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TRANSFORMATION OF LOCAL MARTINGALES UNDER A CHANGE OF LAW[1]

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Girsanov showed that under an absolutely continuous change in probability measure a Wiener process is transformed into the sum of a Wiener process and a second process with sample functions which are absolutely continuous. This result has a natural generalization in the context of local martingales. This generalization is derived in this paper, and some of its ramifications are examined. As a simple application, the likelihood ratio for a single-server queueing process with very general arrival and service characteristics is derived.

1. Introduction. Let $\{W_t, \mathscr{F}_t, 0 \leqq t \leqq 1\}$ be a Wiener process defined on a fixed probability space $(\Omega, \mathscr{A}, \mathscr{P})$. Let $\Lambda(\phi)$ be defined by

$$\Lambda(\phi) = \exp\{\int_0^1 \phi_s \, dW_s - \tfrac{1}{2} \int_0^1 \phi_s^2 \, ds\} \tag{1.1}$$

where $\{\phi_t, 0 \leqq t \leqq 1\}$ is a measurable process adapted to $\{\mathscr{F}_t\}$ and satisfies the condition

$$\int_0^1 \phi_s^2 \, ds < \infty \,, \quad \text{almost surely.} \tag{1.2}$$

If $E\Lambda(\phi) = 1$ then

$$\frac{d\mathscr{P}'}{d\mathscr{P}} = \Lambda(\phi) \tag{1.3}$$

defines a probability measure $\mathscr{P}'$. Girsanov [6] showed that the process

$$W_t' = W_t - \int_0^t \phi_s \, ds \tag{1.4}$$

is a Wiener process with respect to $\mathscr{P}'$ measure. It should be noted that if $\mathscr{P}'$ is any probability measure equivalent to $\mathscr{P}$ then it is necessarily of the form (1.3) [5].

Recent results on continuous parameter martingales and local martingales have made it clear that Girsanov's result has a natural generalization in the context of local martingales, and the statement of the resulting generalization may even be simpler. The objective of this paper is to obtain this generalization and to investigate some of its ramifications. The special case of continuous local martingales was discussed in [10].

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2. Local martingales and stochastic integrals. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space, and consider an increasing family of σ-subfields $\{\mathscr{F}_t, 0 \leqq t < \infty\}$. We shall assume $\{\mathscr{F}_t\}$ to be right-continuous, i.e.,

$$\bigcap_{s>t} \mathscr{F}_s = \mathscr{F}_t, \qquad 0 \leqq t < \infty .$$

A positive random variable τ is said to be a *stopping time* of $\{\mathscr{F}_t\}$ if $\{\omega : \tau(\omega) \leqq t\} \in \mathscr{F}_t$ for every t. For a stopping time τ, $\mathscr{F}_\tau$ will denote the σ-field of all events A for which

$$A \cap \{\omega : \tau(\omega) \leqq t\} \in \mathscr{F}_t .$$

We shall say $\{\mathscr{F}_t\}$ has *no time of discontinuity* if for every increasing sequence of stopping times $\{\tau_n\}$

$$\mathscr{F}_{\lim \tau_n} = \lim \mathscr{F}_{\tau_n} .$$

A process $\{X_t, 0 \leqq t < \infty\}$ is said to be *adapted* to $\{\mathscr{F}_t\}$ if for each $t X_t$ is $\mathscr{F}_t$-measurable. Now let $\mathscr{H}$ denote the σ-field of subsets of $[0, \infty) \times \Omega$ generated by the family of all *left-continuous* processes adapted to $\{\mathscr{F}_t\}$. $\mathscr{H}$-measurable (t, ω) functions will be called *predictable* processes.

We shall adopt the following notations and definitions:

(2.1) $\mathscr{M} =$ {the set of all real-valued right continuous processes M_t with left-limits and adapted to $\{\mathscr{F}_t\}$ such that $M_0 = 0$ and $E(M_{t+s} \mid \mathscr{F}_t) = M_t$, a.s., $\forall\, s > 0$};

(2.2) $\mathscr{M}^c = \{M \in \mathscr{M}$ and M sample continuous$\}$;

(2.3) $\mathscr{M}^2 = \{M \in \mathscr{M}$ and $\sup_{0 \leqq t < \infty} EM_t^2 < \infty\}$;

(2.4) $\mathscr{M}_{\mathrm{loc}} = \{M$: there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that $M_t^{(n)} = M_{t \wedge \tau_n} \in \mathscr{M}$ for each $n\}$;

(2.5) $\mathscr{M}_{\mathrm{loc}}^c = \{M \in \mathscr{M}_{\mathrm{loc}}, M$ sample continuous$\}$;

(2.6) $\mathscr{M}_{\mathrm{loc}}^2 = \{M \in \mathscr{M}_{\mathrm{loc}}, M^{(n)} \in \mathscr{M}^2$ for each $n\}$.

Elements in $\mathscr{M}$, $\mathscr{M}_{\mathrm{loc}}$, etc. will be referred to as martingales and local martingales respectively, with adjectives *continuous* and *squareintegrable* added as appropriate. Note that $\mathscr{M}_{\mathrm{loc}}^c \subset \mathscr{M}_{\mathrm{loc}}^2$, since one can always take $\tau_n = \inf\{t: |M_s| \leqq n, \forall\, s \leqq t\}$ for $M \in \mathscr{M}_{\mathrm{loc}}^c$.

If $X \in \mathscr{M}_{\mathrm{loc}}^2$ then there exists a unique predictable increasing process $\langle X, X\rangle$ such that $X^2 - \langle X, X\rangle \in \mathscr{M}_{\mathrm{loc}}$. If $X, Y \in \mathscr{M}_{\mathrm{loc}}^2$ then $\langle X, Y\rangle$ is defined by

$$\langle X, Y\rangle = \tfrac{1}{4}[\langle X+Y, X+Y\rangle - \langle X-Y, X-Y\rangle] .$$

Kunita and Watanabe [7] introduced the process $\langle X, Y\rangle$ and illuminated its role in the stochastic calculus associated with $\mathscr{M}_{\mathrm{loc}}^2$.

Two local martingales X and Y are said to be orthogonal if their product XY is again a local martingale. Now, denote

(2.7) $\mathscr{M}_{\mathrm{loc}}^d = \{X \in \mathscr{M}_{\mathrm{loc}} : X$ orthogonal to every element of $\mathscr{M}_{\mathrm{loc}}^c\}$.

Then, every $X \in \mathscr{M}_{\mathrm{loc}}$ has a unique decomposition

$$X = X^c + X^d , \qquad X^c \in \mathscr{M}_{\mathrm{loc}}^c,\ X^d \in \mathscr{M}_{\mathrm{loc}}^d . \tag{2.8}$$

We can now define the increasing process $[X, X]$ for every $X \in \mathscr{M}_{\mathrm{loc}}$ by

$$[X, X]_t = \langle X^c, X^c \rangle_t + \sum_{s \le t} (\Delta X_s^d)^2 \tag{2.9}$$

where the summation is taken over all points of discontinuity of X and $\Delta X_s = X_s - X_{s-}$ [9]. For $X, Y \in \mathscr{M}_{\mathrm{loc}}$, we set

$$[X, Y] = \tfrac{1}{4}\{[X + Y, X + Y] - [X - Y, X - Y]\} .$$

If $X, Y \in \mathscr{M}_{\mathrm{loc}}^2$ then both $XY - \langle X, Y \rangle$ and $XY - [X, Y]$ are local martingales. Hence $[X, Y] - \langle X, Y \rangle$ is also a local martingale. The main difference between $\langle X, Y \rangle$ and $[X, Y]$, when they both exist, is that $\langle X, Y \rangle$ is predictable while in general $[X, Y]$ is not. We note that given $[X, Y]$, $\langle X, Y \rangle$ is characterized by the properties:

(a) $\langle X, Y \rangle$ is predictable,
(b) $[X, Y] - \langle X, Y \rangle$ is a local martingale and
(c) $\langle X, Y \rangle$ is of bounded variation.

This allows us to generalize the definition of the process $\langle X, Y \rangle$ to those cases where X, Y are not in $\mathscr{M}_{\mathrm{loc}}^2$. Let $X, Y \in \mathscr{M}_{\mathrm{loc}}$. Then, there exists at most one predictable adapted process of bounded variation $\langle X, Y \rangle$ such that $[X, Y] - \langle X, Y \rangle$ is a local martingale [2]. This serves to define $\langle X, Y \rangle$ whenever it exists. Later, we shall give some examples of cases where $\langle X, Y \rangle$ is well defined, but $X, Y \notin \mathscr{M}_{\mathrm{loc}}^2$.

Let $\mathscr{A}$ denote the set of right-continuous, finite, increasing processes adapted to $\{\mathscr{F}_t\}$ and set $\mathscr{B} = \{B : B = A^+ - A^-,\ A^+, A^- \in \mathscr{A}\}$. *Semi-martingales* are processes of the form

$$X_t = X_0 + M_t + B_t , \qquad M \in \mathscr{M},\ B \in \mathscr{B}. \tag{2.10}$$

If $M \in \mathscr{M}_{\mathrm{loc}}$ instead of $\mathscr{M}$, we shall call X a local semi-martingale. The representation (2.10) is by no means unique. However, the continuous component M^c of M is independent of the decomposition. Because of this we can define for local semi-martingales

$$[X, X]_t = \langle M^c, M^c \rangle_t + \sum_{s \le t} (\Delta X_s)^2 . \tag{2.11}$$

The definition for $[X, Y]$ follows in the usual way.

For a local martingale M a salient feature of the definition of a stochastic integral

$$(H \cdot M)_t = \int_0^t H_s \, dM_s$$

is that $H \cdot M$ is again a local martingale. For this to be possible H must be a predictable process. Define the following classes of integrands:

(2.12) $L_B = \{$all predictable H: there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that $\sup_{t,\omega} I_{(\tau_n > 0)} |H_{t\wedge\tau_n}| < \infty$ for each $n\}$. Processes in L_B are said to be locally bounded.

(2.13) $L^2(M) = \{$all predictable H: $\int_0^\infty H_s^2\, d\langle M, M\rangle_s < \infty$, almost surely$\}$.

THEOREM 2.1 [4]. *If $M \in \mathscr{M}_{\text{loc}}$ and $H \in L_B$, there is one and only one process $H \cdot M \in \mathscr{M}_{\text{loc}}$ such that*

$$[H \cdot M, N]_t = \int_0^t H_s\, d[M, N]_s\,, \qquad \forall N \in \mathscr{M}_{\text{loc}}\,. \tag{2.14}$$

If $M \in \mathscr{M}^2_{\text{loc}}$ and $H \in L^2(M)$, there is one and only one $H \cdot M \in \mathscr{M}^2_{\text{loc}}$ satisfying (2.14). Further, the existence of $\langle M, N\rangle$ implies the existence of $\langle H \cdot M, N\rangle$ and

$$\langle H \cdot M, N\rangle_t = \int_0^t H_s\, d\langle M, N\rangle_s\,. \tag{2.15}$$

We note that if $M \in \mathscr{M}_{\text{loc}} \cap \mathscr{B}$ and $H \in L_B$ then $H \cdot M$ is again in $\mathscr{M}_{\text{loc}} \cap \mathscr{B}$ and the stochastic integral coincides with the Stieltjes integral. If $M \in \mathscr{M}_{\text{loc}} \cap \mathscr{B}$ the Stieltjes integral $\int_0^t H_s\, dM_s$ may well exist even if H is not predictable, but the Stieltjes integral is no longer a local martingale and the stochastic integral is not well defined.

If X is a local semi-martingale

$$X_t = X_0 + M_t + B_t\,, \qquad M \in \mathscr{M}_{\text{loc}},\ B \in \mathscr{B}_{\text{loc}} \tag{2.16}$$

and $H \in L_B$ then we define the stochastic integral $H \cdot X$ by setting

$$(H \cdot X)_t = (H \cdot M)_t + \int_0^t H_s\, dB_s + H_0 X_0 \tag{2.17}$$

and $H \cdot X$ is again a local semi-martingale. For local semi-martingales X and Y, we have

$$[H \cdot X, Y]_t = \int_0^t H_s\, d[X, Y]_s\,. \tag{2.18}$$

If X is a local semi-martingale (by assumption right continuous) then X_{t-} is a locally bounded predictable process. If $\gamma: R \to R$ is a twice continuously differentiable function then $\gamma(X_t)$ is a local semi-martingale and we have the differentiation formula

$$\begin{aligned}\gamma(X_t) = \gamma(X_0) &+ \int_0^t \gamma'(X_{s-})\, dX_s + \tfrac{1}{2} \int_0^t \gamma''(X_{s-})\, d\langle X^c, X^c\rangle_s \\ &+ \textstyle\sum_{s\le t} [\gamma(X_s) - \gamma(X_{s-}) - \gamma'(X_{s-})\Delta X_s]\,.\end{aligned} \tag{2.19}$$

An interesting special case is

$$\begin{aligned}X_t^2 &= X_0^2 + 2\int_0^t X_{s-}\, dX_s + \langle X^c, X^c\rangle_t + \textstyle\sum_{s\le t} (\Delta X_s)^2 \\ &= X_0^2 + 2\int_0^t X_{s-}\, dX_s + [X, X]_t\end{aligned} \tag{2.20}$$

which shows that if $X \in \mathscr{M}_{\text{loc}}$ then $X^2 - [X, X] \in \mathscr{M}_{\text{loc}}$.

Equation (3.14) can be extended to a function of a vector valued X_t. However, the only special case that we will need is

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s + [X, Y]_t\,. \tag{2.21}$$

3. Transformation of local martingales. If $\{W_t, \mathscr{F}_t, 0 \leq t < \infty\}$ is a Wiener process and $\phi \in L^2(W)$ then a simple application of the differentiation formulas (the original Itô version will suffice) shows that

$$\Lambda_t = \exp\left(\int_0^t \phi_s\, dW_s - \tfrac{1}{2} \int_0^t \phi_s^2\, ds\right) \tag{3.1}$$

must satisfy

$$\Lambda_t = 1 + \int_0^t \Lambda_s \phi_s\, dW_s\,. \tag{3.2}$$

McKean [8] showed that (3.2) characterized Λ, and Doléans–Dade [3] has extended the result to local martingales.

THEOREM 3.1 (Doléans–Dade). *Let X be a local semi-martingale such that $X_0 = 0$. Then there is one and only one local semi-martingale Λ which satisfies*

$$\Lambda_t = 1 + \int_0^t \Lambda_{s-}\, dX_s \tag{3.3}$$

and it is given by

$$\Lambda_t = \exp\left(X_t - \tfrac{1}{2}\langle X^c, X^c\rangle_t\right) \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}\,. \tag{3.4}$$

COMMENTS. (i) It is clear from (3.3) that $X \in \mathscr{M}_{\mathrm{loc}}$ implies $\Lambda - 1 \in \mathscr{M}_{\mathrm{loc}}$. However, stronger conditions do not always lead to stronger conditions on Λ. For example, $X \in \mathscr{M}$ does not imply $\Lambda - 1 \in \mathscr{M}$. If $X \in \mathscr{M}^2$, it is not even known if $\Lambda - 1 \in \mathscr{M}^2_{\mathrm{loc}}$.

(ii) It is clear from (3.4) that for Λ to be strictly positive almost surely, it is necessary and sufficient to require

$$\Delta X_t > -1 \qquad \text{with probability 1 for all } t. \tag{3.5}$$

If Λ is a uniformly integrable positive martingale (not merely a local martingale) then

$$\Lambda_\infty = \lim_{t \to \infty} \Lambda_t \tag{3.6}$$

exist and $E(\Lambda_\infty \mid \mathscr{F}_t) = \Lambda_t$, a.s. It follows that $E\Lambda_\infty = E\Lambda_0 = 1$. Given such a positive martingale $(\Lambda_t, \{\mathscr{F}_t\})$, we can define a transformation of the probability measure $\mathscr{P}$ by the formula

$$\frac{d\mathscr{P}'}{d\mathscr{P}} = \Lambda_\infty\,. \tag{3.7}$$

We note that $\mathscr{P}'$ is equivalent to $\mathscr{P}$ if $\Lambda_\infty > 0$ for almost all ω($\mathscr{P}$-measure). Our primary interest in this paper is to investigate the transformation of local martingales when the underlying probability measure undergoes a change of the form given by (3.7).

THEOREM 3.2. *Let $M \in \mathscr{M}_{\mathrm{loc}}(\mathscr{P}, \{\mathscr{F}_t\})$ be such that the solution Λ_t of*

$$\Lambda_t = 1 + \int_0^t \Lambda_{s-}\, dM_s$$

is a uniformly integrable positive martingale with respect to $(\mathscr{P}, \{\mathscr{F}_t\})$. *Let* $\mathscr{P}'$ *be defined by* (3.7). *Let* $X \in \mathscr{M}_{\mathrm{loc}}(\mathscr{P}, \{\mathscr{F}_t\})$. *Suppose the process* $\langle X, M\rangle$ *exists, then*

$$Z_t = X_t - \langle X, M\rangle_t$$

belongs to $\mathscr{M}_{\mathrm{loc}}(\mathscr{P}', \{\mathscr{F}_t\})$. *Furthermore, if* $\langle X, M\rangle_t$ *is sample continuous then* $[Z, Z] = [X, X]$ *under either probability measure.*

PROOF. Because

$$E'(Z_{t+s} \mid \mathscr{F}_t) = \frac{E(\Lambda_{t+s} Z_{t+s} \mid \mathscr{F}_t)}{\Lambda_t}$$

to prove $Z \in \mathscr{M}_{\mathrm{loc}}(\mathscr{P}', \{\mathscr{F}_t\})$, it is sufficient to prove $Z\Lambda \in \mathscr{M}_{\mathrm{loc}}(\mathscr{P}, \{\mathscr{F}_t\})$. Since Z and Λ are both local semi-martingales with respect to $(\mathscr{P}, \{\mathscr{F}_t\})$ we can write

$$\begin{aligned} Z_t\Lambda_t &= \int_0^t Z_{s-}\, d\Lambda_s + \int_0^t \Lambda_{s-}\, dZ_s + [Z, \Lambda]_t \\ &= \int_0^t Z_{s-}\, d\Lambda_s + \int_0^t \Lambda_{s-}\, dX_s - \int_0^t \Lambda_{s-}\, d\langle X, M\rangle_s + [Z, \Lambda]_t\,. \end{aligned}$$

We can now use (3.3) to find $[Z, \Lambda]_t$ and get

$$\begin{aligned} Z_t\Lambda_t = \int_0^t Z_{s-}\, d\Lambda_s &+ \int_0^t \Lambda_{s-}\, dX_s + \int_0^t \Lambda_{s-}\, d\{[X, M]_s - \langle X, M\rangle_s\} \\ &+ \int_0^t \Lambda_{s-}\, d[\langle X, M\rangle, M]_s\,. \end{aligned}$$

It remains only to prove that the last term is a local martingale.

Since the continuous local martingale component of $\langle X, M\rangle$ is zero, we have

$$[\langle X, M\rangle, M]_t = \sum_{s \le t} (\Delta\langle X, M\rangle_s \Delta M_s)\,.$$

We note that $\langle X, M\rangle$ is of bounded variation so that $\Delta\langle X, M\rangle$ is certainly locally bounded. Further, since $\langle X, M\rangle$ is predictable its jumps occur at predictable times. Hence, $\int_0^t \Delta\langle X, M\rangle_s dM_s^d$ is a compensated sum of *predictable* jumps and is equal to $\sum_{s \le t} (\Delta\langle X, M\rangle_s \Delta M_s)$. Therefore, $[\langle X, M\rangle, M]$ is a local martingale with respect to $(\mathscr{P}, \{\mathscr{F}_t\})$ and $\int_0^t \Lambda_{s-}\, d[\langle X, M\rangle, M]_s$ is also a local martingale. The proof that Z is a local martingale with respect to $(\mathscr{P}', \{\mathscr{F}_t\})$ is now complete.

To prove $[Z, Z] = [X, X]$ when $\langle X, M\rangle$ is sample-continuous, we first assume $\mathscr{P}$ to be the underlying measure, and write

$$Z_t = X_t^c + X_t^d - \langle X, M\rangle_t$$

where $X^c \in \mathscr{M}_{\mathrm{loc}}^c$, $X^d \in \mathscr{M}_{\mathrm{loc}}^d$ and $\langle X, M\rangle \in \mathscr{B}_{\mathrm{loc}}$.

By definition (cf. (2.15))

$$[Z, Z]_t = \langle X^c, X^c\rangle_t + \sum_{s \le t} [\Delta X_s^d - \Delta\langle X, M\rangle_s^d]^2\,.$$

If $\langle X, M\rangle$ is continuous, then

$$[Z, Z]_t = \langle X^c, X^c\rangle_t + \sum_{s \le t} (\Delta X_s^d)^2 = [X, X]_t\,.$$

Under $\mathscr{P}'$, the same conclusion follows from writing

$$X_t = Z_t^c + Z_t^d + \langle X, M\rangle_t\,.$$

Since $[Z, Z]_t$ is the L_1-limit of quadratic variations $\sum_\nu (\Delta_\nu^n Z)^2$ for some sequence of partitions of $[0, t]$ and since $\mathscr{P}' \ll \mathscr{P}$, it is easy to show that on the set $\Delta > 0$, $[Z, Z]$ and $[X, X]$ are independent of which probability measure is chosen.

A special case of Theorem 3.2 which is worth isolating is the following. Suppose $\langle X, X\rangle$ exists and $M = \phi \cdot X$. Then we have the result that

$$Z_t = X_t - \int_0^t \phi_s \, d\langle X, X\rangle_s$$

is a local martingale with respect to $(\mathscr{P}', \{\mathscr{F}_t\})$. Specializing still more, we can take X to be a Wiener process with respect $(\mathscr{P}, \{\mathscr{F}_t\})$ then $\langle X, X\rangle_t = t$ and

$$Z_t = X_t - \int_0^t \phi_s \, ds$$

is a continuous local martingale with respect to $(\mathscr{P}', \{\mathscr{F}_t\})$. Since $[Z, Z]_t = [X, X]_t = t$, Z is in fact a Wiener process under $\mathscr{P}'$-measure. This is the theorem of Girsanov [6].

4. Some applications. Let $\{N_t, 0 \leqq t < \infty\}$ be a Poisson process with rate 1 under the probability measure $\mathscr{P}$. Let $\mathscr{F}_t$ denote the σ-field generated by $\{N_s, s \leqq t\}$. The process $X_t = N_t - t$ is a $(\mathscr{P}, \{\mathscr{F}_t\})$ locally square-integrable martingale with $\langle X, X\rangle_t = t$ and $[X, X]_t = N_t$. Let $\mathscr{P}'$ be a probability measure equivalent to $\mathscr{P}$ and set

$$\Lambda_t = E\left(\frac{d\mathscr{P}'}{d\mathscr{P}}\middle|\mathscr{F}_t\right). \tag{4.1}$$

Then Λ is a $(\mathscr{P}, \{\mathscr{F}_t\})$ martingale and has a representation

$$\Lambda_t = 1 + \int_0^t \psi_s \, dX_s \tag{4.2}$$

for a predictable and integrable ψ.[2] Since $\Lambda_t > 0$ with $\mathscr{P}$-measure 1, we can define

$$\phi_s = \Lambda_{s-}^{-1} \psi_s \tag{4.3}$$

and

$$M_t = \int_0^t \phi_s \, dX_s. \tag{4.4}$$

Equation (4.2) now reads

$$\Lambda_t = 1 + \int_0^t \Lambda_{s-} \, dM_s. \tag{4.5}$$

Since from (4.4) we have

$$\begin{aligned} \langle X, M\rangle_t &= \int_0^t \phi_s \, d\langle X, X\rangle_s \\ &= \int_0^t \phi_s \, ds; \end{aligned} \tag{4.6}$$

Theorem 3.2 implies that

$$\begin{aligned} Z_t &= X_t - \int_0^t \phi_s \, ds \\ &= N_t - t - \int_0^t \phi_s \, ds \end{aligned} \tag{4.7}$$

[2] For $\Lambda \in M^2$ this representation is due to Kunita and Watanabe [7]. The generalization to non-square integrable Λ is due to P. P. Varaiya, unpublished.

is a $(\mathscr{P}', \{\mathscr{F}_t\})$ local martingale. Because $\Lambda_t > 0$ for all t requires $\Delta M_t > -1$ for all t, we have

$$\phi_t > -1 \qquad \text{for all } t .$$

Hence, we can define a positive predictable process $\lambda_t = \phi_t + 1$ and rewrite (4.7) as

$$Z_t = N_t - \textstyle\int_0^t \lambda_s\, ds \tag{4.8}$$

which is a local martingale with respect to $(\mathscr{P}', \{\mathscr{F}_t\})$.

Since $\langle X, X\rangle_t = t$ is continuous, the second half of Theorem 3.2 yields

$$[Z, Z]_t = [X, X]_t = N_t . \tag{4.9}$$

Because $N_t - \int_0^t \lambda_s\, ds \in \mathscr{M}_{\text{loc}}(\mathscr{P}', \{\mathscr{F}_t\})$ and $\langle Z, Z\rangle$ is unique, we have

$$\langle Z, Z\rangle_t = \textstyle\int_0^t \lambda_s\, ds . \tag{4.10}$$

These results can be summarized as follows: [1]

THEOREM 4.1. *Let N_t, $t \geqq 0$, be a standard Poisson process with respect to $(\mathscr{P}, \{\mathscr{F}_t\})$ and let $\mathscr{F}_t = \sigma(N_s, s \leqq t)$. Let $\mathscr{P}'$ be a probability measure equivalent to $\mathscr{P}$. Then with respect to $(\mathscr{P}', \{\mathscr{F}_t\})$, N_t has a unique decomposition*

$$N_t = Z_t + \textstyle\int_0^t \lambda_s\, ds \tag{4.11}$$

where $Z \in \mathscr{M}_{\text{loc}}(\mathscr{P}', \{\mathscr{F}_t\})$, λ is a positive predictable process, and $\langle Z, Z\rangle_t = \int_0^t \lambda_s\, ds$. Furthermore, the likelihood ratio is given by

$$\Lambda_t = E\left(\frac{d\mathscr{P}'}{d\mathscr{P}}\,\middle|\,\mathscr{F}_t\right) = \left(\textstyle\prod_{s\le t} \lambda_s\right) \exp\left(-\textstyle\int_0^t (\lambda_s - 1)\, ds\right) \tag{4.12}$$

where the product is taken over all jumps of N.

A straightforward generalization of the above yields the following result. Let $\mathbf{N}_t$ be a vector process with components which are independent Poisson processes with respect to $(\mathscr{P}, \{\mathscr{F}_t\})$, and let $\mathscr{F}_t = \sigma(\mathbf{N}_s, s \leqq t)$. Let $\mathscr{P}'$ be a probability measure equivalent to $\mathscr{P}$. Then

$$\mathbf{N}_t = \mathbf{Z}_t + \textstyle\int_0^t \boldsymbol{\lambda}_s\, ds$$

where the components of $\mathbf{Z}$ belong to $\mathscr{M}_{\text{loc}}(\mathscr{P}', \{\mathscr{F}_t\})$ and the components of $\boldsymbol{\lambda}_t$ are almost surely positive for all t. The likelihood ratio is given by

$$\Lambda_t = \textstyle\prod_{i=1}^n \left[\left(\prod_{s_i\le t} \lambda_{s_i}^i\right)\left(\exp\left(-\int_0^t (\lambda_s^i - 1)\, ds\right)\right)\right] \tag{4.13}$$

where $\prod_{s_i\le t} \lambda_{s_i}^i$ is taken over all jumps of N_t^i.

As a second example, consider a queueing process with a single server. Suppose that under probability measure $\mathscr{P}$ the arrivals are Poisson (rate 1) and the service times are independent and exponentially distributed with rate 1. We also assume that under $\mathscr{P}$ the service and arrivals are independent. If we denote the length of the queue at t by ζ_t then we can write

$$d\zeta_t = d\xi_t - 1(\zeta_{t-})\, d\eta_t \tag{4.14}$$

where ξ and η are a pair of independent standard Poisson processes, and $1(z) = 1$ or 0 according as $z > 0$ or $z \leqq 0$. Now, let $\mathscr{P}'$ be a probability measure equivalent to $\mathscr{P}$. Our interest is to find an expression for the likelihood ratio

$$L_t = E\left(\frac{d\mathscr{P}'}{d\mathscr{P}}\,\middle|\,\mathscr{F}_{\zeta t}\right) \tag{4.15}$$

where $\mathscr{F}_{\zeta t}$ denotes the σ-field generated by $\{\zeta_s, s \leqq t\}$.

Observe that ξ_t is $\mathscr{F}_{\zeta t}$-measurable, since ξ_t counts the positive jumps of ζ up to t. Similarly, $\int_0^t 1(\zeta_{s-})\, d\eta_s$ is $\mathscr{F}_{\zeta t}$-measurable, but not η_t. Now, define

$$X_t^+ = \xi_t - t\,, \qquad X_t^- = \int_0^t 1(\zeta_{s-})[d\eta_s - ds]\,. \tag{4.16}$$

Then, $X^\pm$ are $(\mathscr{P}, \{\mathscr{F}_{\zeta t}\})$ martingales with

$$[X^+, X^+]_t = \xi_t\,, \qquad [X^-, X^-]_t = \int_0^t 1(\zeta_{s-})\, d\eta_s \tag{4.17}$$

and

$$\langle X^+, X^+\rangle_t = t\,, \qquad \langle X^-, X^-\rangle_t = \int_0^t 1(\zeta_{s-})\, ds\,. \tag{4.18}$$

We can also write

$$\begin{aligned} \zeta_t &= X_t^+ + t - [X_t^- + \textstyle\int_0^t 1(\zeta_{s-})\, ds] \\ &= X_t^+ - X_t^- + \textstyle\int_0^t O(\zeta_{s-})\, ds \end{aligned} \tag{4.19}$$

where $O(\zeta) = 1$ if $\zeta \leqq 0$ and is zero otherwise.

Because L is a positive $(\mathscr{P}, \{\mathscr{F}_{\zeta t}\})$ martingale it has a representation (see footnote after (4.2))

$$L_t = 1 + \int_0^t L_{s-}[\phi_s^+\, dX_s^+ + \phi_s^-\, dX_s^-] \tag{4.20}$$

where $\phi^\pm$ are predictable processes with $\phi_t^\pm > -1$ for all t.

Theorem 3.2 now yields the result that

$$\begin{aligned} Z_t^+ &= X_t^+ - \textstyle\int_0^t \phi_s^+\, ds \\ Z_t^- &= X_t^- - \textstyle\int_0^t \phi_s^- 1(\zeta_{s-})\, ds \end{aligned} \tag{4.21}$$

are $(\mathscr{P}', \mathscr{F}_{\zeta t}\})$ local martingale.

From the vector version of Theorem 4.1, we know that under $\mathscr{P}'$ there exist positive predictable processes λ_t and μ_t such that $\xi_t - \int_0^t \lambda_s\, ds$ and $\eta_t - \int_0^t \mu_s\, ds$ are $(\mathscr{P}', \{\mathscr{F}_t\})$ local martingales where $\mathscr{F}_t$ denotes the σ-field generated by $\{\xi_s, \eta_s, s \leqq t\}$. If we define

$$\begin{aligned} \hat{\lambda}_t &= E(\lambda_t \,|\, \mathscr{F}_{\zeta t}) \\ \hat{\mu}_t &= E(\mu_t \,|\, \mathscr{F}_{\zeta t}) \end{aligned}$$

and if we can choose measurable versions for these processes, then $\xi_t - \int_0^t \hat{\lambda}_s\, ds$ and $\int_0^t 1(\zeta_{s-})[d\eta_s - \hat{\mu}_s\, ds]$ are $(\mathscr{P}', \{\mathscr{F}_{\zeta t}\})$ local martingales. If measurable versions cannot be found, the definitions for $\hat{\lambda}$ and $\hat{\mu}$ need to be modified by considering the families of measures on $[0, t] \otimes \mathscr{F}_{\zeta t}$ defined by λ and μ and by finding the Radon–Nikodym derivatives. It follows from (4.16) that

$$X_t^+ - \int_0^t (\hat{\lambda}_s - 1)\, ds\,, \qquad X_t^- - \int_0^t 1(\zeta_{s-})(\hat{\mu}_s - 1)\, ds \tag{4.22}$$

are $(\mathscr{P}', \{\mathscr{F}_{\zeta t}\})$ local martingale. A comparison of (4.21) and (4.22) yields

$$\phi_t^+ = \hat{\lambda}_t - 1\,, \qquad \phi_t^- = \hat{\mu}_t - 1\,. \tag{4.23}$$

The likelihood ratio can now be found by using (3.4). We find

$$L_t = \exp\left(-\int_0^t [(\hat{\lambda}_s - 1) + 1(\zeta_{s-})(\hat{\mu}_s - 1)]\, ds\right) \prod_{s,\tau \le t} \hat{\lambda}_s \hat{\mu}_\tau \tag{4.24}$$

where s and τ in the product denote the positive and negative jumps of ζ respectively.

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