ON THE STABILITY OF THE A-MATRIX
INSIDE THE UNIT CIRCLE

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Summary: In this paper several theorems related to the necessary and sufficient conditions for the characteristic roots (eigenvalues) of a matrix A to lie inside the unit circle are presented. In particular, the following conditions are proved: (i) Linear combinations b_i of the coefficients of λ^i in $(-1)^n\ |A-\lambda I|$ should be positive and (ii) The coefficients of $\mu^i (i=0,1,2,\ldots,m-1)$ in $(-1)^m\ |\hat{K}-\mu I|$, $m = \frac{1}{2}\, n(n-1)$ should be positive, where $\hat{K} \triangleq \hat{A}-I$, and $\hat{A}$ is the bialternate product of A by itself, i.e. $A \cdot A$.

A simple method for generating $A \cdot A$ from the Lyapunov matrix associated with A is indicated.

The formulation of the critical constraints for stability limit in terms of the A matrix and its bialternate product is also discussed.

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Introduction: In a recent paper, Fuller [1] had presented comprehensive methods for testing the stability matrix A for characteristic roots (eigenvalues) in the left half plane. This represents the stability condition for linear continuous systems. Also in this paper, Fuller suggested the need for obtaining such stability criteria for linear discrete systems. Mathematically, this is equivalent for the conditions for the characteristic roots of A to have magnitude less than unity. In a parallel work, Barnett and Storey [2] and Barnett [3] have suggested a solution to the latter problem by applying the bilinear transformation to the A-matrix. This transformation on the A-matrix whose characteristic roots inside the unit circle is presented in terms of $A_c = \frac{A+I}{A-I}$ whose characteristic roots in the left half plane are to be tested. By this bilinear transformation one can apply Fuller's results already established for the continuous case. This bilinear transformation represents a computational burden which unnecessarily complicates the final constraints.

In this paper we avoid this bilinear transformation on the A matrix and obtain the equivalent of Fuller's several theorems directly from the given A matrix. This represents significant computational simplification and also shed some light on certain characteristic matrices associated for characteristic roots inside the unit circle. The number of the final constraints are $\frac{1}{2}$ [n(n+1)]. This number can be reduced to "n" which yields the recently obtained stability criterion [4,5].

Main Results: In the following we present a few theorems which are the discrete analog of Fuller's representation for the continuous case. Some of these theorems are new and thus proofs will be supplied.

Theorem 1: Let,

$$\zeta_n \lambda^n + \zeta_{n-1}\lambda^{n-1} + \ldots + \zeta = 0, \; \zeta_n > 0 \tag{1}$$

have real coefficients. Let,

$$s_m\mu^m + s_{m-1}\mu^{m-1} + \ldots + S_o = 0, \; S_m > 0 \tag{2}$$

have the following $m = \frac{1}{2} n(n-1)$ roots:

$$\mu = \lambda_i\lambda_j-1, \; (i = 2,3,\ldots,n, \; j = 1,2,\ldots i-1) \tag{3}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of eqn. (1).

For the roots of (1) to lie inside the unit circle, it is necessary and sufficient that:

(i) The linear combinations b_i of ζ_i are all positive, where:

$$\text{n-odd,} \quad b_i = \sum_{r=0}^{2s-1} \left[\sum_{i} (-1^{r+i-j+1} \; \zeta_r \binom{r}{j}\binom{2s-1-r}{i-j} \right], n = 2s-1 \tag{4}$$

$$\text{n-even,} \quad b_i = \sum_{r=0}^{2s} \left[\sum_{j} (-1)^{r+1-j} \; \zeta_r \binom{r}{j}\binom{2s-r}{i-j} \right], \; n = 2s \tag{5}$$

In a recent paper [6] a simple combinatorial method for generating the b_i from ζ_i is presented.

(ii) The coefficients $s_0, s_1, \ldots, s_{m-1}$ should all be positive.

Proof: It is wellknown that (i) represents a bilinear transformation on the coefficients of (1) and thus assures that (1) has no real roots outside the unit circle [4]. Condition (ii) covers the complex conjugate roots of (1): It represents the condition that eqn. (2) has all its real roots in the open left half plane [1]. In eqn (3) there are $\frac{1}{2}$ n(n-1) combinations of i and j, and in particular, the combination of each λ_i with its own conjugate exists.

These particular combinations have the interesting property that each of them appears as a real root of eqn. (2).

To show the above, suppose eqn. (1) has r pairs of complex conjugate roots. Let λ_k (k = 1,2,...,r) be one member of each pair and λ_k^* (k = 1,2,...,r) be the conjugate of λ_k. Let,

$$\lambda_k \triangleq \zeta_k e^{j\theta_k} \text{ and } \lambda_k^* \triangleq \zeta_k^* e^{j\theta_k^*}, \; k = 1,2,\ldots,r \tag{6}$$

Using eqn. (6) in eqn. (3) we obtain

$$\bar{\mu} = \zeta_k \zeta_k^* \, e^{j(\theta_k + \theta_k^*)} - 1 \tag{7}$$

Since $\theta_k + \theta_k^* = 2\pi$, $\forall$ k = 1,2,...,r and $\zeta_k = \zeta_k^* \triangleq \zeta$ $\forall$ k = 1,2,...,r, hence

$$\bar{\mu} = \zeta^2 - 1 \tag{8}$$

Since $\bar{\mu}$ are the real roots of eqn. (2), conditions (i) and (ii) are necessary and sufficient that:

$$\bar{\mu} = \zeta^2 - 1 < 0, \text{ or} \tag{9}$$

$$|\zeta| < 1 \tag{10}$$

Since Eqns. (4,5) and (8) include all the roots of (1), i.e. real and complex, hence eqn. (1) has all its roots inside the unit circle.

Remarks:

1. Theorem 1 is the discrete analog of the theorems of Routh [7] proved in his celebrated treatise of June 1877.

2. The stability conditions of Theorem 1 are $\frac{1}{2}n(n+1)$. Hence a certain redundancy appears. This was also conjectured by Fuller for the continuous case. Now if we eliminate these redundancy conditions and obtain only "n" conditions we necessarily obtained the conditions recently derived by Anderson-Jury [8,4]. In a similar fashion one obtains the Liénard-Chipart criterion [9,10] for the continuous case. This can be readily deduced because of eqn. (4) and (5).

3. In later theorems methods for obtaining the polynomial of eqn. (2) will be indicated. In particular this polynomial will be obtained from the given matrix A.

Theorem 2: Let

$$\zeta_n\lambda^n + \zeta_{n-1}\lambda^{n-1} + \ldots + \zeta_0 = 0, \ \zeta_n > 0 \tag{11}$$

be a real polynomial. Also let

$$s_\ell\mu^\ell + s_{\ell-1}\mu^{\ell-1} + \ldots + s_0 = 0, \ \zeta_\ell > 0 \tag{12}$$

have the following $\ell = \frac{1}{2}n(n+1)$ roots:

$$\mu = \lambda_i\lambda_j - 1, \ (i = 1,2,\ldots,n, \ j = 1,2,\ldots,i), \tag{13}$$

where $\lambda_1, \lambda_2, \ldots\lambda_n$ are the roots of eqn. (11).

For the roots of (1) or (11) to lie inside the unit circle, it is necessary and sufficient that the coefficients

$$s_0,\ s_1,\ s_2,\ \ldots\ s_{\ell-1} \text{ be all positive} \qquad (14)$$

Proof: The proof is similar to that of theorem 1 for the complex conjugate roots. The only difference lies in the fact that eqn. (13) includes the values $(\lambda_i\lambda_i-1)$ while eqn. (3) does not. Now if λ is real so is μ, thus the positivity of s_o, s_1, ... $s_{\ell-1}$ is also necessary and sufficient for $\lambda_i^2 -1 < 0 \Rightarrow \lambda_i < 1 \ \forall\ \lambda_i$ real. Combining this with the proof of theorem 1 for complex conjugate roots, theorem 2 follows.

Definition: Let $A'\ P\ A - P = -\ Q$ (15)

be the Lyapunov equation associated with the difference equation

$$x_{k+1} = Ax_k \qquad (16)$$

Let

$$p = \text{col.}[p_{11}, p_{12}, p_{22}, p_{13}, p_{23}, p_{33}, \ldots] \qquad (17)$$

and

$$q = \text{col.}[q_{11}, q_{12}, q_{22}, q_{13}, q_{23}, q_{33}, \ldots] \qquad (18)$$

be vectors framed from P & Q respectively.

The solution of eqn. (15) is given by [11].

$$P = (I-K)^{-1}q \qquad (19)$$

We designate the matrix K as "The Lyapunov Matrix" of dimension $\frac{1}{2}\ n(n+1)$ associated with A. The matrix K is formed from A' as shown below:

For n = 2

$$K = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{21} & a_{21}^2 \\ a_{11}a_{12} & a_{11}a_{22}+a_{12}a_{21} & a_{21}a_{22} \\ a_{12}^2 & 2a_{12}a_{22} & a_{22}^2 \end{bmatrix} \qquad (20)$$

For n = 3

$$K = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{21} & a_{21}^2 & 2a_{11}a_{31} & 2a_{21}a_{31} & a_{31}^2 \\ a_{11}a_{12} & \begin{matrix} a_{11}a_{22}+ \\ a_{12}a_{21} \end{matrix} & \begin{matrix} a_{22} \\ a_{21} \end{matrix} & \begin{matrix} a_{11}a_{32}+ \\ a_{12}a_{31} \end{matrix} & \begin{matrix} a_{21}a_{32}+ \\ a_{22}a_{31} \end{matrix} & a_{31}a_{32} \\ a_{12}^2 & 2a_{12}a_{22} & a_{22}^2 & 2a_{12}a_{32} & 2a_{22}a_{32} & a_{32}^2 \\ a_{11}a_{13} & \begin{matrix} a_{11}a_{23}+ \\ a_{21}a_{13} \end{matrix} & a_{21}a_{23} & \begin{matrix} a_{11}a_{33}+ \\ a_{13}a_{31} \end{matrix} & \begin{matrix} a_{21}a_{33}+ \\ a_{23}a_{31} \end{matrix} & a_{31}a_{33} \\ a_{12}a_{13} & \begin{matrix} a_{12}a_{23}+ \\ a_{22}a_{13} \end{matrix} & a_{22}a_{23} & \begin{matrix} a_{12}a_{33}+ \\ a_{13}a_{32} \end{matrix} & \begin{matrix} a_{22}a_{33}+ \\ a_{23}a_{32} \end{matrix} & a_{32}a_{33} \\ a_{13}^2 & 2a_{13}a_{23} & a_{23}^2 & 2a_{13}a_{33} & 2a_{23}a_{33} & a_{33}^2 \end{bmatrix} \qquad (21)$$

__Remarks__:

(1) The matrix K for any "n" includes along its principle minors arrays all the K's for n = 1,2,... .

(2) Rows (columns) in K numbered 1, 3, 6, 10, ... $\frac{1}{2}n(n+1)$, are formed entirely by row (columns) in A' numbered 1, 2, 3, ... n respectively. Their entiries are obtained by cyclic multiplication of each row (column)

with itself. Entries in the above rows of K that are not in places 1, 3, 6, ... $\frac{1}{2}$ n(n+1) have factor "2".

(3) The rest of the entries of K are formed from A' similar to the bialternate product [1] except that the value of the subdeterminant is in summation form.

Theorem 3:† The characteristic roots of the Lyapunov matrix K associated with the Lyapunov stability equation A'PA-P= -Q, are the $\frac{1}{2}$ n(n+1) values: $\lambda_i\lambda_j$, (i = 1,2,3,...,n; j = 1,2,3,...i) where the λ_i's are the characteristic roots of A.

Proof: We construct a matrix whose characteristic roots are

$$\lambda_i\lambda_j, \ (i = 1,2,\ldots,n, \ j = 1,2,\ldots i) \tag{22}$$

We know the solution of $x_{k+1} = Ax_k$ is

$$x_k = A^k x_0 \tag{23}$$

Hence, any component x_p is a linear combination of λ_i^k (i = 1,2,...,n), and so $x_p x_q$ is a linear combination of $\lambda_i^k\lambda_j^k = (\lambda_i\lambda_j)^k$, (i = 1,2,...n, j = 1,2,...i).

In order to construct the desired matrix we define a new column vector w, such that $w_{pq} = x_p x_q$, (p = 1,...,n, q = 1,...p), and:

$$w = \text{col.}(x_1^2, \ x_2x_1, \ x_2^2, \ x_3x_1, \ldots, \ x_nx_1, \ x_nx_2, \ldots x_n^2) \tag{24}$$

Using eqn. (23), we find:

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†This theorem might have appeared elsewhere, but we were not able to find a reference to it.

$$(w_{pq})_{k+1} = (x_p x_q)_{k+1} = (x_p)_{k+1}(x_q)_{k+1} = \sum_{i=1}^{n} a_{pi}x_i \sum_{i=1}^{n} a_{qi}x_i$$

$$= (a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{p_n}x_n)(a_{q1}x_1 + a_{q2}x_2 + \ldots + a_{qn}x_n) \tag{25}$$

$$= a_{p1}a_{q1}x_1^2 + (a_{p1}a_{q2}+a_{p2}a_{q1})\, x_2x_1 + a_{p2}a_{q2}x_2^2$$
$$+ (a_{p1}a_{q3} + a_{p3}a_{q1})x_3x_1 + (a_{p2}a_{q3} + a_{p3}a_{q2})x_3x_2 + a_{p3}a_{q3}x_3^2 + \ldots$$

or;

$$\begin{bmatrix} \vdots \\ w_{pq} \\ \vdots \end{bmatrix}_{k+1} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & & \\ a_{p1}a_{q1} & (a_{p1}a_{q2} + a_{p2}a_{q1}) & a_{p2}a_{q2} & (a_{p1}a_{q3}+a_{p3}a_{q1}) & (a_{p2}a_{q3}+a_{p3}a_{q2}) & a_{p3}a_{q3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ w_{pq} \\ \vdots \end{bmatrix}_{k} \tag{26}$$

It is noticed from the above that eqns. (26) and (21) for any "n" are identical, except that eqn. (21) is associated with the matrix A, and is formed directly from A'; while eqn. (26) is associated with A' and is formed directly from A. Since the matrix in (26) has the desired roots of eqn. (22), so is the matrix K in eqn. (21).

<u>Corollary 3.1</u>: The matrix (K-I) has the characteristic roots

$$\mu = \lambda_i\lambda_j - 1, \quad (i = 1,2,\ldots n,\ j = 1,2,\ldots 1) \tag{27}$$

<u>Proof</u>: Assume that the similarity transformation of K to the Jordan

Canonical form is given by the matrix T. Then,

$$T^{-1} KT - I = T^{-1}(K-I)T \tag{28}$$

Since the L.H.S of eqn. (28) has eqn. (27) as its characteristic roots, it follows that the R.H.S of eqn. (28) has the same roots.

<u>Theorem 4 (Stephanos [12])</u>: The matrix A·A has the characteristic roots:

$$\lambda_i \lambda_j \quad (i = 2,3,\dots,n, \; j = 1,2,\dots,i-1) \tag{29}$$

where the (·) denotes the bialternate product.

<u>Corollary 4.1</u>: The matrix (A·A-I) has the characteristic roots:

$$\mu = \lambda_i \lambda_j -1, \quad (i = 2,3,\dots,n; \; j = 1,2,\dots,i-1) \tag{30}$$

The proof is similar to corollary 3.1.

<u>Remarks</u>: The matrix A·A is formed from K'by the following process:

(1) Delete rows and columns of K'numbered: $1,3,6,10,\dots, \frac{1}{2} n(n+1)$.

(2) Each entry of the remaining matrix which is of dimension $\frac{1}{2} n(n-1)$ must be written as subtraction of the two terms rather than summation.

The above procedure is similar to that of Barnett and Storey [2] for the continuous case.

After presenting the above four theorems we can present the main results of the paper in the following two theorems.

Theorem 5: Let A be a real square matrix of dimension $n \times n$. Let $\hat{A}$ be the bialternate product of A by itself, which can be generated directly from the Lyapunov matrix K as remarked earlier. Then, for the characteristic roots (eigenvalues) of A to lie inside the unit circle, it is necessary and sufficient that in

$$\text{(i)} \quad (-1)^n |A-\lambda I| \tag{31}$$

$$\text{(ii)} \quad (-1)^m |\hat{K}-\mu I|, \; m = \frac{1}{2} n(n-1), \; \hat{K} = \hat{A}-I, \tag{32}$$

the coefficients of b_i, $i = 0, 1, \ldots, n$, and the coefficients of μ^i ($i = 0, 1, \ldots, m-1$) should all be positive. Note that the b_i's are given in eqns. (4) and (5).

Proof: Direct proof of theorem 1 and corollary 4.1. Note that eqns. (1) and (2) are the characteristic polynomials given in eqns. (31) and (32).

Theorem 6: Let A be a real square matrix of dimension $n \times n$. Let K be the lyapunov matrix of dimension $\ell = \frac{1}{2} n(n+1)$ associated with A. Then, for the characteristic roots of A to lie inside the unit circle, it is necessary and sufficient that in

$$(-1)^{\ell} |\tilde{K}- \mu I|, \text{ where } \tilde{K} \triangleq K-I \tag{33}$$

The coefficients of μ^i ($i = 0,1,\ldots,\ell-1$) should all be positive.

Proof: Direct result of theorem 2 and corollary 3.1.

Critical constraints [13]: In many application of discrete system design it is known that initially for a certain parameter the system is stable. Hence, one would find the maximum value of this parameter for which the system becomes unstable. The constraints for such a situation are much simplified and are given by the following critical values. If we denote the real polynomial in eqn. (1) as $F(\lambda)$, then the constraints are [13].

$$F(1) \geq 0, \ (-1)^n F(-1) \geq 0 \tag{34}$$

$$\Delta^{-}_{n-1} \geq 0 \tag{35}$$

where Δ^{-}_{n-1} is the determinant of an innerwise matrix of dimension $(n-1) \times (n-1)$ obtained from the coefficients of eqn. (1). The equivalent of the critical constraints of eqns. (34) and (35) in terms of theorem 5 are as follows:

$$|I-A| \geq 0, \quad |I+A| \geq 0 \tag{36}$$

and

$$(-1)^m|\hat{K}| \triangleq (-1)^m|A \cdot A-I| \geq 0, \quad m = \frac{1}{2}\, n(n-1) \tag{37}$$

The proof of eqn. (37) can be readily ascertained from the following identity.

$$(-1)^m|\hat{K}| = \Delta^{-}_{n-1} \tag{38}$$

where

$$\Delta^{-}_{n-1} = \prod_{j < i}^{1\ldots n} (1-z_i z_j) = (-1)^m \prod_{j < 1}^{1\ldots n} (z_i z_j - 1) = (-1)^m|\hat{K}| \tag{39}$$

The numbers z_i and z_j are the roots of eqn. (1) or the characteristic roots of the A matrix. Equation (38) can be readily verified.

Remark: From theorem 6, the critical constrains are given in one equation as follows:

$$(-1)^{\ell}|\tilde{K}| = (-1)^{\ell}|K-I| \geq 0, \quad \ell = \frac{1}{2}n(n+1) \tag{40}$$

also, we can readily verify that

$$(-1)^{\ell}|\tilde{K}| = |I+A|\cdot|I-A|(-1)^{m}|\hat{K}| \tag{41}$$

From eqns. (40) and (41) we can obtain the critical constraints as presented in eqns. (36) and (37).

Conclusion: In this paper several theorems related to stability of the A matrix as well as to its characteristic eqn. are given. Of importance are theorems 5 and 6. In theorem (5) the dimension of the highest matrix $\hat{K}$ is $\frac{1}{2}n(n-1)$ while that of theorem (6) is $\frac{1}{2}n(n+1)$. The number of constraints for stability in both cases is $\frac{1}{2}n(n+1)$. This number can be reduced to "n" at the expense of more complicated relationships involving the elements of A. This reduction identically yield the constraint obtained by Anderson-Jury [4] from the characteristic polynomial in equation (1).

In many practical design problems the critical constraints are of importance. These constraints are presented in this paper in terms of the element of the matrix A. They are identical to the critical constraints [13] obtained from the polynomial in equation (1). by obtaining the stability within the unit circle directly simplifies considerably the use of the bilinear transformation on the A-matrix.

Finally, it may be mentioned that theorem 1 represents an alternate form of the stability criteria within the unit circle than those earlier obtained. It represents the discrete version of the classical results of Routh obtained for the continuous case.

The computational aspects of the various theorems presented will be an interesting topic for future research.

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