RELAXED CONTROLS AND THE CONVERGENCE OF OPTIMAL CONTROL ALGORITHMS (Revised)

by

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1. Introduction

Most optimal control algorithms construct a sequence of controls whose corresponding costs form a monotonically decreasing, converging sequence. Because of this, it suffices to require that the sequence of controls and initial states constructed have at least one accumulation point and that any accumulation point of this sequence satisfies an optimality condition, rather than to require that it converges.

In studying the convergence properties of nonlinear programming algorithms, to which the preceding remarks also apply, it is assumed that the sequence of points constructed by the algorithm remains in a compact subset of R^n. This guarantees the existence of an accumulation point. With the exception of penalty function methods (which are not iterative procedures; see, for example [1,3,11]), it has been common among inventors of iterative optimal control algorithms to assure that the sequences of controls constructed remain in L_∞-bounded sets, and to show that any L_2-accumulation point satisfies the Pontryagin maximum principle or some relating necessary condition of optimality. (In the absence of constructive, generally applicable necessary and sufficient conditions,

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one cannot expect proofs of convergence to an optimum.) Unfortunately, there is no mathematical basis for assuming that a sequence of controls in an L_∞-bounded set has an L_2-accumulation point.

The purpose of this paper is to present and illustrate a convergence theory for optimal control algorithms using iteration formulas of the form $u_{i+1} \in A(u_i)$, $i = 0, 1, 2, \ldots$, where the u_i are the successively constructed controls and A is a set valued iteration function. This class of algorithms includes gradient and gradient projection methods, feasible directions methods, strong variations methods and so forth. (It does not include penalty function type methods whose analysis requires a totally different approach). Our theory does not prove that existing optimal control algorithms always construct controls converging to an optimal control. This is clearly false. Instead, our theory examines the properties of accumulation points of control sequences constructed by optimal control algorithms. In particular, it shows that these accumulation points satisfy some optimality condition for the relaxed problem. The optimality condition satisfied differs from algorithm to algorithm. The theory is based on an extension of results in [9] and on the use of a topology, based on relaxed controls [14], [12,12a], which ensures that accumulation points always exist for L_∞-bounded sequences.

The theory found in Young [14], with some minor modifications, seems to be the most appropriate one for analyzing optimal control algorithms. There were two reasons for the modifications. The first is that Young specifies a priori a fixed set U in which all controls must take their value. This is extremely inconvenient in analyzing algorithms for problems without control constraints. We have therefore changed a number of definitions to make them independent of such a set U. The second reason is that we felt it very important to preserve a connection between the old ($L_2 \cap L_\infty$) and new convergence results and have therefore modified slightly Young's definition of convergence of relaxed controls.

We illustrate the manner in which this new convergence theory is to be used by means of two examples: an analysis of a strong variations algorithm due to Mayne and Polak [7] and of the Pironneau-Polak dual method of feasible directions [8]. The latter, as well as gradient methods, require the development of a special directional derivative. Finally, in Appendix A, we give a short discussion of the use of optimality conditions in the construction of optimization algorithms and in Appendix B we establish the relation between the new and the old convergence results.

2. Compactness Properties of the Relaxed Optimal Control Problem

The algorithms which we are about to discuss solve optimal control problems of the form:

1 $$\min g_0(\xi,u) \triangleq \int_0^1 L(x(t,\xi,u), u(t),t)\, dt + h_0(x(1,\xi,u)),$$

subject to the constraints

2 $$\frac{d}{dt} x(t,\xi,u) = f(x(t,\xi,u), u(t),t),\ t \in [0,1], \text{ a.e.}$$

3 $$x(0,\xi,u) = \xi$$

4 $$g_j(\xi,u) \triangleq h_j(x(1,\xi,u)) \leq 0,\ j = 1, 2, \ldots, p,$$

5 $$g_j(\xi,u) \triangleq h_j(\xi) \leq 0,\ j = p+1, \ldots, p+q,$$

6 $$u(t) \in U \subset R^m \text{ for all } t \in [0,1]$$

where $f\colon R^n \times R^m \times [0,1] \to R^n$, and $L\colon R^n \times R^m \times [0,1] \to R^1$. The functions g_j, $j=0,1,\ldots,p+q$ are real valued and u is assumed to be measurable.

The following hypotheses are commonly made, with $T \triangleq [0,1]$.

<u>Assumption 1:</u> The functions $f\colon R^n \times R^m \times T \to R^n$ and $L\colon R^n \times R^m \times T \to R^1$ and their partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial L}{\partial x}$ exist and are continuous on $R^n \times R^m \times T$. The functions $h_j\colon R^n \to R^1$, $j=0,1,\ldots,p+q$, and their derivatives $\frac{\partial h_j}{\partial x}$, $j=0,1,\ldots,p+q$, exist and are continuous on R^n.

<u>Assumption 2:</u> For each compact $\Omega \subset R^m$, there exists an $M > 0$ such that $\|f(x,u,t)\| \leq M\,(\|x\| + 1)$ for all $(x,u,t) \in R^n \times \Omega \times T$ and $\|f(x,u,t) - f(x',u,t)\| \leq M\,\|x-x'\|$ for all $x, x' \in R^n$, $u \in \Omega$, $t \in T$.

With the original problem (1) - (6) we associate a relaxed problem, following Young [14], as will be shown after the necessary definitions have been introduced.

As already pointed out in the introduction, the study of optimization algorithms is substantially simplified when a number of definitions used by Young [14] and Warga [12,12a] are somewhat modified. This is done to avoid the a priori selection of a compact set $U \subset R^m$ such that $u\colon T \to U$, since an a priori selection of a U contradicts the absence of constraints on $u(t)$ in control unconstrained problems. The reader is therefore cautioned that our definitions differ from those of Young and Warga. However, the following results can be deduced directly from those of Young [14] and Warga [12,12a] and are presented here, without claims of originality, so as to make the paper readily accessible to the large number of specialists in computational methods who are not familiar with the theory of relaxed controls.

Definition 1: Let V be the set of non-negative unit measures (probability measures) on R^m and let $T \triangleq [0,1]$. A relaxed control is any function $\underset{\sim}{v}(\cdot)$: $T \to V$ with the property that for some compact set $U \subset R^m$, the measure $\underset{\sim}{v}(t)$ is wholly concentrated on U for all $t \in T$ (this will be referred to as "$\underset{\sim}{v}(\cdot)$ vanishes outside of U").

Throughout the paper a relaxed control will be denoted by a boldface $\underset{\sim}{u}$ or $\underset{\sim}{v}$ and an ordinary control (measurable function) by an ordinary u or v.

Definition 2: Given a continuous function $\phi(\cdot)$ defined on R^m and a measure $\underset{\sim}{v} \in V$, we shall write $\phi_r(\underset{\sim}{v})$ for its integral in the measure $\underset{\sim}{v}$, i.e. $\phi_r(\underset{\sim}{v}) \triangleq \int_{R^m} \phi(u)d\underset{\sim}{v}$, whenever that integral is well defined. More generally, if $\phi(x,u,t)$ is continuous in (x,u,t), the symbol $\phi_r(x,\underset{\sim}{v},t)$ denotes, for fixed (x,t), the integral on R^m of $\phi(x,u,t)$ with respect to the probability measure $\underset{\sim}{v}$, i.e. $\phi_r(x,\underset{\sim}{v},t) \triangleq \int_{R^m} \phi(x,u,t)\, d\underset{\sim}{v}$.

Definition 3: A relaxed control $\underset{\sim}{v}(\cdot)$ will be termed measurable if for every polynomial $p(u)$ in (the components of) u, the function $p_r(\underset{\sim}{v}(t)) \triangleq \int_{R^m} p(u)d\underset{\sim}{v}(t)$ of t is measurable.

Remark: From page 290 in Young, [14], it follows that if $\underset{\sim}{v}(\cdot)$ is a measurable relaxed control and $g(t,u)$ is a continuous function of (t,u), then the function $g_r(t,\underset{\sim}{v}(t)) \triangleq \int_{R^m} g(t,u)d\underset{\sim}{v}(t)$ of t is measurable.

The relaxed problem is obtained from the original problem (1) - (5) by substituting the cost

7 $$g_0(\xi,\underset{\sim}{v}) \triangleq \int_0^1 L_r(x(t,\xi,\underset{\sim}{v}),\ \underset{\sim}{v}(t),\ t)dt + h_0(x(1,\xi,\underset{\sim}{v}))$$

for the cost (1), the differential equation

8 $$\dot{x}(t) = f_r(x(t),\underset{\sim}{v}(t),t) \triangleq \int_{R^m} f(x(t),u,t)d\underset{\sim}{v}(t),$$

for the differential equation (2), and the requirement that

9 $\underset{\sim}{v}(\cdot)$ vanish outside of U

for (6).

We now give an existence and uniqueness theorem for the solution to the relaxed differential equation (8). The proof is found in Young, [14] on pages 291-292 and 298, where the theorem is proved under weaker assumptions.

<u>Theorem 1</u>: Suppose that Assumptions 1 and 2 are satisfied. Then for any measurable relaxed control $\underset{\sim}{v}(\cdot)$, which vanishes outside some compact set $U \subset R^m$, and any initial state x_o, there exists an absolutely continuous function $x(\cdot,x_o,\underset{\sim}{v})$: $T \to R^n$ that is the unique solution to (8), satisfying $x(0,x_o,\underset{\sim}{v}) = x_o$.

In our analysis, in addition to the relaxed optimal control problem, we will also need associated multiplier functions, defined as follows.

<u>Definition 4</u>: For j = 0,1,2,...p, let $\lambda_j(\cdot,\xi,\underset{\sim}{v})$: $T \to R^n$, denote the solution of

10 $$-\dot{\lambda}_j(t,\xi,\underset{\sim}{v}) = \left(\frac{\partial H_j}{\partial x}\right)_r^T (x(t,\xi,\underset{\sim}{v}),\underset{\sim}{v}(t),\lambda_j(t,\xi,\underset{\sim}{v}),t)$$

11 $$\lambda_j(1,\xi,\underset{\sim}{v}) = \left(\frac{\partial h_j}{\partial x}\right)^T (x(1,\xi,\underset{\sim}{v}))$$

where the superscript T denotes transposition and H_j: $R^n \times R^m \times R^n \times T \to R^1$, $j = 0,1,\ldots,p$, is defined by

12 $$H_j(x,u,\lambda,t) \triangleq \lambda^T f(x,u,t) + \delta_{j0} L(x,u,t)$$

where δ_{j0} is the Kronecker delta.

The relaxed optimal control problem leads to two crucial sequential compactness theorems, as we shall shortly see. The first one of these two theorems is due to Young [14], the second one to Warga [12a].

<u>Definition 5</u>: A sequence $\{\underset{\sim}{v}^i(\cdot)\}_{i=0}^{\infty}$ of measurable relaxed controls converges in the sense of control measures (abbreviated i.s.c.m.) to a relaxed control $\bar{\underset{\sim}{v}}(\cdot)$ if for every continuous, real-valued function g(t,u) defined on $T \times R^m$ and every subinterval Δ of T the values $\int_\Delta g_r(t,v^i(t))dt$ converge to $\int_\Delta g_r(t,\bar{v}(t))dt$.

<u>Notation</u>: If $\{\underset{\sim}{v}^i(\cdot)\}$, $i \in K$, converges i.s.c.m to $\bar{\underset{\sim}{v}}(\cdot)$, we denote that by $\underset{\sim}{v}^i(\cdot) \overset{K}{\to} \bar{\underset{\sim}{v}}(\cdot)$.

The first compactness theorem which we need is proved on pages 301-303 in Young, [14].

<u>Theorem 2</u>: Let $\{\underset{\sim}{v}^i(\cdot)\}_{i=0}^{\infty}$ be a sequence of measurable relaxed controls which vanish outside some fixed compact set U. Then there exists a relaxed control $\bar{\underset{\sim}{v}}(\cdot)$ which also vanishes outside of U and a subsequence indexed by a set $K \subset \{0,1,2,\ldots\}$ such that $\underset{\sim}{v}^i(\cdot) \overset{K}{\to} \bar{\underset{\sim}{v}}(\cdot)$.

Notation: Given a sequence of initial states $\{\xi^i\}_{i=0}^{\infty}$ and a sequence of relaxed controls $\{\underset{\sim}{v}^i(\cdot)\}_{i=0}^{\infty}$, we shall denote the corresponding sequences of trajectories and multipliers (determined according to (8) and (3), and (10), (11) respectively) by $\{x^i(\cdot)\}_{i=0}^{\infty}$, $\{\lambda_j^i(\cdot)\}_{i=0}^{\infty}$, $j=0,1,\ldots p$. We shall also use the notation x^u, $x^{\underset{\sim}{u}}$, λ_j^u, $\lambda_j^{\underset{\sim}{u}}$ to denote solutions to (2), (3), (8), (3) and (10), (11) corresponding to a measurable control u or a relaxed control $\underset{\sim}{u}$.[†]

Definition 6: If $\{(\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i,\ldots,\lambda_p^i)\}_{i=0}^{\infty}$ is a sequence of initial states, relaxed controls, corresponding trajectories, and corresponding multipliers such that $\{\xi^i\}$ converges to $\bar{\xi}$, $\{\underset{\sim}{v}^i\}$ converges to $\bar{\underset{\sim}{v}}$ i.s.c.m., $\{x^i\}$ converges to $\bar{x}$ uniformly, and $\{\lambda_j^i\}$ converges to $\bar{\lambda}_j$ uniformly, $j = 0,1,\ldots,p$, then we denote this by $(\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i,\ldots,\lambda_p^i) \to (\bar{\xi},\bar{\underset{\sim}{v}},\bar{x},\bar{\lambda}_0,\ldots,\bar{\lambda}_p)$.

Definition 7: $(\bar{\xi},\bar{\underset{\sim}{v}},\bar{x},\bar{\lambda}_0,\ldots,\bar{\lambda}_p)$ is called an accumulation point of $\{(\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i,\ldots,\lambda_p^i)\}_{i=0}^{\infty}$ if there exists a subsequence, indexed by some $K \subset \{0,1,2,\ldots\}$, such that $(\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i,\ldots,\lambda_p^i) \xrightarrow{K} (\bar{\xi},\bar{\underset{\sim}{v}},\bar{x},\bar{\lambda}_0,\ldots,\bar{\lambda}_p)$.

The second compactness theorem will be established as a consequence of the following lemmas.

Lemma 1: Let C, U be arbitrary compact sets in R^p, R^m, respectively, and let S be the set of measurable relaxed controls which vanish outside

[†] (10) degenerates into an ordinary differential equation when u is an ordinary control.

of U. Let g be a continuous function from $R^p \times R^m \times T$ into R^q. Let $Y^i(\cdot)$, $\bar{Y}(\cdot)$ be continuous functions from T into C such that $Y^i(\cdot)$ converges to $\bar{Y}(\cdot)$ uniformly. Let $\{\underset{\sim}{v}^i(\cdot)\}_{i=0}^{\infty}$ be a sequence of relaxed controls that converges i.s.c.m. to a relaxed control $\bar{\underset{\sim}{v}}(\cdot)$. Then for each subinterval Δ of T,

$$13 \qquad \int_\Delta g_r(Y_i(\tau),\underset{\sim}{v}_i(\tau),\tau)d\tau \to \int_\Delta g_r(\bar{Y}(\tau),\bar{\underset{\sim}{v}}(\tau),\tau)d\tau.$$

Proof: Follows immediately from Definition 5 and the uniform continuity of g on $C \times U \times T$.

The following lemma found in Filippov, [3a], will also be needed to establish the second compactness theorem.

Lemma 2: Let $\{y^i(\cdot)\}_{i\in I}$, where I is some indexing set, be a collection of absolutely continuous functions from T into R^n such that $\{y^i(0)\}_{i\in I}$ or $\{y^i(1)\}_{i\in I}$ is contained in a compact set of R^n. Let functions Y^i: $T \to R$, $i \in I$, be defined by

$$14 \qquad Y^i(t) = \|y^i(t)\|^2 + 1.$$

If there exists an $M > 0$ such that $|\dot{Y}^i(t)| \leq M\, Y^i(t)$, for almost all $t \in T$, $i \in I$, then the set $\{y^i(\cdot)\}_{i\in I}$ is equibounded and equicontinuous. Furthermore, if $I = \{0,1,2,\ldots\}$, then there exists a subsequence indexed by a set $K \subset \{0,1,2,\ldots\}$ and an absolutely continuous function $\bar{y}(\cdot)$ such that $y^i(\cdot)$ converges uniformly to $\bar{y}(\cdot)$ for $i \in K$.

Now making use of Lemmas 1, 2 and Assumption 2 it is straightforward to show that the following compactness result, due to Warga, [12a], holds

Theorem 3: Let C and U be arbitrary compact sets in R^n, R^m, respectively, and let S be the set of measurable relaxed controls which vanish outside of U. If $\{(\xi^i, \underset{\sim}{v}^i, x^i, \lambda_0^i, \ldots, \lambda_p^i)\}_{i=0}^{\infty}$ is a sequence of initial states, relaxed controls, corresponding trajectories, and corresponding multipliers such that $\{\xi^i\}_{i=0}^{\infty} \subset C$, $\{\underset{\sim}{v}^i\} \subset S$, $\{\xi^i\}$ converges to $\bar{\xi}$, $\{v_i\}$ converges to $\bar{\underset{\sim}{v}}$ i.s.c.m.,[†] $\{x^i\}$ converges to $\bar{x}$ uniformly, and $\{\lambda_j^i\}$ converges to $\bar{\lambda}_j$ uniformly, $j = 0,1,\ldots,p$. Then $\bar{x}(\cdot) = x(\cdot,\bar{\xi},\bar{\underset{\sim}{v}})$ and $\bar{\lambda}_j(\cdot) = \lambda_j(\cdot,\bar{\xi},\bar{\underset{\sim}{v}})$, $j = 0,1,\ldots,p$. Furthermore, given a sequence $\{(\xi^i, \underset{\sim}{v}^i, x^i, \lambda_0^i, \ldots, \lambda_p^i)\}_{i=0}^{\infty}$ such that $\{\xi^i\}_{i=0}^{\infty} \subset C$ and $\{\underset{\sim}{v}^i\}_{i=0}^{\infty} \subset S$, there always exists a subsequence that satisfies the above hypotheses and conclusions.

3. Algorithm Prototypes and Convergence Theory

The convergence theorems which we find in [9], as well as in other sources, require that the limit points of sequences constructed by an algorithm lie in the domain of the algorithm. Since this may not be true for optimal control algorithms, it is necessary to modify the existing convergence theory just slightly. We now show how it is done for the simplest case treated in [9]. The more complicated cases discussed in [9] can be modified similarly.

The algorithm prototype below, extends the algorithm prototype 1.3.9 in [9]. Let Z be a topological space, $\bar{W}$ be a subset of Z, and W be a subset of $\bar{W}$.[#] We use two functions, the search function, $A: W \to 2^W$, and the stop function, $c: \bar{W} \to R^1$. Finally, we let the set of desirable points,

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[†] If $\underset{\sim}{v}^i \to \bar{\underset{\sim}{v}}$ i.s.c.m., with $\{\underset{\sim}{v}^i\} \subset S$, it follows that $\bar{\underset{\sim}{v}} \in S$ also.

[#] Prototype 1.3.9 in [9] applies only when $W = \bar{W}$ and its convergence is is established only in terms of a normed topology.

Δ, be a nonempty subset of $\bar{W}$. The problem then is to find any point in Δ, where it is assumed that we have some way of recognizing points in Δ.

Algorithm Prototype:

Step 0: Compute a $z^0 \in W$.

Step 1: Set $i = 0$

Step 2: Compute a point $y \in A(z^i)$.

Step 3: Set $z^{i+1} = y$.

Step 4: If $c(z^{i+1}) \geq c(z^i)$, stop; else, set $i = i+1$ and go to Step 2.

The proof of the following convergence result is the same as that of Theorem 1.3.10 in Polak, [9], except that one uses sequences instead of closed balls.

Theorem 4: Consider the above algorithm. Suppose that

(i) for every nondesirable $\bar{z} \in \bar{W}$ and every sequence $\{z^i\}_{i=0}^{\infty} \subset W$ converging to $\bar{z}$, $\{c(z^i)\}_{i=0}^{\infty}$ converges to $c(\bar{z})$;

(ii) for every nondesirable $\bar{z} \in \bar{W}$ and every sequence $\{z^i\}_{i=0}^{\infty} \subset W$ converging to $\bar{z}$, there exists an infinite subset $K \subset \{0,1,2,...\}$, an integer $N \geq 0$, and a $\delta(\bar{z}) > 0$ such that

15 $$c(z'') - c(z^i) \leq -\delta(\bar{z}) < 0 \qquad \forall i \geq N,\ \forall i \in K,\ \forall z'' \in A(z^i).$$

Then, either the sequence $\{z^i\}_{i=0}^{\infty}$ constructed by the algorithm is finite and its next to last element is desirable, or else it is infinite and every accumulation point in $\bar{W}$ of $\{z^i\}_{i=0}^{\infty}$ is desirable.

With the proper choice of Z, $\bar{W}$, and W this algorithm prototype and convergence theorem can be applied to a large class of optimal control algorithms. This will be demonstrated in the following sections.

4. A Strong Variations Algorithm.

In this section we shall present a proof of convergence for a strong variations algorithm developed by Mayne and Polak [7]. This algorithm is an "$L_\infty \cap L_2$ stabilized" version of a differential dynamic programming algorithm due to Jacobson and Mayne [5]. Differential dynamic programming algorithms are based on fairly complex relationships between changes in Hamiltonians and changes in cost in optimal control problems.
The interested reader is referred to the book by Jacobson and Mayne [5], Mayne [6], and to [7] for background material. The gist of these algorithms is generally as follows. Given a control u_i, an approximation to the optimal control, one computes the corresponding trajectories and multipliers x^{u_i}, $\lambda_j^{u_i}$ by solving (2), (10), (11). Then one constructs a Hamiltonian $H(x^{u_i}(t),w,\lambda^{u_i}(t),t)$, where λ^{u_i} is a certain convex combination of the $\lambda_j^{u_i}$, and is an approximation at the optimal co-state. By minimizing H with respect to $w \in U$, one obtains an intermediate function $\check{u}_i(t)$.[†] For the algorithm to converge, one now has to use a rather complex way of constructing the next control, u_{i+1}, by setting it equal to u_i for some points in T and to $\check{u}_i$ for some other points in T. The specific rule used in [7] is derived from the Armijo [9] step size selection procedure commonly used in nonlinear programming. Fig. 1, reproduced from [7], will perhaps help the reader in understanding the algorithm. Although strong variations (or differential dynamic programming) algorithms are difficult to understand, they have two distinct advantages: (i) they are computationally efficient, and (ii) they solve certain classes of problems which cannot be solved by other algorithms. (In principle all optimal control problems can be solved by means of penalty function methods, but, at least in our experience, penalty function methods have been found to perform unacceptably on quite a few occasions).

[†] Thus, we can think of these algorithms as being derived from the Pontryagin Minimum Principle.

The algorithm to be described solves the problem (1) - (6) under the additional restriction that the system (5) is replaced by $\xi = \xi_0$, that the functions $h_j \equiv 0$ for $j = 1,2,\ldots,p+q$, and that the set U in (6) is compact. In other words, our initial state is fixed and we have no initial or terminal inequality constraints. Because of this we will drop any reference to the initial state. In the discussion below, we shall denote by G the set of measurable functions $u : [0,1] \to U$.

To insure that Algorithm 1 is well defined we need the following theorem which is a consequence of the McShane-Warfield Halfway Principle, [14].

Theorem 5: For any $u \in G$ there exists a $\check{u} \in G$ such that for almost all $t \in T$,

16 $$\check{u}(t) \in \check{U}(u,t) \triangleq \arg\min_{w \in U} H_0(x^u(t),w,\lambda_0^u(t),t)^{\dagger}$$

Next, let $\bar{H}$: $G \times T \to R^1$ be defined by

17 $$\bar{H}(u,t) \triangleq \min_{w \in U} H_0(x^u(t),w,\lambda_0^u(t),t)$$

where H_0 was defined in (12) and let θ: $G \to R^1$ be defined by

18 $$\theta(u) \triangleq \int_0^1 [\bar{H}(u,t)-H_0(x^u(t),u(t),\lambda_0^u(t),t)]dt$$

For any u^1, $u^2 \in G$, let $\Delta g_0(u^2,u^1)$ and $\Delta\hat{g}_0(u^2,u^1)$ be defined by

19 $$\Delta g_0(u^2,u^1) \triangleq g_0(u^2)-g_0(u^1)$$

†Thus $\check{U}(u,t)$ is the set of minimizers.

and

20 $$\Delta\hat{g}_0(u^2,u^1) \triangleq \int_0^1 [H_0(x^1(t),u^2(t),\lambda_0^1(t),t) - H_0(x^1(t),u^1(t),\lambda_0^1(t),t)]dt$$

where g_0 is defined as in (1). (It is shown in [6] that $\Delta\hat{g}_0$ is, in a certain sense, a first order estimate of Δg_0).

Next, for every $u \in G$, let $\check{U}(u)$, $I_u^{H_0}$, and $m(u)$ be defined respectively by

21 $$\check{U}(u) \triangleq \{v \in G:\ v(t) \in \arg\min_{w \in U} H_0(x^u(t),w,\lambda_0^u(t),t) \text{ for almost all } t \in T\}.$$

22 $$I_u^{H_0} \triangleq \{t \in T | \overline{H}(u,t)-H_0(x^u(t),u(t),\lambda_0^u(t),t) \leq \theta(u)\}$$

and

23 $$m(u) \triangleq \mu(I_u^{H_0})$$

where μ is Lebesque measure.

For every $u \in G$ and $\alpha \in [0,1]$, let $I_{\alpha u}$ be any subset of T having the following properties.

24 $$\mu(I_{\alpha u}) = \alpha$$

25 $$\text{If } \alpha \in [0,m(u)],\ I_{\alpha u} \subset I_u^{H_0}$$

26 $$\text{If } \alpha \in (m(u),1],\ I_{\alpha u} \supset I_u^{H_0}$$

27 $$\forall \alpha \in [0,m(u)],\ \{t \in I_u^{H_0},\ t' \in I_{\alpha u},\ t < t'\} \Rightarrow \{t \in I_{\alpha u}\}$$

28 $$\forall \alpha \in (m(u),1],\ \{t \in T,\ t' \in I_{\alpha u}\setminus I_u^{H_0}, t < t'\} \Rightarrow \{t \in I_{\alpha u}\}.$$

Next, for any $u \in G$, for any $\alpha \in [0,1]$, $u_\alpha \in G$ will denote a function with the following properties

29 $$u_\alpha(t) \in \check{U}(u,t) \qquad \forall t \in I_{\alpha u}$$

30 $$u_\alpha(t) = u(t) \qquad \forall t \in T \setminus I_{\alpha u}$$

Finally, let α: $G \to 2^{[0,1]}$ be defined by

31 $$\alpha(u) = \{\alpha \mid \alpha = \max \{\beta \in [0,1] \mid \Delta g_0(u_{\beta'},u) \leq \beta'\theta(u)/2, \quad \forall \beta' \in [0,\beta]\}\}$$

where $u_{\beta'} \in G$ is any control that satisfies (29), (30).

Algorithm 1: (Mayne and Polak [7]).

Step 0: Select a $u^0 \in G$.

Step 1: Set $i = 0$.

Step 2: Compute x^i by solving (2), with $\xi = \xi_0$.

Step 3: Compute λ_0^i by solving the ordinary control versions of (10) and (11).

Step 4: Compute $\check{u}^i$ such that $\check{u}^i(t) \in U(\check{u}^i,t)$.

Step 5: Compute $\theta(u^i) \triangleq \Delta\hat{g}_0(\check{u}^i,u^i)$ using (20). If $\theta(u^i) = 0$ stop.
Else go to Step 6.

Step 6: Compute an $\alpha^i \in \alpha(u^i)$.

Step 7: Set $u^{i+1} = u^i_{\alpha^i}$. Set $i = i+1$. Go to Step 2.

Algorithm 1 constructs a sequence of ordinary controls. However, in proving convergence, we must use relaxed controls. Therefore, with each ordinary control u we associate a relaxed control $\underset{\sim}{u}$ which has the property that the measure $\underset{\sim}{u}(t)$ is wholly concentrated at u(t), i.e. $\int_{\{u(t)\}} d\underset{\sim}{u}(t)$

$= 1$ for all $t \in T$. We then see that Algorithm 1 defines a map $A : W \in 2^W$ where W is defined by

32 $$W \triangleq \{(\underset{\sim}{u}, x^u, \lambda_0^u) \mid u \in G\}.$$

In other words, for any $(\underset{\sim}{u}^i, x^i, \lambda_0^i) \in W$ the set $A((\underset{\sim}{u}^i, x^i, \lambda_0^i))$ consists of the possible $(\underset{\sim}{u}^{i+1}, x^{i+1}, \lambda_0^{i+1})$ which the algorithm can construct from the given point $(\underset{\sim}{u}^i, x^i, \lambda_0^i)$. We will now establish our convergence result for Algorithm 1 using the theory developed in sections 2 and 3.

As before, let S be the set of measurable relaxed controls which vanish outside of U. We also have to make the straightforward extension of the domain of definition of functions such as θ, $\overline{H}$, etc. to include relaxed controls.

For example,

33 $$\theta(\underset{\sim}{u}) \triangleq \int_0^1 [\overline{H}(\underset{\sim}{u}, t) - H_{0_r}(x^{\tilde{u}}(t), \underset{\sim}{u}(t), \lambda_0^{\tilde{u}}(t), t)]dt$$

$$\triangleq \int_0^1 [\min_{w \in U} H_0(x^{\tilde{u}}(t), w, \lambda_0^{\tilde{u}}(t), t) - H_{0_r}(x^{\tilde{u}}(t), \underset{\sim}{u}(t), \lambda_0^{\tilde{u}}(t), t)]dt.$$

The following lemma is proved in Mayne and Polak [7].

<u>Lemma 3</u>: Let A: $W \to 2^W$ be the map defined by Algorithm 1. Then there exists a $c > 0$ such that for all $u \in G$

34 $$\Delta g_0(u', u) \leq -[\theta(u)]^2/c, \quad \forall (\underset{\sim}{u}', x^{\tilde{u}'}, \lambda_0^{\tilde{u}'}) \in A(\underset{\sim}{u}, x^{\underset{\sim}{u}}, \lambda_0^{\tilde{u}})).$$

Lemma 4: Let $(\underset{\sim}{u}^i, x^i, \lambda_0^i) \to (\overline{\underset{\sim}{u}}, \overline{x}, \overline{\lambda}_0)$ where $\{\underset{\sim}{u}^i\}_{i=0}^{\infty} \subset S$. Then $\theta(\underset{\sim}{u}^i) \to \theta(\overline{\underset{\sim}{u}})$.

Proof: This follows from Lemmas 3 and 4, the continuity of $\min_{w \in U} H_0(x,w,\lambda,t)$ in (x,λ,t), Lemma 1 and Theorem 3.

We can now prove the convergence result. Let W be as in (32) and let Z and $\overline{W}$ be defined by

35 $$Z = S \times C_n[T] \times C_n[T]$$

and

36 $$\overline{W} = \{(\underset{\sim}{u}, x^{\underset{\sim}{u}}, \lambda_0^{\underset{\sim}{u}}) \mid \underset{\sim}{u} \in S\}$$

where $C_n[T]$ is the space of continuous n vector valued functions on T, with the uniform convergence topology. Let the set of desirable points, Δ, be defined by

37 $$\Delta = \{(\underset{\sim}{u}, x^{\underset{\sim}{u}}, \lambda_0^{\underset{\sim}{u}}) \in \overline{W} \mid \theta(\underset{\sim}{u}) = 0\}.^{\dagger}$$

Theorem 6: Suppose Algorithm 1 generates a sequence $\{(u^i, x^i, \lambda_0^i)\}_{i=0}^{\infty}$, then either the corresponding sequence $\{(\underset{\sim}{u}^i, x^i, \lambda_0^i)\}_{i=0}^{\infty}$ is finite, in which case the last element is desirable, or it is infinite and every accumulation point in $\overline{W}$ (at least one exists) is desirable.

Proof: The above Algorithm 1 is obviously of the form of our Algorithm Prototype. Letting W, Z, $\overline{W}$, and Δ be defined respectively as in (32), (35), (36), and (37) and c be g_0, we only need to verify conditions (i) and (ii) of Theorem 4 in order to invoke this theorem: (i) If $(\underset{\sim}{u}^i, x^i, \lambda_0^i) \to$

†It is shown in Appendix A that $\theta(\underset{\sim}{u}) = 0$ is an optimality condition.

$(\bar{u},\bar{x},\bar{\lambda}_0)$, Lemma 1 immediately implies $g_0(u^i) \to g_0(\bar{u})$. (ii) If $(u^i,x^i,\lambda_0^i) \to (\bar{u},\bar{x},\bar{\lambda}_0)$ with $\theta(\bar{u}) < 0$, Lemmas 3 and 4 immediately imply that there exists an $N > 0$ such that

38 $$g_0(u^{i'}) - g_0(u^i) \leq -\frac{[\theta(u^i)]^2}{2c} \leq -\frac{[\theta(\bar{u})]^2}{4c} < 0 \ ,$$

$$\forall i \geq N \ ,$$
$$\forall (u^{i'},x^{u^{i'}},\lambda^{u^{i'}}) \in A((u^i,x^i,\lambda_0^i))$$

Thus Theorem (4) can be applied. The existence of at least one accumulation point follows from the second half of Theorem 3.

5. A Dual Method of Centers.

We shall now consider an algorithm due to Pironneau and Polak [8]. Unlike the algorithm presented in the preceeding section, this one cannot be treated by simply cannibalizing its convergence proof in $L_2 \cap L_\infty$. A special directional derivative must be developed for its analysis.

The algorithm in [8] solves the problem (1) - (6) under the restriction that $h_0 \equiv 0$ and $U = R^m$.

Assumption 3: We will assume that f, L, and h_i, $i = 1,\dots,p+q$, are such that their partials up to second order with respect to x and u exist and are continuous in (x,u,t) on the sets on which they are defined.

The following algorithm is derived from the F. John condition of optimality, as explained in detail in [8]. It is called a "dual" method of feasible directions because it uses multipliers. The "primal", Zoutendijk type methods of feasible directions [9] are derived from the F. John condition in multiplier free form (see [9]), and do not extend to optimal control problems, because the direction finding problems become as difficult as the original problems.

Algorithm 2: (Pironneau-Polak [6]): ($\beta \in (0,1)$ is a step size parameter).

Step 0: Compute a $\xi^0 \in R^n$ and a measurable ordinary control u^0 such that $h_j(\xi^0) \leq 0$ for $j = p+1, \ldots, p+q$, $h_j(x(1,\xi^0,u^0)) \leq 0$ for $j = 1, \ldots, p$. Set $i = 0$.

Step 1: Compute $z^i = (\xi^i,u^i,x^i,\lambda_0^i,\ldots,\lambda_p^i)$ according to (2), and the ordinary control versions of (10), and (11).

Step 2: Compute $\nabla g_j(\xi^i,u^i)$, $j = 0, \ldots, p+q$, according to

39a $$\nabla g_j(\xi^i,u^i) = \left(\lambda_j(0,\xi^i,u^i), \frac{\partial H_j}{\partial u}^T (x(\cdot,\xi^i,u^i), u^i(\cdot), \lambda(\cdot,\xi^i,u^i),\cdot)\right), \quad j = 1,2,\ldots,p$$

39b $$\nabla g_j(\xi^i,u^i) = (\nabla h_j(\xi^i),0), \; j = p+1, p+2,\ldots,p+q.$$

Step 3: Compute $\mu(z^i) \triangleq (\mu_0(z^i), \ldots, \mu_{p+q}(z^i)) \in R^{p+q+1}$ as a solution of

40 $$\phi(z^i) = \max_\mu \Big\{ \sum_{j=1}^{p} \mu_j h_j(x(1,\xi^i,u^i)) + \sum_{j=p+1}^{p+q} \mu_j h_j(\xi^i) - (1/2)\Big\| \sum_{j=0}^{p+q} \mu_j \nabla g_j(\xi^i,u^i)\Big\|_2^2 \;\Big|\; \sum_{j=0}^{p+q} \mu_j = 1, \mu \geq 0, j = 0,1, \ldots, p+q\Big\},$$

where $\|\cdot\|_2$ denotes the $L_2^n[0,1]$ norm.

Step 4: If $\phi(z^i) = 0$, set $\bar{\xi} = \xi^i$ and $\bar{u} = u^i$ and stop; else, go to Step 5.

Step 5: Set

41 $$\omega^i = -\sum_{j=0}^{p} \mu_j(z^i)\, \lambda_j(0,\xi^i,u^i) - \sum_{j=p+1}^{p+q} \mu_j(z^i)\, \frac{\partial h_j}{\partial x}^T (\xi^i)$$

42a $$\nu^i(t,u) = -\sum_{j=0}^{p} \mu_j(z^i)\, \frac{\partial H_j}{\partial u}^T (x(t,\xi^i,u^i),u,\, \lambda_j(t,\xi^i,u^i),t)$$

for all $(t,u) \in T\times R^m$.

42b $$\bar{\nu}^i(\cdot) = \nu^i(\cdot,u^i(\cdot)).$$

Step 6: Compute the smallest integer k, such that

$$\max\left\{\int_0^1 \left\{L\left(x(t,\xi^i + \beta^k\omega^i,u^i + \beta^k\bar{\nu}^i),u^i(t) + \beta^k\bar{\nu}^i(t),\ t\right) - L\left(x(t,\xi^i,u^i),u^i(t),t\right)\right\} dt;\right.$$

$$h_j\left(x(1,\xi^i + \beta^k\omega^i,u^i + \beta^k\bar{\nu}^i)\right), \quad j = 1,\ldots,p;$$

$$\left. h_j(\xi^i + \beta^k\omega^i),\ j = p+1,\ \ldots,\ p+q\right\} - \frac{\beta^k}{2}\,\phi(z^i) \leq 0.$$

Step 7: Set $\xi^{i+1} = \xi^i + \beta^k\omega^i$, set $u^{i+1}(\cdot) = u^i(\cdot) + \beta^k\bar{\nu}^i(\cdot)$, and go to Step 1.

Before proving any convergence results for the above algorithm, we must develop some more theory to make the transition from ordinary controls to relaxed controls. Again this is necessary because we want to study relaxed controls which are accumulation points of a sequence of ordinary controls. In particular, we need to construct a special directional differential, and we develop a variational equation for this purpose. We first define the following functions which are generalizations of the differentials for functions of ordinary controls.

Definition 8: For any $\xi \in R^n$ and $\underset{\sim}{v}$ a measurable relaxed control, let

$\nabla g_j(\xi,\underset{\sim}{v})$: $T\times R^m \to R^n\times R^m$, $j = 0, \ldots, p+q$, be defined by

43a $$\nabla g_j(\xi,\underset{\sim}{v})(t,u) = \left(\lambda_j(0,\xi,\underset{\sim}{v}), \frac{\partial H_j}{\partial u}^T (x(t,\xi,\underset{\sim}{v}),u,\lambda_j(t,\xi,\underset{\sim}{v}),t)\right)$$

for $j = 0,1,\ldots,p$, and

43b $$\nabla g_j(\xi,\underset{\sim}{v})(t,u) = \left(\frac{\partial h_j}{\partial x}^T (\xi),0\right)$$

for $j = p+1,\ldots,p+q$ where λ_j and H_j are as defined in Definition 4.

It will be shown in Theorem 9 that the $\nabla g_j(\xi,\underset{\sim}{v})$ are analogous to L_∞ gradients, (see (56), (57)).

Definition 9: Let $\underset{\sim}{v}$ be a measurable relaxed control, let ξ, $\xi' \in R^n$, and let y, y' be continuous functions from $T \times R^m$ into R^m. Then $\langle (\xi,y), (\xi',y')\rangle_{\underset{\sim}{v}}$ and $|(\xi,y)|_{\underset{\sim}{v}}$ will denote

44 $$\langle (\xi,y), (\xi',y')\rangle_{\underset{\sim}{v}} = \langle \xi,\xi'\rangle + \int_0^1 \left(\int_{R^m} \langle y(t,u),y'(t,u)\rangle \, d\underset{\sim}{v}(t)\right)dt$$

and

45 $$|(\xi,y)|_{\underset{\sim}{v}} = \left[\langle \xi,\xi\rangle + \int_0^1 \left(\int_{R^m} \|y(t,u)\|^2 d\underset{\sim}{v}(t)\right)dt\right]^{1/2}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product and $\|\cdot\|$ denotes the Euclidean norm.

Definition 10: Let W and $\bar{W}$ be defined by

46 $$W \triangleq \{\xi,\underset{\sim}{v},x^v,\lambda_0^v,\ldots,\lambda_p^v) \mid \text{where } \underset{\sim}{v} \text{ is a relaxed control associated with the ordinary control } v,\ \xi \in R^n \text{ and } g_j(\xi,v) \leq 0,\ j = 1, \ldots, p+q\}$$

and

47 $$\bar{W} \triangleq \{(\xi,\underset{\sim}{v},x^{\overset{v}{\sim}},\lambda_0^{\overset{v}{\sim}},\ldots,\lambda_p^{\overset{v}{\sim}}) \mid \underset{\sim}{v} \text{ is a measurable relaxed control, } \xi \in R^n \text{ and } g_j(\xi,\underset{\sim}{v}) \leq 0, j = 1, \ldots, p+q\}.$$

<u>Definition 11</u>: Let Δ, the set of desirable points, be defined by $\Delta = \{(\xi,\underset{\sim}{v},x^{\overset{v}{\sim}},\lambda_0^{\overset{v}{\sim}},\ldots,\lambda_p^{\overset{v}{\sim}}) \in \bar{W} \mid$ there exists multipliers μ_j, $j = 0,1,\ldots,p+q$ such that, (i) $\mu_j \geq 0$, $j = 0,1,\ldots,p+q$, (ii) $\sum_{j=0}^{p+q} \mu_j = 1$, (iii) $\mu_j g_j(\xi,\underset{\sim}{v}) = 0$ for $j = 1,\ldots,p+q$, (iv) $|\sum_{j=0}^{p+q} \mu_j \nabla g_j(\xi,\underset{\sim}{v})|_{\underset{\sim}{v}}^2 = 0\}$.

<u>Assumption 4</u>: The set $\{(\xi,\underset{\sim}{u},x^u,\lambda_0^u,\ldots,\lambda_p^u) \in W \mid g_j(\xi,u) < 0, j = 1,\ldots, p+q\} \neq \emptyset$.

The following definition is an extension of ϕ in (40) to relaxed controls.

<u>Definition 12</u>: Let $z^i = (\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i,\lambda_1^i,\ldots,\lambda_p^i)$. Then

48 $$\phi(z^i) = \max\{\sum_{j=1}^{p} \mu_j h_j(x(\cdot,\xi^i,\underset{\sim}{v}^i)) + \sum_{j=p+1}^{p+q} \mu_j h_j(\xi^i) - (1/2)|\sum_{j=0}^{p+q} \mu_j \nabla g_j(\xi^i,\underset{\sim}{v}^i)|_{\underset{\sim}{v}^i}^2 \mid \sum_{j=0}^{p+q} \mu_j = 1, \mu_j \geq 0, j = 0, 1, \ldots, p+q\}.$$

Thus Algorithm 2 defines a map A: $W \to W$.

We can establish convergence properties only for bounded infinite sequences $\{\xi^i, u^i, x^i, \lambda_0^i, \ldots, \lambda_{p+q}^i\}$ constructed by Algorithm 2. We therefore introduce an arbitrary compact set $C \subset R^n$ which will be assumed to contain $\{\xi^i\}$ and an arbitrary compact set $U \subset R^m$ which will be assumed to contain $\{u^i(t)\}$, $t \in T$. In addition, we shall make use of an arbitrary compact set D containing C in its interior, and we shall denote by S the set of measurable relaxed controls which vanish outside of U.

<u>Lemma 7</u>: Let $\{\xi^i\}_{i=0}^{\infty} \subset C$, $\{\underset{\sim}{u}^i\}_{i=0}^{\infty} \subset S$ be such that $(\xi^i, \underset{\sim}{u}^i, x^i, \lambda_0^i, \ldots, \lambda_p^i)$ $\to (\bar{\xi}, \bar{\underset{\sim}{u}}, \bar{x}, \bar{\lambda}_0, \ldots, \bar{\lambda}_p)$. Then there exists a subsequence indexed by $K \subset \{0,1,2,\ldots\}$ such that $\phi(z^i) \overset{K}{\to} \phi(\bar{z})$, where $z^i = (\xi^i, \underset{\sim}{u}^i, x^i, \lambda_0^i, \ldots, \lambda_p^i)$ and $\bar{z} = (\bar{\xi}, \bar{\underset{\sim}{u}}, \bar{x}, \bar{\lambda}_0, \ldots, \bar{\lambda}_p)$.

<u>Proof</u>: Let μ^i be a solution to (47). Since $\{\mu^i\}_{i=0}^{\infty}$ is contained in a compact set, there exists a subsequence indexed by $K \subset \{0,1,2,\ldots\}$ and a $\bar{\mu} \in R^{p+q+1}$ such that $\mu^i \to \bar{\mu}$, $\sum_{j=0}^{p+q} \bar{\mu}_j = 1$, and $\bar{\mu}_j \geq 0$ for $j = 0,1,\ldots,p+q$. By Lemma (1) we obtain $\phi(z^i) \overset{K}{\to} (\sum_{j=1}^{p} \bar{\mu}_j h_j(x(1,\bar{\xi},\bar{\underset{\sim}{u}}))$

$$+ \sum_{j=p+1}^{p+q} \bar{\mu}_j h_j(\bar{\xi}) - (1/2) \left| \sum_{j=0}^{p+q} \bar{\mu}_j \nabla g_j(\bar{\xi},\bar{\underset{\sim}{u}}) \right|_{\underset{\sim}{u}}^2).$$

$$\text{Now } \phi(\bar{z}) \geq \{ \sum_{j=1}^{p} \bar{\mu}_j h_j(x(1,\bar{\xi},\bar{\underset{\sim}{u}})) + \sum_{j=p+1}^{p+q} \bar{\mu}_j h_j(\bar{\xi}) - (1/2) \left| \sum_{j=0}^{p+q} \bar{\mu}_j \nabla g_j(\bar{\xi},\bar{\underset{\sim}{u}}) \right|_{\underset{\sim}{u}}^2 \}.$$

Suppose the inequality is strict and let $\bar{\bar{\mu}} = (\bar{\bar{\mu}}_0, \ldots, \bar{\bar{\mu}}_{p+q})$ be a solution of (47), for $z^i = \bar{z}$, that gives $\phi(\bar{z})$. Then we must have that

$$(\sum_{j=1}^{p} \bar{\bar{\mu}}_j h_j(x(1,\xi^i,\underset{\sim}{u}^i)) + \sum_{j=p+1}^{p+q} \bar{\bar{\mu}}_j h_j(\xi^i) - (1/2)\Big|\sum_{j=0}^{p+q} \bar{\bar{\mu}}_j \nabla g_j(\xi^i,\underset{\sim}{u}^i)\Big|^2_{\underset{\sim}{u}^i})$$

$\xrightarrow{K} \phi(\bar{z})$. But this implies that for sufficiently large $i \in K$,

$$(\sum_{j=1}^{p} \bar{\bar{\mu}}_j h_j(x(1,\xi^i,\underset{\sim}{u}^i)) + \sum_{j=p+1}^{p+q} \bar{\bar{\mu}}_j h_j(\xi^i) - (1/2)\Big|\sum_{j=0}^{p+q} \bar{\bar{\mu}}_j \nabla g_j(\xi^i,\underset{\sim}{u}^i)\Big|^2_{\underset{\sim}{u}^i}) > \phi(z^i).$$

This is a contradiction of (47). Therefore

49 $$\phi(\bar{z}) = \sum_{j=1}^{p} \bar{\mu}_j h_j(x(1,\bar{\xi},\bar{\underset{\sim}{u}})) + \sum_{j=p+1}^{p+q} \bar{\mu}_j h_j(\bar{\xi}) - (1/2)\Big|\sum_{j=0}^{p+q} \bar{\mu}_j \nabla g_j(\bar{\xi},\bar{\underset{\sim}{u}})\Big|_{\bar{\underset{\sim}{u}}}$$

Thus $\phi(z^i) \xrightarrow{K} \phi(\bar{z})$.

The following lemma can be deduced from an analogous result in [8].

<u>Lemma 8</u>: Let $\phi: \bar{W} \to R^1$ be defined as in (47) and let $z \in \bar{W}$ be arbitrary. Then $\phi(z) \leq 0$, and $\phi(z) = 0$ if and only if $z \in \Delta$.

<u>Lemma 9 (see [8])</u>: Suppose that $z \in W$ is such that $\phi(z) < 0$ and that $\mu(z) = (\mu_0(z), \ldots, \mu_{p+q}(z))$ is a solution to (40) for $z^i = z$. Then

50 $$\max\{ \langle \nabla g_0(\xi,u), -\sum_{j=0}^{p+q} \mu_j(z)\nabla g_j(\xi,u) \rangle_2;$$
$$g_j(\xi,u) + \langle \nabla g_j(\xi,u), -\sum_{j=0}^{p+q} \mu_j(z)\nabla g_j(\xi,u) \rangle_2,$$
$$j = 1,\ldots,p+q\} \leq \phi(z) - (1/2)\Big\| \sum_{j=0}^{p+q} \mu_j(z)\nabla g_j(\xi,u)\Big\|_2^2 < 0.$$

The following corollary to Lemma 9 is obtained by application of Lemma 1.

Corollary 1: Suppose that $z \in \overline{W}$ is such that $\phi(z) < 0$, then there exists a $\mu(z)$ such that

$$51 \qquad \max\{ \langle \nabla g_0(\xi,\underset{\sim}{v}), - \sum_{j=0}^{p+q} \mu_j(z)\nabla g_j(\xi,\underset{\sim}{v}) \rangle_{\underset{\sim}{v}};$$

$$g_j(\xi,\underset{\sim}{v}) + \langle \nabla g_j(\xi,\underset{\sim}{v}), - \sum_{j=0}^{p+q} \mu_j(z)\nabla g_j(\xi,\underset{\sim}{v})\rangle_{\underset{\sim}{v}},$$

$$j = 1,\dots,p+q\} \leq \phi(z) - (1/2) \Big| \sum_{j=0}^{p+q} \mu_j(z)\nabla g_j(\xi,\underset{\sim}{u})\Big|^2_{\underset{\sim}{v}} < 0.^{\dagger}$$

At this point we develop a set of variational equations defining a special directional differential which we shall need to show that Algorithm 2 satisfies (ii) of Theorem 4.

Definition 13: For any $\xi \in R^n$, any measurable relaxed control $\underset{\sim}{v}$, any $\alpha \in [-1,1]$ and any $y \in C_m[T \times R^m]$, let $x(t,\xi,\underset{\sim}{v},\alpha,y)$ denote the solution of

$$52 \qquad \frac{d}{dt} x(t,\xi,\underset{\sim}{v},\alpha,y) = \int_{u\in R^m} f(x(t,\xi,\underset{\sim}{v},\alpha,y),\; u + \alpha y(t,u),t)d\underset{\sim}{v}(t)$$

$$x(0,\xi,\underset{\sim}{v},\alpha,y) = \xi.$$

† Note: If $z^i \to z$ where $z^i \in W$, then a $\mu(z)$ that is an accumulation point of $\{\mu(z^i)\}$ satisfies (50).

The following results can be established by lengthy, but straightforward calculations. For a proof see [13].

Definition 14: For any $\xi \in R^n$, any measurable relaxed control $\underset{\sim}{v}$, any $\alpha \in [-1,1]$, any $y \in C_m[T\times R^m]$, and any $\delta\xi \in R^n$, let $x(t) = x(t,\xi,\underset{\sim}{v},0,y)$ and let $\delta x(x,\delta\xi,y,\alpha)(\cdot)$ denote the solution of

53 $$\delta\dot{x}(x,\delta\xi,y,\alpha)(t) = \int_{R^m} \left[\frac{\partial f}{\partial x}(x(t),u,t)\delta x(x,\delta\xi,y,\alpha)(t) + \frac{\partial f}{\partial u}(x(t),u,t)\cdot \alpha y(t,u)\right] d\bar{\underset{\sim}{v}}(t),\ t \in T,$$

and

54 $$\delta x(x,\delta\xi,y,\alpha)(0) = \delta\xi$$

Theorem 8: There exists a $K > 0$ such that $\|x(t,\xi+\delta\xi,\underset{\sim}{v},\alpha,y) - x(t,\xi,\underset{\sim}{v},0,y)\| \leq K(|\alpha|+\|\delta\xi\|)$ for all $t \in T$, for all $\alpha \in [-1,1]$, for all $\bar{\xi} \in C$, for all $\bar{\underset{\sim}{v}} \in S$ and for all $\delta\xi \in R^n$ such that $\bar{\xi} + \delta\xi \in D$.

Theorem 9: Let $y(\cdot)$ and $x(\cdot)$ be as in Definition 14. Then there exists an $M > 0$ such that $\left\|\delta x(x,\delta\xi,y,\alpha)(t) - \left(x(t,\xi+\delta\xi,\underset{\sim}{v},\alpha,y) - (x(t,\xi,\underset{\sim}{v},0,y)\right)\right\| \leq M(\|\delta\xi\|+|\alpha|)^2$ for all $t \in T$, for all $\alpha \in [-1,1]$, for all $\xi \in C$, for all $\underset{\sim}{v} \in S$, and for all $\delta\xi \in R^n$ such that $\xi + \delta\xi \in D$.†

The following theorem is a consequence of Theorem 9.

Theorem 10: Let L: $R^n \times R^m \times T \to R^1$, and ϕ: $R^n \to R^1$, h_j: $R^n \to R^1$, $j = 1,2,\ldots,p$, be functions whose partial derivatives with respect to x and u exist and are continuous in (x,u,t) up through second order, then

†Thus, $\delta x(x,\delta\xi,y,\alpha)(\cdot)$ is a kind of directional differential.

there exists a $P > 0$ such that for all $\alpha \in [-1,1]$, $v \in S$, $\xi \in C$, and $\delta\xi$ such that $\bar{\xi} + \delta\xi \in D$,

$$55 \qquad \left| \int_0^1 \left(\int_{R^m} L(x(t,\xi+\delta\xi,\underset{\sim}{v},\alpha,y),u+\alpha y(t,u),t)\,d\underset{\sim}{v}(t)\right) dt \right.$$

$$+ \phi(x(1,\xi+\delta\xi,\underset{\sim}{v},\alpha,y)) - \int_0^1 \left(\int_{R^m} L(x(t),u,t)\,d\underset{\sim}{v}(t)\right) dt$$

$$- \phi(x(1)) - \alpha \int_0^1 \left(\int_{R^m} \left\langle \frac{\partial H_0^T}{\partial u} (x(t),u,\lambda(t),t),\right.\right.$$

$$\left.\left. y(t,u) \right\rangle d\bar{\underset{\sim}{v}}(t)\right) dt - \langle \lambda_0(0),\delta\xi \rangle \Big| \leq P(|\alpha| + \|\delta\xi\|)^2 ,$$

$$56 \qquad \Big| h_j(x(1,\xi+\delta\xi,\underset{\sim}{v},\alpha,y)) - h_j(x(1,\xi,\underset{\sim}{v},0,y)) - \alpha \int_0^1 \left(\int_{R^m} \left\langle \frac{\partial H_j^T}{\partial u} (x(t),u,\right.\right.$$

$$\left.\left. \lambda_j(t),t),y(t,u) \right\rangle d\underset{\sim}{v}(t)\right) dt - \langle \lambda_j(0),\delta\xi \rangle \Big| \leq P(|\alpha| + \|\delta\xi\|)^2$$

where the λ_j, $j = 0,1,2,\ldots,p$ and H_j are defined as in Definition 4.

To relate Theorem 10 to Gateaux differentials in $L_2 \cap L_\infty$, we observe that Theorem 10 implies that there exists a P' such that with g_j defined as in (4), (5) and ∇g_j as in (43a), (43b),

$$57 \qquad \Big| h_j(x(1,\xi+\alpha\delta\xi,\bar{\underset{\sim}{v}},\alpha,y)) - h_j(x(1,\xi,\bar{\underset{\sim}{v}},0,y)$$

$$- \alpha \langle \nabla g_j(\xi,\underset{\sim}{v}), (\delta\xi,y) \rangle_{\underset{\sim}{v}} \Big| \leq P'\alpha^2, \quad j = 1,2,\ldots,p+q .$$

and

$$\left|\int_0^1 \left(\int_{R^m} L(x(t,\xi+\alpha\delta\xi,\underset{\sim}{v},\alpha,y)d\underset{\sim}{v}(t)\right)dt + \phi(x(1,\xi+\alpha\delta\xi,\underset{\sim}{v},\alpha,y)\right.$$

$$\left. - \int_0^1 \left(\int_{R^m} L(x(t,\xi,v,0,y)d\underset{\sim}{v}(\xi)\right)dt - \phi(x(1)) - \alpha\langle \nabla g_0(\xi,\underset{\sim}{v}),(\delta\xi,y)\rangle_{\underset{\sim}{v}}\right|$$

$$\leq P'\alpha^2.$$

<u>Definition 15</u>: Let $\theta: R^1 \times (R^n \times C_m[T]) \times C \times S \to R^1$ be defined by

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$$\theta\left(\alpha,(\delta\xi,y),\xi,\underset{\sim}{v}\right) = \max\left\{\int_0^1 \left[\int_{R^m} \left(L\left(x(t,\xi+\alpha\delta\xi,\underset{\sim}{v},\alpha,y),\ u+\alpha y(t),t\right)\right.\right.\right.$$

$$\left.\left.\left. - L\left(x(t,\xi,\underset{\sim}{v},0,0),u,t\right)\right)d\underset{\sim}{v}(t)\right]dt;\ h_j(x(1,\xi+\alpha\delta\xi,\underset{\sim}{v},\alpha,y)),\right.$$

$$\left. j=1,\dots,p;\ h_j(\xi+\alpha\cdot\delta\xi),\ j=p+1,\dots,p+q\right\}$$

<u>Proposition 1</u>: Let $\{z^i\}_{i=0}^{\infty} \triangleq \{(\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i)\} \subset \overline{W}$ be a sequence converging to $\bar{z} = (\bar{\xi},\bar{\underset{\sim}{v}},\bar{x},\bar{\lambda}_0,\dots,\bar{\lambda}_0)$, with $\{\underset{\sim}{v}^i\} \subset S$ and $\phi(\bar{z}) < 0$. Then there exists an integer $k(\bar{z})$ such that

60
$$\theta(\beta^{k(\bar{z})}, -\sum_{j=0}^{p+q} \mu_j(\bar{z})\ \nabla g_j(\bar{\xi},\bar{\underset{\sim}{v}}),\ \bar{\xi},\ \bar{\underset{\sim}{v}}) -3/4\ \beta^{k(\bar{z})}\phi(\bar{z}) \leq 0$$

where $\mu(\bar{z})$ is an accumulation point of a sequence $\{\mu(z^i)\}$ corresponding to $\{z^i\}$.

<u>Proof</u>: This result follows directly from the definition of $\nabla g_j(\bar{z})$, $j = 0,\dots,p+q$, Theorem 10 and the fact that by inequality (50), $\langle \nabla g_0(\bar{z}), -\sum_{j=0}^{p+q} \mu_j(\bar{z})\nabla g_j(\bar{\xi},\bar{\underset{\sim}{v}})\rangle_{\bar{\underset{\sim}{v}}} \leq \phi(\bar{z})$, and $\langle \nabla g_j(\bar{z}), -\sum_{j=0}^{p+q} \mu_j(\bar{z})\nabla g_j(\bar{\xi},\bar{\underset{\sim}{v}})\rangle_{\bar{\underset{\sim}{v}}} \leq \phi(\bar{z})$

for all $j \in \{1,\dots,p+q\}$ such that $g_j(\bar{\xi},\bar{\underset{\sim}{v}}) = 0$.

The following lemma is obtained by repeated utilization of Lemma 1.

<u>Lemma 10</u>: Let $\{z^i\}_{i=0}^{\infty} \triangleq \{(\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i,\dots,\lambda_p^i)\}_{i=0}^{\infty} \subset W$ be a sequence converging to $\bar{z} = (\bar{\xi},\bar{\underset{\sim}{v}},\bar{x},\bar{\lambda}_0,\dots,\bar{\lambda}_p)$, where $\{\underset{\sim}{v}^i\}_{i=0}^{\infty} \subset S$, $\{\xi^i\}_{i=0}^{\infty} \subset C$, and suppose that a corresponding sequence of solutions to (47), $\{\mu(z^i)\}_{i=0}^{\infty}$, converges to a $\mu(\bar{z})$. Then for any $\alpha \in [-1,1]$, there exists an infinite subset $J(\alpha) \subset \{0,1,2,\dots\}$ such that

61 $$\theta\left(\alpha, -\sum_{j=0}^{p+q} \mu_j(z^i)\nabla g_j(\xi^i,\underset{\sim}{v}^i),\ \xi^i,\underset{\sim}{v}^i\right) \xrightarrow{J(\alpha)} \theta\left(\alpha, -\sum_{j=0}^{p+q} \mu_j(\bar{z})\nabla g_j(\bar{\xi},\bar{\underset{\sim}{v}}),\bar{\xi},\bar{\underset{\sim}{v}}\right).$$

<u>Lemma 11</u>: Let $\{(\xi^i,\underset{\sim}{v}^i,x^i,\lambda_0^i,\dots,\lambda_p^i)\}_{i=0}^{\infty}$ be a sequence in W converging to $\bar{z} \triangleq (\bar{\xi},\bar{\underset{\sim}{v}},\bar{x},\bar{\lambda}_0,\dots,\bar{\lambda}_p)$, that satisfies the hypotheses of Lemma 10. Suppose that $\phi(\bar{z}) < 0$. Then there exist a $\delta(\bar{z}) < 0$, an integer $M > 0$, and an infinite subset $K \subset \{0,1,2,\dots\}$ such that

62 $$g_0(\xi^{i+1},\underset{\sim}{v}^{i+1}) - g_0(\xi^i,\underset{\sim}{v}^i) \le \delta(\bar{z}) \qquad \begin{array}{l} \forall i \in K \\ \forall i \ge M \end{array}$$

where ξ^{i+1} and v^{i+1} are respectively the initial state and the measurable control that Algorithm 2 would construct from the control v^i and initial state ξ^i.

<u>Proof</u>: For $j = 1,2,\dots,p$, let $y_j(t,u) = \frac{\partial H_j}{\partial u}^T (x(t,\bar{\xi},\bar{v}),u,\lambda_j(t,\bar{\xi},\bar{v}),t)$. Then by Proposition 1 there exists an integer $k(\bar{z}) \ge 0$ such that

63 $$\int_0^1 \left[\left(\int_{R^m}\left(L\left(x(t,\bar{\xi}+\beta^{k(\bar{z})}\left[-\sum_{j=0}^{p}\mu_j(\bar{z})\lambda_j(0,\bar{\xi},\bar{v})-\sum_{j=p+1}^{p+q}\mu_j(\bar{z})\frac{\partial h_j}{\partial x}^T(\bar{\xi})\right],\bar{v},\beta^{k(\bar{z})},\right.\right.\right.\right.$$

$$-\sum_{j=0}^{p}\mu_j(\bar{z})\, y_j\Big),\ u+\beta^{k(\bar{z})}\left[-\sum_{j=0}^{p}\mu_j(\bar{z})y_j(t,u)\right],t\Big)$$

$$\left.\left.-L\left(x(t,\bar{\xi},\bar{v},0,0),u,t\right)\right)d\bar{v}(t)\right]dt$$

$$\leq \theta\left(\beta^{k(\bar{z})},\ -\sum_{j=0}^{p+q}\mu_j(\bar{z})\nabla g_j(\bar{\xi},\bar{v})\ ,\bar{\xi},\bar{v}\right)\ \leq 3/4\ \beta^{k(\bar{z})}\phi(\bar{z})\ <0.$$

By Lemmas 10 and 1 and inequality (63), there exists an integer $M' > 0$ such that

64 $$\int_0^1 L\Big(x(t,\xi^i+\beta^{k(\bar{z})}\omega^i,v^i+\beta^{k(\bar{z})}\bar{v}^i),$$

$$v^i(t)+\beta^{k(\bar{z})}\bar{v}^i(t)),t\Big)dt-\int_0^1 L\left(x(t,\xi^i,v^i),v^i(t),t\right)dt$$

$$=\int_0^1\left[\int_{R^m}\left(L\left(x(t,\xi^i+\beta^{k(\bar{z})}\left[-\sum_{j=0}^{p}\mu_j(z^i)\lambda_j(0,\xi^i,v^i)\right.\right.\right.\right.$$

$$\left.-\sum_{j=p+1}^{p+q}\mu_j(z^i)\frac{\partial h_j}{\partial x}(\xi^i)\right],v^i,\beta^{k(\bar{z})},\ -\sum_{j=0}^{p}\mu_j(z^i)\ y_j\Big),$$

$$u+\beta^{k(\bar{z})}\left[-\sum_{j=0}^{p}\mu_j(z^i)y_j(t,u)\right],t\Big)$$

$$\left.-L\left(x(t,\xi^i,v^i,0,0),u,t\right)\Big)dv^i(t)\right]dt\leq\theta\Big(\beta^{k(\bar{z})},$$

$$-\sum_{j=0}^{p+q}\mu_j(z^i)\nabla g_j(\xi^i,v^i),\xi^i,v^i\Big)$$

$$\leq 1/2\ \beta^{k(\bar{z})}\phi(z^i)<0 \qquad \forall i\geq M'$$

$$\forall i\in J(\beta^{k(\bar{z})}).$$

Now consider the control v^i and initial state ξ^i and the control v^{i+1} and initial state ξ^{i+1} which Algorithm 2 constructs. The control v^{i+1} is given by

$$65 \qquad v^{i+1}(\cdot) = v^i(\cdot) + \beta^{k(z^i)}\bar{v}^i(\cdot)$$

and initial state by

$$66 \qquad \xi^{i+1} = \xi^i + \beta^{k(z^i)}w^i,$$

where $k(z^i)$ is the integer computed in Step 6 of Algorithm 2. It follows from (64) that $\beta^{k(z^i)} \geq \beta^{k(\bar{z})}$. Therefore by construction we get

$$67 \qquad g_0(\xi^{i+1},v^{i+1}) - g_0(\xi^i,v^i) \leq \theta\left(\beta^{k(z^i)}, -\sum_{j=0}^{p+q} \mu_j(z^i)\nabla g_j(\xi^i,\underset{\sim}{v}^i),\xi^i,v^i\right)$$

$$\leq 1/2\ \beta^{k(z^i)}\phi(z^i) \leq 1/2\beta^{k(\bar{z})}\phi(z^i) \qquad \forall i \geq M' \quad \forall i \in J(\beta^{k(\bar{z})}).$$

Since $\phi(z^i) \to \phi(\bar{z})$, there exists an integer $M \geq M'$ such that

$$67 \qquad g_0(\xi^{i+1},v^{i+1}) - g_0(\xi^i,v^i) \leq 1/4\beta^{k(\bar{z})}\phi(\bar{z}) \qquad \forall i \geq M \quad \forall i \in J(\beta^{k(\bar{z})}).$$

which completes our proof.

We now give the main result of this section.

<u>Theorem 11</u>: Let $\{(\xi^i,v^i,x^i,\lambda_0^i,\ldots,\lambda_p^i)\}_{i=0}^{\infty}$ be a sequence of initial states, measurable controls, corresponding trajectories and corresponding multipliers constructed by Algorithm 2. If there exist compact sets $C \subset R^m$, $U \subset R^m$ such that $\xi^i \in C$ and $u^i(t) \in U$, $t \in T$, for all $i = 0, 1, 2, \ldots$, then either the sequence is finite, in which case the

last element is desirable, or it is infinite and every accumulation point of this sequence is desirable. Furthermore, at least one accumulation point exists.

Proof: The above Algorithm 2 is basically of the form of our Algorithm Prototype. With W, $\bar{W}$, and Δ defined as in definition (10 and (11), $c \equiv g_0$, we only need verify conditions (i) and (ii) of Theorem 4 in order to invoke Theorem 4. Lemma 2 immediately implies (i) and Lemmas 7 and 11 immediately imply (ii). The existence of at least one accumulation point is guaranteed by Theorem 3.

Conclusion

The two examples we have included in this paper illustrate the use of the new convergence results for optimal control algorithms. Many other and much more complex algorithms can be analyzed in a similar way. The interested reader can find further results in [9]. The net effect of our work is to show that optimal control algorithms are very well behaved, contrary to the misgivings felt by some theoreticians.

APPENDIX A: Optimality Conditions in Optimization Algorithms.

1) A careful examination of nonlinear programming algorithms (see e.g. Ch. 4 in [9]) shows that they are frequently derived from variants of some basic optimality condition. For example, Rosen's gradient projection method is based on the Kuhn-Tucker conditions in standard form. The Zukhovidskii-Polyak-Primak method of feasible directions is based on the Kuhn-Tucker conditions stated as a multiplier free constrained optimization problem. The Zoutendijk and Demyanov methods of feasible directions are based on the F. John condition stated as a multiplier free min max problem, and the Pironneau-Polak method is based on the F. John conditions stated as a max problem with multipliers, which also happens to be the dual of a multiplier free min max problem. Many more such examples can be cited.

The same phenomenon holds true in optimal control algorithms, as illustrated by the two algorithms presented in this paper. We shall now show the relationship between the optimality conditions $\theta(z_i) = 0$, $\phi(z_i) = 0$ used in Algorithms 1 and 2 with the relaxed minimum principle.

Theorem A1: The Relaxed Minimum Principle:

If $\underset{\sim}{u}$ is optimal for the relaxed optimal control problem (7), (8), (9), (4), (5), and $x^{\underset{\sim}{u}}$ is the corresponding optimal trajectory, then $x^{\underset{\sim}{u}}(1)$ satisfies (4), $x^{\underset{\sim}{u}}(0) = \xi$ satisfies (5), $\underset{\sim}{u}(\cdot)$ satisfies (9), and there exist a scalar λ^0 and a co-state trajectory $\lambda^{\underset{\sim}{u}}$, with $(\lambda^0, \lambda^{\underset{\sim}{u}}(t)) \neq 0$, such that $\lambda^0 \geq 0$ and

$$\text{A1} \quad \frac{d}{dt}\lambda^{\underset{\sim}{u}}(t) = -\lambda^0 \left(\frac{\partial L}{\partial x}\right)_r (x^{\underset{\sim}{u}}(t), \underset{\sim}{u}(t), t) - \left(\frac{\partial f}{\partial x}\right)_r^T (x^{\underset{\sim}{u}}(t), \underset{\sim}{u}(t), t)^T \lambda^{\underset{\sim}{u}}(t), \quad t \in T,$$

with

$$\text{A2} \qquad \lambda^{u}(1) = \sum_{j=0}^{p} \mu_j \nabla h_j(x^{u}(1))$$

$$\text{A3} \qquad \lambda^{u}(0) = - \sum_{j=p+1}^{p+q} \mu_j \nabla h_j(x^{u}(0))$$

where $\mu_j \geqq 0$ for $j = 0, 1, 2, \ldots, p+q$, $\mu_j h_j(x^{u}(1)) = 0$ for $j = 1, 2, \ldots, p$, $\mu_j h_j(x^{u}(0)) = 0$ for $j = p+1, \ldots, p+q$ and for all admissible relaxed controls $\hat{u}$,

$$\text{A4} \qquad \Delta H(x^{u}(t),\lambda^{u}(t),u(t),\hat{u}(t),t) \triangleq \lambda^0 L_r(x^{u}(t),u(t)t) + \langle \lambda^u(t), f_r(x^{u}(t),u(t),t)\rangle$$
$$-\left(\lambda^0 L_r(x^{u}(t),\hat{u}(t),t) + \left\langle \lambda^u(t), f_r(x^{u}(t),\hat{u}(t),t)\right\rangle\right) \leq 0.$$

Now consider Algorithm 1, which solves the fixed initial state, free terminal state problem. Since (see (33)) $\theta(z) \leq 0$ for all admissible z and since whenever $\theta(z) < 0$, the algorithm will construct a z' resulting in a lower cost, it is clear that if z is optimal, then $\theta(z) = 0$. The relationship of $\theta(z) = 0$ to the relaxed maximum principle is as follows.

<u>Theorem A2</u>: Consider the optimal control problem solved by Algorithm 1 with the accompanying assumptions. Suppose that $\overline{W}$ is defined as in (36) and that $z = (u, x^{u}, \lambda_0^{u})$ is such that $\theta(z) = 0$, then λ_0^{u} satisfies A1, with $\lambda^0 = 1$, $\lambda_0^u(1)$ satisfies A2 with $\mu^0 = 1$, $\mu_j = 0$, $j = 1, \ldots, p$ and A3 with $\mu^j = 0$, $j = p+1, \ldots, q$, and for all admissible relaxed controls $\hat{u}$,

$$\text{A5} \qquad \int_0^1 \Delta H(x^{u}(t),\lambda^{u}(t),u(t),\hat{u}(t),t)dt \leq 0.$$

Thus, $\theta(z) = 0$ is seen as an integral form of the relaxed maximum principle.

Now consider the problem solved by Algorithm 2. Again, by construction, it is clear that $\phi(z) = 0$ (see (40)) is a necessary condition of optimality. Its relation to the relaxed minimum principle is as follows.

<u>Theorem A3</u>: Consider the optimal control problem solved by Algorithm 2, with the accompanying assumption. Suppose that $\bar{W}$ is defined as in (45) and that $z = (\tilde{u}, \tilde{x}^u, \tilde{\lambda}_0^u, \ldots, \tilde{\lambda}_p^u)$ is such that $\phi(z) = 0$, and let $\mu_j(z)$, $j = 0, 1, \ldots, p+q$ be computed according to (40). Then the co-state $\tilde{\lambda}^u(t) = \sum_{j=0}^{p} \mu_j(z)\tilde{\lambda}_j^u(t)$ satisfies (A1) with $\lambda^0 = \mu_0(z) \geq 0$, (A2) with $\mu_j = \mu_j(z)$, $j = 0, 1, .., p$, and (A3) with $\mu_j = \mu_j(z)$, $j = p+1, .., p+q$. Furthermore,

$$\text{A6} \qquad \int_0^1 \left\| \lambda^0 \left(\frac{\partial L}{\partial u} \right)_r^T (\tilde{x}^u(t), \tilde{u}(t), t) + \left(\frac{\partial f}{\partial x} \right)_r^T (\tilde{x}^u(t), \tilde{u}(t), t) \tilde{\lambda}^u(t) \right\|^2 dt = 0.$$

Thus, the condition $\phi(z) = 0$, is seen as a weak, or "differential," form of the relaxed minimum principle.

APPENDIX B: Convergence in L_2 and I.S.C.M.

We will now present the link between the convergence of a sequence $\{u^i\}$ of controls in $L_2^m\,[0,1] \cap L_\infty^m\,[0,1]$ in the L_2 norm, and the convergence of the associated sequence of measurable relaxed controls $\{\underset{\sim}{u}^i\}$, in he sense of control measures. We first give the definition of almost uniform convergence of measurable function defined on a closed interval T.

<u>Definition B1</u>: A sequence of measurable functions, $\{u^i(\cdot)\}_{i=0}^{\infty}$, is said to be almost uniformly convergent to a measurable function $\bar{u}(\cdot)$ if for each $\delta > 0$ there is a set E_δ in T with $\mu(E_\delta) < \delta$ such that $u^i(\cdot)$ converges uniformly to $\bar{u}(\cdot)$ on T/E_δ.

The following theorem is found on page 75 in Chapter 7 of Bartle, [9].

<u>Theorem B1</u>: If a sequence of measurable functions, $\{u^i(\cdot)\}_{i=0}^{\infty}$, converges to a measurable function $\bar{u}(\cdot)$ in the L_2 norm, then there exists a subsequence which converges almost uniformly to $\bar{u}(\cdot)$.

In the standard L_2 theory of convergence of optimal control algorithms, one assumes that the sequence of measurable controls $\{u^i(\cdot)\}_{i=0}^{\infty}$ constructed by an optimal control algorithm, has a subsequence which converges in the L_2 norm to a measurable function $\bar{u}(\cdot)$. Theorem B1 shows that when the above assumption is made, it is automatically assumed that there exists a subsequence of $\{u^i(\cdot)\}_{i=0}^{\infty}$ which converges almost uniformly to the function $\bar{u}(\cdot)$.

The following theorem shows that almost uniform convergence of measurable controls implies i.s.c.m. convergence of the associated measurable relaxed controls.

<u>Theorem B2</u>: Let $\{u^i\}_{i=0}^{\infty} \subset L_2^m [0,1] \cap L_\infty^m [0,1]$ be a sequence of uniformly bounded measurable controls which converges almost uniformly to $\bar{u}$, and let $\{v^i\}_{i=0}^{\infty}, \bar{v}$ be associated measurable relaxed controls. Then v^i converges i.s.c.m. to $\bar{v}$.

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