(P_\epsilon) = \int_0^t \langle P_\epsilon \psi_s dB_s \rangle . \tag{5}$$

Let $\epsilon_n > 0$, $n = 1, 2, \ldots$ be a sequence decreasing to 0 and let P_{ϵ_n} be an increasing sequence of orthogonal projections with $P_{\epsilon_n} H \subset jF^*$ satisfying inequalities corresponding to (4). By Doob's inequality [10, p. 353] the sequence of martingales $(Y_t(P_{\epsilon_n}))_{t \in R_+}$ converges uniformly and for each t the sequence of random variables $Y_t(P_{\epsilon_n})$ converges in $L^2(\Omega, \mathcal{F}_t, P)$. The result follows.

□

Corollary 1. Let $(\psi_t)_{t \in R_+}$ be a predictable H-valued process with

$$E \int_0^t |\psi_t|_H^2 dt < \infty \quad a.s. \tag{5}$$

Then the real-valued process Z_t defined by

$$Z_t = \int_0^t \langle \psi_s, dB_s \rangle \tag{6}$$

is a locally[2] square-integrable martingale which has a modification with continuous sample paths.

The following representation of square integrable functionals on the F-valued Brownian motion probability space will be useful subsequently. For R^n-valued Brownian motion K. Itô [11] has obtained this representation by describing results of Wiener [12] and Cameron-Martin [13] in terms of stochastic integrals.

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[2] m_t is said to be a locally square-integrable martingale if there exists a sequence of stopping times $T_n \uparrow \infty$ a.s. such that $m_{t \wedge T_n}$ is a square-integrable martingale for each n.

<u>Proposition 1.</u> Let $(\Omega, \mathcal{F}, P)$ be the probability space for a F-valued

Brownian motion $(B_t, \mathcal{F}_t)_{t \in R_+}$. Let f be a real-valued square-integrable

functional on $(\Omega, \mathcal{F}, P)$. Then f can be represented as

$$f = c + \int_0^\infty \langle \psi_s, dB_s \rangle \tag{7}$$

where $c = Ef$ and $(\psi_t)_{t \in R_+}$ is a predictable H-valued process with

$$E \int_0^\infty |\psi_t|_H^2 dt < \infty$$

<u>Proof.</u> Since H is separable so is F, hence by the Hahn-Banach theorem

there is a countable family $\Gamma \subset F^*$ that separates points of F. Consider

the random variables $\langle \gamma, \int_0^t dB_s \rangle$ where $\gamma \in \Gamma$, $t \in R_+$ and let $\mathcal{A}$ be the

algebra of real-valued random variables formed from these and the constant

random variables. Since the random variables $\langle \gamma, \int_0^t dB_s \rangle$ are jointly

Gaussian it follows that $\mathcal{A} \subset L^2(P)$ and furthermore

$$\exp\left(\sum_{i=1}^n a_i |f_i| \right) \in L^2(P)$$

for $a_i \in R$ and $f_i \in \mathcal{A}$. Using a result of Segal ([14], Lemma 2.1) it

follows that the family $\mathcal{A}$ which generates the σ-algebra $\mathcal{F}$ is dense in

$L^2(P)$.

Let $f \in L^2(P)$. There exist a sequence g_n in $\mathcal{A}$ such that $E |f-g_n|^2 \to 0$.

By properties of finite-dimensional Brownian motion g_n has a representation

of the form (7) (see [15]) i.e.,

$$g_n = c_n + \int_0^\infty \langle \psi_s^n, dB_s \rangle \qquad (8)$$

where $c_n = Eg_n$ and ψ_t^n is a predictable process with values in some finite-dimensional subspace L of H and with $L \subset jF^*$. Since $L^2(P)$-convergence implies $L^1(P)$-convergence it follows that c_n converges to c. Furthermore the stochastic integrals in (8) must be Cauchy hence

$$\int_0^\infty E|\psi_t^n - \psi_t^m|_H^2 dt \to 0,$$

as m, n $\to \infty$. Since the sequence of processes ψ_t^m are predictable there must exist a predictable process ψ_t such that

$$\int_0^\infty |\psi_t^n - \psi_t|_H^2 \to 0,$$

and evidently (7) is satisfied.   ¤

3. Transformation of measures.

Theorem 1 below describes how Fréchet-valued Brownian motion is transformed by changing the probability measure by an absolutely continuous substitution of the measure. This result was first established by Girsanov [16] for the case of R^n-valued Brownian motion. The result has been presented in [17] and a related result is given in [18].

Theorem 1. Let $(B_t, \mathcal{F}_t, P)_{t \in [0,1]}$ be an F-valued Brownian motion and let $(\psi_t)_{t \in [0,1]}$ be a predictable H-valued process such that

$$\int_0^1 |\psi_t|_H^2 dt < \infty \text{ a.s. } P. \qquad (9)$$

Define the non-negative process $(M_t, \mathcal{F}_t, P)_{t \in [0,1]}$ by

$$M_t = \exp[\int_0^t \langle \psi_s, dB_s \rangle - \frac{1}{2} \int_0^t |\psi_s|_H^2 ds] \qquad (10)$$

then

$$E(M_1) \leq 1 \qquad (11)$$

Suppose that

$$E(M_1) = 1, \qquad (12)$$

then the process $(\tilde{B}_t, \mathcal{F}_t, \tilde{P})_{t \in [0,1]}$ is an F-valued Brownian motion where,

$$\tilde{B}_t = B_t - \int_0^t i\psi_s ds,$$

and the probability measure $\tilde{P}$ is given by

$$\frac{d\tilde{P}}{dP} = M_1$$

Proof. Define the increasing sequence of stopping times T_n,

$$T_n = \begin{cases} \inf \{t | M_t > n\} \\ 1 \quad \text{if the set above is empty} \end{cases}$$

Because of (9) M_t has (a modification with) continuous sample paths, so $T_n \uparrow 1$ a.s. P, and in particular

$$\lim_{n \to \infty} M_{t \wedge T_n} = M_t \text{ as P.} \qquad (13)$$

By the main result in [19], the processes M_t and $M_{t \wedge T_n}$ satisfy

$$M_t = 1 + \int_0^t M_s \langle \psi_s, dB_s \rangle \qquad (14)$$

$$M_{t \wedge T_n} = 1 + \int_0^{t \wedge T_n} M_s \langle \psi_s, dB_s \rangle \qquad (15)$$

So that since $M_{t \wedge T_n}$ is bounded (15) implies that it is a martingale and hence

$$E(M_{t \wedge T_n}) = E(M_{T_n}) = 1$$

An application of Fatou's lemma to this result and (13) proves (11).

From now on suppose that (12) holds, so that from (14) it follows that $(M_t, \mathcal{F}_t, P)_{t \in [0,1]}$ is in fact a martingale. Let $\ell \in H$ be fixed and consider the process $(N_t, \mathcal{F}_t, \tilde{P})_{t \in [0,1]}$ where

$$N_t = \langle \ell, \tilde{B}_t \rangle = \langle \ell, B_t \rangle - \int_0^t \langle \ell, \psi_s \rangle \, ds$$

The theorem will be proved once it is shown that

$$(N_t, \mathcal{F}_t, \tilde{P}) \text{ is a martingale,} \qquad (16)$$

$$\tilde{E}(N_t^2 - N_s^2 | \mathcal{F}_s) = |\ell|_H^2 (t-s), \qquad (17)$$

where $\tilde{E}$ denotes expectation with respect to the measure $\tilde{P}$.

Now to prove (16) it is sufficient to show instead that

$$(M_t N_t, \mathcal{F}_t, P) \text{ is a martingale.} \qquad (18)$$

Because suppose that (18) is true. Then using the fact that (M_t, P) is a martingale,

$$\tilde{E}(N_t \mid \mathcal{F}_s) = \frac{E(M_t N_t \mid \mathcal{F}_s)}{E(M_t \mid \mathcal{F}_s)} \quad \text{by [20, p. 345]}$$

$$= \frac{M_s N_s}{M_s} \quad \text{by (18)}$$

$$= N_s$$

which is equivalent to (16). Now

$$M_t N_t = M_t \langle \ell, B_t \rangle - M_t \int_0^t \langle \ell, \psi_s \rangle \, ds.$$

Applying the differentiation fromula for continuous martingales [15] gives

$$M_t N_t - M_0 N_0 = \int_0^t M_s \langle \ell, dB_s \rangle - \int_0^t M_s \langle \ell, \psi_s \rangle \, ds + \int_0^t N_s dM_s$$

$$+ \int_0^t M_s \langle \ell, \psi_s \rangle \, ds$$

$$= \int_0^t M_s \langle \ell, dB_s \rangle + \int_0^t N_s dM_s$$

which clearly implies (18).

It remains to prove (17). To this end note that

$$\tilde{E}(N_t^2 - N_s^2 | \mathcal{F}_s) = \frac{E(M_t(N_t^2 - N_s^2) | \mathcal{F}_s)}{M_s}, \qquad (19)$$

and apply the differentiation formula to $M_t N_t^2$ to obtain

$$M_t N_t^2 - M_0 N_0^2 = \int_0^t N_s^2 dM_s + 2\int_0^t M_s N_s \langle \ell, dB_s \rangle - 2\int_0^t M_s N_s \langle \ell, \psi_s \rangle ds$$

$$+ 2\int_0^t N_s M_s \langle \ell, \psi_s \rangle ds + \int_0^t M_s |\ell|_H^2 ds.$$

Substitution of this into (19) gives

$$\tilde{E}(N_t^2 - N_s^2 | \mathcal{F}_s) = \frac{E\left(\int_s^t M_\tau |\ell|_H^2 d\tau | \mathcal{F}_s \right)}{M_s} = |\ell|_H^2 (t-s)$$

and so (17) is proved.   ◻

In many applications of the transformation of measures technique of Theorem 1 the crucial difficulty is to verify that $\tilde{P}(\Omega) = 1$ The next result gives a sufficient condition for $\tilde{P}(\Omega) = 1$.

Lemma 3. Let $(B_t, \mathcal{F}_t, P)_{t \in [0,1]}$ be an F-valued Brownian motion. Let $(\psi_t)_{t \in [0,1]}$ be a predictable H-valued process such that

$$|\psi(t,B)|_H \le K + K' \sup_{t \in [0,1]} |B_t| \qquad (20)$$

where $|\cdot|$ is a seminorm on F and K' is a sufficiently small constant. Then there is $\alpha > 1$ and $M < \infty$, depending only on K', such that

$$E \exp \alpha \left[\int_0^1 \langle \psi_s, dB_s \rangle - \frac{1}{2} \int_0^1 |\psi_s|_H^2 ds \right] < M \qquad (21)$$

In particular,

$$E \exp \left[\int_0^1 \langle \psi_s, dB_s \rangle - \frac{1}{2} \int_0^1 |\psi_s|_H^2 ds \right] = 1.$$

<u>Proof</u>. From (20) it is immediate that

$$|\psi_s|_H^2 dt < \infty \quad \text{a.s.}$$

so that the terms in the exponential in (21) are well-defined. Let $\alpha > 1$. It is clear from the way that (11) was proved that for any real number β $Eh_1(\beta) \le 1$, where

$$h_t(\beta) = \exp \left[\beta \int_0^t \langle \psi_s, dB_s \rangle - \frac{1}{2} \beta^2 \int_0^t |\psi_s|_H^2 ds \right]$$

In particular for $\beta = \alpha^2$ this gives $E(f(\alpha))^\alpha \le 1$ where

$$f(\alpha) = \exp \left[\alpha \int_0^1 \langle \psi_s, dB_s \rangle - \frac{1}{2} \alpha^3 \int_0^1 |\psi_s|_H^2 ds \right].$$

Let

$$g(\alpha) = \exp \left[\frac{1}{2} (\alpha^3 - \alpha) \int_0^1 |\psi_s|_H^2 ds \right].$$

Then (21) is equivalent to $E f(\alpha)g(\alpha) < M$. By Hölder's inequality

$$E f(\alpha) g(\alpha) \le \left[E(f(\alpha))^\alpha \right]^{\frac{1}{\alpha}} \left[E(g(a))^{\frac{\alpha}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha}}$$

Now on the space $\mathcal{C}$ of all F-valued continuous function on [0,1] the real-valued function on $\mathcal{C}$ defined by $z \to \|z\|$, where

$$\|z\| = \sup_{t \in [0,1]} |z(t)|,$$

defines a seminorm on $\mathcal{C}$. The result of Fernique [7] implies the existence of $\gamma > 0$ such that

$$E \exp \gamma \|B\|^2 < \infty.$$

From (20) one immediately obtains

$$\int_0^1 |\psi(s,B)|_H^2 ds \leq 2K^2 + 2(K'\|B\|)^2.$$

Also

$$(g(\alpha))^{\frac{\alpha}{\alpha-1}} = \exp [\frac{1}{2} \alpha^2 (\alpha+1) \int_0^1 |\psi|_H^2 ds],$$

hence if K' is sufficiently small so that

$$\alpha^2(\alpha+1)K' \leq \gamma$$

then $E(g(\alpha))^{\frac{\alpha}{\alpha-1}} < \infty$ and (21) follows. The final assertion is then immediate because (21) implies that

$$\exp [\int_0^t \langle \psi_s, dB_s \rangle - \frac{1}{2} \int_0^t |\psi_s|_H^2 ds]$$

is a martingale.   ◻

Corollary 2. If in the lemma above $|\psi(t,B)| \leq K$ then $E \exp \alpha [\int_0^1 \langle \psi_s, dB_s \rangle - \frac{1}{2} \int_0^1 |\psi_s^2|_H ds] < \infty$

4. Preliminaries for optimization.

The system to be controlled is represented by the stochastic differential equation

$$dX(t) = if(t,X,u(t,X))dt + dB(t), \quad t \in [0,1] \tag{22}$$

where B_t is an F-valued Brownian motion, $X_t \in F$ is the state with $X_0 = 0$ a.s., and u is the control law taking values in a prespecified compact subset $U \subset F$ called the <u>control set</u>. The function f takes values in H and i: $H \to F$ is the canonical injection. The first difficulty to be resolved is to define the solution of the differential equation (22) for a large class of control laws. This is achieved in the following manner. One starts with a process $(X_t, \mathcal{F}_t, P_0)_{t \in [0,1]}$ which is an F-valued Brownian motion. For a given control law u an F-valued process B_t^u is defined by

$$B_t^u = X_t - \int_0^t if(s,X,u(s,X))ds.$$

Next the probability measure P_0 is replaced by another probability measure P^u such that the process $(B_t^u, \mathcal{F}_t, P^u)_{t \in [0,1]}$ is an F-valued Brownian motion. The process $(X_t, \mathcal{F}_t, P^u)_{t \in [0,1]}$ is then regarded as the solution of (22) corresponding to the control law u. To make this procedure precise the following notations and definitions are useful.

<u>Definition 4.</u> (a) $\mathcal{C}$ is the linear space of all F-valued continuous functions, denoted by z, on [0,1].

(b) For $t \in [0,1]$ $\mathcal{A}_t$ is the smallest σ-algebra of subsets of $\mathcal{C}$ which contain all sets of the form $\{z \in \mathcal{C} | z(s) \in A\}$ where $s \in [0,t]$ and

A is a (topological) Borel subset of F. $\mathcal{B} = \mathcal{B}_1$.

Throughout the remainder of this paper Ω is a fixed probability space with an increasing family of σ-algebras $\mathcal{F}_t$, $t \in [0,1]$. $\mathcal{F} = \mathcal{F}_1$. It will be necessary to consider different probability measures on the space $(\Omega, \mathcal{F})$. If Y_t is a family of measurable functions on $(\Omega, \mathcal{F}_t)$ and if P is a probability measure on $(\Omega, \mathcal{F})$ the stochastic process corresponding to P and the family Y_t will be denoted by $(Y_t, \mathcal{F}_t, P)_{t \in [0,1]}$. Then the same family Y_t generates different stochastic processes corresponding to different probability measures. Finally let P_0 be a distinguished probability measure, and let X_t be a distinguished family of F-valued measurable functions on $(\Omega, \mathcal{F}_t)$, $t \in [0,1]$ such that the process $(X_t, \mathcal{F}_t, P_0)_{t \in [0,1]}$ is an F-valued Brownian motion with continuous sample paths. Unless mentioned otherwise the process X_t refers to this process. Also E_0 will denote expectation with respect to P_0. The measure induced on $(\mathcal{C}, \mathcal{B})$ by the process X_t is denoted by μ and will be called the <u>Wiener measure</u> on $(\mathcal{C}, \mathcal{B})$.

The following conditions are imposed on the function f in (22).

<u>f1</u> f is a map from $[0,1] \times \mathcal{C} \times U$ into H and f is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}_u$ where $\mathcal{B}(\mathcal{B}_u)$ is the family of Borel subsets of $[0,1]$ (U).

<u>f2</u> For $t \in [0,1]$ $f(t,\cdot,\cdot)$ is $\mathcal{B}_t \otimes \mathcal{B}_u$-measurable.

<u>f3</u> For $(t,z) \in [0,1] \times \mathcal{C}$ $f(t,z,\cdot):$ $U \to H$ is continuous.

<u>f4</u> There is an increasing function $f_0:$ $R_+ \to R_+$, and a seminorm $|\cdot|$ on F such that for $(t,z,u) \in [0,1] \times \mathcal{C} \times U$

$$|f(t,z,u)|_H \leq f_0(\|z\|)$$

where $\|z\| = \max\limits_{t \in [0,1]} |z(t)|$. Throughout the remainder the symbols $|\cdot|$ and $\|\cdot\|$ will denote the seminorms assumed here.

$\underline{f5}$ For $(t,z) \in [0,1] \times \mathcal{C}$ $f(t,z,U) = \{f(t,z,u)|u \in U\}$ is a closed and convex subset of H.

Definition 5. (a) An <u>admissible control</u> (law) is a map u: $[0,1] \times \mathcal{C} \to U$ which is $\mathcal{B} \otimes \mathcal{S}$-measurable and, further, is such that for each $t \in [0,1]$ $u(t,\cdot)$ is $\mathcal{S}_t$-measurable. $\mathcal{U}$ denotes the set of all admissible controls.

(b) The <u>drift</u> corresponding to $u \in \mathcal{U}$ is the function g_u given by g_u given by $g_u(t,z) = f(t,z,u(t,z))$. $\mathcal{Y} = \{g_u | u \in U\}$.

(c) For $g \in \mathcal{Y}$ and positive integer n, g^n is the function given by

$$g^n(t,z) = \begin{cases} g(t,z) & \text{if } |z(s)| \leq n \text{ for } s \in [0,t] \\ 0 & \text{otherwise} \end{cases}$$

Definition 6. A function ϕ: $[0,1] \times \mathcal{C} \to H$ is <u>causal</u> if it is $\mathcal{B} \otimes \mathcal{S}$-measurable and if $\phi(t,\cdot)$ is $\mathcal{S}_t$-measurable for $t \in [0,1]$.

Definition 7. Φ is the collection of all causal functions such that $|\phi(t,z)|_H \leq f_0(\|z\|)$ for $(t,z) \in [0,1] \times \mathcal{C}$. $\Phi^n = \{\phi \in \Phi | |\phi(t,z)|_H \leq n$ for all $(t,z)\}$.

The next result which follows immediately from ([21], Lemma 1) gives a very useful characterization of $\mathcal{Y}$.

Lemma 4. A causal function g is in $\mathcal{Y}$ if and only if $g(t,z) \in f(t,z,U)$ for all t, z.

<u>Definition 8.</u> Let ϕ be a causal function such that

$$\int_0^1 |\phi(t,z)|_H^2 dt < \infty \quad \text{for all } z \in \mathscr{C}. \tag{23}$$

Then $(\zeta_t(\phi), \mathcal{F}_t, P_0)_{t \in [0,1]}$ denotes the continuous real-valued process defined by

$$\zeta_t(\phi) = \int_0^t \langle \phi(s,X), dX_s \rangle - \frac{1}{2} \int_0^t |\phi(s,X)|_H^2 ds$$

Let $\zeta(\phi) = \zeta_1(\phi)$. Note that (23) is always satisfied for $\phi \in \Phi$.

The problem of the existence of solutions of (22) is resolved in the following result which follows immediately from Theorem 1.

<u>Theorem 2.</u> Let $u \in \mathcal{U}$ be such that

$$E_0 \exp \zeta(g_u) = 1, \tag{24}$$

where E_0 denotes expectation with respect to P_0. Define the probability measure P_u by

$$dP_u = \exp \zeta(g_u) dP_0.$$

Then the process $(B_t, \mathcal{F}_t, P_u)_{t \in [0,1]}$ defined by

$$B_t = X_t - \int_0^t if(s,X,u(s,X)) ds$$

is an F-valued Brownian motion.

The next result shows that (24) must be satisfied by every solution

of (22).

Lemma 5. Let $\phi \in \Phi$. Let $(Y_t, \mathcal{F}_t, P)_{t \in [0,1]}$ be any process with continuous sample paths such that the stochastic process $(B_t, \mathcal{F}_t, P)_{t \in [0,1]}$ defined by

$$dB_t = dY_t - i\phi(t,Y)dt$$

is an F-valued Brownian motion. Then

$$\int_\Omega \exp\left[- \int_0^1 \langle \phi(s,Y), dB_s \rangle - \frac{1}{2} \int_0^1 |\phi(s,Y)|_H^2 ds \right] dP = 1$$

Proof. Define the function ϕ_n: $[0,1] \times \mathcal{C} \to H$ by

$$\phi_n(t,z) = \begin{cases} \phi(t,z) & \text{if } |z(s)| \leq n \text{ for } s \in [0,t] \\ 0 & \text{otherwise} \end{cases}$$

From the definition of Φ it follows that

$$|\phi_n(t,Y(\omega))|_H \leq f_0(n) \quad \text{for } t \in [0,1], \omega \in \Omega.$$

By Corollary 2

$$\int_\Omega \exp \zeta(-\phi_n) dP = 1 ,$$

where

$$\zeta(-\phi_n) = - \int_0^1 \langle \phi_n(s,Y), dB_s \rangle - \frac{1}{2} \int_0^1 |\phi_n(s,Y)|_H^2 ds$$

By Theorem 1 the process $(Y_n(t), \mathcal{F}_t, \tilde{P}_n)_{t \in [0,1]}$, where,

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$$Y_n(t) = \int_0^t i\phi_n(s,Y)ds + B(t),$$

$$d\tilde{P}_n = \exp \zeta(-\phi_n)dP,$$

is a Brownian motion with continuous sample paths and hence induces Wiener measure μ on $(\mathcal{C}, \mathcal{S})$. Let $\epsilon > 0$, and n be so large that

$$\tilde{P}_n(\Omega_n) = \tilde{P}_n(\| Y_n \| < n) = \mu(z \in \mathcal{C} \mid \| z \| < n) \geq 1-\epsilon.$$

Now it is clear that $\| Y_n(\omega) \| < n$ only if $\| Y(\omega) \| < n$ and hence

$$Y_n(\omega,t) = Y(\omega,t), \quad \phi_n(\omega,t) = \phi(\omega,t) \text{ for } \omega \in \Omega_n, \ t \in [0,1],$$

so that

$$\tilde{P}_n(\Omega_n) = \int_{\Omega_n} \exp \zeta(-\phi_n)dP = \int_{\Omega_n} \exp \zeta(-\phi)dP \geq 1-\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows. $\qquad\qquad$ ¤

<u>Corollary 3</u>. Let $\phi \in \Phi$ and $(Y_t, \mathcal{F}_t, P)_{t \in [0,1]}$ satisfy the hypothesis of Lemma 5. Let ν be the measure induced by the process $(Y_t, \mathcal{F}_t, P)$ on the measurable space $(\mathcal{C}, \mathcal{S})$. Then ν is mutually absolutely continuous with respect to μ and

$$\frac{d\mu}{d\nu} (Y(\omega)) = \exp \zeta(-\phi)(\omega).$$

<u>Remark</u>. This corollary implies that the solutions of (22) are unique in a weak sense i.e., all solutions of (22) which have continuous sample paths must induce the same measure on $(\mathcal{C}, \mathcal{S})$. The qualification "weak" is

inserted because only the uniqueness of the probability law has been proved.

Recall that $(X_t, \mathcal{F}_t, P_0)_{t \in [0,1]}$ is an F-valued Brownian motion with continuous sample paths. Also recall Definition 8.

__Definition 9__. For any subset $\Lambda \subset \Phi$ define $\mathcal{D}(\Lambda) \subset L^1(\Omega, \mathcal{F}, P_0)$ by

$$\mathcal{D}(\Lambda) = \{\exp \zeta(\phi) \mid \phi \in \Lambda\}$$

The corollary to Lemma 3 implies the next assertion.

__Proposition 2__. $\mathcal{D}(\Phi^n)$ is a bounded subset of $L^2(\Omega, \mathcal{F}, P_0)$.

__Lemma 6__. $\mathcal{D}(\Phi^n)$ is a closed subset of $L^2(\Omega, \mathcal{F}, P_0)$.

__Proof__. Let $\phi_1, \phi_2, \ldots$ be a sequence from Φ^n and let ρ be such that

$$\lim_{n \to \infty} E_0 |\rho - \exp \zeta(\phi_n)|^2 = 0, \tag{25}$$

and

$$\lim_{n \to \infty} \exp \zeta(\phi_n) = \rho \text{ a.s. } P_0 \tag{26}$$

Since $E_0 \exp \zeta(\phi_n) = 1$ for all n, $E_0 \rho = 1$, and so by Proposition 1 there is a functional ψ such that $E_0 \int_0^1 |\psi(t,x)|_H^2 dt < \infty$ and

$$\rho = 1 + \int_0^1 \langle \psi(t,X), dX_s \rangle \text{ a.s. } P_0. \tag{27}$$

Define the martingale ρ_t by

$$\rho_t = E_0\{\rho \mid \mathcal{F}_t\}, \quad t \in [0,1].$$

By taking modifications if necessary it can be assumed that the martingales ρ_t and $\zeta^t(\phi_n)$ have continuous sample paths so that, by Doob's inequality [10, p. 353], it follows from (25) that

$$\rho_t = \lim_{n \to \infty} \exp \zeta^t(\phi_n) \quad \text{uniformly on } [0,1] \text{ a.s. } P_0. \tag{28}$$

Next

$$\exp \zeta(\phi_n) = 1 + \int_0^1 \exp \zeta^t(\phi_n) \langle \phi_n(t), dX(t) \rangle$$

so that from (25), (26)

$$\lim_{n \to \infty} E_0 \int_0^1 |\exp \zeta^t(\phi_n) \phi_n(t) - \psi(t)|_H^2 dt = 0$$

and hence by taking subsequences if necessary it can be assumed that

$$\psi(t) = \lim_{n \to \infty} \exp \zeta^t(\phi_n) \phi_n(t) \quad \text{a.s. } \ell \otimes P_0. \tag{29}$$

where ℓ denotes Lebesque measure on $[0,1]$. Now $\rho > $ a.s. P_0, because if $P_0(A) = P_0\{\rho = 0\} > 0$ then (26), together with the fact that $|\phi_n|_H \leq n$, implies that

$$\lim_{n \to \infty} \int_0^1 \langle \phi_n(t), dX(t) \rangle = -\infty \text{ on } A.$$

But then

$$\infty = E_0 \left| \int_0^1 \langle \phi_n(t), dX(t) \rangle \right|^2 \leq E_0 \int_0^1 |\phi_n(t)|_H^2 dt \leq N^2,$$

so that to avoid the contradiction one must have $\rho > 0$ a.s. P_0. It

follows that $\rho_t > 0$ a.s., and hence combining (28) and (29) gives

$$\frac{\psi(t)}{\rho(t)} = \lim_{n \to \infty} \phi_n(t) \quad \text{a.s. } \ell \otimes P_0.$$

Thus there is a causal map $\phi \in \Phi^n$ such that

$$\phi(t,X) = \lim_{n \to \infty} \phi_n(t,X) \quad \text{a.s. } \ell \otimes P_0$$

and evidently $\rho = \exp \zeta(\phi)$. $\square$

The proof of the next two results is identical respectively with the proof of [2, Lemma 4] and [2, Theorem 2] with some obvious notational changes. Hence the proofs are omitted.

Lemma 7. $\mathcal{O}(\Phi^n)$ is a convex subset of $L^2(\Omega, \mathcal{F}, P_0)$.

Theorem 3. Let

$$\mathcal{Y}^0 = \{g \in \mathcal{Y} \mid E_0 \exp \zeta(g) = 1\}$$

Then $\mathcal{D}(\mathcal{Y}^0)$ is a closed, convex subset of $L^1(\Omega, \mathcal{F}, P_0)$.

5. Applications.

   The results developed above immediately imply the existence of optimal control laws for a broad class of problems. Consider the control system

$$dX(t) = if(t,X,u(t,X))dt + dB(t), \quad t \in [0,1]$$

with $X(0) = 0$ a.s.   Suppose that f satisfies the assumptions f1 to f5 and in addition the function f_0 in f4 satisfies assumption f6.

$\underline{f6}$. There exists K, K' and $\alpha > 1$ such that $f_0(n) \leq K + K'n$ and

$$\alpha^2(\alpha+1) \, K' \leq \gamma$$

where γ is defined in Lemma 3.

Note that $\underline{f6}$ is automatically satisfied if f_0 has less than linear growth i.e., if $\lim_{n \to \infty} \dfrac{f_0(n)}{n} = 0$.

Let L: $\mathcal{G} \to R$ be a fixed bounded $\mathcal{J}$-measurable function. For each $u \in \mathcal{U}$ the $\underline{\text{cost incurred by } u}$ is defined to be

$$J(u) = E_0[(\exp \zeta(g_u))L(X)] = \int_\Omega L(X(\omega))\exp (g_u(\omega)dP_0. \qquad (30)$$

$\underline{\text{Theorem 4}}$. Under assumptions $\underline{f1}$ to $\underline{f6}$ there exists an optimal control $u^* \in \mathcal{N}$ i.e.,

$$J(u^*) \leq J(u), \ u \in \mathcal{N}.$$

$\underline{\text{Proof}}$. From Lemma 3 it follows that $E_0 \exp \zeta(g_u) = 1$ for all $u \in \mathcal{U}$ and furthermore there exists $\alpha > 1$ such that

$$\sup_{u \in} E_0 \exp \alpha\zeta(g_u) < \infty.$$

Hence by [6, Chapter 2], $\mathcal{D}(\mathcal{U})$ is a uniformly integrable subset of $L^1(\Omega, \mathcal{J}, P_0)$. By Theorem 3 $\mathcal{D}(\mathcal{U})$ is convex and strongly closed in $L^1(\Omega, \mathcal{J}, P_0)$. So that by [6, Chapter 2] $\mathcal{D}(\mathcal{U})$ is weakly compact in $L^1(\Omega, \mathcal{J}, P_0)$. Hence the linear functional $\exp \zeta(g_u) \mapsto E_0 L(X)\exp \zeta(g_u)$ attains a minimum.

In a manner corresponding exactly to the argument developed in [2], the results presented above can be used to obtain the existence of a saddle point for a class of two-person zero-sum games. Since there is nothing new here the details are omitted.

Remark on [2]. It is necessary to make the assumption of uniform integrability of the family of densities to obtain the results in Theorems 4 and 5 of [2]. One sufficient condition for this uniform integrability is that the growth of the drift term is at most linear. This fact is shown by Beneš [1] by verifying that the family of densities have a bounded αth moment for some $\alpha > 1$.

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