TWO SPECIAL CASES OF THE ASSIGNMENT PROBLEM

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The assignment problem may be stated as follows: given finite sets of points S and T, with $|S| \geq |T|$, and given a "metric" which assigns a distance $d(x,y)$ to each pair (x,y) such that $x \in T$ and $y \in S$, find a 1 - 1 function $Q: T \rightarrow S$ which minimizes $\sum_{x \in T} d(x,Q(x))$. We consider the two special cases in which the points lie (1) on a line segment and (2) on a circle, and the metric is the distance along the line segment or circle, respectively. In each case, we show that the optimal assignment Q can be computed in a number of steps (additions and comparisons) proportional to the number of points. The problem arose in connection with the efficient rearrangement of desks located in offices along a corridor which encircles one floor of a building.

1. The linear case

Suppose we are given two disjoint finite sets S (the sources) and T (the destinations) of points in the open interval $(0,X)$, with

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$|S| \geq |T|$. Each destination $x \in T$ must receive a desk from some source $Q(x) \in S$. A source can supply at most one desk, so the function $Q: T \rightarrow S$ is one-to-one. We wish to choose Q to minimize $\sum_{x \in T} |x-Q(x)|$, which is just the total distance that the desks must travel.

Define $H(x) = |S \cap (0,x]| - |T \cap (0,x)|$, $x \in [0, X]$

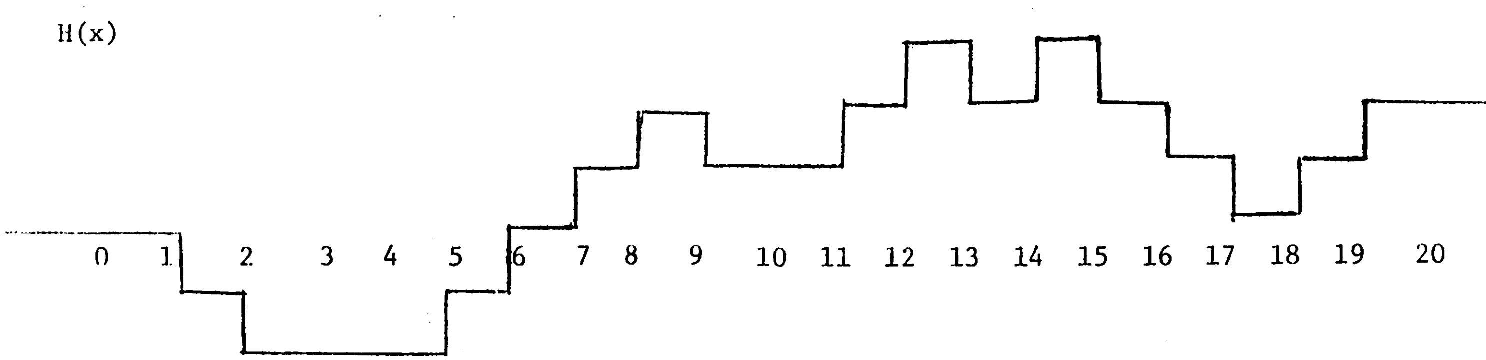
and define

$$e = |S| - |T| = H(X) .$$

Then

$$e \geq 0 ;$$

e is just the excess of sources over destinations, and the "height function" H(x) gives the excess of sources over destinations up to x.



$X = 20$, $S = \{5,6,7,8,11,12,14,18,19\}$, $T = \{1,2,9,13,15,16,17\}$

<u>FIGURE 1</u> : The function H(x)

The case $e = 0$

Here $|S| = |T|$, and we seek a one-to-one assignment of sources to destinations. This problem is trivial to solve, but we discuss it in order to extract the following theorem.

Theorem 1: The cost of an optimal solution is $\int_0^X |H(x)|dx$.

Proof: Consider any assignment Q. For any $x \in [0,X]$ such that $x \notin S \cup T$, define

$$L_Q(x) = |\{y \in T \mid Q(y) < x < y\}|$$

and

$$R_Q(x) = |\{y \in T \mid y < x < Q(y)\}|$$

Thus, using the terminology of the desk-moving application, $L_Q(x)$ is the number of desks passing x from left to right, and $R_Q(x)$ is the number of desks passing x from right to left. Define $f_Q(x) = L_Q(x) - R_Q(x)$; $f_Q(x)$ may be interpreted as a flow equal to the net number of desks passing x from left to right.

Now observe that

$$f_Q(0) = 0$$

f_Q is constant in each interval which does not contain an element of $S \cup T$,

$$f_Q(x^+) = f_Q(x^-) + 1 , \qquad x \in S$$

and

$$f_Q(x^+) = f_Q(x^-) - 1, \qquad x \in T$$

where the notation $f_Q(x^+)$ stands for $\lim_{\epsilon \to 0^+} f_Q(x + \epsilon)$, etc.

But these are the same properties that determine $H(x)$ on $[0,X] - (S \cup T)$, so $f_Q(x)$ is identically equal to $H(x)$ on this domain.

Now

$$\sum_{x \in T} |x - Q(x)| = \int_{x=0}^{X} L_Q(x)dx + \int_{x=0}^{X} R_Q(x)dx \tag{1}$$

also

$$\int_{x=0}^{X} L_Q(x)\,dx + \int_{x=0}^{X} R_Q(x)\,dx \geq \int_{x=0}^{X} |f_Q(x)|dx = \int_{x=0}^{X} |H(x)|dx \tag{2}$$

with equality holding in (2) if and only if, at every x, either $L(x) = 0$ or $R(x) = 0$; i.e., if and only if there is no cancellation of left-to-right flow against right-to-left flow.

Hence, $\int_{x=0}^{X} |H(x)|$ is a lower bound on the cost of any assignment, and this lower bound is achieved by any assignment in which no flow cancellation occurs. Such an assignment is easy to construct; one way is by the following "left-to-right ordering rule":

Write

$$T = \{x_1, x_2, \ldots, x_n\} \quad \text{where} \quad x_1 < x_2 < \ldots < x_n$$

and

$$S = \{y_1, y_2, \ldots, y_n\} \quad \text{where} \quad y_1 < y_2 < \ldots < y_n \ .$$

Then set

$$Q(x_k) = y_k \ , \qquad k = 1, 2, \ldots, n.$$

□

<u>The case e > 0</u>

Now suppose that supply exceeds demand; i.e., $|S| - |T| = e > 0$. Then the problem is to decide which e elements of S are to be left unused; once this is determined, the optimal assignment is obtained by the method used when $e = 0$.

Let E be a subset of X such that $|E| = e$.

Define

$$H^E(x) = |(S-E) \cap (0,x]| - |T \cap (0,x)| \qquad x \in [0,X] .$$

Equivalently,

$$H^E(x) = H(x) - |E \cap (0,x]|$$

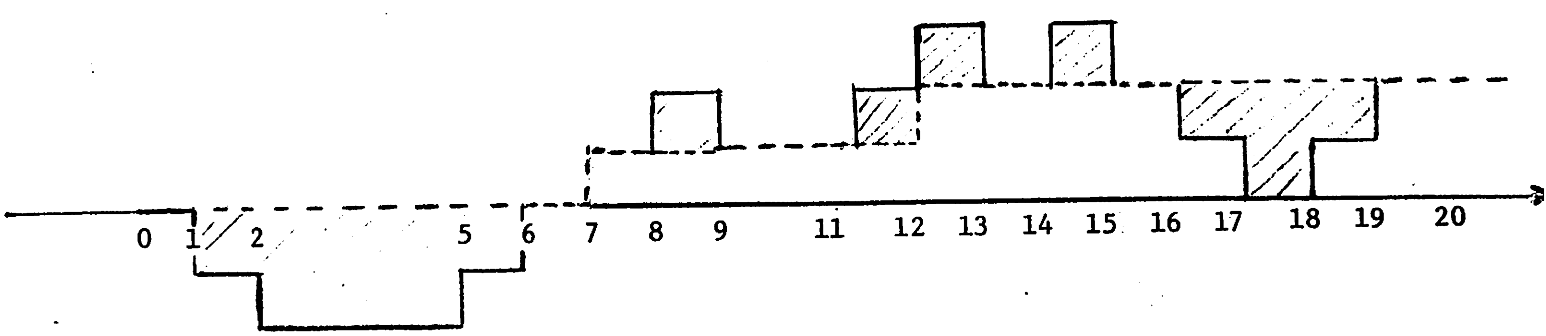
Then it follows from Theorem 1 that the cost of the best solution which omits the sources in E is

$$\int_{x=0}^{X} |H^E(x)|dx .$$

Our problem is to determine

$$\min_{\{E \mid E \subseteq S \text{ and } |E| = e\}} \int_{x=0}^{X} |H^E(x)|dx \tag{3}$$

An E yielding the minimum in (3) will be called <u>optimal</u>.



<u>FIGURE 2</u>: An optimal choice of E for a linear assignment problem.

$E = \{7,11\}$ is optimal for the example of Figure 1. The heavy curve describes the function H and the dotted stair-shaped curve has jumps at 7 and 11. The total area between the curves is

$$\int_0^X |H^E(x)|dx$$

<u>Theorem 2</u>: Let E be optimal. Let $y_{(k)}$ denote the k^{th} smallest element of E. Then

$$H(y_{(k)}) = k \ , \qquad k = 1, 2, \ldots, e.$$

<u>Proof</u>: We show by contradiction that $H^E(y_{(k)}) = 0$, $k = 1,2,\ldots,e$. Suppose $H^E(y_{(k)}) > 0$ (the case $H^E(y_{(k)}) < 0$ being similar). Then the optimal flow pattern includes some left-to-right flow past $y_{(k)}$; i.e., there is an $x \in T$ such that $Q(x) < y_{(k)} < x$; but this contradicts optimality, since it would be better to eliminate $Q(x)$ instead of $y_{(k)}$, and ship a desk from $y_{(k)}$ to x.

Thus $H^E(y_{(k)}) = 0$. But $H^E(y_{(k)}) = H(y_{(k)}) - k$; so that $H(y_{(k)}) = k$. □

We shall be interested in sets $E \subseteq S$ such that

$$|E| = e \tag{4}$$

and the necessary condition for optimality

$$H(y_{(k)}) = k, \quad k = 1, 2, \ldots, e \tag{5}$$

is satisfied.

We give a useful expression for $\int_{x=0}^{X} (|H^E(x)| dx$. For $y \in S$, define

$$P(y) = \int_{x=y}^{X} |H(x) - H(y) + 1| - |H(x) - H(y)|dx$$

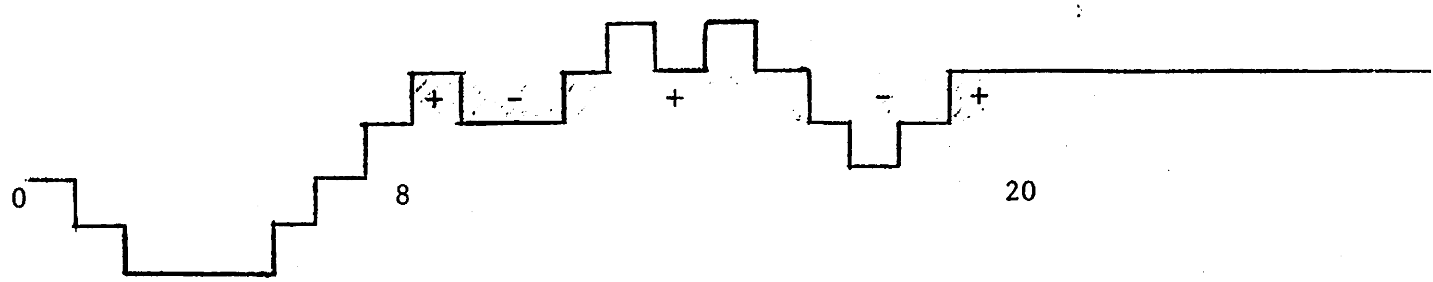


FIGURE 3: Expression of P(y) as an alternating sum of areas of unit-height rectangles (y = 8).

The following theorem justifies considering P(y) as the profit associated with including y in the set E.

Theorem 3: Let $E = \{y_1, y_2, \ldots, y_e\} \subseteq S$ be a set satisfying (4) and (5). Then

$$\int_{x=0}^{X} |H^E(x)|dx = \int_{x=0}^{X} |H(x)|dx - \sum_{y \in E} P(y) \tag{6}$$

Proof: Consider an interval $I = (t, t+\Delta)$ containing no element of $S \cup T$. Assume that $y_{(\ell)} < t < t+\Delta < y_{(\ell+1)}$. We compute the contributions of the interval I to each side of (6).

$$\int_{x=t}^{t+\Delta} |H^E(x)|dx = \int_{x=t}^{t+\Delta} |H(x) - \ell|dx = \Delta|H(t) - \ell|$$

The contribution to the right-hand-side is:

From $$\int_{x=0}^{X} |H(x)| \quad , \qquad \Delta\ |H(t)|$$

From $$- P(y_{(k)}) \qquad \begin{cases} \Delta\ (-|H(t) - k + 1| + |H(t) - k|), \\ \qquad k = 1, 2, \ldots, \ell \\ 0, \ k = \ell+1, \ldots, e. \end{cases}$$

Here we have used the fact (from Theorem 2) that $H(y_{(k)}) = k$. Thus the total contribution to the right-hand side is the telescoping sum

$$\Delta\ (|H(t)| + \sum_{k=1}^{\ell} |H(t) - k| - |H(t) - (k-1)|) = \Delta|H(t) - \ell|$$

□

From (6) we see that an optimal E is a set which, among all sets satisfying (4) and (5), maximizes $\sum_{y \in E} P(y)$. We shall show that such a set is easily determined. For any integer k such that $\{y|y \in S \text{ and } H(y) = k\}$ is nonempty, define y_k^* by:

(i) $y_k^* \in S$

(ii) $H(y_k^*) = k$

(iii) $P(y_k^*) = \max \{P(y)|y \in S \text{ and } H(y) = k\}$

and (iv) If $y \in S$, $H(y) = k$ and $P(y) = P(y_k^*)$

then

$$y \geq y_k^* \ .$$

Thus y_k^* is leftmost among points in S of height k that give a maximum profit. In particular, y_k^* is defined for $k = 1, 2, \ldots, e$.

Theorem 4: The set $E^* = \{y_1^*, y_2^*, \ldots, y_e^*\}$ is optimal.

Proof: The only point that is not obvious is that E^* satisfies (5); i.e., that $y_k^* < y_{k+1}^*$, $k = 1, 2, \ldots, e - 1$.

We shall prove this by contradiction. First we remark that, for any fixed t, the function

$$|t - w + 1| - |t - w|$$

is a monotone nondecreasing function of w;

hence, for any fixed H, a and b, $a < b$,

$$\int_a^b (|H(x) - w + 1| - |H(x) - w|)dx$$

is a monotone nondecreasing function of w.

Now assume for contradiction that there is a q, $1 \leq q \leq k - 1$, such that $y_{q+1}^* < y_q^*$.

Let

$$\bar{y}_q = \max \{y | y \in S \text{ and } H(y) = q \text{ and } y < y_{q+1}^*\}$$

and let $$\bar{y}_{q+1} = \min \{y | y \in S \text{ and } H(y) = q + 1 \text{ and } y > y_q^*\}$$

Then, for $$x \in [\bar{y}_q, y_{q+1}^*] , \quad H(x) \geq q \tag{7}$$

and for $$x \in [y_q^*, \bar{y}_{q+1}) , \quad H(x) \leq q \tag{8}$$

Then

$$P(y_{q+1}^*) - P(\bar{y}_{q+1}) = \int_{y_{q+1}^*}^{y_q^*} (|H(x)-q|-|H(x)-q-1|)dx + \int_{y_q^*}^{\bar{y}_{q+1}} (|H(x)-q|-|H(x)-q-1|)dx \geq 0$$

and

$$P(\bar{y}_q) - P(y_q^*) = \int_{\bar{y}_q}^{y_{q+1}^*} (|H(x)-q+1|-|H(x)-q|)dx + \int_{y_{q+1}^*}^{y_q^*} (|H(x)-q+1|-|H(x)-q|)dx < 0$$

By (8),
$$\int_{y_q^*}^{\bar{y}_{q+1}} (|H(x) - q| - |H(x) - q - 1|)\, dx < 0$$

By (7),
$$\int_{\bar{y}_q}^{y_{q+1}^*} (|H(x) - q + 1| - |H(x) - q|)\, dx \geq 0$$

and finally, by the monotonicity property stated at the beginning of the proof,

$$\int_{y_{q+1}^*}^{y_q^*} (|H(x)-q+1|-|H(x)-q|)dx \geq \int (|H(x)-q|-|H(x)-q-1|)dx \ .$$

But these inequalities are mutually inconsistent, and the required contradiction is reached. □

For any integer k, define

$$\Pi(k) = \begin{cases} -\infty & \text{if } y_k^* \text{ is undefined} \\ P(y_k^*) & \text{if } y_k^* \text{ is defined.} \end{cases}$$

<u>Corollary 1</u>: The cost of an optimal solution is

$$\int_{x=0}^{X} |H(x)|dx - \sum_{k=1}^{e} \Pi(k) \ .$$

2. The circular case

Now we suppose that the sources and destinations lie on a circle of arc length X; each point in $S \cup T$ is assigned a coordinate $x \in (0,X)$ given by its clockwise displacement from an arbitrary zero point which is not in $S \cup T$. The distance $d(x,y) = \min (r(x,y), r(y,x))$, where $r(x,y)$ is the clockwise displacement from x to y; i.e.,

$$r(x,y) = \begin{cases} y - x, & x \leq y \\ y + X - x, & x > y \end{cases}$$

The case e = 0

The discussion of this case closely parallels the discussion of the linear case when e = 0. Exactly as in the linear case, define

$$H(x) = |S \cap (0,x]| - |T \cap (0,x)| \quad , \quad x \in (0,X).$$

Theorem 5: The cost of an optimal solution is

$$\min_h \int_{x=0}^{X} |H(x) - h| \, dx \tag{9}$$

where h ranges through all integers.

Proof: First we prove that (9) gives a lower bound on the cost of an optimal solution. Let Q be any assignment. For any $x \in (0,X)$ such that $x \notin S \cup T$, define

$$R_Q(x) = |\{y \in T \mid r(y,x) < r(y,Q(y)) \leq r(Q(y),y)\}|$$

$$L_Q(x) = |\{y \in T \mid r(Q(y),x) < r(Q(y),y) < r(y,Q(y))\}|$$

and

$$f_Q(x) = L_Q(x) - R_Q(x) \quad .$$

Thus, $R_Q(x)$ is the number of desks that pass x in a counterclockwise direction, and $L_Q(x)$ is the number that pass x in a clockwise direction. Now observe that:

f_Q is constant in any interval which does not contain an element of $S \cup T$

$$f_Q(x^+) = f_Q(x^-) + 1 \quad , \quad x \in S$$

and

$$f_Q(x^+) = f_Q(x^-) - 1 \quad , \quad x \in T \ .$$

Hence,

$$f_Q(x) = H(x) + f_Q(0)$$

and

$$\sum_{x \in T} d(x,Q(x)) = \int_{x=0}^{X} L_Q(x) + \int_{x=0}^{X} R_Q(x) \geq \int_{x=0}^{X} |f_Q(x)|dx$$

$$\int_{x=0}^{X} |H(x) + f_Q(0)|dx \geq \min_h \int_{x=0}^{X} |H(x) - h|dx.$$

Now we show that there is a solution which achieves the lower bound (9). Let h^* be a minimizing h. Define $f(x) = H(x) - h^*$. Also, there is at least one point x where $f(x) = 0$. For,otherwise, f never changes sign; and if, for instance, $f(x) \geq 1$ for all x, then

$$\int_{x=0}^{X} |H(x) - (h^* + 1)|dx = \int_{x=0}^{X} |f(x)-1|dx < \int_{x=0}^{X} |f(x)|dx$$

$$= \int_{x=0}^{X} |H(x) - h^*|dx \quad ,$$

contradicting the optimality of h^*.

Choose any $\bar{x} \in S \cup T$ such that $f(\bar{x}) = 0$.

Let the elements of T, in clockwise order starting at $\bar{x}$, be $x_1, x_2, \ldots, x_n$, and let the elements of S, in clockwise order starting at $\bar{x}$, be $y_1, y_2, \ldots, y_n$. Let Q^* be the assignment given by $Q^*(x_k) = y_k$, $k = 1, 2, \ldots, n$; i.e., Q^* matches elements in clockwise order starting at $\bar{x}$. Then for all x,

$$L_{Q^*}(x) = \max(f(x), 0)$$

$$R_{Q^*}(x) = -\min(f(x), 0)$$

and the cost of the assignment Q^* is

$$\int_{x=0}^{X} |f(x)|\,dx \qquad \square$$

The case $e > 0$

Just as in Section 1, define

$$H^E(x) = |(S-E) \cap (0,x]| - |T \cap (0,x)| , \qquad x \in (0,X] ,$$

where E denotes a subset of S such that $|E| = e$.

Then it is immediate from Theorem 5 that the cost of an optimal solution is

$$\min_h \min_E \int_{x=0}^{X} |H^E(x) - h|\,dx \tag{10}$$

Theorem 6: The optimal value of (10) is

$$\min_{h} \left[\int_{x=0}^{X} |H(x) - h|\,dx - \sum_{\ell=h+1}^{h+e} \Pi(\ell) \right] \tag{11}$$

Proof: The proof parallels the proofs of Theorems 3 and 4, which deal with the linear case when $e > 0$. We give the proof in a somewhat telegraphic style, since no essentially new ideas are involved. First we show that (11) gives a lower bound on the cost of an optimal solution. Let $(\bar{h},\bar{E})$ be the minimizing pair in (10). Then the optimal assignment $\bar{Q} : T \to (S-\bar{E})$ satisfies

$$L_{\bar{Q}}(x) - R_{\bar{Q}}(x) = f_{\bar{Q}}(x) = H^{\bar{E}}(x) - \bar{h}, \qquad x \in [0,X] - (S \cup T)$$

and

$$R_{\bar{Q}}(x) \cdot L_{\bar{Q}}(x) = 0 \qquad x \in [0,X] - (S \cup T) \tag{12}$$

Here $L_{\bar{Q}}(x)$ denotes the clockwise flow past x, and $R_{\bar{Q}}(x)$, the counterclockwise flow past x. Equation (12) asserts that there is no cancellation of clockwise flow against counterclockwise flow. For each $y \in \bar{E}$, $f_{\bar{Q}}(y) = 0$; else we could improve $\bar{Q}$ by using y in place of some element of $S - \bar{E}$.

Thus, if $\bar{y}_{(k)}$ denotes the k^{th} smallest element of $\bar{E}$, then

$$H(\bar{y}_{(k)}) = \bar{h} + k.$$

A calculation like the one used in the proof of Theorem 3 yields

$$\int_{x=0}^{X} |H^{\bar{E}}(x) - \bar{h}|\,dx = \int_{x=0}^{X} |H(x) - \bar{h}|\,dx - \sum_{k=1}^{e} P(\bar{y}_{(k)}) \quad .$$

Since $P(\bar{y}_{(k)}) \leq \Pi(\bar{h} + k)$, it follows that (11) gives a lower bound.

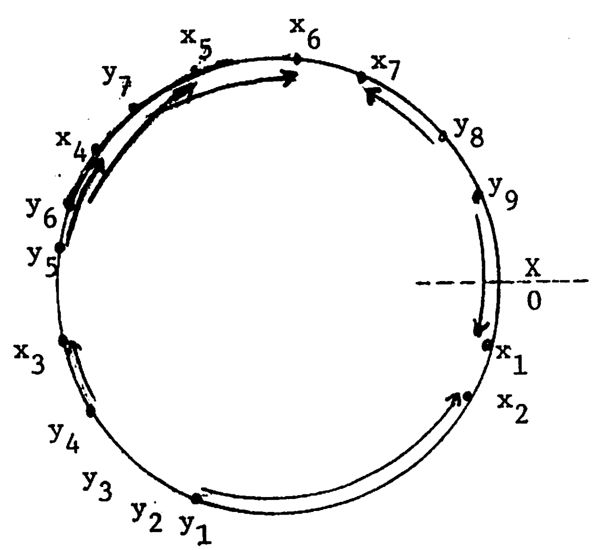
Now we show that the lower bound given by (11) can be realized. Let h^* be the minimizing h in (11). Let

$$E^* = \{ y_\ell^* \mid \ell = h^* + 1,\ h^* + 2,\ \ldots,\ h^* + e\} .$$

An easy calculation similar to the proof of Theorem 4 shows $y^*_{h^*+1} < y^*_{h^*+2} < \ldots < y^*_{h^*+e}$, and

$$\int_{x=0}^{X} |H^{E^*}(x) - h^*|\,dx = \int_{x=0}^{X} |H(x) - h^*|\,dx - \sum_{\ell=h^*+1}^{h^*+e} P(y_\ell^*) =$$

$$\int_{x=0}^{X} |H(x) - h^*|\,dx - \sum_{\ell=h^*+1}^{h^*+e} \Pi(\ell) = \min_h \left[\int_{x=0}^{X} |H(x) - h|\,dx - \sum_{\ell=h+1}^{h+e} \Pi(\ell) \right] \qquad \square$$



$E^* = \{y_2, y_3\}$ $\qquad$ h^* = optimal h = -1

(same data as in previous figures)

FIGURE 4: An optimal solution to an assignment problem on a circle

3. Computational complexity

We describe the computation of the minimizing h and E in the circular case with $e > 0$. Enough detail is given to demonstrate that the entire computation can be performed in at most $20|S|$ arithmetic steps (additions and comparisons of numbers). One can show similarly that, in the linear case, the entire computation can be carried out in a number of steps proportional to $|S|$.

Define

$$h_{max} = \max_{y \in S} H(y) = \max_{x \in [0,X]} H(x)$$

and

$$h_{min} = \min_{y \in T} H(y) - 1 = \min_{x \in [0,X]} H(x)$$

For $k = h_{min}, h_{min} + 1, \ldots, h_{max}$,

Let

$$S(k) = \{y \mid y \in S \text{ and } H(y) = k\}$$

and $$T(k) = \{y \mid y \in T \text{ and } H(y) = k\} .$$

Define

$$\tilde{S}(k) = \begin{cases} S(k) \cup \{0\} & k \leq 0 \\ S(k) & k > 0 \end{cases}$$

and

$$\tilde{T}(k) = \begin{cases} T(k) \cup \{X\} & k \leq e \\ T(k) & k > e \end{cases} .$$

One can regard $\tilde{S}(k)$ as the set of abscissa values at which the function H reaches the value k from below, and $\tilde{T}(k)$ as the set of abscissa values where it decreases from k (with the convention that $H(0^-) = H(X^+) = -\infty$). Clearly, these crossings from below and above alternate; thus $|\tilde{S}(k)| = |\tilde{T}(k)|$, and the pth smallest element of $\tilde{S}(k)$ is less than the pth smallest element of $\tilde{T}(k)$, but greater than the (p-1)th smallest element of $\tilde{T}(k)$.

For any $x \in \tilde{S}(k)$, let x' denote $\min \{y | y \in \tilde{T}(k) \text{ and } y > x\}$; similarly, for $x \in \tilde{T}(k)$, let $x' = \min \{y | y \in \tilde{S}(k) \text{ and } y > x\}$; also, let $X' = X$, and for any x, write $(x')'$ as x''.

In what follows,

$$B(k) \text{ is the measure of } \{x | H(x) \geq k\} \quad ,$$

$$A(k) \text{ is the measure of } \{x | H(x) = k\} \quad ,$$

$$I(h) = \int_{x=0}^{X} |H(x) - h| dx$$

and

$$\mathrm{Prof}(h) = \sum_{k=h+1}^{h+e} \Pi(k) \quad .$$

The computation is determined by the following formulas:

$$P(X) = 0$$

$$P(y) = 2y' - y - y'' + P(y'') \;, \qquad y \in S$$

$$\Pi(k) = \max_{y \in S(k)} P(y) \;, \qquad k = h_{min}+1, \ldots, h_{max}$$

$$y_k^* = \min \{y | y \in S(k) \text{ and } P(y) = \Pi(k)\}, \quad k = h_{min}+1, \ldots h_{max}$$

$$B(k) = \sum_{y \in \tilde{S}(k)} (y' - y) \;, \qquad k = h_{min}, \ldots, h_{max}$$

$$\begin{cases} A(h_{max}) = B(h_{max}) \\ A(k) = B(k) - B(k+1) \qquad k = h_{min}, h_{min}+1, \ldots, h_{max}-1 \end{cases}$$

$$\begin{cases} I(h_{min}) = \sum_{k=h_{min}+1}^{h_{max}} B(k) \\ I(h) = I(h-1) + X - 2B_h \qquad h = h_{min}+1, h_{min}+2, \ldots, h_{max} \end{cases}$$

$$\text{Prof }(h_{min}) = \sum_{k=h_{min}+1}^{h_{min}+e} \Pi(k)$$

$$\text{Prof }(h+1) = \text{Prof }(h) + \Pi(h+e) - \Pi(h) \qquad h = h_{min}, \ldots, h_{max}-1$$

$$\text{Val }(h) = I(h) - \text{Prof }(h), \qquad h = h_{min}, \ldots, h_{max}.$$

$$V = \min_h \text{Val }(h)$$

$$h^* = \min \{h \mid \text{Val }(h) = V\}$$

$$E^* = \{y^*_{h^*+1}, \ldots, y^*_{h^*+e}\}$$

Note that h^* is especially easy to determine when $e = 0$. In that case, h^* is characterized by

$$B(h^*) \geq \frac{X}{2} \qquad \text{and} \qquad B(h^*+1) < \frac{X}{2} \quad .$$

In other words, h^* is the median value of $H(x)$.

Finally, we can drop the assumption that one desk is available at each source and one desk is required at each destination. If source y_j supplies a_j desks, and destination x_i requires b_i desks, with $\Sigma\, a_j \geq \Sigma\, b_i$, then the entire theory carries through with trivial changes, and the computational work for the circular problem is as follows:

$n^2 + 0(n \log n)$	additions
$\frac{1}{4} n^2 + 0(n \log n)$	comparisons
$3n$	multiplications

where $n = |S \cup T|$.