UNI-MOMENT METHOD OF SOLVING ANTENNA AND SCATTERING PROBLEMS

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Abstract

It has been shown by this investigator and numerous others [6] [7] [8] that exterior boundary value problems involving localized inhomogeneous media are most conveniently solved using finite difference or finite element technique together with integral equations or harmonic expansions, which satisfy the radiation conditions. The methods result in large matrices which are partly full and partly sparse, and methods to solve them, such as iteration or banded matrix methods are not very satisfactory. The uni-moment method alleviates the difficulties by decoupling enterior problems from the interior boundary value problems. This is done by solving the interior problem many times so that N linearly independent solutions are generated. The continuity conditions are then enforced by a linear combination of the N independent solutions, which may be done by solving much smaller matrices. Methods of generating solutions of the interior problems are discussed.

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Introduction

Since computer aided analysis became a part of electromagnetic research the integral equation [1] and the moment methods [2] have been the prima donas of numerical methods used in antenna and scattering problems. Finite difference and finite element methods, stars of research in structure engineering and many other engineering disciplines have been essentially overshadowed for lack of appeal to the antenna experts. The choice so made by the antenna specialists has indeed been a natural one in that the radiation condition, vital to all antenna problems, cannot be easily enforced in difference or finite element equations, while it is automatically satisfied in the integral equation formulations, and in that the nodal points of a discretized integral equation are distributed only over the surface of the obstacle while those of a finite difference or finite element equation are over the volume of the obstacle and beyond. Integral equation formulations, however, are limited by the availability of Green's functions, which are simple functions only in linear isotropic media. Indeed, any excursion of the integral equation or moment methods to inhomogeneous or anisotropic media problems requires massive analytical and programming efforts, which can be attested to by those few who have so tried [3] [4] [5]. Since the finite difference or finite element equations are easily formulated regardless of the complexity of the medium, they would be more attractive than the integral equations particularly in the inhomogeneous media problems, if the above mentioned difficulties could be resolved. Recent work by Mei, Stovall and Tremain [6], Silvester and Hsieh [7], and McDonald and Wexler [8] has essentially pointed the way to incorporate the radiation condition in finite difference and finite element methods, yet they still have not brought those methods within competitive range to the moment methods either in speed or storage requirement. The matrix operators resulted from the above references are large, although they are partly full and partly sparse. Direct inversion of such matrices are impractical and iterative methods are slow, and always diverge when the source frequency is higher than a critical value, usually the lowest resonant frequency of the finite difference or finite element region.

The concept of uni-moment which this paper is to present promises to solve the radiation conditioned finite difference or finite element equations with great speed, less storage and simple programming. With all these conditions secured the solutions to antennas or scatterings in complex and inhomogeneous media are within reach. Hopefully this elementary treatment of the subject will eventually open the door to

"A land so fascinating
Gathering countless heroes
Working diligently to show their gallantry"*.

Moment and Uni-moment

Consider the problem of scattering by a dielectric obstacle in a two dimensional space as shown in Fig. 1. The mathematical problem is to solve the scalar Helmholtz equation subject to the illuminating field ϕ^{inc}, the continuity conditions on the boundary c, and radiation condition at infinity. The integral equation formulation of the problem results

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* Verse from a Chinese poem.

in the following two coupled integral equations:

$$\frac{1}{2}\,\phi(\bar{r}) = \phi^{inc}(\bar{r}) - \lim_{\sigma \to 0} \int_{C-\sigma} \{G_o(\bar{r},\bar{r}')\,\frac{\partial \phi(\bar{r}')}{\partial n'} - \phi(\bar{r}')\,\frac{\partial G_o(\bar{r},\bar{r}')}{\partial n'}\,dc'\} \quad (1)$$

$$\frac{1}{2}\,\phi(\bar{r}) = \lim_{\sigma \to 0} \int_{C-\sigma} \{G_1(\bar{r},\bar{r}')\,\frac{\partial \phi(\bar{r}')}{\partial n'} - \phi(\bar{r}')\,\frac{\partial G_1(\bar{r},\bar{r}')}{\partial n'}\}\;dc' \quad (2)$$

where $\bar{r}$ and $\bar{r}'$ are both points on C;

$$G_i(\bar{r},\bar{r}') = \frac{-j}{4}\;H_o^{(2)}\;(k_i|\bar{r}-\bar{r}'|)$$

is the Green's function in the respective regions, and σ denotes an infinitesimal segment of C surrounding the singularity of $G_i(\bar{r},\bar{r}')$.

Both $\phi(\bar{r})$ and $\frac{\partial \phi(\bar{r})}{\partial n}$ are unknowns to be found. The moment method of solving (1) and (2) starts with a set of trial functions, ψ_1, ψ_2, ... ψ_N, and the solutions are represented by two sets of linear combinations of the trial functions,

$$\phi(\bar{r}) = \sum_{i=1}^{N} a_i\;\psi_i(\bar{r}) \quad (3)$$

$$\frac{\partial \phi}{\partial n}\,(\bar{r}) = \sum_{i=1}^{N} b_i\;\psi_i(\bar{r}) \quad (4)$$

The coefficient vectors a_i and b_i may be found by substituting (3) and (4) into eq's (1) and (2) and each equation is enforced either by collocation or by weighting integrals.

The uni-moment method of solving the same problem starts with trial function pairs,

$$\left\{ \begin{array}{llll} \psi_1 , & \psi_2 , & \cdots & \psi_N \\ \frac{\partial \psi_1}{\partial n} , & \frac{\partial \psi_2}{\partial n} , & \cdots & \frac{\partial \psi_N}{\partial n} \end{array} \right\} \qquad (5)$$

We have assumed that for each trial function ψ_i on C, there is available an associated unique function $\frac{\partial \psi_i}{\partial n}$. Assuming such pairs have been found, the solutions of the integral equations may be obtained by a single coefficient vector a_i, and only eq. (1) is needed to obtain the solution by substituting,

$$\left. \begin{array}{l} \phi(\bar{r}) = \sum_{i=1}^{N} a_i \, \psi_i(\bar{r}) \\ \frac{\partial \phi}{\partial n} (\bar{r}) = \sum_{i=1}^{N} a_i \frac{\partial \psi_i}{\partial n} (\bar{r}) \end{array} \right\} \qquad (6)$$

In the way the problem is formulated by eqs (1) and (2), however, the uni-movement approach as suggested by (5) and (6) really does not offer any advantage, since the trial function pairs are actually solutions of eq. (2). The advantage should become obvious if eq. (2) could be replaced by other formulas, whose solutions can be easily generated. Finite difference and finite element formulations are good candidates for the task. Because, their solutions can be generated relatively easily in C, and their formulations are relatively indifferent to the complexities of the medium and geometry. Indeed, the integral equation (2) would not be available if the medium inside C is inhomogeneous.

In many instances when finite difference or finite element formulations are used, it is more advantageous to extend C to a separable boundary C', which is a circle in the 2-dimensional problem as described by the dashed curve in Fig. 1. In such cases, of course, the trial functions ψ_i will be assigned on C'. Assuming the medium outside C' is uniform and isotropic we may expand $\psi^o(\bar{r})$ outside C' by cylindrical harmonics.

$$\phi^o(\bar{r}) = \sum_{n=0}^{N} (A_n^e \cos n\theta + A_n^o \sin n\theta)\, H_n^{(2)}(k_o r) \tag{7}$$

Eq. (1) can now be replaced by the continuity conditions on C', which may be enforced either by collocation or by the method of weighting functions. Using the latter we should have,

$$\int_0^{2\pi} \{\sum_{n=1}^{2N} a_n \psi_n(a,\theta)\}\, \xi_i(\theta)\, d\theta = \int_0^{2\pi} \{ \sum_{n=0}^{N} (A_n^e \cos n\theta + A_n^o \sin n\theta)\, H_n^{(2)}(k_o a) + \phi^{inc}(A,\theta)\}\, \xi_i(\theta)\, d\theta \tag{8}$$

$$\int_0^{2\pi} \{\sum_{n=0}^{2N} a_n \frac{\partial\psi_n}{\partial n}(a,\theta)\}\, \xi_i(\theta)\, d\theta = \int_0^{2\pi} \{\sum_{n=0}^{N} (A_n^e \cos n\theta + A_n^o \sin n\theta)\, \frac{\partial H_n^{(2)}}{\partial r}(k_o a) + \frac{\partial\phi^{in}}{\partial r}(a,\theta)\}\, \xi_i(\theta)\, d\theta \tag{9}$$

where $\xi_i(\theta)$ is the weighting function set which may be chosen quite arbitrary. For i = 1, ... 2N, (8) and (9) represent 4N linear equations, which may be solved for A_i^e, A_i^o and a_i. Because, equations (8) and (9) are easier to generate than (1), much computer time can be saved by this extension. Furthermore, once the coefficients A_i^e and A_i^o of the exterior expansion are found, it is quite a trivail computational matter to obtain either the near fields or far fields outside C'.

Generation of Trial Function Pairs

The successful application of the uni-moment methods depends on how fast the trial function pairs can be generated. Actually, the function pairs cannot be found without solving the field equations inside C'. In the following we shall discuss the methodology of generating solutions of the Helmholtz equation in a closed region. We shall assume that the reader is already familiar with the finite difference approximations of partial differential equations. For convenience in the ensuing discussions we shall use only rectangular mesh as shown in Fig. 2a. The finite difference form of Helmholtz equation at a point i in Fig. 3a is,

$$\phi_{i+N} + \phi_{i+1} + (k^2h^2 - 4)\,\phi_i + \phi_{i-1} + \phi_{i-N} = 0 \tag{10}$$

The finite element formulation of the same problem requires the minimization of the functional,

$$I = \int_S (|\nabla\phi|^2 - 2k^2\phi)\,dS \tag{11}$$

In general the mesh in the finite element method is triangular. Triangularizing the rectangular mesh of Fig. 2a, we get that of Fig. 2b.

In the special case of Fig. 3b, the finite element formulation, i.e., minimization of (11) gives exactly the same equation as (10). This is not to say, however, finite difference and finite element formulations are the same. In fact, the finite element has definite advantages over the finite difference formulations in dealing with complex geometries. Using the simple mesh, we shall illustrate the following methods of generating solutions to equation (10).

(A) Shooting Method

We note that trial functions in (5) are rather arbitrary, although they need to be complete and linearly independent. Instead of solving the boundary value problem as suggested by (10), we may generate solutions to (10) by specifying one row of $\phi(\bar{r})$ next to a boundary with known boundary values, such as the bottom boundary of Fig. 2a. Specifying two adjacent rows of ϕ is equivalent to specifying 4 values of ϕ in the five point equation of (10). Consequently the values of ϕ on a new row can be found. This step-marching towards the top of Fig. 2a, gives the trial function pair at the top boundary. The one dimensional equivalence of this method is to solve a second order ordinary differential equation with specified $f(t_o)$ and $f'(t_o)$, which are respectively the initial position and slope of a trajectile, and hence the name shooting. As may be seen from the simple algebra that the shooting method is fast and requires little memory space. The results of computing the scattering by a periodic metal groove using the shooting method is shown in Fig. 3, together with the results of other methods. The computation time for this problem using integral equation (Kalhor and Neureuther [9]) is 32 sec. on a CDC-6400 computer, it only takes 2 sec. using the uni-moment method (with shooting technique) on the same computer. Furthermore, the uni-moment method is easily modified to accommodate inhomogeneous dielectric loading of the groove, or grooves on dielectric substrate.

Attractive as it may seem, however, the shooting method is basically unstable, i.e., errors accumulate quickly in the step marching calculations. Furthermore, one cannot shoot from a singular point or its vicinity, such as the origin of a polar or spherical coordinate, otherwise high order modes would overwhelm the results. Where it is applicable, such as secattering by aperture of finite thickness, scattering by inhomogeneous dielectric shell, etc., the uni-moment method using shooting technique offers a very attractive alternative to existing methods.

(B) Ricatti Transformation

The equations of (10) for i = 1, 2, ..., mN, where m is the total number of rows of the mesh, may be written in a matrix form,

$$\underline{\phi}_{k+1} + Q\underline{\phi}_k + \underline{\phi}_{k-1} = 0 \qquad (12)$$

for k = 2, 3, ..., m-1. The vector $\underline{\phi}_j$ is defined as,

$$\underline{\phi}_j = (\phi_{(j-1)N+1}, \phi_{(j-1)N+2}, \ldots, \phi_{(j-1)N+N}) \qquad (13)$$

which represents the jth row of Fig. 3a. And,

$$
Q = \begin{bmatrix} k^2h^2-4 & 1 & 0 & 0 & 0 & \cdots \\ 1 & k^2h^2-4 & 1 & 0 & 0 & \cdots \\ 0 & 1 & k^2h^2-4 & 1 & 0 & \cdots \\ 0 & 0 & 1 & k^2h^2-4 & 1 & \\ \vdots & \vdots & \vdots & & \ddots & \ddots \end{bmatrix} \tag{14}
$$

is a tridaigonal matrix. It is evident that for each k, (12) represents N equations of (10). The Ricatti transformation of (12) assumes a solution of the form,

$$
\underline{\phi}_{j+1} = R_j \underline{\phi}_j + \underline{s}_j \tag{15}
$$

where the matrices R_j and vectors $\underline{s}_j$ are independent of $\underline{\phi}_j$. If R_j and $\underline{s}_j$ could be found for all j, it would be possible to find all $\underline{\phi}_j$ starting from the known boundary values. We substitute (15) in (12) to get,

$$
(R_k + Q + R_{k-1}^{-1})\ \underline{\phi}_k + (\underline{s}_k - R_{k-1}^{-1}\ s_{k-1}) = 0 \tag{16}
$$

It is noticed that (16) like (12), represents the Helmholtz equation without the complete boundary conditions, hence it has to be satisfied by many ϕ_i which represent solutions of the Helmholtz equation. Since ϕ_i, in this case, is not unique, (16) can only be true if the coefficients of $\underline{\phi}_k$ vanish, i.e.,

$$
R_k + Q + R_{k-1}^{-1} = 0 \tag{17}
$$

hence

$$\underline{s}_k - R_{k-1}^{-1} \underline{s}_{k-1} = 0 \tag{18}$$

Referring to Fig. 3a, let us consider (15) for $j = m - 1$,

$$\underline{\phi}_m = R_{n-1} \underline{\phi}_{m-1} + \underline{s}_{m-1} \tag{19}$$

where $\underline{\phi}_m$ is the specified boundary value, which should be independent of $\underline{\phi}_{m-1}$. Hence,

$$R_{m-1} \equiv 0 \tag{20}$$

and

$$\underline{s}_{m-1} \equiv \underline{\phi}_m \tag{21}$$

The recurrence relation (17) can now be solved starting with $k = m - 1$ in

$$R_{k-1} = - (R_k + Q)^{-1} \tag{22}$$

After the R_i's are found $\underline{s}_i$ may be computed from (18) and (21). With all R_i and $\underline{s}_i$ determined, we can calculate $\underline{\phi}_2$ from the boundary value $\underline{\phi}_1$, $\underline{\phi}_3$ from $\underline{\phi}_2$ etc., using (19).

Using the above procedure the solution of the difference equations (10) actually involves inversion of $N \times N$ matrices $m - 2$ times. Since in lossless time harmonic wave equations, real boundary values result in real solutions, all R_i and $\underline{s}_i$ will be real if we use only real trial functions. Furthermore, if we are only interested in the expansion coefficients outside, there is no need to store all the R_i, only $\underline{R}_1$ (and R_2 for quadratic approximation) is needed to find $\frac{\partial \psi_i}{\partial n}$ from ψ_i. Therefore, the trial function pairs of (5) can be generated speedily without large storage spaces.

We notice in the above development that R_i and $\underline{s}_i$ are found from $i = m - 1$ to $i = 1$, and $\underline{\phi}_i$ are found from $i = 1$ to $i = m - 1$. This procedure is known as a two sweep method, and numerous discussions may be found on a one sweep method [10], [11]. In our application where only the solutions next to the boundaries are needed the one sweep method does not seem to offer us much advantage.

The stability of the Ricatti transformation on Laplaces equations has been studied by Angel [12]. Our computations using Ricatti transformation on wave equations have not yet experienced any instability even when the region exceeds several wavelengths.

(C) Sparse Matrix Algorithm

The Ricatti transformation of the scalar Helmholtz equation in difference form offers a simple algorithm for direct solutions. It is easy to formulate and easy to program. It is, however, by no means the best method, especially when the dimension of $\underline{\phi}_k$ is large. Methods of the Elimination Form of Inverse using optimum pivoting† should result in even greater speed, but its storage and algorithmic schemes are more sophisticated and a self-contained discussion of the subject is beyond the scope of this paper. Interested readers may

† Optimum in preserving sparsity rather than reducing round off errors.

consult the text by Tewarson [13], which also contains up to date references. In the following we shall discuss Ricatti transformation with reference to sparse matrix algorithms. It is intended to present a different viewpoint and to show where a more sophisticated approach may be used to advantage.

The name Ricatti transformation is often used by authors working with potential problems. It is so named because (15) transforms an elliptical differential equation to a nonlinear, Ricatti type equation. The discussion of the method in the context of elliptical partial differential equation is really not necessary. We have adopted that approach because of the readers' familiarity with the scalar Helmholtz equation. Indeed, eqs. (10) is a banded matrix, where the matrix elements

$$a_{ij} = 0 \quad \text{for} \quad |j - 1| > N \tag{23}$$

A banded matrix may be partition as illustrated in Fig. 4. The partitioned matrix equations take the form,

$$A_{k,k-1}\underline{\phi}_{k-1} + A_{k,k}\underline{\phi}_k + A_{k,k+1}\,\underline{\phi}_{k+1} = 0 \tag{24}$$

which is a generalized form of (12). The transformation (15) applies equally well to the 3-term recurrence formual of (24), and hence the solution [14]. If one wishes to use a 9-point difference equation instead of the 5-point of (10) the formulation of (12) suggests a 5-term recurrence formula which may not be solved with the transformation (15). The banded matrix, however, immediately suggests that, if $\underline{\phi}_i$ now represents two rows of Fig. 3a, i.e.,

$$\underline{\phi}_i = (\phi_{2(j-1)N+1}, \phi_{2(j-1)N+2}, \ldots, \phi_{2(j-1)N+2N}), \qquad (25)$$

the three term recurrence formula should prevail and transformation (15) may still apply, yet the matrices to be inverted would be twice in rank but half as many.

As we now see that the Ricatti transformation is essentially an algorithm for solving banded matrices, and its application is not limited to the elliptical difference equations. Of special interest is formula (22), which shows the relation between two adjacent R_i's. In the special case of the scalar Helmholtz equation Q is a tridiagonal matrix, but the R_i's are in general full matrices. Therefore, calculations of (22) should in general involve $N \times N$ full matrices. In other words, the Ricatti transformation does not take advantage of the sparsity of Q, except in the special case where Q is diagonal. In order to take full advantage of the sparsity within the band, the direct sparse matrix algorithm seems to be the only solution.

Results

The uni-moment method using Ricatti transformation to generate the trial function pairs has been successfully applied to solve biconical antennas. In this application we have used finite difference inside the bicone region. Typical results of the impedances at the antenna-freespace interface is shown in Fig. 5, together with the results obtained by Tai [15]. Our curve is interpolated from 40 computations and the total running time is 40 seconds using a CDC 6400 computer. The results of scattering of E_z^{inc} fields by dielectric cylindrical bodies are given in Fig. 6 and 7. These bistatic scattering patterns are computed at 10 sec. for seven different incident angles*. We have used finite element equations inside the circular regions of the above two calculations. The speed and simple programming of the uni-moment method in solving the above problems are not matched by any existing methods in field computations.

Remarks

Throughout the discussion of the uni-moment method, we have used the scalar Helmholtz equation to illustrate the principle techniques. However, the reader should not be misled to believe that solving the scalar Helmholtz equation is equivalent to solving Maxwell's equations. It would be true only in a homogeneous medium. But, in inhomogeneous media, where the uni-moment method finds its greatest usefulness, Maxwell's equations can not be reduced to scalar Helmholtz type equations. The only exceptions are the two dimensional scattering by dielectric body in TE mode, and scattering by ferrite in TM mode. The TM wave scattering by dielectric cylinder, should result in the differential equation,

$$\nabla^2\phi + \frac{\nabla\varepsilon}{\varepsilon}\cdot\nabla\phi + k^2\phi = 0 \qquad (26)$$

The variational formulation of (26) results in the optimization of the functional,

*Since the method always results in inverses of the matrices the running time is not proportional to the number of incident angles.

$$I = \int_S \{\log \varepsilon |\nabla\phi|^2 - 2k^2\phi\} \, ds \tag{27}$$

In three dimensional inhomogeneous media the Maxwell's equations may be formulated in coupled scalar partial differential equations, which are more complicated. In the special case of rotational symmetry, it is possible to reduce Maxwell's equations to two coupled scalar partial differential equations, and corresponding functional for the variational formulation can also be found. We shall present these details when we discuss specific applications of uni-moment methods in future papers.

Conclusion

The concept of uni-moment separates the exterior boundary value problem of an antenna or scattering problem from an interior one. The separation is made possible by generating many linearly independent solutions both in the exterior and the interior regions. The final solution is then obtained by enforcing continuity conditions. This appraoch, while new in numerical methods, is a standard one in the classical methods of separation of variables, (such as, scattering by a dielectric circular cylinder) where the exterior fields are expanded in series of Hankel's functions, and the interior parts in Bessel's functions. The uni-moment method uses computer generated solutions in the interior region. With the problem limited to a finite closed region, it becomes possible to replace integral equations by finite difference or finite element equations. The advantage of this change is the easy formulation for inhomogeneous media problems, but we have also traded a stable system of (integral) equations for an unstable system of (difference or finite element) equations. Because of the instability of the equations, direct solutions are necessary. The methods of generating solutions, as discussed in this paper are rather elementary ones, yet they are applicable to many problems, which hitherto were unsolvable. The power of the uni-moment method will be further enhanced when better techniques of solving sparse banded matrices are uncovered by antenna engineers. Hopefully, this introductory discussion of the uni-moment method will influence many antenna engineers to join our colleagues in industrial, structure, power and circuit engineering in the fascinating land of sparse matrix research.

Acknowledgement

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Figure Captions

Fig. 1 Schematic of scattering by a dielectric cylindrical obstacle

Fig. 2 (a) A rectangular mesh for the finite difference method

(b) A triangular mesh for the finite element method

Fig. 3 Results of computation of scattering of a plane wave by a metal grating

Fig. 4 Partition of a banded matrix

Fig. 5 The impedance s a biconical antenna at the cone-free space interface

Fig. 6 Scattered E-field by a dielectric semi-circular cylinder of radius $a = 0.3\lambda$, $\varepsilon = 5.0$

Fig. 7 Scattered E-field by a middle section of a dielectric circular cylinder of radius $a = 0.3\lambda$, $\varepsilon = 3.0$

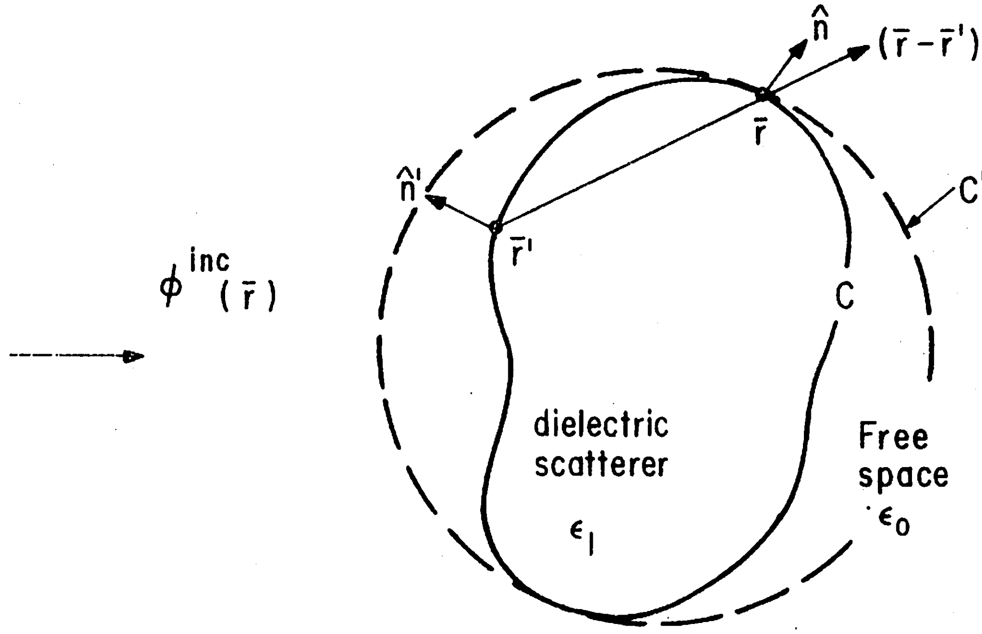

n̂
$(\bar{r}-\bar{r}')$
$\bar{r}$
n̂'
C'
$\bar{r}'$
$\phi^{inc}(\bar{r})$
C
dielectric
scatterer
Free
space
ϵ_1
ϵ_0

Figure 1.

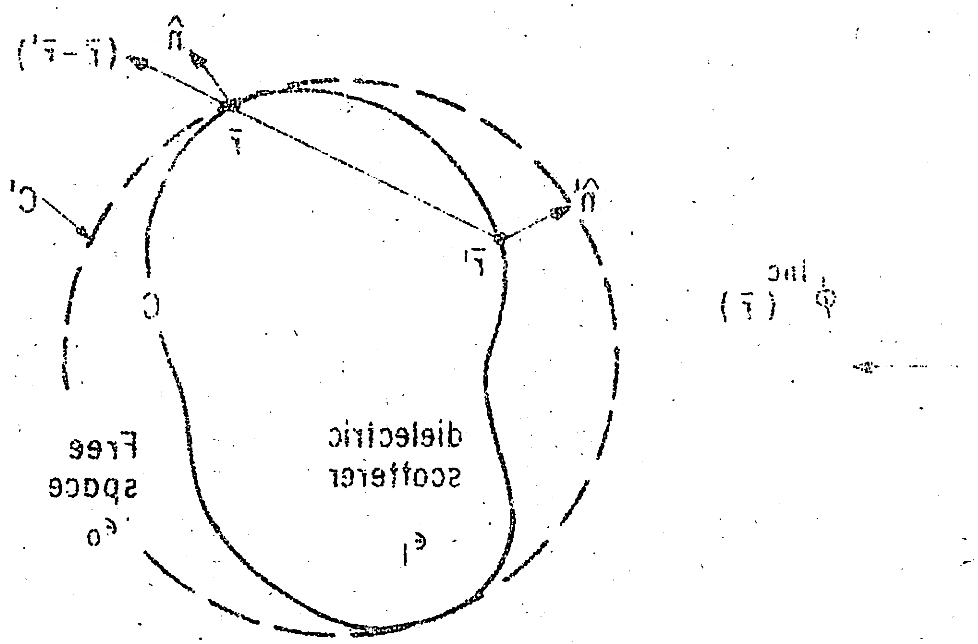


Figure 1.

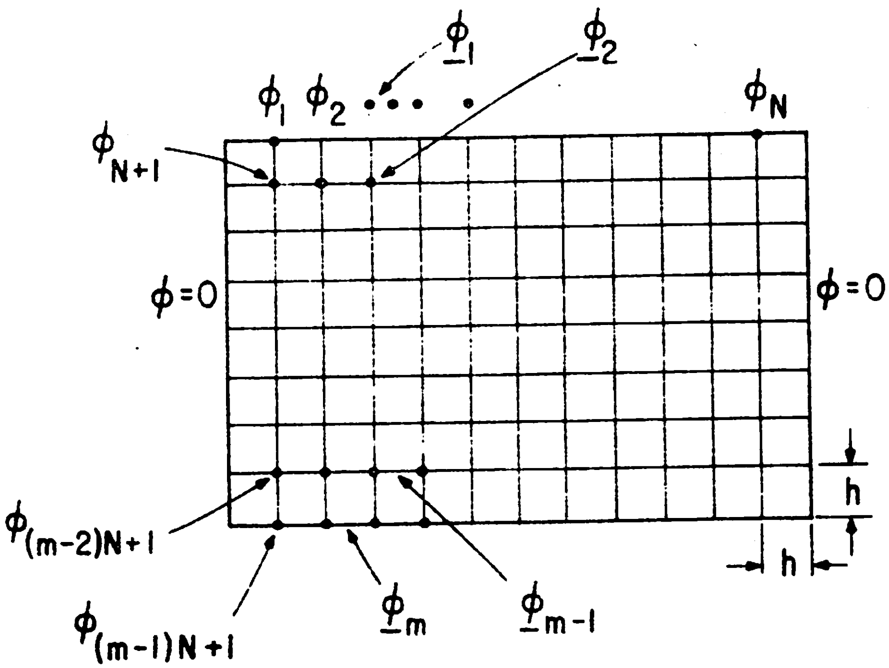


Figure 2(a).

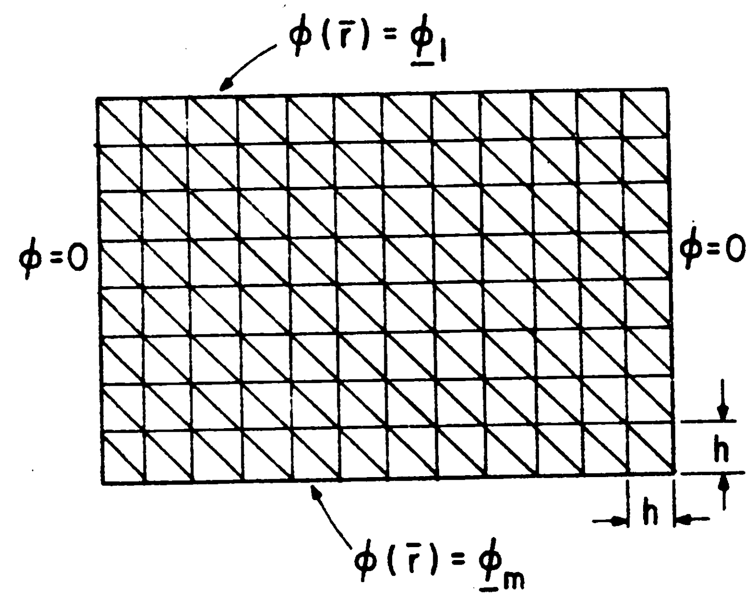
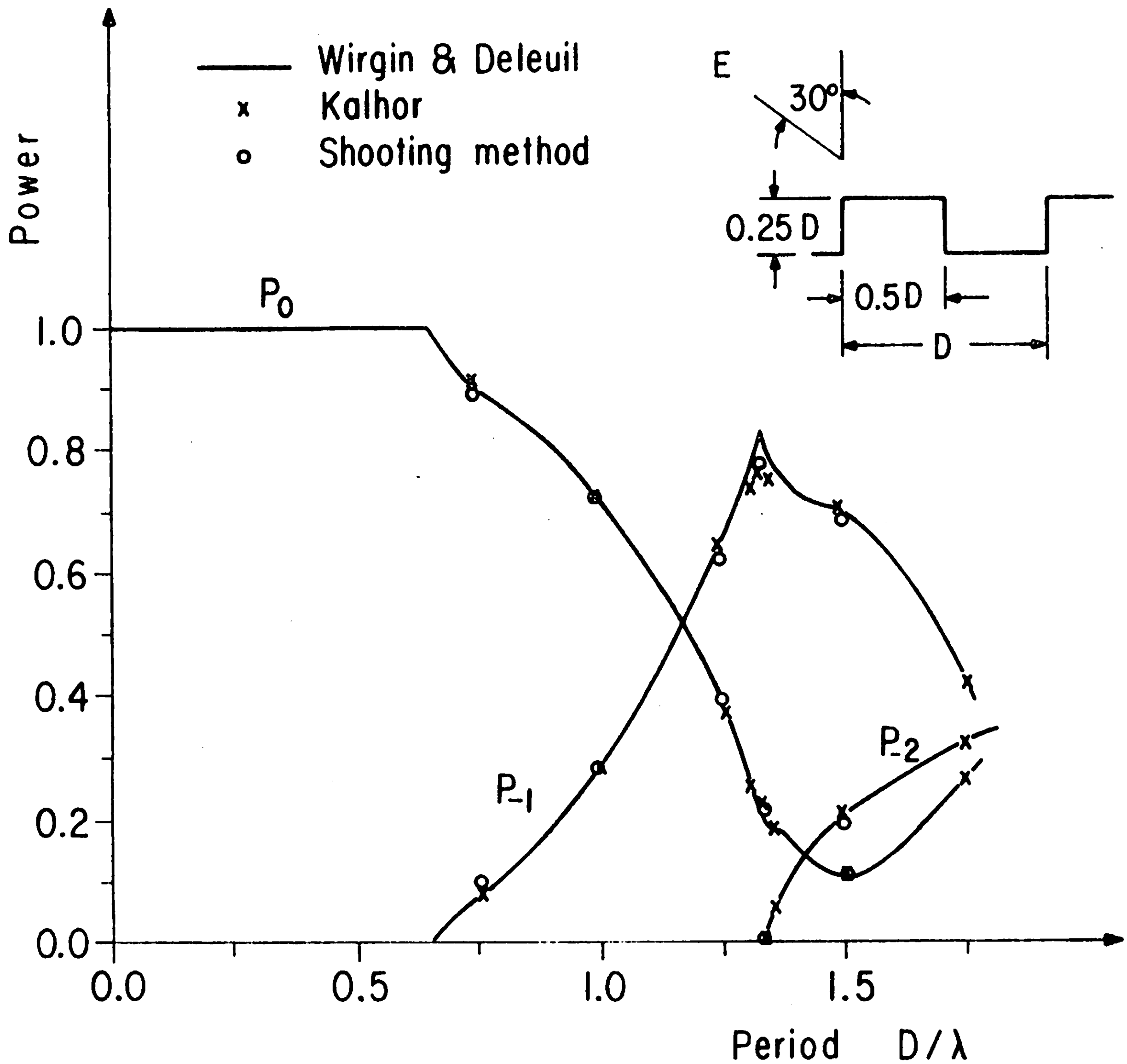


Figure 2(b).



Wirgin & Deleuil
x Kalhor
o Shooting method
E
30°
0.25 D
0.5 D
D
Power
P_0
P_{-1}
P_{-2}
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
Period D/λ

Figure 3.

$$
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & 0 & 0 & \cdots \\
A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & \cdots \\
0 & A_{32} & A_{33} & A_{34} & 0 & 0 & \cdots \\
0 & 0 & A_{43} & A_{44} & A_{45} & 0 & \cdots \\
\vdots & \vdots & & & \ddots & & \\
 & & & & & A_{m,m-1} & A_{m,m}
\end{bmatrix}
$$

Figure 4.

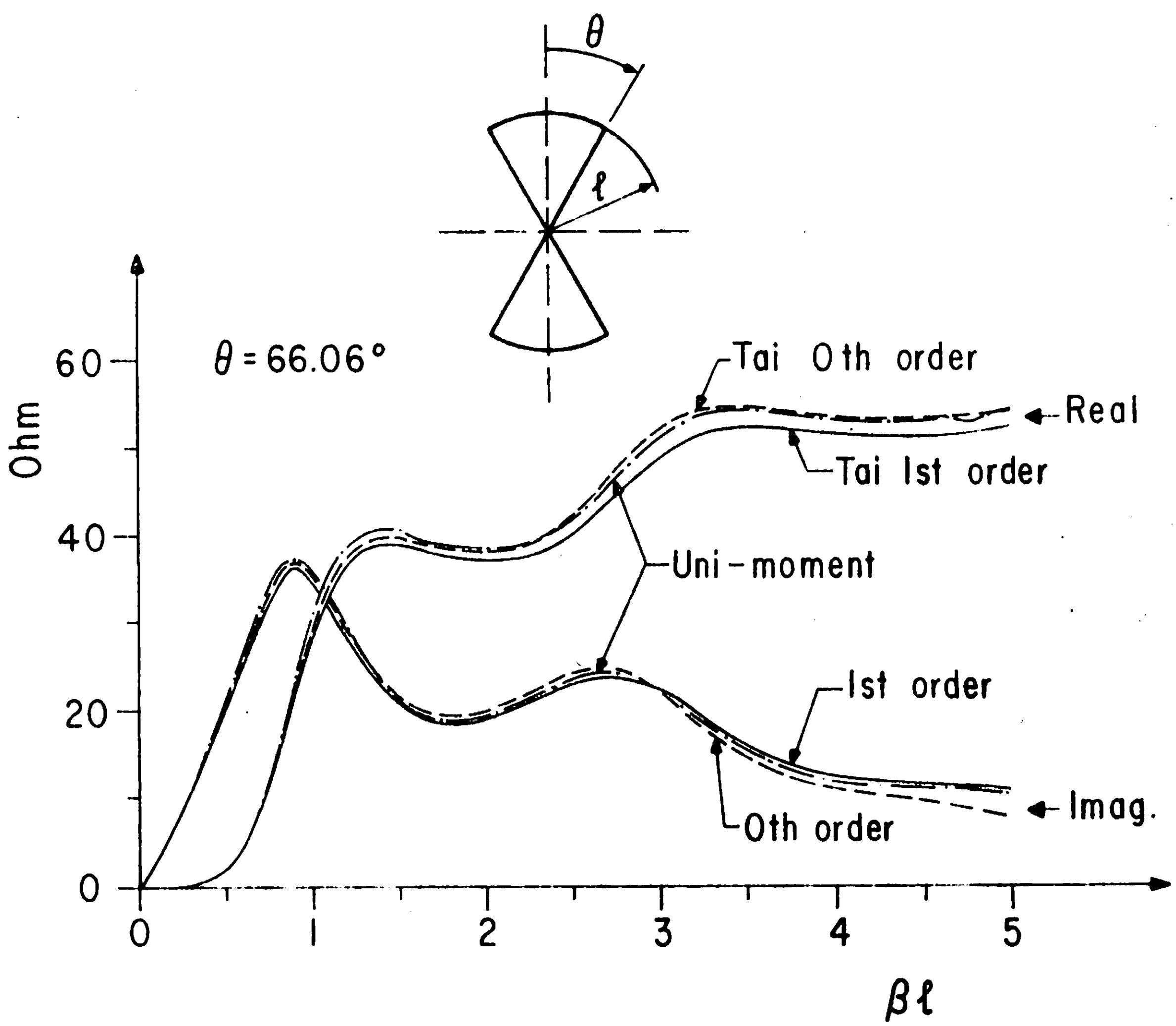

θ
ℓ
60
40
20
0
Ohm
θ = 66.06°
Tai 0th order
Real
Tai Ist order
Uni-moment
Ist order
Imag.
0th order
0
1
2
3
4
5
βℓ

Figure 5.

Scattering far field pattern

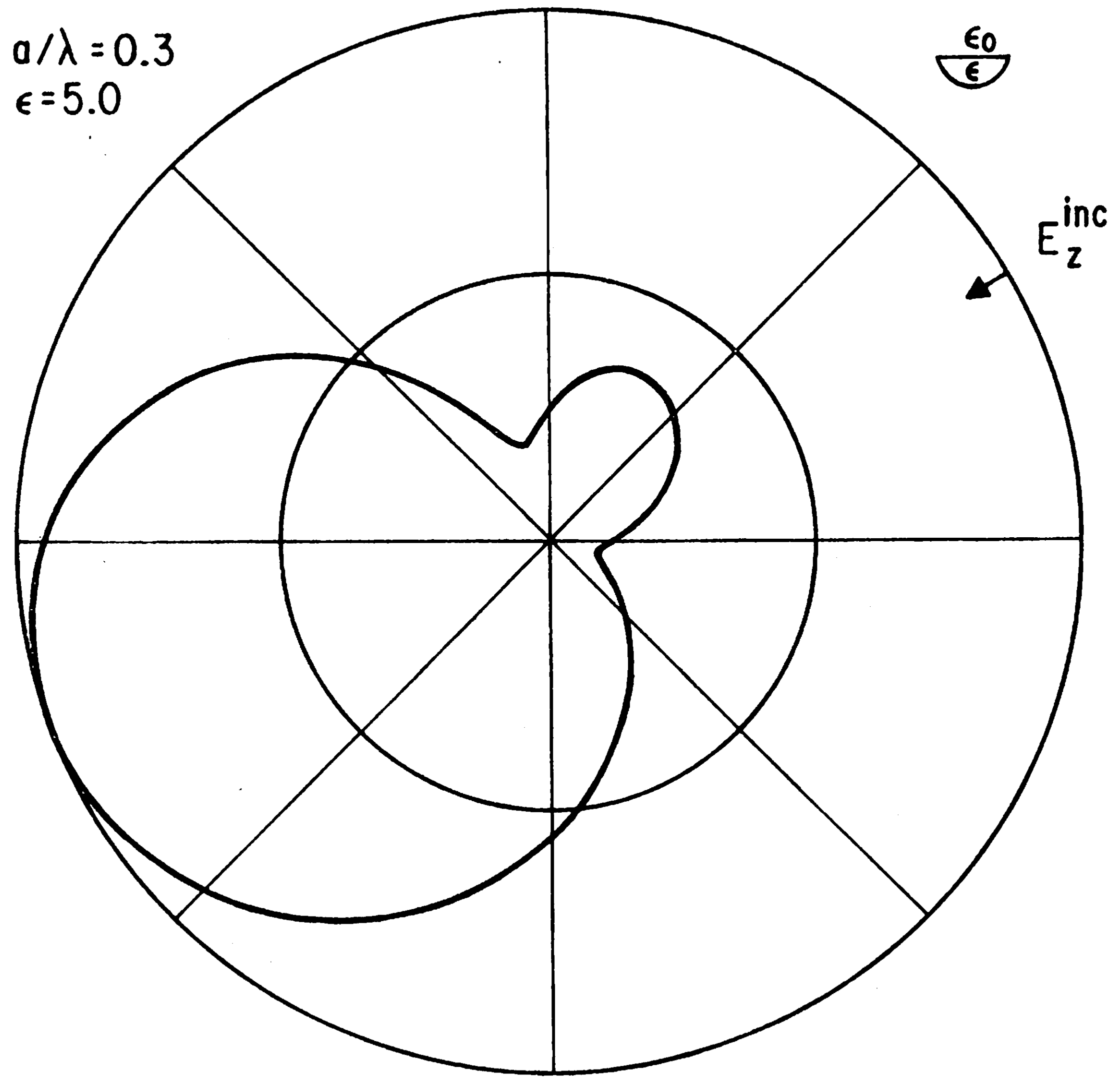


Figure 6.

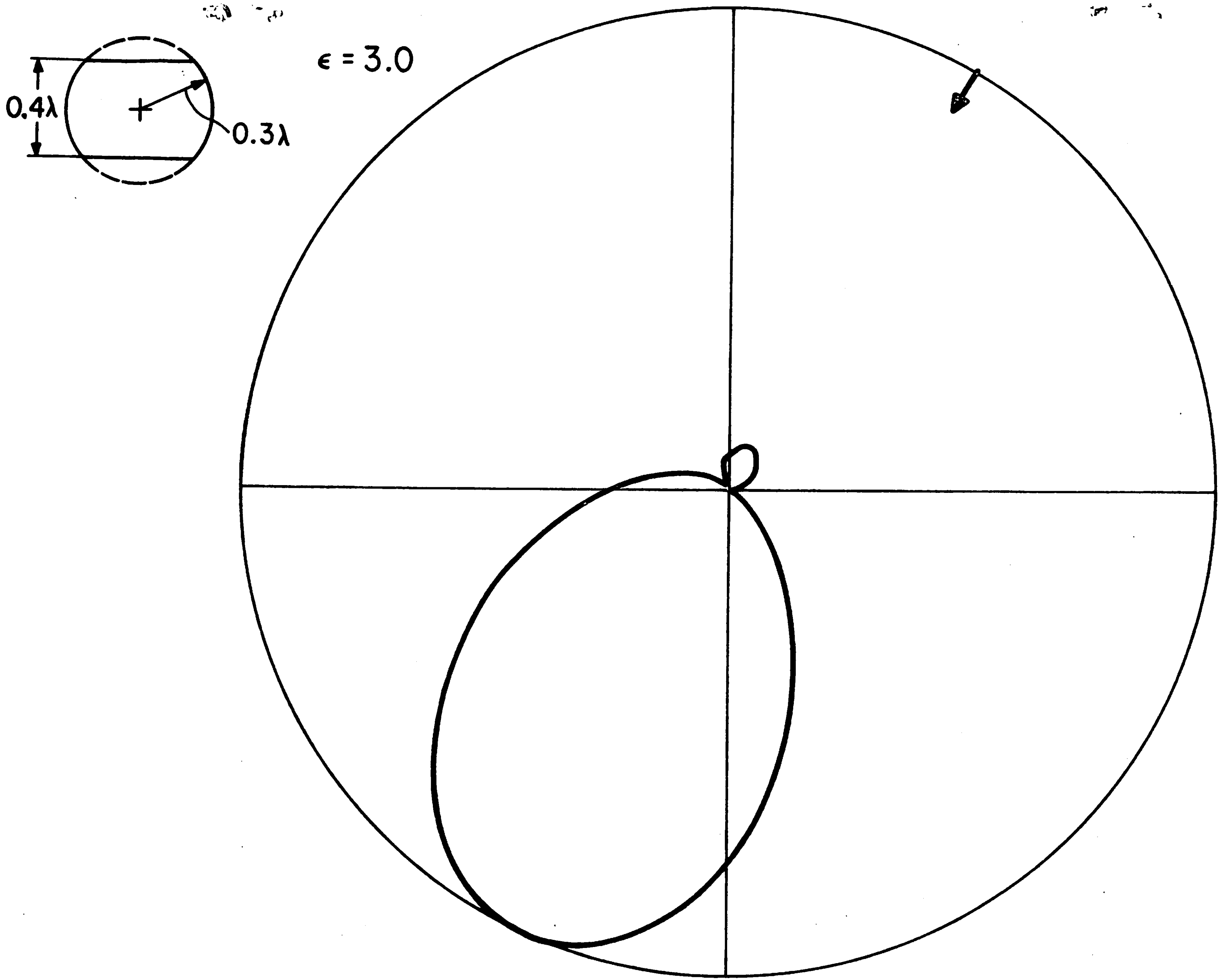


Figure 7.