MARTINGALES ON JUMP PROCESSES I: REPRESENTATION RESULTS

by

R. Boel, P. Varaiya and E. Wong

Memorandum No. ERL-M407

6 September 1973

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

MARTINGALES ON JUMP PROCESSES I: REPRESENTATION RESULTS

R. Boel, P. Varaiya and E. Wong

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Abstract

The paper is a contribution to the theory of martingales of processes whose sample paths are piecewise constant and have finitely many discontinuties in a finite time interval. The assumption is made that the jump times of the underlying process are totally inaccessible and necessary and sufficient conditions are given for this to be true. It turns out that all martingales are then discontinuous, and can be represented as stochastic integrals of certain basic martingales. This representation theorem is used in a companion paper to study various practical problems in communication and control. The results in the two papers constitute a sweeping generalization of recent work on Poisson processes.

Research sponsored by the U.S. Army Research Office - Durham, Contract DAHC 04-67-C-0046 and the National Science Foundation Grant GK-10656X3.

The authors are very grateful to J. M. C. Clark, M. H. A. Davis and J. H. Van Schuppen for many helpful suggestions and discussions.

I. Introduction and Summary

The theory of martingales has proved to be successful as a framework for formulating and analyzing many issues in stochastic control, and in detection and filtering problems [2,4,5,10,11,12,32,33,34]. Three sets of results in the abstract or general theory of martingales seem to be the most useful ones in these applications. The first set consists of the optional sampling theorem and the classical martingale inequalities [17]. The second set consists of the locus of results culminating in the decomposition theorem for supermartingales [24]. The third set includes the calculus of stochastic integrals [16,22] and the differentiation formula and its application to the so-called "exponentiation formula" [15].

In applications one is concerned with martingales which are functionals of a basic underlying process such as a Wiener or Poisson porcess, and in order to use the abstract theory one needs to know how to represent these martingales usefully and explicitly in terms of the underlying process. Thus the "martingale representation theorems" serve as a bridge linking the abstract theory and the concrete applications. Their role is quite analogous to that of matrix representations of linear operators which serve as the instrument with which one can apply the abstract theory of linear algebra.

The most familiar of all the basic processes which can arise in practice is the Wiener process. It is known that every martingale of a Wiener process can be represented as a stochastic integral of the Wiener process [6,22]. This fundamental representation theorem, together with the exponentiation formula, has been used to derive solutions of stochastic differential equations [2,19,20], to obtain recursive equations for filters [5,21,30,31] and the likelihood ratios for some detection problems [10,18], to mention just a few applications. These very results combined with the decomposition theorem for supermartingales form the foundation of an approach to one family of stochastic optimal control problems [12]. It turns out that every martingale of a Wiener process has continuous sample paths. This is fortunate because it implies that the martingale is locally square integrable, and hence most of the questions about martingales can be posed within the Hilbert space structure of the space of square integrable random variables.

However, for many processes, e.g. Poisson process, one can have martingales which are not locally square integrable. As Meyer and his co-workers have pointed out [16,26] the L^2 structure is no longer appropriate and one needs to be more careful in defining stochastic integrals and in obtaining the differentiation formula. Indeed the current theory of stochastic integration with respect to such martingales is still not completely satisfactory.

This paper is a contribution to the abstract theory and to its applications for the relatively simple case where the sample functions of the underlying process are step functions which have only a finite number of jumps in every finite time interval. In some ways this is the polar opposite of the Wiener process case since all the martingales are discontinuous, that is, all the continuous martingales have constant sample paths. The most important special cases covered by this paper include the Poisson process, Markov chains and extensions of these, such as processes arising in queueing theory. To some extent the results for some of these special cases are also covered in [4,5,10,11,29,30,31]

The next section gives a precise definition of the underlying process and exhibits some of the important properties of the generated σ-fields. Conditions are derived which guarantee that the jump times of the process are totally inaccessible stopping times. These preliminary results are used in Section 3 to show first that there are no non-constant continuous martingales and then to obtain an integral representation of all martingales. A particular example, which includes most of the special cases mentioned above, is presented in Section 4. Applications of the results are given in the companion paper [3].

II. The Basic Process and Its Stopping Times

Let $(Z,\mathcal{Z})$ be a Blackwell space, that is a measurable space such that $\mathcal{Z}$ is a separable σ-field and every measurable function $f: Z \to R$ maps Z onto an analytic subset of R (see [24, p. 61]). Let Ω be a family of functions on $R_+ = [0,\infty)$ with values in Z, such that each $\omega \in \Omega$ is a step function with only a finite number of jumps in every finite interval, and such that for all $\omega \in \Omega$, $t \in R_+$, $\omega(t) = \omega(t+\varepsilon)$ for all ε less than some sufficiently small $\varepsilon_0 > 0$. If Z is also a topological space, then each function ω is right-continuous and has left-hand limits. Let x_t be the evaluation process on Ω i.e. $x_t(\omega) = \omega(t)$, $t \in R_+$. Let $\mathcal{F}_t$ be the σ-field on Ω generated by sets of the form $\{x_s \in B\}$, $B \in \mathcal{Z}$, $s \leq t$. Let $\mathcal{F} = \bigvee_{t \in R_+} \mathcal{F}_t$.[1]

Because the positive rationals are dense in R_+, it is clear that $\mathcal{F}$ can also be written as $\bigvee_n \sigma(x_{r_n})$ where $\sigma(x_{r_n})$ is the σ-field generated

[1] If A_α is a family of subsets then $\bigvee_\alpha A_\alpha$ denotes the smallest σ-field containing all the A_α.

by the function x_{r_n} and r_n is rational. Hence the separability of $\mathcal{Z}$ implies the separability of $\mathcal{F}$. Moreover, as will be shown, every real-valued $\mathcal{F}$-measurable function on Ω will map Ω onto an analytic subset, hence $(\Omega,\mathcal{F})$ is a Blackwell space. The assertion follows from considering approximations for any measurable $f: \Omega \to R$ of the form $f^n = g^n \cdot h^n \cdot i$, where $i: (\Omega,\mathcal{F}) \to (Z^{\mathbb{N}}, \mathcal{Z}^{\mathbb{N}})$ is the natural isomorphism ($\mathbb{N}$ is the set of natural numbers), and $h^n: (Z^{\mathbb{N}}, \mathcal{Z}^{\mathbb{N}}) \to (R^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}})$ ($\mathcal{B}$ is the Borel field on R) consists of measurable components $h_1^n, h_2^n, \ldots$ and $h^n(z_1,z_2,\ldots) = (h_1^n(z_1), h_2^n(z_2),\ldots)$, and finally g^n is a measurable mapping from $(R^{\mathbb{N}},\mathcal{B}^{\mathbb{N}})$ into $(R,\mathcal{B})$. Since the Cartesian product of analytic sets is analytic (see [1]), the image of $Z^{\mathbb{N}}$ in $R^{\mathbb{N}}$ under h^n is an analytic set which is in turn mapped into an analytic subset of R by g^n. Since analytic sets form a class closed under countable unions and intersections, this limiting procedure shows that every measurable function $f: \Omega \to R$ maps Ω onto an analytic set. Since $(\Omega,\mathcal{F})$ is a Blackwell space it follows from [24 , II-T16] that $(\Omega,\mathcal{F})$ is isomorphic to $(A,\mathcal{B}(A))$ where A is an analytic subset of R. Hence the results of [28] can be applied without assuming a topological structure on Z itself.

A Z-valued or $R \cup \{\infty\}$-valued function f on Ω is a <u>random variable</u> (r.v.) if $f^{-1}(B) \in \mathcal{F}$ whenever $B \in \mathcal{Z}$ or whenever B is a Borel subset of $R \cup \{\infty\}$. Unless otherwise stated a r.v. is $R \cup \{\infty\}$-valued. A non-negative r.v. T is said to be a <u>stopping time</u> (s.t.) if for every $t \in R_+$, $\{T \leq t\} \in \mathcal{F}_t$. If T is a s.t. then $\mathcal{F}_T$ consists of those sets $A \in \mathcal{F}$ for which $A \cap \{T \leq t\} \in \mathcal{F}_t$ for each $t \in R_+$, whereas $\mathcal{F}_{T-}$ is the σ-field generated by $\mathcal{F}_0$ and sets of the form $A \cap \{t < T\}$ where $A \in \mathcal{F}_t$, and finally $\mathcal{F}_{T+} = \bigcap_{n>0} \mathcal{F}_{T + \frac{1}{n}}$.

Define inductively the functions T_n:

$$T_0 \equiv 0, \; T_{n+1}(\omega) = \inf\{t \mid t \geq T_n(\omega) \text{ and } x_t(\omega) \neq x_{T_n(\omega)}(\omega)\},$$

where the infimum over an empty set is taken to be $+\infty$. The next few results characterize the σ-field $\mathcal{F}_t$ and demonstrate that the T_n are indeed s.t.s. The key results, Corollary 2.2 and Proposition 2.3, which are the only ones used subsequently, can in fact be proven from first principles assuming only the separability of $\mathcal{Z}$, but it is much more intuitive and easier to rely on the results of [7] and [28].

Let H: $\Omega \to [0,\infty]$ be any function. Then H defines three equivalence relations on Ω as follows:

$$\omega \overset{H}{\sim} \omega' \Longleftrightarrow H(\omega) = H(\omega') \text{ and } x_t(\omega) = x_t(\omega') \text{ for } t \leq H(\omega).$$

$$\omega \overset{H+}{\sim} \omega' \Longleftrightarrow H(\omega) = H(\omega') \text{ and there is } \varepsilon > 0 \text{ such that } x_t(\omega) = x_t(\omega') \text{ for } t \leq H(\omega) + \varepsilon$$

$$\omega \overset{H-}{\sim} \omega' \Longleftrightarrow H(\omega) = H(\omega') \text{ and } x_t(\omega) = x_t(\omega') \text{ for } t < H(\omega).$$

A set $A \subset \Omega$ is said to be <u>saturated for H</u>, respectively H_+, H_-, if $\omega \in A$, and $\omega \overset{H}{\sim} \omega'$, respectively $\omega \overset{H}{\sim}+ \omega'$, $\omega \overset{H}{\sim}- \omega'$, implies $\omega' \in A$. Let $\mathcal{S}_H$, $\mathcal{S}_{H_+}$, $\mathcal{S}_{H_-}$ denote the family of subsets of Ω which are saturated for H, H_+, H_- respectively.

<u>Proposition 2.1</u> $\mathcal{F}_t = \mathcal{S}_t \cap \mathcal{F}$, where $\mathcal{S}_t = \mathcal{S}_H$ for $H \equiv t$.

<u>Proof</u> Follows from [28, proposition 1]. ¤

__Corollary 2.1__ A non-negative r.v. T is a s.t. if and only if $\{T \leq t\} \in \mathcal{G}_t$ for all $t \in R_+$.

__Corollary 2.2__ T_n is a s.t. for all n.

__Proof__ T_n is obviously a non-negative r.v. and $\{T_n \leq t\} \in \mathcal{G}_t$ by definition. ¤

__Proposition 2.2__ Let T be a s.t. then

$$\mathcal{F}_T = \mathcal{G}_T \cap \mathcal{F}, \ \mathcal{F}_{T+} = \mathcal{G}_{T+} \cap \mathcal{F}, \ \mathcal{F}_{T-} = \mathcal{G}_{T-} \cap \mathcal{F}.$$

__Proof__ This follows from [28 , Propositions 1, 2]. ¤

For a s.t. T, $\mathcal{F}_\infty (x_{t\wedge T})$ denotes the σ-field generated by the Z-valued r.v.s. $X_{t\wedge T}$, $t \in R_+$. (If S, T are r.v.s. then $S \wedge T = \{\min S,T\}$,)

__Proposition 2.3__ Let T be a s.t. then $\mathcal{F}_T = \mathcal{F}_\infty(x_{t\wedge T})$.

__Proof__ Since $\mathcal{F}_\infty(x_{t\wedge T})$ and $\mathcal{F}_T = \mathcal{G}_T \cap \mathcal{F}$ are sub-σ-fields of $\mathcal{F}$ they are separable. Hence the spaces $(\Omega, \mathcal{F}_\infty(x_{t\wedge T}))$ and $(\Omega, \mathcal{F}_T)$ are Blackwell spaces by [1, Corollary 3]. They also have the same atoms, namely, $\bigcap_n \{x_{r_n \wedge T} \in B_n\}$ where r_n is a rational and B_n an atom of Z. The result then follows from [1, Corollary 1] and Proposition 2.2. ¤

__Corollary 2.3__ $\mathcal{F}_{T_n} = \sigma(x_{T_i}, T_i; 0 \leq i \leq n)$.

__Proof__ Follows from Proposition 2.3 since

$$\mathcal{F}_\infty(x_{t\wedge T_n}) = \sigma(x_{T_i \wedge T_n}, T_i \wedge T_n; 0 \leq i \leq \infty)$$

$$= \sigma(x_{T_i}, T_i; 0 \leq i \leq n)$$ ¤

__Corollary 2.4__ $\mathcal{F}_t = \sigma(x_{T_i \wedge t}, T_i \wedge t, 0 \leq i < \infty)$

__Corollary 2.5__ Let T be a s.t. then $\mathcal{F}_{T_+} = \mathcal{F}_T$

__Proof__ Since the sample functions are piecewise constant and $\omega(t) = \omega(t+)$ it follows that $\mathcal{G}_T = \mathcal{G}_{T_+}$ and then the result follows from Proposition 2.2. ¤

__Proposition 2.4__ $\mathcal{F}_{T_{n-}} = \sigma(x_{T_i}, T_{i+1}, 0 \leq i \leq n-1)$

__Proof__ Similar to the proof of Proposition 2.3, with both σ-fields having the atoms $\{x_{T_i} \in A_i, T_{+1} \in B_i ; 0 \leq i \leq n-1\}$ where A_i is an atom of Z and B_i is an atom of R.

__Proposition 2.5__ Let $n \geq 1$, and $\delta > 0$. Let $T = (T_{n-1} + \delta) \wedge T_n$, and let $A \in \mathcal{F}_T$. Then there exists $A^0 \in \mathcal{F}_{T_{n-1}}$ such that $A \cap \{T < T_n\} = A^0 \cap \{T < T_n\}$.

__Proof__ By Proposition 2.3 $\mathcal{F}_T = \mathcal{F}_\infty(x_{t\wedge T})$ and it is easy to see that the latter coincides with the σ-field generated by the r.v.s. $\{x_{T_i\wedge T}, T_i \wedge T; i = 0,1,2,\ldots\}$. Hence there exists a function g, measurable in its arguments such that

$$I_A(\omega) = g(x_{T_0\wedge T}(\omega), T_0\wedge T(\omega), \ldots, x_{T_{n-1}\wedge T}(\omega), T_{n-1}\wedge T(\omega), x_{T_n\wedge T}(\omega), T_n\wedge T(\omega), \ldots)$$

$$= g(x_{T_0}(\omega), T_0(\omega), \ldots, x_{T_{n-1}}(\omega), T_{n-1}(\omega), x_{T_n\wedge T}(\omega), T_n\wedge T(\omega), \ldots).$$

Define the measurable function g^0 by

$$g^o(x_0,t_0,\ldots,x_{n-1},y_{n-1}) = g(x_0,t_0,\ldots,x_{n-1},t_{n-1},x_{n-1},t_{n-1}+\delta,\ x_{n-1},\ t_{n-1}+\delta,\ \ldots.).$$

Now if $T_{n-1} \le T < T_n$, then $x_{T_{n+k}\wedge T}(\omega) = x_{T_{n-1}}(\omega)$ and

$T_{n+k}\wedge T(\omega) = T_{n-1}(\omega) + \delta$ for all $k \ge 0$. Therefore,

$$I_A(\omega)\, I_{\{T<T_n\}}(\omega) = g^o(x_{T_0}(\omega),\ T_0(\omega),\ \ldots,\ x_{T_{n-1}}(\omega),\ T_{n-1}(\omega))\, I_{\{T<T_n\}}(\omega),$$

So that the set $A^o = \{\omega | g^o(x_{T_0}(\omega),\ \ldots,\ T_{n-1}(\omega)) = 1\}$ satisfies the assertion ¤

<u>Lemma 2.1</u> Let $n \ge 1$, and let S be a s.t. then there exists a r.v.f , measurable with respect to $\mathcal{F}_{T_{n-1}}$ such that $S\, I_{\{S<T_n\}} = f\, I_{\{S<T_n\}}$.

<u>Proof</u> $S\, I_{\{S<T_n\}} = S\, I_{\{S<T_{n-1}\}} + S\, I_{\{T_{n-1}\le S<T_n\}}$, and

$S\, I_{\{S<T_{n-1}\}}$, $I_{\{S<T_{n-1}\}}$ are $\mathcal{F}_{T_{n-1}}$-measurable so that by replacing S by $S \vee T_{n-1}$ if necessary, one can assume that $S \ge T_{n-1}$. Let $\Gamma = \{S < T_n\}$. Then $\Gamma = \bigcup_m \Gamma_m$ where

$$\Gamma_m = \bigcup_k \{S \le T_{n-1} + k2^{-m}\} \cap \{T_{n-1} + k2^{-m} < T_n\}$$

Fix $\delta = 2^{-m}$. By Proposition 2.5 there exist sets $A_k \in \mathcal{F}_{T_{n-1}}$ such that

$$\{S \le T_{n-1} + k\delta\} \cap \{T_{n-1} + k\delta < T_n\} = A_k \cap \{T_{n-1} + k\delta < T_n\},\ k \ge 1.$$

Define sets B_k by

$$B_1 = A_1 \text{ and } B_k = \{\omega \in A_k | \omega \notin A_i \text{ for } i < k\} \text{ for } k > 1,$$

and then define the function $f_m\colon \Omega \to [0,\infty]$ by

$f_m(\omega) = T_{n-1} + k\delta$ if $\omega \in B_k$ and $f_m(\omega) = T_{n-1}(\omega)$ if $\omega \notin \bigcup_k B_k$.

Certainly f_m is $\mathcal{F}_{T_{n-1}}$-measurable. Also

$$f_m(\omega) - \delta \leq S(\omega) \leq f_m(\omega) < T_n(\omega) \text{ for } \omega \in \Gamma_m. \tag{2.1}$$

To see this note first that if $\omega \in A_1 \cup \{T_{n-1} + \delta < T_n\}$ then clearly $T_{n-1}(\omega) = f_m(\omega) - \delta \leq S(\omega) < f_m(\omega) < T_n(\omega)$. Next, as induction hypothesis, suppose that the inequalities in (2.1) hold for

$\omega \in \bigcup_{k=1}^{N} A_k \cap \{T_{n-1} + k\delta < T_n\}$, and let

$$\omega \in A_{N+1} \cap \{T_{n-1} + (N+1)\,\delta < T_n\},\ \omega \notin \bigcup_{k=1}^{N} A_k \cap \{T_{n-1} + k\,\delta < T_n\}. \tag{2.2}$$

Let $k \leq N + 1$ be the smallest integer such that $\omega \in B_k$. Suppose $k \leq N$. Then, since $B_k \subset A_k$, and since from (2.2) $T_n > T_{n-1} + k\delta$, it follows that $\omega \in A_k \cap \{T_{n-1} + k\delta < T_n\}$ which contradicts the second condition of (2.2). Hence $\omega \in B_{N+1}$ and so $T_{n-1}(\omega) + N\delta \leq S(\omega) \leq T_{n-1}(\omega) + (N+1)\delta = f_m(\omega) < T_n(\omega)$. Therefore (2.1) holds by induction. Finally, define the $\mathcal{F}_{T_{n-1}}$-measurable function f by $f(\omega) = \liminf_m f_m(\omega)$. The obvious inclusion $\Gamma_m \subset \Gamma_{m+1}$ implies that if $\omega \in \Gamma_m$ then $f_{m+k}(\omega) - 2^{-(m+k)} \leq S(\omega) \leq f_{m+k}(\omega)$ for all $k \geq 0$. Hence $f(\omega) = S(\omega)$ and the assertion is proved. ¤

To proceed further it is convenient to introduce a probability measure on $(\Omega, \mathcal{F})$[2]. <u>Throughout this paper let P denote a fixed</u>

[2]It may be of interest to note that Lemmas 2.2, 2.3 and 2.4 below can be proven without imposing a probability measure P by using the algebraic definition of a predictable s.t. of [28]. Then a predictable s.t. in the sense used here is simply a non-negative r.v. which is a.s. P equal to a predictable s.t. in the sense of [28].

probability measure on $(\Omega,\mathcal{F})$. Recall the following important classification of stopping times [25].

Let T be a s.t. T is said to be totally inaccessible if $T > 0$ a.s. and if for every increasing sequence of s.t.s. $S_1 \leq S_2 \leq \ldots,$

$$P\{S_k(\omega) < T(\omega) \text{ for all } k \text{ and } \lim_{k\to\infty} S_k(\omega) = T(\omega) < \infty\} = 0;$$

whereas T is said to be predictable if there exists an increasing sequence of s.t.s $S_1 \leq S_2 \leq \ldots$ such that

$$P\{T = 0, \text{ or } S_k < T \text{ for all } k \text{ and } \lim_{k\to\infty} S_k = T\} = 1.$$

The next three lemmas relate this classification to the properties of the jump times T_n of the process x.

Lemma 2.2 Let T be a totally inaccessible s.t. Then

$$T\, 1_{\{T < \infty\}} = [\sum_{n=1}^{\infty} T_n\, I_{\{T = T_n\}}]\, I_{\{T < \infty\}} \text{ a.s.}$$

Proof The equality above holds if and only if $P\{T_{n-1} < T < T_n\} = 0$ for each $n \geq 1$. Let n be fixed. By Lemma 2.1 there exists a $\mathcal{F}_{T_{n-1}}$-measurable function f such that $f(\omega) = T(\omega)$ for $\omega \in \{T_{n-1} < T < T_n\}$. Let $S_k = T_{n-1} \vee (f - \frac{1}{k})$. Then $S_k \geq T_{n-1}$ and S_k is $\mathcal{F}_{T_{n-1}}$-measurable so that it is a s.t. Also S_k is increasing and clearly

$$\{T_{n-1} < T < T_n\} \subset \{S_k < T \text{ for all } k \text{ and } \lim_{k\to\infty} S_k = T < \infty\}.$$

Since T is totally inaccessible, the set on the right has probability measure zero. The assertion is proved. ¤

Lemma 2.3 Let T be a s.t. such that for all $n \geq 1$, $P\{T = T_n < \infty\} = 0$. Then T is predictable.

Proof Let h be a function measurable in its arguments and taking values in the set $\{0,1\}$ such that the process $I_{T \leq t}$ has the representation

$$I_{T \leq t} = h(t, x_{T_0 \wedge t}, T_0 \wedge t, \ldots, x_{T_n \wedge t}, T_n \wedge t, \ldots).$$

By modifying h if necessary it can be assumed that

$$h(t,\xi) = \max_{s \leq t} h(s,\xi).$$

Because of this property the r.v. T_ε defined by

$$T_\varepsilon(\omega) = \inf\{t \mid h(t+\varepsilon, x_{T_0 \wedge t}, T_0 \wedge t, \ldots\ldots) = 1\}$$

in a s.t., and it is immediate that for $\varepsilon > 0$

$$T_\varepsilon(\omega) < T(\omega) \text{ for } \omega \in \{0 < T < \infty\}.$$

Furthermore $T_\varepsilon \leq T_{\varepsilon'}$ if $\varepsilon' \leq \varepsilon$. Define then s.t.s S_k by $S_k = T_{\frac{1}{k}} \wedge k$.

It will now be shown that

$$\lim_{k \to \infty} S_k(\omega) = T(\omega) \text{ for } \omega \in \bigcup_{n=1}^{\infty} \{T_{n-1} < T < T_n\}.$$

Let $\omega \in \{T_{n-1} < T < T_n\}$. Then

$$h(t, x_{T_0 \wedge t}(\omega), T_0 \wedge t(\omega), \ldots, x_{T_n \wedge t}(\omega), T_n \wedge t(\omega) \ldots) = \begin{cases} 0 \text{ for } T_{n-1}(\omega) < t < T(\omega) \\ 1 \text{ for } t \geq T(\omega) \end{cases}$$

so that

$$h(t + \tfrac{1}{k}, x_{T_0 \wedge t}(\omega), T_0 \wedge t(\omega), \ldots) = \begin{cases} 0 \text{ for } T_{n-1}(\omega) < t + \frac{1}{k} < T(\omega) \text{ or } T_{n-1}(\omega) < t < T(\omega) \\ 1 \text{ for } t \geq T(\omega) \end{cases}$$

Hence $T_{\frac{1}{k}}(\omega) = T(\omega) - \frac{1}{k}$ for $\frac{1}{k} < T(\omega) - T_{n-1}(\omega)$. It follows that $S_k(\omega)$ converges to $T(\omega)$ and the assertion follows. ¤

<u>Lemma 2.4</u> T_n is totally inaccessible if and only if for every $\mathcal{F}_{T_{n-1}}$-measurable function f, $P\{T_n = f < \infty\} = 0$.

<u>Proof</u> Suppose $P\{T_n = f < \infty\} > 0$. Let $S_k = T_{n-1} \vee (f - \frac{1}{k})$. Then S_k is an increasing sequence of s.t.s and

$$\{T_n = f < \infty\} \subset \{S_k < T \text{ for all } k \text{ and } \lim_{k\to\infty} S_k = T < \infty\}$$

so that T_n cannot be totally inaccessible thereby proving necessity. To prove sufficiency suppose that T_n is not totally inaccessible so that there is an increasing sequence of s.t.s S_k such that

$$P\{\Gamma\} = P\{S_k < T_n \text{ for all } k \text{ and } \lim_{k\to\infty} S_k = T_n < \infty\} > 0. \tag{2.3}$$

By Lemma 2.1 there exist functions f_k, measurable with respect to $\mathcal{F}_{T_{n-1}}$, such that $S_k(\omega) = f_k(\omega)$ for $\omega \in \{S_k < T_n\}$. Let $f = \liminf f_k$. Then from (2.3) it follows that $f(\omega) = T_n(\omega)$ for $\omega \in \Gamma$ so that $P\{f = T_n < \infty\} > 0$ and sufficiency is proved. ¤

From the lemma above the following intuitive sufficient condition follows immediately.

<u>Theorem 2.1</u> Let $F(t_n|x_0,t_0,\ldots,x_{n-1},t_{n-1})$ be the conditional probability distribution of T_n given $x_{T_0},T_0,\ \ldots,\ x_{T_{n-1}},T_{n-1}$. Suppose that F is continuous in t_n for all values of $(x_0,t_0,\ldots,x_{n-1},t_{n-1})$. Then T_n is totally inaccessible.[3]

---

[3]If Z is a Borel subset of $\mathcal{R}^p$ and $\mathcal{Z}$ contains all Borel subsets of Z then the conditional probability F exists by [23,p.361].

As an application of Theorem 2.1 note that if x_t is a Poisson process, then $F(t_n | x_0, t_0, \ldots, x_{n-1}, t_{n-1}) = (1 - \exp - (t_n - t_{n-1})) I_{t_n \geq t_{n-1}}$ is continuous. Hence the jump times of a Poisson process are totally inaccessible.

III. The Martingale Representation Theorem

It will be necessary from now on to complete the σ-fields $\mathcal{F}_t$ and $\mathcal{F}$ with respect to the measure P. An additional condition is also imposed.

Assumptions (i) The σ-fields $\mathcal{F}_t$, $\mathcal{F}$ are augmented so as to be complete with respect to P. (ii) The stopping times T_n are totally inaccessible for $n \geq 1$.

Note that after completion of the space $(\Omega, \mathcal{F})$ it ceases to be a Blackwell space. But, of course, the results of Section II continue to hold if the relevant equalities are interpreted as being true almost surely P.

The family $\mathcal{F}_t$ is said to be free of times of discontinuity if for every increasing sequence of s.t.s S_k, $\mathcal{F}_{\lim S_k} = \bigvee_k \mathcal{F}_{S_k}$.

Proposition 3.1 The family $\mathcal{F}_t$ is free of times of discontinuity.

Proof By Lemma 2.2 and Assumption (ii) a s.t. T is totally inaccessible if and only if its graph[4] [T] is contained in the union $\bigcup_n [T_n]$ of the graphs of T_n, whereas by Lemma 2.3 T is predictable if $[T] \cap \bigcup_n [T_n] = \phi$. The assertion follows from [14,III-T51,p. 62]. ¤

It will be useful to recall some definitions at this time. This will

[4] $[T] = \{(\omega, T(\omega)) | \omega \in \Omega\} \subset \Omega \times [0,\infty]$.

be followed by some remarks and a reproduction of some known results which will be used in the discussion to follow.

A process y_t is said to be <u>adapted</u> (to the family $\mathcal{F}_t$) if y_t is $\mathcal{F}_t$-measurable for all t. Two processes y_t and y_t' are said to be <u>indistinguishable</u>, and are written $y_t \equiv y_t'$, if for almost all ω $y_t(\omega) = y_t'(\omega)$ for all $t \in R_+$.

Let m_t be a martingale with respect to $(\Omega, \mathcal{F}_t, P)$. It is said to be <u>uniformly integrable</u> (u.i.), and one writes $m_t \in \mathcal{M}^1$, if $\{m_t | t \in R_+\}$ is a u.i. set of r.v.s. It is said to be <u>square integrable</u> (s.i.), and one writes $m_t \in \mathcal{M}^2$, if $\sup\{Em_t^2 | t \in R_+\} < \infty$.

Let m_t be a process. It is said to be a <u>locally integrable martingale</u> [locally square integrable martingale], and one writes $m_t \in \mathcal{M}^1_{loc}$ [$m_t \in \mathcal{M}^2_{loc}$], if there is an increasing sequence of s.t.s S_k with $S_k \to \infty$ a.s. such that for each k $m_{t\wedge S_k} I_{\{S_k > 0\}} \in \mathcal{M}^1$[$m_{t\wedge S_k} I_{\{S_k > 0\}} \in \mathcal{M}^2$].

An adapted process a_t is said to be an <u>increasing process</u> if $a_0 = 0$ and if its sample paths are non-decreasing and right continuous. It is said to be <u>integrable</u>, and one writes $a_t \in \mathcal{A}^+$ if $\sup\{Ea_t | t \in R_+\} < \infty$. $\mathcal{A}^+_{loc}$ is defined in a manner analogous to the previous definition. Finally let $\mathcal{A} = \mathcal{A}^+ - \mathcal{A}^+ = \{a_t - a_t' | a_t \in \mathcal{A}^+,\ a_t' \in \mathcal{A}^+\}$ and $\mathcal{A}_{loc} = \mathcal{A}^+_{loc} - \mathcal{A}^+_{loc}$.

It will be assumed throughout that all the local martingales have sample paths which are right-continuous and have left-hand limits. It is known that since the σ-fields $\mathcal{F}_t$ are complete and since by Corollary 2.5 $\mathcal{F}_{t+} = \mathcal{F}_t$ for all $t \in R_+$ therefore one can always choose a modification of a local martingale so that its sample paths have the above mentioned property [see 24, VI-T4]. Two modifications with this property are indistinguishable.

It can be immediately verified that $\mathcal{M}^2 \subset \mathcal{M}^1$ and so $\mathcal{M}^2_{loc} \subset \mathcal{M}^1_{loc}$, and if $m_t \in \mathcal{M}^1$ has continuous sample paths then $m_t \in \mathcal{M}^2_{loc}$. However if the sample paths of $m_t \in \mathcal{M}^1$ are not continuous then m_t may not belong to $\mathcal{M}^2_{loc}$. Thus in dealing with discontinuous martingales one may be unable to use the Hilbert space structure of square integrable r.v.s.

The next result follows from Proposition 3.1 and [22, Theorem 1.1].

Theorem 3.1 Let m_t and m'_t be in $\mathcal{M}^2_{loc}$. Then there exists a unique[5], continuous process $\langle m,m'\rangle_t \in \mathcal{A}$ such that $m_t m'_t - \langle m,m'\rangle_t \in \mathcal{M}^1_{loc}$.

Definition 3.1 Let $B \in \mathcal{Z}$. Let

$$P(B,t) = \sum_{s \le t} I_{\{x_{s-} \neq x_s\}} I_{\{x_s \in B\}}$$

be the number of jumps of x which occur prior to t and which end in the set B.

Proposition 3.2 There is a unique continuous process $\tilde{P}(B,t) \in \mathcal{A}^+_{loc}$ such that the process $Q(B,t) = P(B,t) - \tilde{P}(B,t)$ is in $\mathcal{M}^2_{loc}$.

Proof Let $P_n(B,t) = P(B,t \wedge T_n)$. Then $P_n(B,t) \le n$ so that it is square integrable. Furthermore the jumps of $P_n(B,t)$ occur at the s.t.s T_n, $1 \le i \le n$, and these s.t.s are totally inaccessible by assumption. It follows from [24, VIII-T31, p. 210] that there is a unique, continuous, integrable, increasing process $\tilde{P}_n(B,t)$ such that

[5] Throughout "unique" means unique up to modification.

$Q_n(B,t) = P_n(B,t) - \tilde{P}_n(B,t) \in \mathcal{M}^2$. From this last relation and the uniqueness of $\tilde{P}_n$ one can conclude that $\tilde{P}_{n+1}(B,t\wedge T_n) \equiv \tilde{P}_n(B,t)$, $Q_{n+1}(B,t\wedge T_n) \equiv Q_n(B,t)$. Hence the processes $\tilde{P},Q$ defined by

$$\tilde{P}(B,t\wedge T_n) \equiv \tilde{P}_n(B,t),\ Q(B,t\wedge T_n) \equiv Q_n(B,t)$$

satisfy the assertion ¤

Two processes m_t, m'_t in $\mathcal{M}^2_{loc}$ are said to be <u>orthogonal</u> if $m_t m'_t \in \mathcal{M}^1_{loc}$ or equivalently if $\langle m,m'\rangle_t \equiv 0$.

<u>Lemma 3.1</u> Let $B_i \in \mathcal{Z}$, $i = 1, 2$. Then $Q(B_1,t)\ Q(B_2,t) - \tilde{P}(B_1\cap B_2,t) \in \mathcal{M}^1_{loc}$ i.e., $\langle Q(B_1,\cdot),\ Q(B_2,\cdot)\rangle_t \equiv \tilde{P}(B_1\cap B_2,t)$. In particular $Q(B_1,t)$ and $Q(B_2,t)$ are orthogonal if $B_1 \cap B_2 = \phi$.

<u>Proof</u> $Q(B_1,t\wedge T_n) = Q(B_1\cap B_2,t\wedge T_n) + Q(B_1-B_2,t\wedge T_n)$ and $Q(B_2,t\wedge T_n) = Q(B_1\cap B_2,t\wedge T_n) + Q(B_2-B_1,t\wedge T_n)$ where $B - B' = \{z|z \in B,\ z \notin B'\}$. The s.i. martingales $Q(B_1\cap B_2,t\wedge T_n)$, $Q(B_1-B_2,t\wedge T_n)$ and $Q(B_2-B_1,t\wedge T_n)$ have no discontinuities in common so that they are pairwise orthogonal by [24, VIII-T31, p. 210]. The assertion follows then if one can show that for any $B \in \mathcal{Z}$

$$Q^2(B,t\ T_n) - \tilde{P}(B,t\ T_n) \in \mathcal{M}^1. \tag{3.1}$$

Let $Q(t) = Q(B,t\wedge T_n)$, $P(t) = P(B,t\wedge T_n)$ and $\tilde{P}(t) = \tilde{P}(B,t\wedge T_n)$. Let $\varepsilon > 0$ and $s < t$ be arbitrary. Let $S_0 \leq S_1 \leq S_2 \leq \ldots$ be a sequence of s.t.s such that $S_0 \equiv s$, $\lim_{k\to\infty} S_k = t$ a.s. and such that $0 \leq \tilde{P}(S_k) - \tilde{P}(S_{k-1}) \leq \varepsilon$ a.s. Such a sequence exists since $\tilde{P}$ is continuous. Then

$$\sum_{k=1}^{\infty} (Q(S_k) - Q(S_{k-1}))^2 = \sum_{k=1}^{\infty} (P(S_k) - P(S_{k-1}) - \tilde{P}(S_k) + \tilde{P}(S_{k-1}))^2$$

$$= \sum_{k=1}^{\infty} (P(S_k) - P(S_{k-1}))^2 - 2\sum_{k=1}^{\infty} (P(S_k) - P(S_{k-1}))(\tilde{P}(S_k) - \tilde{P}(S_{k-1})) +$$

$$\sum_{k=1}^{\infty} (\tilde{P}(S_k) - \tilde{P}(S_{k-1}))^2.$$

The first term in the last expression is equal to $P(t) - P(s)$ so that

$$|E\{\sum_{k=1}^{\infty} (Q(S_k) - Q(S_{k-1}))^2 - (P(t) - P(s))|\mathcal{F}_s\}|$$

$$\leq 2\varepsilon\, E\{P(t) - P(s)|\mathcal{F}_s\} + \varepsilon\, E\{\tilde{P}(t) - \tilde{P}(s)|\mathcal{F}_s\}.$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$E\sum_{k=1}^{\infty} (Q(S_k) - Q(S_{k-1}))^2 - P(t) - P(s))|\mathcal{F}_s\} = 0 \tag{3.2}$$

Now $Q_t \in \mathcal{M}^2$ so that $E\{(Q(S_k) - Q(S_{k-1}))^2|\mathcal{F}_s\} = E\{Q^2(S_k) - Q^2(S_{k-1})|\mathcal{F}_s\}$.

Also

$P_t - \tilde{P}_t \in \mathcal{M}^1$ so that $E\{P(t) - P(s)|\mathcal{F}_s\} = E\{\tilde{P}(t) - \tilde{P}(s)|\mathcal{F}_s\}$.

Substituting these relations in (3.2) one obtains

$$E\{\sum_{k=1}^{\infty} (Q^2(S_k) - Q^2(S_{k-1})) - (\tilde{P}(t) - \tilde{P}(s))|\mathcal{F}_s\} = E\{Q^2(t) - Q^2(s)$$

$$- (\tilde{P}(t) - \tilde{P}(s))|\mathcal{F}_s\} = 0.$$

which is the same as (3.1). ¤

For fixed t $Q(B,t)$, $P(B,t)$ and $\tilde{P}(B,t)$ can be regarded as set functions on $\mathcal{J}$. In order to define stochastic integrals and Lebesgue-Stieltjes integrals with respect to these set functions it is necessary to show that they are countably additive.

<u>Lemma 3.2</u> Let B_k, $k \geq 1$, be a decreasing sequence in $\mathcal{J}$ such that $\bigcap_k B_k = \phi$. Then for almost all $\omega \in \Omega$, $Q(B_k,t) \to 0$, $P(B_k,t) \to 0$, $\tilde{P}(B_k,t) \to 0$ for all $t \in R_+$ as $k \to \infty$. Furthermore for all $t \in R_+$ and $n \geq 0$. $E\,Q^2(B_k\ ,t\wedge T_n) \to 0$ as $k \to \infty$.

<u>Proof</u> Fix $t \in R_+$. The non-negative r.v.s $P(B_k,t)$ and $\tilde{P}(B_k,t)$ decrease as k increases so that they converge to some r.v.s $P(t)$ and $\tilde{P}(t)$ respectively. Hence $Q(B_k,t) = P(B_k,t) - \tilde{P}(B_k,t)$ converges to $Q(t) = P(t) - \tilde{P}(t)$. From the definition of $P(B_k,t)$ it is clear that $P(t) = 0$ a.s. and from Lemma 3.1 it follows that $Q_t \in \mathcal{M}^2_{loc}$. Thus $Q(t) = -\tilde{P}(t) \in \mathcal{M}^2_{loc}$. But $\tilde{P}(t)$ is an increasing process and $\tilde{P}(0) = 0$ so that this is possible only if $Q(t) = -\tilde{P}(t) = 0$ a.s. Thus $P(t) = \tilde{P}(t) = Q(t) = 0$ for ω not belonging to a null set $N \in \mathcal{F}$. The monotonicity of the sample functions of P, $\tilde{P}$ implies that $P(s) = \tilde{P}(s) = 0$, hence $Q(s) = 0$ for $\omega \notin N$ and $s \leq t$. To prove the remaining assertion it is enough to note that by Lemma 3.1 and by what has just been shown

$E\,Q^2(B_k,t\wedge T_n) = E\,\tilde{P}(B_k,t\wedge T_n) \to 0$ as $k \to \infty$. ¤

The following definition relates to the different classes of integrands for which a satisfactory theory of integration is available.

Let $\mathcal{H}$ denote the set of all processes $h(t) = h(\omega,t)$ of the form

$$h(t) = h_0\, I_{(t_0,t_1]} + h_1\, I_{(t_1,t_2]} + \dots + h_k\, I_{(t_k,t_{k+1}]}$$

where h_i is a bounded r.v. measurable with respect to $\mathcal{F}_{t_i}$ and $0 \leq t_0 \leq \ldots \leq t_{k+1} < \infty$. Let $\mathcal{P}_0$ denote the set of all functions $f(z,t) = f(z,\omega,t)$ of the form

$$f(z,\omega,t) = \sum_{i=0}^{k} \phi_i(z)\ h_i(\omega,t)$$

where ϕ_i is a bounded function measurable with respect to $\mathcal{Z}$ and $h_i \in \mathcal{H}$.

<u>Definition 3.2</u> A function $f(z,t) = f(z,\omega,t)$ is said to be <u>predictable</u> if there exists a sequence f_k in $\mathcal{P}_0$ such that

$$\lim_{k\to\infty} f_k(z,\omega,t) = f(z,\omega,t) \text{ for all } (z,\omega,t) \in \mathcal{Z} \times \Omega \times R_+.$$

Let $\mathcal{P}$ denote the set of all predictable functions and let $\mathcal{F}^P$ be the sub-σ-field of $\mathcal{Z} \otimes \mathcal{F} \otimes \mathcal{B}$ generated by $\mathcal{P}$.

If $f(z,t) = f(z,\omega,t)$ is measurable with respect to $\mathcal{Z} \otimes \mathcal{F} \otimes \mathcal{B}$ and if for all fixed (z,ω) $f(z,\omega,t)$ is left-continuous in t then $f \in \mathcal{P}$.

<u>Definition 3.3</u> $L^2(\tilde{P}) = \{f \in \mathcal{P} \mid (\|f\|_2^{\sim})^2 = E \int_Z \int_{R^+} f^2(z,t)\ \tilde{P}(dz,dt) < \infty\}$.
$L^1(\tilde{P}) = \{f \in \mathcal{P} \mid \|f\|_1^{\sim} = E \int_Z \int_{R^+} |f(z,t)| \tilde{P}(dz,dt) < \infty\}$. Similarly
$L^1(P) = \{f \in \mathcal{P} \mid \|f\|_1 = E \int_Z \int_{R^+} |f(z,t)| P(dz,dt) < \infty\}$. $L^2_{loc}(\tilde{P})$ is the set of all $f \in \mathcal{P}$ for which there exists a sequence of s.t.s $S_k \uparrow \infty$ a.s. such that $f\, I_{t \leq S_k} \in L^2(\tilde{P})$ for all k. $L^1_{loc}(\tilde{P})$ and $L^1_{loc}(P)$ are defined in an analogous manner. The integrals in this definition are to be interpreted as Lebesgue-Stieltjes integrals. Finally let $L^1(Q) = L^1(P) \cap L^1(\tilde{P})$, $L^1_{loc}(Q) = L^1_{loc}(P) \cap L^1_{loc}(\tilde{P})$. If $f(z,t) \in L^1(Q)$ then the integral

$\int_Z \int_{R^+} f(z,t)\, P(dz,dt) - \int_Z \int_{R^+} f(z,t)\, \tilde{P}(dz,dt)$ is denoted $\int_Z \int_{R^+} f(z,t)$ $Q(dz,dt)$.

Lemma 3.3 To each $f \in L^2(\tilde{P})$ there corresponds a unique process $(f\circ Q)_t \in \mathcal{M}^2$, called the stochastic integral of f with respect to Q with the following properties:

(i) if $f(z,\omega,t) = I_B(z)\, I_A(\omega)\, I_{(t_0,t_1]}(t) \in L^2(\tilde{P})$ where $B \in \mathcal{Z}$ and $A \in \mathcal{F}_{t_0}$, then

$$(f\circ Q)_t = \begin{cases} I_A(\omega)\, [Q(B,t\wedge t_1) - Q(B,t\wedge t_0)] \text{ for } t > t_0 \\ 0 \text{ for } t \le t_0. \end{cases}$$

(ii) if f, g are in $L^2(\tilde{P})$ and α, β are in R, then

$$(\alpha f + \beta g) \circ Q \equiv \alpha(f\circ Q) + \beta(g\circ Q).$$

Furthermore the stochastic integral satisfies the following relations

$$\langle f\circ Q, g\circ Q\rangle_t = \int_Z \int_{R^+} f(z,s)\, g(z,s)\, I_{(0,t]}(s)\, \tilde{P}(dz,ds), \tag{3.3}$$

and in particular

$$E(f\circ Q)_\infty^2 = (\|f\|_2^{\sim})^2 \tag{3.4}$$

Proof The proof follows quite closely that of [22, Proposition 5.1].

Let $f^j = \sum_{i=0}^{k} \alpha_i^j\, I_{B_i^j}(z)\, I_{A_i^j}(\omega)\, I_{(t_i,t_{i+1}]}(t)$, j = 1, 2, be simple functions in $L^2(\tilde{P})$ with $B_i^j \in \mathcal{Z}$, $A_i^j \in \mathcal{F}_{t_i}$ and $0 = t_0 < t_1 < \dots < t_{k+1} < \infty$.

Then from (i), (ii) and Lemma 3.1 it can be verified directly that

$$(f^1 oQ)_t (f^2 oQ)_t - \int_Z \int_{R^+} f^1(z,s)\ f^2(z,s)\ I_{(0,t]}(s)\ \tilde{P}(dz,ds) \in \mathcal{M}^1$$

so that (3.3) and (3.4) hold for all simple functions in $L^2(\tilde{P})$. Since such simple functions are dense in $L^2(\tilde{P})$ (3.4) implies that there is a unique extension of the map $f \to (foQ)$ to all of $L^2(\tilde{P})$. Evidently (3.3) and (3.4) will hold for the extension. ◻

Lemma 3.4 Let $m_t \in \mathcal{M}^2$ have continuous sample paths. Then $m_t \equiv m_0$.

Proof By replacing the martingale m_t by $m_t - m_0$ it can be assumed that $m_0 = 0$. It will be shown that $m_t \equiv 0$. Suppose $m_{T_{n-1}} = 0$ for some $n \geq 1$ so that in fact $m_{t\wedge T_{n-1}} = E\{m_{T_{n-1}} | \mathcal{F}_{t\wedge T_{n-1}}\} = 0$ for all t, and consider the continuous martingale $\mu_t = m_{t\wedge T_n}$. By Corollary 2.2 there exists a function h, measurable in its arguments, such that $\mu_t \equiv h(t, x_{T_0\wedge t}, T_0\wedge t, \ldots, x_{T_n\wedge t}, T_n\wedge t)$. The process $\mu'_t = h(t, x_{T_0\wedge t}, \ldots, x_{T_{n-1}\wedge t}, T_{n-1}\wedge t, x_{T_{n-1}\wedge t}, t)$ is then measurable with respect to $\mathcal{F}_{T_{n-1}}$. Since $x_{T_n\wedge t} = x_{T_{n-1}\wedge t}$ and $t = T_n\wedge t$ for $t < T_n$ it follows that $\mu_t = \mu'_t$ for $t < T_n$ and so by continuity of μ_t, $\mu_t = \mu'_t$ for $t \leq T_n$. For $\alpha \in R_+$ define S_α by

$$S_\alpha(\omega) = \sup\{s \leq \alpha |\ \mu'_s(\omega) \geq 0\}.$$

Then since $\mu'_s = \mu_s = 0$ for $s \leq T_{n-1}$ it follows that $S_\alpha \geq T_{n-1}$ and since S_α is measurable with respect to $\mathcal{F}_{T_{n-1}}$ therefore S_α is a s.t. for every α. Now let

$$T_\alpha(\omega) = \sup\{s \leq \alpha\wedge T_n(\omega) |\ \mu'_s(\omega) \geq 0\}.$$

It will be shown that T_α is a s.t. Fix t: If $\alpha \le t$ then $\{T_\alpha \le t\} = \Omega \in \mathcal{F}_t$ since $T_\alpha \le \alpha$. Suppose then that $\alpha > t$. Now

$$\{T_\alpha \le t\} = (\{T_\alpha \le t\} \cap \{T_n \le t\}) \cup (\{T_\alpha \le t\} \cap \{T_n > t\}). \tag{3.5}$$

Since $T_\alpha \le T_n$ therefore $\{T_n \le t\} \subset \{T_\alpha \le t\}$ so that the first set on the right in (3.5) is equal to $\{T_n \le t\}$ which is in $\mathcal{F}_t$ since T_n is a s.t. It will be shown now that

$$\{T_\alpha \le t\} \cap \{T_n > t\} = \{S_\alpha \le t\} \cap \{T_n > t\} \tag{3.6}$$

Since $S_\alpha \ge T_\alpha$ the set on the right is at least as large as the one on the left. Suppose $\omega \in \{S_\alpha \le t\} \cap \{T_n > t\}$. Then $\mu'_s(\omega) < 0$ for $s \in [t,\alpha]$ and $t < T_n(\omega)$ so that $T_\alpha(\omega) \le t$ which proves (3.6).

Thus $\{T_\alpha | \alpha \in R_+\}$ is a family of s.t.s. and furthermore the sample paths $T_\alpha(\omega)$ are non-decreasing functions of α. By the Optional Sampling Theorem [17, Theorem 11.8, p. 376] the process $\eta_\alpha(\omega) = \mu_{T_\alpha(\omega)}(\omega)$, $\alpha \in R_+$, is a martingale. But $\eta_0 = 0$ and $\eta_\alpha \geqq 0$ so that one must have $\eta_\alpha \equiv 0$. In turn this can happen only if $\mu_t \leqq 0$ which together with $\mu_0 = 0$ implies $\mu_t \equiv 0$. The lemma is proved. ¤

<u>Theorem 3.2</u> Let $m_t \in \mathcal{M}^1_{loc}$ have continuous sample paths. Then $m_t \equiv m_0$.

<u>Proof</u> The s.t.s. $S_k(\omega) = \inf\{t \,|\, |m_t(\omega)| > k\}$ converge to ∞ and $m_{t\wedge S_k} I_{\{S_k > 0\}} \in \mathcal{M}_2$ so that by Lemma 3.4 $m_{t\wedge S_k} \equiv m_0$.

Thus there are no non-trivial continuous martingales. On the other hand if m_t is a martingale then its discontinuities occur at the jump times T_n of the process x_t as shown below.

Lemma 3.5 Let S be a predictable s.t. and let $m_t \in \mathcal{M}^2$. Then $\Delta m_S = m_S - m_{S-} = 0$ a.s.

Proof By [24, VIII-T29, p. 209] the process $\Delta m_S I_{t \geq s}$ is a martingale. By [25, Prop. 7, p. 159] $E\{\Delta M_S | \mathcal{F}_{S-}\} = 0$ a.s. But by Proposition 3.1 and [14, III-T51, p. 62] $\mathcal{F}_{S-} = \mathcal{F}_S$ so that $\Delta M_S = 0$ a.s. ◻

The next result gives the first martingale representation theorem. It should be compared with [22, Thm. 4.2 and Prop. 5.2].

Theorem 3.3 Let $m_t \in \mathcal{M}^2$. Then $m_t - m_0 \in \{f o Q | f \in L^2(\tilde{P})\}$.

Proof It can be assumed without losing generality that $m_0 = 0$. The space $\mathcal{M}_0^2 = \{m_t \in \mathcal{M}^2 | m_0 = 0\}$ is a Hilbert space under the norm $\|m\|^2 = E\, m_\infty^2$ by [16, Thm. 1], and by Lemma 3.3 the set $\mathcal{N} = \{f o Q | f \in L^2(\tilde{P})\}$ is a closed linear subspace of $\mathcal{M}_0^2$. Furthermore $\mathcal{N}$ is closed under stopping i.e., if $(f o Q)_t \in \mathcal{N}$ and T is a s.t. then $(f o Q)_{t\wedge T} \in \mathcal{N}$. This is clear because $(f o Q)_{t \wedge T} = (f_T o Q)_t$ where $f_T(t) = f_t\, I_{\{t \leq T\}}$. Thus by [27, Thm. 2 and the remark following Definition 4] the theorem is proved if it can be shown that $m_t \equiv 0$ when it is orthogonal to foQ for every $f \in L^2(\tilde{P})$. By [16, Thm. 4] m_t can be decomposed uniquely as

$$m_t = m_t^c + m_t^d$$

where $m_t^c \in \mathcal{M}_0^2$ is continuous and $m_t^d \in \mathcal{M}_0^2$ is orthogonal to every continuous martingale. By Theorem 3.2 $m_t^c \equiv 0$. By Lemmas 2.2 and 3.5 the discontinuities of m_t^d occur during the stopping times $T_n, n \geq 1$. Therefore, by [16, Thm. 4] again, $m_t = m_t^d$ can be further decomposed as

$$m_t = \sum_{n=1}^{\infty} (M_n I_{t \geq T_n} - a_n(t)) = \sum_{n=1}^{\infty} \mu_{nt} \text{ say,}$$

where $M_n = \Delta m_{T_n} = m_{T_n} - m_{T_n^-}$, $a_n(t) \in \mathcal{A}$ has continuous sample paths, and $\mu_{nt} \in \mathcal{M}_0^2$. Furthermore the martingale μ_{nt} is orthogonal to every martingale which has no discontinuities at T_n.

To prove that $m_t \equiv 0$ it suffices to show that $M_n = 0$ for each n. Fix n and suppose that $P\{M_n \neq 0\} > 0$. Since M_n is measurable with respect to $\mathcal{F}_{T_n}$ therefore by Corollary 2.2 there must exist sets $A \in \mathcal{F}_{T_{n-1}}$, $B \in \mathcal{Z}$, and $C \in \mathcal{B}[0,\infty)$ such that

$$E\{M_n(\omega)\, I_A(\omega)\, I_{\{x_{T_n} \in B\}}\, I_{\{T_n \in C\}}\} \neq 0. \tag{3.6}$$

Consider the function $f(z,\omega,t)$ defined by

$$f(z,\omega,t) = I_B(z)\, I_A(\omega)\, I_C(t)\, I_{\{T_{n-1} < t \leq t_n\}}$$

The function $g(z,\omega,t) = I_B(z)\, I_A(\omega)\, I_{\{T_{n-1} < t \leq T_n\}}$ has left-continuous paths for fixed (z,ω) and for each fixed z, t the set $\{I_A(\omega)\, I_{\{T_{n-1} < t \leq T_n\}} = 1\} = A \cap \{T_{n-1} < t\} \cap \{t \leq T_n\} \in \mathcal{F}_t$ since $A \in \mathcal{F}_{T_{n-1}}$. Therefore $g(z,t)$ is adapted, so that $g \in \mathcal{P}$ and hence $f = g\, I_C(t)$ is also predictable. Also $|f| \leq 1$ and $f(z,t) = 0$ for $t > T_n$ so that $f \in L^2(\tilde{P}) \cap L^1(\tilde{P}) \cap L(P)$. Therefore by Lemma 3.6 below it follows that

$$\eta_t = (f \circ Q)_t = \int_Z \int_{R^+} f(z,s)\, I_{(0,t]}(s)\, P(dz,ds) - \int_Z \int_{R^+} f(z,s)\, I_{(0,t]}(s)\, \tilde{P}(dz,ds)$$

$$= I_A(\omega)\, I_{\{x_{T_n} \in B\}}\, I_{\{T_n \in C\}}\, I_{\{t \geq T_n\}} - a(t)$$

where $a(t)$ is a continuous process. Thus the discontinuities of $(f \circ Q)_t$ occur at T_n. Since m_t is orthogonal to η_t therefore

$$0 \equiv \langle m,\eta\rangle_t = \sum_{k\neq n} \langle \mu_k,\eta\rangle_t + \langle \mu_n,\eta\rangle_t$$

Also $\langle \mu_k,\eta\rangle_t \equiv 0$ for $k \neq n$, hence $\langle \mu_n,\eta\rangle_t \equiv 0$ so that $\mu_n \cdot \eta \in \mathcal{M}^1$. By the Corollary in [16, p. 106] and the Definition in [16, p. 87] it follows that $\Delta\mu_{nT_n} \cdot \Delta\eta_{T_n} \cdot I_{t\geq T_n}$ is a martingale so that

$$E\{M_n(\omega)\, I_A(\omega)\, I_{\{x_{T_n} \in B\}}\, I_{\{T_n \in C\}}\} = 0$$

which contradicts (3.6). The theorem has been proved. ∎

Lemma 3.3 provides an obvious extension of the definition of the stochastic integral $(f \circ Q)_t$ to $f \in L^2_{loc}(\tilde{P})$ and so Theorem 3.3 extends in the following manner.

Corollary 3.1 $\{m_t - m_o \mid m_t \in \mathcal{M}^2_{loc}\} = \{(f \circ Q)_t \mid f \in L^2_{loc}(\tilde{P})\}$.

To obtain the representation for martingales in $\mathcal{M}^1_{loc}$ two preliminary results are needed.

Lemma 3.6 i) Let $f \in \mathcal{P}$. Then $f \in L^1(P)$ if and only if $f \in L^1(\tilde{P})$. In fact $\|f\|_1 = \|f\|^{\sim}_1$. In particular $L^1(P) = L^1(\tilde{P}) = L^1(Q)$.

ii) Let $f \in L^2(\tilde{P})$. Then $f \in L^1(\tilde{P})$ and

$$(f \circ Q)_t = \int_Z \int_{R^+} f(z,s)\, I_{(0,t]}(s)\, Q(dz,ds) \qquad (3.7)^6$$

iii) If $f \in L^1(\tilde{P})$ then

$$m_t = \int_Z \int_{R^+} f(z,s)\, I_{(0,t]}(s)\, Q(dz,ds) \in \mathcal{M}^1 \cap \mathcal{A}.$$

Proof By an argument which is almost identical to the proof of [16,Prop. 3] it can be shown that (3.7) holds for $f \in L^2(\tilde{P}) \cap L^1(\tilde{P}) \cap L^1(P)$.

Since $L^2(\tilde{P}) \subset L^1(\tilde{P})$ the second assertion will then follow from the first one. Now let Φ consist of all bounded functions $f(z,t) \in \mathcal{P}$ such that $f(z,t) \equiv 0$ for $t \geq T_n$ for some $n < \infty$. Then certainly $\Phi \subset L^2(\tilde{P}) \cap L^1(\tilde{P}) \cap L^1(P)$. So $(|f| \circ Q)_t \in \mathcal{M}_2$ for $f \in \Phi$ and in particular by (3.7)

$$0 = E(|f| \circ Q)_\infty = \|f\|_1 - \|f\|_1^{\sim}.$$

Then the identity map, restricted to Φ, from $L^1(P)$ to $L^1(\tilde{P})$ preserves norms. Since Φ is dense in $L^1(P)$ and $L^1(\tilde{P})$ the first assertion follows. To prove the last assertion let f_k, $k \geq 1$ be a sequence in $L^2(\tilde{P})$ such that $\|f - f_k\|_1$ converges to zero. Then $m_{kt} = (f_k \circ Q)_t \in \mathcal{M}^2$ and by (3.7) $E|m_{kt} - m_t| \leq 2\|f - f_k\|_1$ converges to zero uniformly in t

[6] It may be worth repeating, to clarify the content of (3.7), that the integral on the right in (3.7) is a Lebesgue-Stieltjes integral whereas that on the left is the stochastic integral as defined in Lemma 3.3.

so that $m_t \in \mathcal{M}^1$. ◻

<u>Proposition 3.3</u> Let M be a $\mathcal{F}_{T_n}$-measurable r.v. for some $n \geq 1$. Suppose $E|M| < \infty$. Then there is a unique $f(z,t) \in L^1(\tilde{P})$ such that

$$M\, I_{t \geq T_n} = \int_Z \int_{R^+} f(z,s)\, I_{(0,t]}(s)\, P(dz,ds). \tag{3.8}$$

Furthermore $f(z,s) = 0$ for $s \leq T_{n-1}$ and $s > T_n$, and

$$E|M\, I_{\{T_n < \infty\}}| = \|f\|_1. \tag{3.9}$$

<u>Proof</u> Since $M\, I_{t \geq T_n} = M\, I_{\{T_n < \infty\}}\, I_{\{t \geq T_n\}}$ it can be assumed that $M = M\, I_{\{T_n < \infty\}}$. By Corollary 2.2 there exist r.v.s M^k of the form

$$M^k(\omega) = \sum_i \alpha_i\, I_{\{x_{T_n} \in B_i\}} I_{A_i}(\omega)\, I_{\{T_n \in C_i\}}$$

where $\alpha_i \in R$, $B_i \in \mathcal{Z}$, $A_i \in \mathcal{F}_{T_{n-1}}$ and $C_i \in \mathcal{B}[0,\infty)$, such that $E|M - M^k| \to 0$. If f^k is defined by

$$f^k(z,\omega,t) = \sum_i \alpha_i\, I_{B_i}(z)\, I_{A_i}(\omega)\, I_{C_i}(t)\, I_{\{T_{n-1} < t \leq T_n\}}$$

then it is clear that (3.8) and (3.9) hold for M^k and f^k. The assertion now follows by taking limits. ◻

<u>Lemma 3.7</u> Let $m_t \in \mathcal{M}^1 \cap \mathcal{A}$. Then there exists $f \in L^1(\tilde{P})$ such that

$$m_t - m_o = \int_Z \int_{R^+} f(z,s)\, I_{(o,t]}(s)\, Q(dz,ds) \tag{3.10}$$

and
$$E \int_0^\infty |dm_t| = 2\|f\|_1 \tag{3.11}$$

Proof m_t has the representation

$$m_t - m_o = \sum_{n=1}^{\infty} (M_n I_{t \geq T_n} - a_n(t)) = \sum_{n=1}^{\infty} \mu_{nt}$$

where $M_n = \Delta m_{T_n}$, $a_n(t) \in \mathcal{A}$ is continuous, and $\mu_{nt} \in \mathcal{M}$. Since $m_t \in \mathcal{A}$

$$\infty > E \int_0^\infty |dm_t| > \sum_{n=1}^{\infty} E|M_n|,$$

so that by Proposition 3.3 there exist functions $f_n(z,t) \in L^1(\tilde{P})$ which vanish outside of $\{T_{n-1} < t \leq T_n\}$ such that $E|M_n| = \|f_n\|_1$ and

$$M_n I_{t \geq T_n} = \int_Z \int_{R^+} f_n(z,s)\, I_{(0,t]}(s)\, P(dz,ds)$$

By Lemma 3.6

$$\eta_n(t) = a_n(t) - \int_Z \int_{R^+} f_n(z,s)\, I_{(0,t]}(s)\, \tilde{P}(dz,ds) \in \mathcal{M}^1$$

But $\eta_n(t)$ is continuous so that $\eta_n(t) \equiv 0$ by Theorem 3.2. Therefore

(3.10) holds for $f(z,t) = \sum_{n=1}^{\infty} f_n(z,t)$ and (3.11) follows from Lemma 3.6 and the fact that $f_k(z,t)\, f_n(z,t) \doteq 0$ for $k \neq n$

<u>Theorem 3.4</u> $m_t \in \mathcal{M}^1_{loc}$ if and only if there exists $f \in L^1(\tilde{P})$ such that

$$m_t - m_o \doteq \int_Z \int_{R^+} f(z,s)\, I_{(0,t]}(s)\, Q(dz,ds) \tag{3.12}$$

<u>Proof</u> The sufficiency follows readily from Lemma 3.6 (iii). To prove the necessity one starts by noting that by [16, Lemma 3 and Proposition 4] there exists an increasing sequence of s.t.s S_k converging to ∞ such that for each k $m_{t\, S_k} - m_o$ has a decomposition

$$m_{t\wedge S_k} - m_o = \mu^k_t + \eta^k_t$$

where $\mu^k_t \in \mathcal{M}^2_0$ and $\eta^k_t \in \mathcal{M}^1_0 \cap \mathcal{A}$. By Lemmas 3.6 (ii) and 3.7 there exists $f^k \in L^1(\tilde{P})$ such that

$$m_{t\wedge S_k} - m_o = \int_Z \int_{R^+} f^k(z,s)\, I_{(0,\infty]}(s)\, Q(dz,ds).$$

It is clear that $f^k(z,t) = f^{k+1}(z,t)$ for $t \leq S_k$. Thus (3.12) holds for $f \in L^1_{loc}(\tilde{P})$ defined by $f(z,t) = f^k(z,t)$ for $t \leq S_k$. ◻

The results above give a characterization of the classes $\mathcal{M}^2$, $\mathcal{M}^2_{loc}$, $\mathcal{M}^1 \cap \mathcal{A}$ and $\mathcal{M}^1_{loc}$. It seems much more difficult to obtain a useful characterization of the class $\mathcal{M}^1$.

The (local) martingales with respect to $(\Omega, \mathcal{F}_t, P)$ have been represented as sums or integrals of the 'basic' martingales $Q(B,t)$. The latter are associated in a one-to-one manner with the counting processes $P(B,t)$ which count those jumps of the underlying process x_t which <u>end</u> in the set B. Thus jumps are distinguished by their final values. Now it is also possible to distinguish jumps by their values. The corresponding counting processes will be of the form $p(A,t)$ which counts those jumps of the x_t process which have values in the set A. The martingales $q(A,t)$ associated with the $p(A,t)$ also form a 'basis' for the set of all martingales on $(\Omega, \mathcal{F}_t, P)$ as will be shown below. The alternative representation obtained with this basis can sometimes be more useful since the description of the x_t process is, in practice, often given in terms of a statistical characterization of the jumps of x_t.

For simplicity of notation it will be assumed in the remainder of this section that the x_t process starts at time 0 in a fixed state i.e., $x_0(\omega) = x_0(\omega')$ for all ω, ω' in Ω [7]. Next it is assumed that there is given a set Σ of transformations σ: $Z \to Z$ with the following properties:

i) Σ contains the jumps of the x_t process i.e., if $x_{s-}(\omega) \neq x_s(\omega)$ for some $s \in R_+$, $\omega \in \Omega$ then there is a <u>unique</u> $\sigma \in \Sigma$ such that $\sigma(x_{s-}(\omega)) = x_s(\omega)$,

ii) Σ contains a distinguished element σ_0 corresponding to the identity transformation i.e., $\sigma_0(z) = z$ for all $z \in Z$.

To each sample function $\omega \in \Omega$ of the x_t process is associated a function $\gamma(\omega)$: $R_+ \to \Sigma$ defined as follows:

[7] It should be noted however that the results below continue to hold in the absence of this simplification.

$$\gamma_t(\omega) = \sigma_0 \text{ if } t = 0 \text{ or if } x_t(\omega) = x_{t-}(\omega)$$
$$= \sigma \text{ if } x_t(\omega) \neq x_{t-}(\omega)$$

where $\sigma \in \Sigma$ is the unique element for which $\sigma(x_{t-}(\omega)) = x_t(\omega)$.

Remark i) Given a sample path $x_s(\omega)$, $0 \leq s \leq t$, there corresponds in a one-to-one manner a sample path $\gamma_s(\omega)$, $0 \leq s \leq t$.

ii) The functions $\gamma(\omega)$ are not right continuous. However if $\gamma_t(\omega) = \sigma_0$ then $\gamma_{t-}(\omega) = \sigma_0$. This observation will be used later in an example.

The following 'regularity' assumption appears to be necessary. In practice it is readily verifiable.

Assumption. There is a σ-field Ξ on Σ such that $\mathcal{F}_t$ coincides with the σ-field generated by subsets of the form $\{\omega | \gamma_s(\omega) \in A\}$ where $s \leq t$ and $A \in \Xi$.

With the assumptions above it is clear that the processes x_t and γ_t are equivalent alternative descriptions of the same process. In particular they generate the same σ-fields, so that the two processes have the same martingales. The representation theorems derived earlier for the x_t process can be applied to the γ_t process but there is a minor point to be cleared up. Recall that it was assumed that the x_t process was right-continuous whereas γ_t is not. However the assumption of right-continuity was used only to establish the right-continuity of the family $\mathcal{F}_t$. This continues to hold of course since γ_t and x_t generate the same σ-fields $\mathcal{F}_t$. Hence one can apply the representation theorems.

Definition 3.4 Let $A \in \Xi$. Let

$$p(A,t) = \sum_{s \leq t} I_{\{\gamma_{s-} \neq \gamma_s\}} I_{\{\gamma_s \in A\}} = \sum_{s \leq t} I_{\{x_{s-} \neq x_s\}} I_{\{\gamma_s \in A\}}$$

be the number of jumps of the x_t process with 'values' in A and which occur prior to t.

By Proposition 3.2 there is a unique continuous process $\tilde{p}(A,t) \in \mathcal{A}^+_{loc}$ such that the process $q(A,t) = p(A,t) - \tilde{p}(A,t)$ is in $\mathcal{M}^2_{loc}$. In analogy with Definitions 3.2 and 3.3 one can define the subsets of $\mathcal{P}_\Sigma$: $L^2(\tilde{p})$, $L^2_{loc}(\tilde{p})$, $L^1(\tilde{p})$, $L^1(p)$ etc.[8] Lemma 3.3 describes the stochastic integrals (foq) for $f \in L^2(\tilde{p})$. An application of Theorem 3.3, Corollary 3.1, Lemma 3.7 and Theorem 3.4 yields the following representation theorem.

Theorem 3.5 i) $m_t \in \mathcal{M}^2(\mathcal{M}^2_{loc})$ if and only if $m_t - m_0 = (foq)_t$ for some $f \in L^2(\tilde{p})$ $(L^2_{loc}(\tilde{p}))$.
ii) $m_t \in \mathcal{M}^1 \cap \mathcal{A}$ $(\mathcal{M}^1_{loc})$ if and only if $m_t - m_0 = \int_\Sigma \int_{R^+} f(\sigma,s)\, I_{(0,t]}(s)\, q(d\sigma,ds)$ for some $f \in L^1(\tilde{p})$ $(L^1_{loc}(\tilde{p}))$.

IV An example

This section consists of a simple example showing how Theorem 3.5 can be applied. The example will be further elaborated in [3].
Let Z be countable and let $\mathcal{Z}$ consist of all subsets of Z. Let x_t be a process with values in Z and satisfying the assumptions listed at the beginning of Section III. Suppose that from each state z the process x_t

8 $\mathcal{P}_\Sigma$ is the set of predictable functions of $(\sigma,\omega,t) \in \Sigma \times \Omega \times R_+$ defined in analogy with Definition 3.2.

can jump to one of n states. In terms of a state-transition diagram (see Figure 1) there are n transitions or links emanating from each state or node. Label these transitions by the symbols $\sigma_1, \ldots, \sigma_n$. Let $\Sigma = \{\sigma_0, \ldots, \sigma_n\}$. Thus each $\sigma \in \Sigma$ corresponds to a transformation in Z, σ_0 is the identity transformation. Let Ξ be the set of all subsets of Σ. The x_t process defines the process of transitions, γ_t. Evidently Σ, Ξ satisfy the assumptions made above.

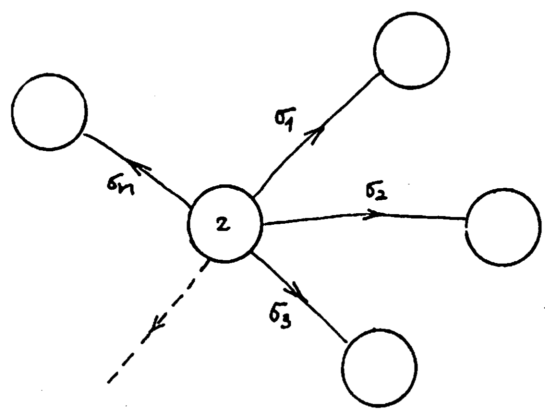

Figure 1: State-transition diagram for example.

Let $p_i(t) = p(\{\sigma_i\},t)$, $\tilde{p}_i(t) = \tilde{p}_i(\{\sigma_i\},t)$ and $q_i(t) = q_i(\{\sigma_i\},t)$, $0 \leq i \leq n$. From a remark made in the last section $I_{\{x_{s-} \neq x_s\}} \cdot I_{\{\gamma_s = \sigma\}} \equiv 0$. Hence $p_0(t) \equiv 0$ and so $q_0(t) \equiv 0$. Theorem 3.5 simplifies to the following. Here the predictable integrands are functions of (ω,t) only.

<u>Theorem 4.1</u> i) $m_t \in \mathcal{M}^2(\mathcal{M}^2_{loc})$ if and only if $m_t - m_0 \equiv \sum_{i=1} (f_i \circ q_i)_t$ for some $f_i \in L^2(\tilde{p}_i)(L^2_{loc}(\tilde{p}_i))$, $1 \leq i \leq n$.

ii) $m_t \in \mathcal{M}^1 \cap \mathcal{A}\ (\mathcal{M}^1_{loc})$ if and only if $m_t - m_0$

$$= \sum_{i=1}^{n} \int_{(0,t]} f_i(s)\, q_i(ds) \text{ for some } f_i \in L^1(\tilde{p}_i)(L^1_{loc}(\tilde{p}_i)),\ 1 \leq i \leq n.$$

__Example__ Let x_t be a process taking values in a countable state space and of the type described immediately above. From each state the process can make n transitious $\sigma_1, \ldots, \sigma_n$ as sketched in Figure 1. Let $p_i(t)$, $\tilde{p}_i(t)$, $q_i(t)$ be as in Theorem 3.6.

Let $\lambda(t)$, $\rho_1(t), \ldots, \rho_n(t)$ be non-negative predictable processes such that

$$\sum_{i=1}^{n} \rho_i(t) \equiv 1 \tag{4.1}$$

$$p_i(t\wedge T_k) - \int_0^{t\wedge T_k} \rho_i(s)\lambda(s)\, ds \in \mathcal{M}^1, \; k = 1, 2, \ldots, n \tag{4.2}$$

Then the processes $\lambda(t)$, $\rho_i(t)$ have the following interpretation: since from (4.1) and (4.2)

$$\left(\sum_{i=1}^{n} p_i(t\wedge T_k) - \int_0^{t\wedge T_k} \lambda(s)\, ds\right) \in \mathcal{M}^1 \tag{4.3}$$

and since $\sum_{i=1}^{n} p_i(t)$ is just the total number of jumps of the process occuring prior to t, therefore the probability that the process x_t makes a transition in the time interval $[t, t + h]$, conditioned on the past $\mathcal{F}_t$ of the process, is equal to $\lambda(t)h + o(h)$. Similarly $\rho_i(t)$ is the probability that the process makes a transition represented by σ_i, conditioned on $\mathcal{F}_t$ __and__ conditioned on the fact that a transition does occur at t.

Now since the process represented by the indefinite integral in

(4.2) has continuous sample paths it follows quite readily (see e.g. [25, p. 153]) that the jump times of the process are totally inaccessible. Hence from Theorem 4.1 it can be concluded that every $m_t \in \mathcal{M}^1_{loc}$ has a representation

$$m_t - m_o = \sum_{i=1}^{n} [\int_0^t f_i(s)\, d\, p_i(s) - \int_0^t f_i(s)\, \rho_i(s)\, \lambda(s)\, ds] \tag{4.4}$$

for some predictable processes $f_i \in L^1_{loc}\ (\rho_i \lambda)$ i.e., for which

$$\int_0^t f_i(s)\ \rho_i(s) \lambda(s)\ ds < \infty \text{ a.s. for all } t \in R_+.$$

This result indicates how one can immediately write down the representation results if the process x_t is described in terms of the 'rate' processes λ and the 'transition' probabilities ρ_i. It should be kept in mind, however, that it has not been proven that given processes $\lambda(t)$ and $\rho_i(t)$ there exists a process x_t for which (4.2) holds. This question of existence will be pursued in [3]. The next remark relates to the representation (4.4), which asserts that the n local martingales in (4.2) indeed from a "basis" for the space of all local martingales $\mathcal{M}^1_{loc}$. The question is whether n is the minimum number of martingales in every basis of $\mathcal{M}^1_{loc}$. For the case where x_t is a Gaussian process the minimum number of martingales has been called the "multiplicity" of the process by Cramer [8 ,9]. It turns out that this notion of multiplicity extends in a very natural way to arbitrary processes [13]. From the results of [13] the following sufficient condition can be obtained: Suppose that the processes $\rho_i(s)\lambda(s)$ satisfy

$$\rho_i(s)\ \lambda(s) > 0 \Leftrightarrow \rho_j(s)\ \lambda(s) > 0 \quad \text{all } i,j.$$

Then n is the minimum number of martingales in a representation of $\mathcal{M}^1_{loc}$.

Finally, specialize the example still further and assume that x_t is a counting process i.e., $x_o = 0$, x_t takes integer values and has unit positive jumps. Then x_t is a direct extension of a Poisson process. The state-transition diagram then simplifies to that of Figure 2 and since $n = 1$ in (4.1), (4.2) and (4.4) therefore $\rho_1(t) \equiv 1$ and can be omitted. Also $p_1(t) \equiv x_1(t)$ and so the representation (4.4) simplifies to (4.5). Every $m_t \in \mathcal{M}^1_{loc}$ can be written as

$$m_t - m_o = \int_0^t f(s)\ dx_s - \int_0^t f(s)\ \lambda(s)\ ds \tag{4.5}$$

where f is a predictable function such that

$$\int_0^t f(s)\ \lambda(s)\ ds < \infty \text{ a.s. for all } t \in R_+.$$

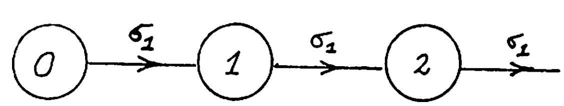

Figure 2. Transition diagram for counting process.

This representation result has been obtained by very different techniques from several authors [4, 5, 11, 12]. However even here the cited references prove (4.5) for the special case where the probability law of the x_t process is mutually absolutely continuous with respect to the probability law of a standard Poisson process. Hence even for this special case (4.5) is a strict generalization of the available results.

Appendix: The increasing processes $\tilde{P}(A,t)$ and the Lévy system.

This section attempts to give an intuitive interpretation of the increasing processes $\tilde{P}(B,t)$ and shows the connection with the Lévy system for Hunt processes.

Begin with the observation that for all $B \in \mathcal{Z}$ the measure $P(B,t)$ is absolutely continuous with respect to the measure $\tilde{P}(Z,t)$ i.e., there exists a predictable function $(\omega,t) \to n(B,\omega,t)$ such that

$$\tilde{P}(B,t) = \int_0^t n(B,\omega,s)\ P(Z,ds) \tag{A-1}$$

To see this it is enough to demonstrate that for all predictable functions $\phi(\omega,s) = \phi^2(\omega,s)$ (i.e., all indicator functions)

$$E \int_0^\infty \phi(\omega,s)\ \tilde{P}(Z,ds) = 0 \tag{A-2}$$

implies

$$E \int_0^\infty \phi(\omega,s)\ \tilde{P}(B,ds) = 0 \tag{A-3}$$

Suppose (A-2) holds, then

$$\left\langle \int_0^t \phi(s)\ dQ(Z,s), \int_0^t \phi(s)\ dQ(Z,s) \right\rangle \equiv \int_0^t \phi^2(s)\ \tilde{P}(Z,ds) \equiv 0,$$

and so

$$0 \equiv \left\langle \int_0^t \phi(s)\, dQ(B,s),\ \int_0^t \phi(s)\, dQ(Z,s) \right\rangle$$

$$= \int_0^t \phi^2(s)\ \tilde{P}(B \cap Z, ds) \qquad \text{by Lemma 3.1}$$

$$= \int_0^t \phi^2(s)\ \tilde{P}(B, ds)$$

which proves (A-3).

In exactly the same way as Lemma 3.2 was proved it can be shown that the $n(B,\omega,s)$ considered as a set function in $\mathcal{Z}$ is countably additive in the sense that if B_1, B_2, ... is a disjoint sequence of sets in $\mathcal{Z}$ then

$$\tilde{P}(\bigcup_i B_i,\ t) \equiv \sum_i \int_0^t n(B_i, s)\ \tilde{P}(Z, ds)$$

Hence if one sets $\tilde{P}(Z,t) \equiv \Lambda(t) \in \mathcal{A}^+_{loc}$, then the system $\{n(B,t,\omega),\ \Lambda(t)\}$ is analogous to a Lévy system for Hunt processes (see [22]), and has a similar interpretation : the probability of x_t having a jump in $[t, t + dt)$ is $d\Lambda(t) + o(dt)$, while $n(A,t,\omega)$ is the chance that $x_t \in A$ given $\mathcal{F}_t$ and given that a jump occurs at t.

References

[1] D. Blackwell: "On a class of probability spaces," Third Symposium on Math. Stat. and Prob., vol. II, Univ. of Calif. Press, Calif., 1-6 (1956).

[2] V. Beneš, "Existence of optimal stochastic control laws," SIAM J. Control 9, 446-475 (1971).

[3] R. Boel, P. Varaiya and E. Wong, "Martingales on Jump Processes Part II: Applications," in preparation.

[4] P. Brémaud, "A martingale approach to point processes," Electronics Res. Lab., Univ. of Calif., Berkeley, Memo #M-345 (August 1972).

[5] P. Brémaud, "Filtering for point processes," preprint, submitted to Information and Control, June 1973.

[6] J. M. C. Clark, "The representation of functionals of Brownian motion by stochastic integrals," Ann. Math. Statist. 41, 1285-1295 (1970).

[7] P. Courrège and P. Priouret, "Temps d'arrét d'une fonction aléatoire, théorèmes de décomposition," Publ. Inst. Stat. Univ. Paris 14, 242-274(1965).

[8] H. Cramer, "Stochastic Processes as Curves in Hilbert Space," Theory of Prob. Appl. 9, 195-204, (1964)

[9] H. Cramer, "A Contribution to the Multiplicity Theory of Stochastic Processes," Fifth Symposium on Math. Stat. and Prob., Vol. II, Univ. of Calif. Press, Calif., 215-221 (1967).

[10] M. H. A. Davis, "Detection of signals with point process observations," Dept. of Computing and Control, Imperial College, London, May 1973.

[11] M. H. A. Davis, "Nonlinear filtering with point process observations," preprint (1972).

[12] M. H. A. Davis and P. Varaiya, "Dynamic Programming Conditions for partially observable stochastic systems," SIAM J. Control, 11(2), 226-261 (1973).

[13] M. H. A. Davis and P. Varaiya, "The Multiplicity of an Increasing Family of σ-fields," Electronics Res. Lab., Univ. of Calif., Berkeley, Calif., Memo #402, (August 1973).

[14] C. Dellacherie, Capacités et processus stochastiques, Berlin: Springer-Verlag, 1972.

[15] C. Doléans-Dade, "Quelques applications de la formule de changement de variables pour les semimartingales," Z. Wahrscheinlichkeits theorie verw. Greb 16, pp. 181-194, (1970).

[16] C. Doléans-Dade and P. A. Meyer, "Intégrales stochastiques par rapport aux martingales locales," in Séminaire de Probabilités: IV, Lecture notes in Mathematics, Berlin: Springer-Verlag, p. 77-107, (1970).

[17] J. L. Doob, Stochastic Processes, New York: Wiley (1953).

[18] T. Duncan, "Evaluation of likelihood functions," Information and Control 13, 62-74, (1968).

[19] T. E. Duncan and P. Varaiya, "On the solutions of a stochastic control system," SIAM J. Control 9, 354-371, (1971).

[20] Girsanov, "On transforming a certain class of stochastic processes by absolutely continuous subtitution of measures," Theory of Prob. and Appl. 5, 285-301, (1960).

[21] T. Kailath and M. Zakai, "Absolute continuity and Radon-Nikodym derivatives for certain measures relative to Wiener measure," Ann. Math. Statist. 42, 130-140 (1971).

[22] H. Kunita and S. Watanabe, "On square integrable martingales," Nagoya Math. J. 30, 209-245, (August 1967).

[23] M. Loève, Probability theory, 2nd edition, New York: Van Nostrand (1960).

[24] P. A. Meyer, Probabilités et potentiel, Paris: Hermann, (1966); (English translation: Probability and Potentials, Waltham, Mass: Blaisdell (1966)).

[25] P. A. Meyer, "Guide détailé de la théorie 'générale' des processus," in Séminaire de Probabilité, II, Lecture Notes in Mathematics Vol. 51, Berlin: Springer-Verlag, 140-165, (1968).

[26] P. A. Meyer, "Non-square integrable martingales etc.," in Lecture Notes in Mathematics, No. 190, Berlin: Springer-Verlag, 38-43 (1971).

[27] P. A. Meyer, "Square integrable martingales, a survey," in Lecture Notes in Mathematics, No. 190, Berlin: Springer Verlag, 32-37 (1971).

[28] P. A. Meyer:"Temps d'arret algebriquement prévisibles," Séminaire de Probabilités VI, Lecture Notes in Mathematics, No. 258, Berlin: Springer-Verlag, 158-163 (1972).

[29] D. L. Snyder, "A representation theorem for observed jump processes," Proc. IEEE Conf. on Decision and Control, New Orleans, Louisiana, p. 218, (Dec. 13-15, 1972).

[30] D. L. Snyder, "Information Processing for Observed Jump Processes," Information and Control 22 (1), 69-78, (1973).

[31] D. L. Snyder, "Statistical Analysis of Dynamic Tracer Data," IEEE Trans. on Biomedical Engr. BME-20(1), 11-20, (1973).

[32] J. H. Van Schuppen and E. Wong, "Transformations of Local Martingales under a Change of Law," Electronics Res. Lab., Univ. of Calif., Berkeley, Memo #M-385 (May 18, 1973).

[33] E. Wong "Martingale Theory and Applications to Stochastic Problems in Dynamical Systems," Publication 72/19, Imperial College, Dept. of Computing and Control, (June 1972).

[34] E. Wong, "Recent Progress in Stochastic Processes - A Survey," IEEE Trans Info. Theory IT-19(3), 263-275 (May 1973).