A UNIFIED FORMULATION FOR THE QUADRATIC INTEGRAL (SUM)

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ABSTRACT

A unified method for the time weighted square integral for both continuous and discrete free linear system is presented. In both cases, the value of the integral is the quadratic form of the initial condition $x_o'P_ox_o$, where P_o is a single matrix which is obtained either by recursive matrix equations or by single matrix transformation. The strange b_i's coefficients that appear in Man's paper [5], have a natural meaning in the present work. Moreover, a simple formula for these coefficients is developed.

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1. INTRODUCTION

In a series of articles [1]-[4], MacFarlane developed the method to evaluate the total square integral of linear continuous free systems. The analysis was done in the time domain for various types of functionals. The value of the integral is given in a quadratic form of the initial condition: $x_o'P_ox_o$, where P_o is the solution of a sequence of Lyapunov's type equations. Later [5], Man evaluated the infinite sum:

$$J_r = \sum_{j=0}^{\infty} k^r x_k' Q_k x_k \text{ , subject to:}$$

$$x_{k+1} = Ax_k$$

His solution is given by $x_o'P_ox_o$, where $P_o = \sum_{j=0}^{r} b_jQ_j$, and Q_j is given recursively by Lyapunov's equations.

However, in his words, "the coefficients b_j must be determined sequentially for r = 1,2... and no general formula is available which enables the systematic determination of these coefficients."

We show in this work, that for both continuous and discrete time systems, P_o is a solution of a sequence of Lyapunov's type equations, such that in the discrete case, the b_j's are hidden in the sequence and are given by a simple equation. The method is based on the general solution of the integral which is a special case of either random process [7] or optimal regulator.

2. CONTINUOUS TIME SYSTEM

(a) The General Solution

The infinite integral of linear continuous autonomous system is given by [6 p.110-111]

$$\begin{cases} \dot{x} = Ax \\ J = \int_o^\infty x'Q(t)x\,dt = x_o'P_ox_o, \text{ where: } P_o \triangleq P(0), \text{ and:} \\ -\dot{P}(t) = P(t)A + A'P(t) + Q(t)\ , \lim\limits_{T\to\infty} P(T) = 0 \end{cases} \tag{1}$$

(b) Matrix Laplace Transform

We are interested in J such that $Q(t) = f(t) \cdot Q$, where Q is a constant matrix and f(t) is the time weight function.

We can write (1) as follows:

$$\dot{p} = MP - q\ f(t) \tag{2}$$

For instance, n = 2:

$p = \text{col. } [p_{11}p_{12}p_{22}]$; $q = \text{col. } [q_{11}q_{12}q_{22}]$

$$M = -\begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & a_{11} + a_{22} & a_{21} \\ 0 & a_{12} & 2a_{22} \end{bmatrix}$$

The solution of (2) is:

$$p(t) = e^{Mt}p(o) - \int_o^t e^{M(t-\tau)}q\ f(\tau)d\tau = e^{Mt}[p(o) - \int_o^t e^{-M\tau}\ q\ f(\tau)d\tau] \tag{3}$$

The boundary condition for finite transfer time T is:

$p(T) = 0$, hence:

$$p(o) = [\int_o^\infty e^{-M\tau}\ f(\tau)d\tau]\ q \triangleq \mathcal{L}_M[f(\tau)] \cdot q \tag{4}$$

This "matrix laplace transform" was obtained earlier [2] from another point of view. As we shall see soon, the extension of (4) to discrete time system is "$\mathcal{Z}$ matrix transform".

(c) Recursive Matrix Formulation

Here we operate directly on (1)

$$\begin{cases} J = \int_o^\infty f(t)x'Qx\,dt = x_o'P_ox_o \ , \text{ where} \\ -\dot{P}(t) = P(t)A + A'P(t) + f(t)\,Q \end{cases} \tag{5}$$

(i) $f(t) = \sum_{i=o}^{r} \alpha_i t^i$, r is a positive integer

Differentiate (5) at $P = P_o$:

$$\begin{cases} P_o^{(i+1)} = -P_o^{(i)}A - A'P_o^{(i)} - Q\alpha_i \cdot i! \ , \ 0 \le i \le r \\ P_o^{(i+1)} = -P_o^{(i)}A - A'P_o^{(i)} \ , \ i > r \end{cases} \tag{6}$$

Equation (6) has the following property:

$$P_o^{(i)} \equiv 0 \ \forall \ i > r, \quad (P_o^{(i)} \triangleq \frac{d^iP}{dt^i} \ \text{ at } \ t = 0) \tag{7}$$

Proof: See Appendix I.

Solving (6) recursively (backward), we obtain P_o.

(ii) $f(t) = t^r$.

Substituting in (6): $\alpha_i = 0, \forall\ i < r, \alpha_r = 1$, one obtains:

$$\begin{cases} P_o^{(i)}A + A'P_o^{(i)} = -P_o^{(i+1)} \ , \ 0 \le i \le r \\ P_o^{(r)}A + A'P_o^{(r)} = -r!Q \end{cases} \tag{8}$$

The last result agrees with [1].

(iii) $f(t) = e^{2\alpha t}$

Using the same transformation as in feedback optimal regulator [6]:

$\tilde{x} \triangleq xe^{\alpha t}$, one obtains (9)

$$J = \int_o^\infty e^{2\alpha t} x'Qx \, dt = \int_o^\infty \tilde{x}' \, Q \, \tilde{x} \, dt, \text{ and:}$$

$$\frac{d}{dt} \tilde{x} = (A + \alpha I)\tilde{x} \text{ (by: (1) and (9)).} \tag{10}$$

Hence:

$$\begin{cases} J = x_o{}'\tilde{P} \, x_o \quad \text{(note: by (9): } \tilde{x}_o = x_o\text{), where:} \\ \tilde{p}(A + \alpha I) + (A' + \alpha I)\tilde{P} = -Q \end{cases} \tag{11}$$

(iv) $f(t) = e^{2\alpha t} \sum_{i=0}^{r} \alpha_i t^i$

Using (9) we obtain (10), and: $\tilde{f}(t) = \sum_{i=0}^{r} \alpha_i t^i$.

Hence: $J = x_o{}'P_o x_o$, where P_o is obtained from (6) replacing A by $(A + \alpha I)$.

3. DISCRETE TIME SYSTEM

(a) The General Solution

The infinite quadratic sum of linear discrete autonomous system is given by [6 p.471-472]:

$$\begin{cases} x_{k+1} = Ax_k \\ J = \sum_{k=o}^{\infty} x_k{}'Q_k x_k = x_o{}'P_o x_o \\ P_k = A'P_{k+1}A + Q_k, \quad \lim_{kT\to\infty} P(kT) = 0 \end{cases} \tag{12}$$

(b) Matrix Z Transform

For $Q_k = f_k \cdot Q$ (Q-constant), we can write (12) as:

$$Kp_{k+1} = p_k - qf_k \tag{13}$$

For instance, n = 2:

$p = \text{col.}[p_{11}p_{12}p_{22}]$; $q = \text{col.}[q_{11}q_{12}q_{22}]$

$$K = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{21} & a_{21}^2 \\ a_{11}a_{12} & a_{11}a_{22}+a_{12}a_{21} & a_{21}a_{22} \\ a_{12}^2 & 2a_{12}a_{22} & a_{22}^2 \end{bmatrix}$$

The solution of (13) is:

$$p_o = [f_o + Kf_1 + \ldots + K^{n-1}f_{n-1}]q + k^n p_n \tag{14}$$

By boundary condition:

p(nT) = 0, hence:

$$p(0) = [\sum_o^\infty K^j f_j]\ q \triangleq Z_{K^{-1}}[f_j]\cdot q \tag{15}$$

(15) represents "Matrix Z Transform." It should be noted that p(0) is a function of K.

<u>Example</u>: $f_k = k^{(r)} \triangleq k(k-1)(k-2)\ldots(k-r+1)$

$$p(0) = r!K^r(I-K)^{-(r+1)}\cdot q \tag{16}$$

(c) Recursive Matrix Formulation

Here we operate directly on (12)

$$\left\{\begin{array}{l} J = \sum_o^\infty f_k x_k{}'Qx_k = x_o{}'P_o x_o\text{, where} \\ \\ P_k = A'P_{k+1}A + f_k Q \end{array}\right. \tag{17}$$

(i) $f_k = \sum_{\ell=o}^{r} \alpha_\ell k^\ell$, r is a positive integer.

Let $E(\cdot)$ be shift operator:

$$E^{i}(P_k) \triangleq P_{k+i} \tag{18}$$

The "derivative" in discrete time for fixed sampling time interval is given by the following "difference":

$$\Delta_k^i \triangleq (E-I)^i P_k \tag{19}$$

Δ_k^i is the i^{th} difference at the k^{th} sampling.

Another way of writing (19) is:

$$\Delta_k^i = \Delta_{k+1}^{i-1} - \Delta_k^{i-1} \tag{20}$$

Operating with (19) on the R.H.S. of P_k in (17), one obtains:

$$\begin{cases} P_o = \Delta_o^o \\ \Delta_o^o = A'(\Delta_o^o + \Delta_o^1)A + Q\alpha_o \\ \Delta_o^1 = A'(\Delta_o^1 + \Delta_o^2)A + Q\sum_{\ell=1}^{r}\alpha_\ell \\ \Delta_o^2 = A'(\Delta_o^2 + \Delta_o^3)A + Q\sum_{\ell=2}^{r}\alpha_\ell(2^\ell-2) \\ \vdots \\ \Delta_o^i = A'(\Delta_o^i + \Delta_o^{i+1})A + Qb_i \\ \vdots \\ \Delta_o^r = A'\Delta_1^r A + Q\, r!\alpha_r \end{cases} \tag{21}$$

In our case, (21) has the following property:

$$\Delta_o^i \equiv 0 \;\forall\; i > r, \text{ and in particular } \Delta_1^r = \Delta_0^r \tag{22}$$

Proof: See Appendix II.

The coefficients b_i's in (21) are given by:

$$b_i \triangleq \sum_{\ell=i}^{r} \alpha_\ell [\sum_{j=0}^{i} (-1)^j (1-j)^\ell \binom{i}{j}], \quad 0 \le i \le r \tag{23}$$

Solving (21) recursively (backward), we obtain P_o.

(ii) $f_k = k^r$

Substituting in (21): $\alpha_i = 0,\ \forall\ i < r,\ \alpha_r = 1$, one obtains:

$$\left\{\begin{array}{l} P_o = \Delta_o^o \\ \Delta_o^o = A'(\Delta_o^o + \Delta_o^1)A + 1\cdot Q \\ \Delta_o^1 = A'(\Delta_o^1 + \Delta_o^2)A + 1\cdot Q \\ \Delta_o^2 = A'(\Delta_o^2 + \Delta_o^3)A + (2^r - 2)Q \\ \vdots \\ \Delta_o^i = A'(\Delta_o^i + \Delta_o^{i+1})A + b_i\cdot Q \\ \vdots \\ \Delta_o^r = A'\Delta_1^r A + r!\ Q \\ \Delta_1^r = \Delta_o^r \end{array}\right. \tag{24}$$

where:

$$b_i \triangleq \sum_{j=o}^{i} (-1)^j (i-j)^r \binom{i}{j} \tag{25}$$

(25) has a very simple pattern - See Appendix III.

(iii) $f_k = k^{(r)} \triangleq k(k-1)(k-2)\ \ldots\ (k-r+1)$

It is easy to verify that $(E - I)^i f_o \equiv 0\ \forall\ \ 0 \le i < r$,

or equivalently: $b_i \equiv 0\ \forall\ 0 \le i < r$, and $b_r = r!$

Hence:

$$\sum_{o}^{\infty} k^{(r)} x'_k Q x_k = r! x_o' P_r x_o, \text{ where: } P_j - A'P_j A = \left\{\begin{array}{ll} Q & , j = 0 \\ A'P_{j-1}A, & j > 0 \end{array}\right. \tag{26}$$

(iv) $f_k = e^{2\alpha k}$

Let $\tilde{x}_k \triangleq x_k e^{\alpha k} \Rightarrow \tilde{x}_{k+1} = (Ae^{\alpha})\tilde{x}_k$ (... by $x_{k+1} = Ax_k$),

Noting that $\tilde{x}_o = x_o$, we obtain:

$$\begin{cases} J = \sum_o^{\infty} e^{2\alpha k} x_k' Q x_k = \sum_o^{\infty} \tilde{x}_k' Q \tilde{x}_k = \tilde{x}_o' \tilde{P}_o \tilde{x}_o = x_o' \tilde{P}_o x_o, \text{ where:} \\ \tilde{P}_o = (Ae^{\alpha})\tilde{P}_o(Ae^{\alpha}) + Q \end{cases} \tag{27}$$

(v) $f_k = e^{2\alpha k} \sum_{\ell=o}^{r} \alpha_\ell k^\ell$

As in the continuous time case, replace A in (21) by (Ae^{α}).

4. REMARKS

1. In the discrete time, K is not necessarily nonsingular, however, for a stable system (I-K) is nonsingular.
2. The advantages and disadvantages of the recursive and transform forms:
   (i) The transform form has answer for all transformable weight functions.
   (ii) We can use Laplace and Z transforms for the transform forms.
   (iii) If the system's dimension is n, M and K's dimension is $m = \frac{1}{2} n(n+1)$
3. The matrix M in (2) is equal to -B, where B is given in [1].
   The general form of the matrix K is given in [8].
4. Equations (4) and (15) show that if discrete time system $x_{k+1} = Ax$ is obtained from continuous time system $\dot{x} = Ax$, then K is related to B by:

   $K = e^{BT}$

   where T is the sampling interval.

5. If we use the simple transformation:

$$\hat{P}_K = P_K - Q_K$$

we obtain equivalent to equations (24), (25):

$$\begin{cases} J_o = x_o'(P_o + Q)x_o \;;\; J_r = x_o'P_ox_o \;,\; r \geq 1, \text{ where:} \\ \Delta_o^i = A'[\Delta_o^i + \Delta_o^{i+1}]A + b_iA'QA \\ b_i \triangleq \sum_{j=o}^{i} (-1)^j(i+1-j)^r\binom{i}{j} \end{cases} \qquad (28)$$

By <u>inspection</u>, the $|b_i|$'s in [5] are identical to those of (28) (in the direction of the sequence), however, no general formula is presented in [5] for $|b_i|$'s

In this paper (25) or (28) are natural as a result of the binomial pattern of the discrete differentiation.

6. It is important to note that alternative form of (15) is:

$$P_o = \sum_{j=o}^{\infty} f_j(A')^jQ(A)^j.$$

7. The difference between MacFarlane's recursive point of view and the one presented here is that the first is integral form, and the second is differential form. Therefore, it is difficult to extend his procedure to the discrete time case (See Appendix IV).

5. CONCLUSIONS

The main contribution of this work is in introducing unified method to evaluate the total square integral (sum) for linear continuous (discrete) free system.

New results are reported for the discrete case, while for the continuous case they are identical to [1], [2].

The coefficients that appear in [5] without general formula, have natural meaning in this work with simple pattern.

6. ACKNOWLEDGMENT

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7. REFERENCES

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APPENDIX

I. Proof of equation (7): By differentiation (2) r+1 times, we obtain:

$$\begin{cases} \dot{p} = Mp - q \sum_{i=0}^{r} \alpha_i t^i \\ \ddot{p} = M\dot{p} - q \sum_{i=1}^{r} i\alpha_i t^{i-1} \\ \vdots \\ p^{(r+1)} = Mp^{(r)} - q\alpha_r r! \\ p^{(r+2)} = Mp^{(r+1)} \\ \vdots \end{cases}$$

and at $t = 0$

$$\begin{cases} \dot{p}_o = Mp_o - q\alpha_o \cdot 0! \quad (0! \triangleq 1) \\ \ddot{p}_o = M\dot{p}_o - q\alpha_1 \cdot 1! = M^2 p_o - Mq\alpha_o - q\alpha_1 \\ \vdots \\ p_o^{(r+1)} = Mp_o^{(r)} - q\alpha_r r! = M^{r+1} p_o - M^r q\alpha_o - M^{r-1} q\alpha_1 - M^{r-2} q\alpha_2 2! - \dots - q\alpha_r r! \\ p_o^{(r+2)} = Mp_o^{(r+1)} \\ \vdots \end{cases}$$

But: $p(o) = [\int_o^{\infty} e^{-M\tau} \sum_{i=0}^{r} \alpha_i \tau^i d\tau]q = (\alpha_r r! M^{-(r+1)} + \alpha_{r-1}(r-1)! M^{-r} + \dots)q$

or: $M^{r+1} p_o = (\alpha_r r! + \alpha_{r-1}(r-1)! M^1 + \dots + \alpha_o M^r)q$ (29)

substituting (29) in $p_o^{(r+1)}$, we obtain $p_o^{(r+1)} \equiv 0$, and so:

$p_o^{(i)} \equiv 0 \quad \forall\, i > r$.

II. Proof of equation (22).

We wish to show that $\Delta_o^{r+1} = 0$.

For this end, we proceed in equation (21) one step further.

Using the fact $(E-I)^{r+1} f_o = 0$, one obtains

$$\Delta_o^{r+1} = A'(\Delta_o^{r+1} + \Delta_o^{r+2})A$$

$$\vdots$$

etc.

The last equation can be written as

$$\Delta_o^{r+1} = A'\Delta_o^{r+2} A + A'^2 \Delta_o^{r+2} A^2 + \ldots + A'^k \Delta_o^{r+2} A^k + \ldots$$

Likewise

$$\Delta_o^{r+2} = A' \Delta_o^{r+3} A + \ldots$$

Or: $\Delta_o^{r+1} = A'^2 \Delta_o^{r+3} A^2 +$ (higher order of A).

Likewise

$$\Delta_o^{r+1} = A'^3 \Delta_o^{r+4} A^3 + \text{(higher order of A)}.$$

$$\vdots$$

$$\Delta_o^{r+1} = \lim_{K\to\infty} (A'^k \Delta_o^{k+1} A^k) + 0(A) = A'^\infty \Delta_o^\infty A^\infty + 0(A) \qquad (30)^*$$

Using (19): $\Delta_o^o = P_o,\ \Delta_1^o = P_1, \ldots \Delta_K^o = P_K \ldots \Delta_\infty^o = P_\infty = 0$

Using (20): $\Delta_\infty^i = \Delta_\infty^{i-1} - \Delta_\infty^{i-1} = 0 \quad \forall\ i$, so that: $\Delta_\infty^\infty = 0$

Once more by (20): $\Delta_1^\infty = \Delta_o^\infty + \Delta_o^\infty = 2\Delta_o^\infty \Rightarrow \Delta_o^\infty = \frac{1}{2}\Delta_1^\infty = \frac{1}{4}\Delta_2^\infty = \lim_{k\to\infty} (\frac{1}{k}\Delta_k^k) = 0,$

---

*$0(A)$ is the rest of the series ("higher" order of A), where each component is simply $A'^\infty \Delta_o^\infty A^\infty$. Thus, if $A'^\infty \Delta_o^\infty A^\infty = 0$, so is Δ_o^{r+1}.

so finally, if the matrix A is stable, equation (30) becomes

$$\Delta_o^{r+1} = 0$$

III. A simple pattern for equation (25).

Step 1: Write "Pascal Triangle" for $(x-y)^n$:

i	
$i = 0$	1
1	1 -1
2	1 -2 1
3	1 -3 3 -1
4	1 -4 6 -4 1
5	$\boxed{1\ -5\ 10\ -10\ 5\ -1}$
⋮	⋮

Step 2: Multiply the i^{th} row, from right to left, term by term by $0^r, 1^r, 2^r$ .... .

Step 3: b_i equal to the sum of these terms.

Example: $i = 5$: Pick the dashed rectangular and form:

$b_5 = 1(5^r) - 5(4^r) + 10(3^r) - 10(2^r) + 5(1^r) - 1(0^r)$.

Note: For the b_i in (28), in step 2, multiply by $1^r, 2^r, 3^r, \ldots$.

IV. The discrete analogy to [1]:

r = 0: Let: $x_k'Qx_k \triangleq (E-I)\{x_k'P_1x_k\} = x_{k+1}'P_1x_{k+1} - x_k'P_1x_k$

By: $x_{k+1} = Ax_k$: $\qquad = x_k'A'P_1Ax_k - x_k'P_1x_k$,

$$\text{or: } Q = A'P_1A - P_1 \text{ , and:} \tag{31}$$

$$\sum_{k=0}^{\infty} x_k'Qx_k = x_\infty'P_1x_\infty - x_o'P_1x_o = - x_o'P_1x_o \tag{32}$$

$$\text{r = 1: Let: } (E-I)\{k(x_k'P_1x_k)\} \triangleq x_k'P_1x_k + x_k'Qx_k + kx_k'Qx_k \tag{33}$$

$$0 = [k(x_k'P_1x_k)]_o^\infty \equiv \sum_{k=0}^{\infty} x'_k(P_1+Q)x_k + \sum_{k=0}^{\infty} kx_k'Qx_k \text{ , or:}$$

$$\sum_{k=0}^{\infty} kx_k'Qx_k = -\sum_{k=0}^{\infty} x_k'(P_1+Q)x_k$$

Let: $x_k'(P_1+Q)x_k \triangleq (E-I)\{x_k'P_2x_k\} =$

$$= x_{k+1}'P_2x_{k+1} - x_k'P_2x_k = x_k'A'P_2Ax_k - x_k'P_2x_k, \text{ or:}$$

$$Q + P_1 = A'P_2A - P_2 \text{ , and:} \tag{34}$$

$$\sum_{k=0}^{\infty} x_k'(Q+P_1)x_k = -x_o'P_2x_o \text{ , and by (33):}$$

$$(k+1)x_{k+1}'P_1x_{k+1} - kx_k'P_1x_k = (k+1)x_k'A'P_1Ax_k - kx_k'P_1x_k = x_k'P_1x_k + x_k'Qx_k + kx_k'Qx_k,$$

hence: $(k+1)A'P_1A - kP_1 = P_1 + (k+1)Q$, or:

$$Q = A'P_1A - P_1 \tag{35}$$

and: $$\sum_{k=0}^{\infty} kx_k'Qx_k = x_o'P_2x_o \tag{36}$$

(Note: We solve (35) and (36) recursively.)

<u>r = 2</u>: Let: $(E-I)\{k^2(x_k'P_1x_k)\} = x_k'P_1x_k + 2kx_k'P_1x_k + x_k'Qx_k + 2kx_k'Qx_k + k^2x_k'Qx_k$

and so on ...