FLEMING'S RANDOMIZED GAME AND HIS PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT

Proceeding from first principles and making no use of existing PDE theory, this paper gives a direct and elementary proof that the value of Wendell Fleming's 1964 randomized mixed-strategy game exists and satisfies his parabolic PDE. Fleming required the terminal function to have Lipschitzian first and second partial derivatives; this paper requires only that the terminal function and its gradient be Lipschitzian. The solution is given a new representation, which allows precise Lipschitz and Hölder estimates to be made. A trick of Fleming allows the deduction of a uniqueness and existence theorem for a class of parabolic equations with Laplacian operator under the lightened terminal conditions.

CONTENTS

CONTENTS (CONTINUED)

1. Introduction

In 1964 Wendell Fleming, in his paper [5] which we shall call Convergence II, made a quite remarkable contribution to the theory of differential games. He showed that if the game were "randomized" by the superposition of a Brownian motion onto the deliberate moves of the players and a certain notion of value adopted, that value would exist and satisfy a certain partial differential equation of parabolic type.

Fleming was considering N-stage games of prescribed duration, with mixed strategies at each stage. For his randomization he supposed that the position vector was displaced at each stage, just before the choices of the players, by a random variable drawn from a certain probability distribution with expectation at the origin and of standard deviation $\theta(1-t)^{\frac{1}{2}}/N^{\frac{1}{2}}$, θ being a fixed positive number and (1-t) being the duration of the game.

Fleming knew, from results of Friedman and his student Kaplan [9], and of Oleĭnik and Kružkov [10], that the parabolic PDE in question had a sufficiently smooth solution provided that the data, which in differential games mean the terminal function, had Lipschitzian first and second partial spatial derivatives. By "sufficiently smooth" here we mean that there exists a $\beta \leq 1$ such that the first and second spatial derivatives, and the first time derivative, of the solution satisfy uniform Hölder conditions wiun exponent β relative to the space variable x and exponent $\beta/2$ relative to the time t. Using these facts, he proved (Lemma 1 of <u>Convergence II</u>) that the values of the N-stage randomized games converge uniformly to the solution of the PDE.

Fleming's objective in <u>Convergence II</u> was to prove the convergence of the value of his unrandomized mixed-strategy game. He accomplished this by proving, in a special case whichwas adequate to his needs, that the values of the N-stage unrandomized and randomized games differ by less than $C\theta$, where C is an absolute constant. This was his Lemma 2 in <u>Convergence II</u>. It follows from this and his Lemma 1 that there is an $N(\theta)$ such that if $N, N' \geq N(\theta)$ the values of the N-stage and N'-stage unrandomized games differ by less than $(2C+1)\theta$. Since this is true for any positive θ, the values of the N-stage unrandomized games form a uniform Cauchy sequence and so converge uniformly to a limit, the Fleming mixed-strategy value of the unrandomized game.

The parabolic equation when applied to other values, such as Fleming's upper perfect-information value of [4] or Friedman's upper perfect-information value of [8], or to their own values defined in terms of "relaxed controls", has been a powerful tool in the hands of Robert Elliott and Nigel Kalton, who in a series of papers, particularly [2] and [3], have used it to prove a number of existence and equality results. The Fleming randomization, and the parabolic PDE associated with it, appear therefore to be important objects in themselves.

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Our objective in the present paper is to turn Fleming's approach, as it applies to his randomized game, around. First we prove (Theorem I) in §§4,5, without mentioning any PDE, that the value of Fleming's N-stage randomized mixed-strategy game converges uniformly to a limit, the Fleming mixed-strategy value of the randomized game. The techniques here are the Kolmogorov inequality for partial sums of independently distributed random variables, the ϵ-construction from Chapter IV of the author's book [1], and a simple observation (§3) on the effect of a Gaussian smoothing of a Lipschitzian function, used earlier in Chapter V of [1] in proving the existence of the Ω-value. Although it must be admitted that the ϵ-construction is complicated, all three of these tools are directly accessible without special theory and may be accounted elementary.

In §6 there is a diversion; for completeness we give a now very easy proof of Fleming's theorem from Convergence II of the uniform convergence of the value of his N-stage unrandomized mixed-strategy game to a limit. This is our Theorem II. We gave a different elementary proof in [1], Chapter V, §10.

Next we consider a particular decomposition, (2.11), of the N-stage randomized value function into the sum of a smoothing of the terminal function and components $\psi_n(t,x)$ which are smoothings of the increments $\iota_n(t,x)$, given by (2.9), resulting from the play. By a rather lengthy calculation, taking up §§7-15, we estimate Lipschitz and Hölder coefficients for the first and second spatial derivatives of the $\psi_n(t,x)$, and thus obtain Lipschitz and Hölder estimates for the first and second spatial derivatives of $\sum \psi_n(t,x)$ and thus of $\Phi\lambda_N^\theta(t,x)$. The calculation in question mostly amounts to estimating a determinant, given by (9.3), and is entirely elementary. These results are summarized in Theorems III and V.

The Lipschitz and Hölder estimates on $\Phi\lambda_N^\theta(t,x)$ and its first and second spatial derivatives persist in the limit; this is Theorem VI in §16. In §17 we directly calculate the time derivative, and thus show that the limiting value function $\Phi\lambda^\theta(t,x)$ satisfies Fleming's parabolic PDE. That is the main result of this paper, Theorem VIII.

In §19 we present an integral equation which is a generalization of the parabolic equation: any uniformly Lipschitzian solution of the latter which has bounded generalized second spatial partial derivatives satisfies the former. We then show in §20 that $\Phi\lambda^{\theta}(t,x)$ is the only uniformly Lipschitzian solution of the integral equation (Theorem XI) .

Our methods, combined with a trick of Fleming which exhibits certain parabolic equations as the Fleming equation of a randomized game, allow us in §22 to prove an existence and uniqueness theorem for a special class of parabolic equations with Laplacian operators, under light conditions on the terminal data. This is Theorem XIII.

Our randomization is different from Fleming's; we show that this makes no difference in the limit in §18. This is Theorem IX.

If the terminal data are of Hölder class $2+\alpha$, i.e if in addition to the hypothesis of §2 that the terminal function φ and its gradient $\nabla\varphi$ are uniformly Lipschitzian we assume that the second partial derivatives $\partial^2\varphi/\partial x_i \partial x_j$ exist everywhere and satisfy a Hölder condition with exponent $\alpha \in (0,1]$, we retrieve *) Hölder estimates on $\Phi\lambda^\theta(t,x)$ which

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*) Interim draft note: Approximately; see the note on page 139.

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were already known for a much more general parabolic equation; see Theorem XIV in Oleĭnik-Kružkov [10]. Our versions of these facts appear in Theorems IV, V, VII, VIII, and XIII.

Other values are discussed in §21.

Except where explicitly stated, we stick to one set of hypotheses, stated in the second paragraph of the following §2.

2. Assumptions. Definition of Fleming's randomized value function $\Phi\lambda_N^\theta(t,x)$

We shall have to refer frequently to the book [1]. References to it will be given in the form: abbreviated title, chapter number, section number. E. g., secion 2 of Chapter I becomes Value , I, §2. Section references to the present paper will be given simply by a § followed by the section number; this section is §2.

The <u>elements of the randomized differential game</u> are x_t, U, V, $\mathfrak{f}$, φ, θ. The game is of prescribed duration, played on the interval $[t,1)$, starting at the <u>starting point</u> $x = x_t \in R^p$. U and V are arbitrary compact topological spaces, called the <u>control spaces</u> for the maximizer and minimizer respectively. The <u>control function</u> $\mathfrak{f}(x,t,u,v)$ is continuous and bounded by μ on $R^p \times [0,1] \times U \times V$, and Lipschitzian in x and t with constants A and a respectively. The <u>terminal function</u> $\varphi(x)$ is uniformly Lipschitzian throughout R^p, with constant L. It has a gradient $\nabla\varphi(x)$ everywhere, which is uniformly Lipschitzian in R^p with constant λ. θ, the <u>standard deviation of the randomization</u>, is a fixed positive number. There is no integrand function *) **) .

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*) For the classical elimination of the integrand function in the deterministic problem by passing to p+1 dimensions, see e.g. <u>Value</u>, I, §12. The reader will easily see how to eliminate the integrand in the present randomized problem by passing to p+2 dimensions.

**) In <u>Value</u>, I, §2, we assumed that $\mathfrak{f} \equiv 0$ for x outside some compact box $\hat{B}$; this assumption is not needed here. Also, we assumed there only that $\varphi(x)$ was uniformly Lipschitzian in x with constant L; here we add the assumption that $\nabla\varphi(x)$ is uniformly Lipschitzian with constant λ, an assumption which we do not need before §7. Otherwise the hypotheses concerning U, V, $\mathfrak{f}$, and φ are the same as those stated in <u>Value</u>, I, §2, and adhered to in the first five chapters there.

Except where explicitly noted, we do not make any other hypotheses concerning U, V, $\mathfrak{f}$, or φ. When we use the expression "Under the hypotheses of the second paragraph of §2" in the statement of a theorem, this is what we mean.

In the following definition we use a Gaussian distribution for the shocks, while Fleming in Convergence II used a particular discrete distribution. As we show (Theorem IX) in §18, this makes no difference in the limit.

We denote by $\mathcal{g}^{\varkappa}$ the spherically symmetric Gaussian distribution in R^p given by the density

$$(\varkappa/2\pi)^{p/2}\, e^{-\varkappa(z_1^2 + \dots + z_p^2)/2} \,. \qquad (2.1)$$

The expectation of this distribution is at the origin, and its standard deviation is $\sqrt{p/\varkappa}$.

We put

$$\varkappa_t^{(n)} = pN/n\theta^2(1-t) \,, \qquad (2.2)$$

$n=1,\dots,N$. Until we reach formula (2.10) we are only concerned with $\varkappa_t^{(1)} = pN/\theta^2(1-t)$; the standard deviation of the distribution $\mathcal{g}^{\varkappa_t^{(1)}}$ is then $\theta(1-t)^{\frac{1}{2}}/N^{\frac{1}{2}}$ *).

*) The reason for the factor 1-t in (2.2) is that we wish the total variance of the shocks in a game of duration 1-t to be proportional to 1-t. Because Fleming in <u>Convergence II</u> referred his time-subdivision to a fixed time interval $[0,T_0]$, this factor did not enter explicitly there.

In the present formulation, the shocks occur at the times $\tau_n = t + (1-t)n/N$, $n=0,\dots,N-1$. At time τ_n, the first thing that occurs is a shock, i.e. the position vector $\bar{x}_n$ is displaced to a new point $x_n = \bar{x}_n + z_n$, the shock z_n being drawn from $\mathcal{G}^{\varkappa}{}_t^{(1)}$. Then the players choose controls $u_n \in U$ and $v_n \in V$ respectively. The position vector $\mathfrak{r}(\tau)$, starting at time τ_n at x_n, now follows the differential equation

$$\dot{\mathfrak{r}}(\tau) = \mathfrak{f}(\mathfrak{r}(\tau), \tau, u_n, v_n) \qquad (2.3)$$

across the interval $[\tau_n, \tau_{n+1})$, reaching a point

$x_{n+1}^{-} = \xi(\tau_{n+1}^{-})$ at time τ_{n+1}. We shall denote this last point by $X_{n+1}(x_n, u_n, v_n)$. If $n+1 < N$, there is now a shock, and the game proceeds. When the point $x_N^{-} = X_N(x_{N-1}, u_{N-1}, v_{N-1})$ is reached at time $\tau_N = 1$, the game terminates without a shock, and the payoff is $\varphi(x_N^{-})$.

We define the value of this randomized game in terms of before and after position functions, as follows. We write

$$\varphi\lambda_N^{\theta}(t, x_N, \tau_N) = \varphi(x_N) \ , \qquad (2.4)$$

and call $\varphi\lambda_N^{\theta}(t, x_N, \tau_N)$ the <u>before position function at time</u> $\tau_N = 1$. There is no after position function at this time, there being no shock. Now suppose that $0 \leq n \leq N-1$ and that the before position function $\varphi\lambda_N^{\theta}(t, x_{n+1}, \tau_{n+1})$ has been defined and is continuous in x_{n+1}. Let $x_n \in R^p$. Put

$$f_n(x_n, u_n, v_n) = \varphi\lambda_N^{\theta}(t, X_{n+1}(x_n, u_n, v_n), \tau_{n+1}) \ . \qquad (2.5)$$

The function f_n defines a continuous game over $U \times V$ at x_n. Put

$$\Phi\lambda_N^\theta(t,x_n,\tau_n) = \operatorname*{Value}_{U\times V} f_n(x_n,u_n,v_n) , \qquad (2.6)$$

the notation $\operatorname*{Value}_{U\times V}$ denoting the operation of taking the ordinary mixed-strategy value of the continuous game at x_n. We call $\Phi\lambda_N^\theta(t,x_n,\tau_n)$ the <u>after position function at time</u> τ_n. By the Principle of the Transmission of Continuity (<u>Value</u>, II, §3), $\Phi\lambda_N^\theta(t,x_n,\tau_n)$ is continuous in x_n. We then put

$$\varphi\lambda_N^\theta(t,x_n,\tau_n) = \int \Phi\lambda_N^\theta(t,x_n+z_n,\tau_n)\,dg^{\varkappa}{}_t^{(1)}(z_n) . \qquad (2.7)$$

This is the <u>before position function at time</u> τ_n.

We carry this process down to time $\tau_0=0$. We then put

$$\Phi\lambda_N^\theta(t,x) = \varphi\lambda_N^\theta(t,x,\tau_0) . \qquad (2.8)$$

This is the <u>value of the</u> N-<u>stage Fleming randomized</u>

game, *starting at time* t *at the point* x. Evidently $\Phi\lambda_N^\theta(t,x)$ is a C^∞ function of x. Also, by Lemma A in *Value*, II, §4, as it applies to randomized games, $\Phi\lambda_N^\theta(t,x)$ and all the before and after position functions defined above are Lipschitzian in the position variables, with constant Le^A.

In what follows we shall need another representation for $\Phi\lambda_N^\theta(t,x)$. For $n=1,\ldots,N$ put

$$\iota_n(t,x) = \Phi\lambda_N^\theta(t,x,\tau_{n-1}) - \varphi\lambda_N^\theta(t,x,\tau_n) \quad . \tag{2.9}$$

Then put

$$\psi_n(t,x) = \int \iota_n(t,x+z)\,dg^{\varkappa}t^{(n)}(z) \quad , \tag{2.10}$$

$n=1,\ldots,N$. The reader will then easily verify that

$$\Phi\lambda_N^\theta(t,x) = \int \varphi(x+z)\,dg^{\varkappa}t^{(N)}(z) + \sum_{n=1}^{N} \psi_n(t,x) \quad . \tag{2.11}$$

This decomposition, and its generalization at (8.2), constitute the key to our method.

3. Notes on the normal distribution

Let $\chi(x)$ be a function in R^p with Lipschitz constant ι, and put

$$\psi(x) = \int \chi(x+z)\, d\mathcal{G}^{\kappa}(z) \quad , \tag{3.1}$$

the distribution $\mathcal{G}^{\kappa}$ being the one defined at (2.1). Evidently ψ is C^{∞}. We are interested in the Lipschitz constants of its first and second partial derivatives. We did this easy calculation for the first derivative in Value, V, §2; the Lipschitz constant of $\nabla\psi$ turned out to be $\sqrt{2\kappa p/\pi}\ \iota$.

If $i \neq j$ we calculate

$$\frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} = \kappa^2 (\kappa/2\pi)^{p/2} \int \chi(x+z) z_i z_j \; e^{-\kappa(z_1^2+\dots+z_p^2)/2} \, dz_1 \cdots dz_p \; . \tag{3.2}$$

The Lipschitz constant of $\dfrac{\partial^2 \psi(x)}{\partial x_i \partial x_j}$ is therefore not more than

$$\iota \cdot \varkappa^2 (\varkappa/2\pi)^{p/2} \int |z_i||z_j| \; e^{-\varkappa(z_1^2+\ldots+z_p^2)/2}$$

$$dz_1 \cdots dz_p \;=\; 2\varkappa\iota/\pi \;. \qquad (3.2)$$

For the case i=j the constant is somewhat different; we find that $\partial^2\psi(x)/\partial x_i{}^2$ has the Lipschitz constant $2\varkappa\iota$. Thus in any case $\frac{\partial^2\psi(x)}{\partial x_i \partial x_j}$ has the Lipschitz constant $2\varkappa\iota$.

Next suppose that $\tilde{\chi}(x,t)$ is measurable in x and Lipschitzian in t with constant $\tilde{\iota}$, and put

$$\tilde{\psi}(x,t) = \int \tilde{\chi}(x+z\,,\,t)\,d\mathfrak{g}^{\varkappa}(z) \;. \qquad (3.4)$$

Then in just the same way as above we see that the Lipschitz constant of $\nabla\hat{\psi}(x,t)$ in t does not exceed $\sqrt{2\varkappa p/\pi}\;\tilde{\iota}$, and that the Lipschitz constant of $\frac{\partial^2\tilde{\psi}(x,t)}{\partial x_i \partial x_j}$ in t does not exceed $2\varkappa\tilde{\iota}$.

These trivial principles extend to Hölder coefficients as well, and they are applied extensively in that form in §15.

4. The special MP-stage game

We must assume at this point that the reader is thoroughly familiar with Chapter IV of Value, particularly §§12,13.

We index the stages of the MP-stage game by double subscripts (m,r), $m=0,\ldots,M-1$, $r=0,\ldots,P-1$.

The special MP-stage game differs from the Fleming MP-stage game of §2 in that there are only M shocks, of standard deviation $\theta/M^{\frac{1}{2}}$ each, at the times $\tau_{m,0} = t + (m/M)(1-t)$, $m=0,\ldots,M-1$. The shocks are Gaussian and spherically symmetric as before. The reader will readily see how to define the value $\Phi\lambda^*_{MP}(t,x)$ of the special game in analogy with §2; or see Value, II, §5, where we did it for general shocks (though there we had no shock at the outset). We shall denote the various position functions for this game by using a star where with Fleming's game we had a θ. We wish to compare $\Phi\lambda^*_{MP}(t,x)$ with $\Phi\lambda^{\theta}_{MP}(t,x)$; this is accomplished at (4.29).

The shocks in the Fleming MP-stage game will be denoted by $z_{m,r}$, $m=0,\ldots,M-1$, $r=0,\ldots,P-1$. We will denote a succession of these shocks by $\zeta=(z_{0,0},\ldots,z_{M-1,P-1})$; this is a <u>sample point for the</u> MP-<u>stage Fleming game</u>. For each m we define the partial sums

$$Z_{m,r} = z_{m,0} + \cdots + z_{m,r} , \tag{4.1}$$

$r=0,\ldots,P-1$. Each sample point ζ in this way uniquely determines a sample point $Z(\zeta) = (Z_{0,P-1},\ldots,Z_{M-1,P-1})$ for the special game. The shocks $Z_{m-1,P-1}$, $m=0,\ldots,M-1$, applied in the special game at the times $\tau_{m,0}$, are evidently spherically symmetric Gaussian and of standard deviation $\theta/M^{\frac{1}{2}}$, as required for the special game. With this understanding, we may take the underlying sample space for the two games to be the same, the space of the ζ.

We wish to say what we mean by a "good" sample point ζ. First we consider a group $(z_{m,0},\ldots,z_{m,P-1})$ of shocks in ζ. This group is said to be <u>good</u> if the partial sums $Z_{m,r}$ given by (4.1) all satisfy the inequality

$$|z_{m,r}| \leq \theta M^{-1/3} . \tag{4.2}$$

Otherwise the group is said to be <u>bad</u>. Since the standard deviation of $z_{m,P-1}$ is $\theta M^{-\frac{1}{2}}$, it follows from the Kolmogorov inequality (see e.g. <u>Value</u>, IV, §3) that the probability, calculated over the space of the P-tuples $(z_{m,0},\ldots,z_{m,P-1})$, that the group is bad, does not exceed $1/(M^{1/6})^2 = M^{-1/3}$.

The sample point ζ is now said to be <u>good</u> if the number of groups $(z_{m,0},\ldots,z_{m,P-1})$ in it which are bad does not exceed $M^{5/6}$. Otherwise ζ is said to be <u>bad</u>. Since the expected number of bad groups, calculated over the sample space of all the ζ, does not exceed $M \cdot M^{-1/3} = M^{2/3}$, the probability that ζ is bad does not exceed $M^{2/3}/M^{5/6} = M^{-1/6}$.

We first suppose that ζ is good, and that the controls chosen by the players are $u_{0,0},\ldots,u_{M-1,P-1}$ and $v_{0,0},\ldots,v_{M-1,P-1}$ respectively. We wish in this case to calculate the deviation of the terminal point of the trajectory of the special game from that of the Fleming game.

We now carry out a calculation somewhat analogous to that in Value, IV, §6. This time $\mathfrak{r}$ will denote the trajectory for the special game, $\mathfrak{r}'$ the trajectory for the Fleming game, and $\mathfrak{r}''$ a trajectory running parallel to $\mathfrak{r}$, but taking the shocks of the Fleming game. More precisely:

(i) the trajectory $\mathfrak{r}$, starting at $\mathfrak{r}(\tau_{0,0}^-) = x_t$ at time $\tau_{0,0} = t$, is displaced at each time $\tau_{m,0}$, $m=0,\ldots,M-1$, by the shock $Z_{m,P-1}$ given by (4.1) with $r=P-1$. Otherwise it follows on the stage (m,r), $m=0,\ldots,M-1$, $r=0,\ldots,P-1$, the differential equation

$$\dot{\mathfrak{r}}(\tau) = \mathfrak{f}(\mathfrak{r}(\tau), \tau, u_{m,r}, v_{m,r}) . \qquad (4.3)$$

(ii) the trajectory $\mathfrak{r}'$, starting at $\mathfrak{r}'(\tau_{0,0}^-) = x_t$ at time $\tau_{0,0} = t$, is displaced at each time $\tau_{m,r}$, $m=0,\ldots,M-1$, $r=0,\ldots,P-1$, by the component $z_{m,r}$ of ζ. Otherwise it follows on the stage (m,r), $m=0,\ldots,M-1$, $r=0,\ldots,P-1$, the differential equation

$$\dot{\mathfrak{r}}'(\tau) = \mathfrak{f}(\mathfrak{r}'(\tau), \tau, u_{m,r}, v_{m,r}) . \qquad (4.4)$$

(iii) <u>the trajectory</u> ξ'', starting at $\xi''(\tau_{0,0}) = x_t$ at time $\tau_{0,0} = t$, is displaced at each time $\tau_{m,r}$, $m=0,\ldots,M-1$, $r=0,\ldots,P-1$, by the component $z_{m,r}$ of ζ. Otherwise it follows on the stage (m,r), $m=0,\ldots,M-1$, $r=0,\ldots,P-1$. the differential equation

$$\dot{\xi}''(\tau) = \mathfrak{f}(\xi(\tau), \tau, u_{m,r}, v_{m,r}). \tag{4.5}$$

Put

$$\delta(\tau) = |\xi'(\tau) - \xi''(\tau)| . \tag{4.6}$$

Clearly $\delta(\tau)$ is absolutely continuous on $[t,1]$, and $\delta(\tau_{0,0}) = 0$. Our task is to estimate $\delta(1)$.

Suppose that $(z_{m,0},\ldots,z_{m,P-1})$ is a good group in ζ, so that all the partial sums $Z_{m,r}$ given by (4.1) satisfy (4.2). Now the before-shock positions of ξ'' and ξ at time $\tau_{m,0}$ coincide: $\xi''(\tau^-_{m,0}) = \xi(\tau^-_{m,0})$, $m=0,\ldots,M-1$. Hence for all $\tau\in(\tau_{m,0}, \tau_{m+1,0})$ we have

$$|\xi''(\tau) - \xi(\tau)| \le \theta M^{-1/3} . \tag{4.7}$$

It follows that

$$|r'(\tau) - r(\tau)| \leq \delta(\tau) + \theta M^{-1/3} \tag{4.8}$$

there. Hence from (4.4) and (4.5) we have

$$\dot{\delta}(\tau) \leq |\dot{r}'(\tau) - \dot{r}''(\tau)| \leq A[\delta(\tau) + \theta M^{-1/3}] . \tag{4.9}$$

By making the substitution $\eta(\tau) = \delta(\tau) e^{-A(\tau - \tau_{m,0})}$, we deduce immediately from (4.9) that

$$\delta(\tau_{m+1,0}) \leq [\delta(\tau_{m,0}) + A\theta M^{-4/3}] \, e^{A/M}. \tag{4.10}$$

On the other hand, if $(z_{m,0},\ldots,z_{m,P-1})$ is a bad group we have the crude estimate

$$\delta(\tau_{m+1,0}) \leq \delta(\tau_{m,0}) + 2\mu/M . \tag{4.11}$$

It follows that for any m=0,...,M-1

$$\delta(\tau_{m+1,0}) \leq \delta(\tau_{m,0}) \, e^{A/M} + \omega_m , \tag{4.12}$$

where $\omega_m = A\theta M^{-4/3} e^{A/M}$ if the group is good, and

$\omega_m = 2\mu/M$ if the group is bad. By induction we now deduce from (4.12) that

$$\delta(\tau_{m,0}) \leq \omega_0 e^{-(m-1)A/M} + \omega_1 e^{-(m-2)A/M} + \ldots + \omega_{m-1} , \quad (4.13)$$

m=1,...,M. Hence in particular

$$\delta(1) = \delta(\tau_{M,0}) \leq (\omega_0 + \ldots + \omega_{M-1}) e^{A} . \quad (4.14)$$

Now since ζ is good, at most $M^{5/6}$ groups are bad. Hence it follows from the definition of the ω_m that

$$\delta(1) \leq (A\theta M^{-1/3} e^{A/M} + 2\mu M^{-1/6}) e^{A}. \quad (4.16)$$

For typographical convenience we replace this by the cruder estimate

$$\delta(1) < (A\theta M^{-1/3} + 2\mu M^{-1/6}) e^{2A} . \quad (4.16)$$

Now observing that $\mathfrak{r}''(1) = \mathfrak{r}(1)$, we get

$$|\mathfrak{r}(1) - \mathfrak{r}'(1)| < (A\theta M^{-1/3} + 2\mu M^{-1/6}) e^{2A} . \quad (4.17)$$

For a good ζ and the successions $u_{0,0},\ldots,u_{M-1,P-1}$ and $v_{0,0},\ldots,v_{M-1,P-1}$, this is the desired estimate of the deviation between the terminal points of the special and Fleming games.

If ζ is bad, (4.17) is replaced by the crude estimate

$$|\mathfrak{x}(1) - \mathfrak{x}'(1)| \leq 2u \, , \qquad (4.18)$$

holding for any pair of successions $u_{0,0},\ldots,u_{M-1,P-1}$ and $v_{0,0},\ldots,v_{M-1,P-1}$.

In order to assemble the above estimates we now make use of an ε-construction similar to the one we used in <u>Value</u>, IV, §§12,13 in comparing the value of the unrandomized Fleming mixed-strategy game with that of a general randomized Fleming mixed-strategy game. Here we have two randomizations of his game to compare. We will have the maximizer play the special randomized MP-stage game and the minimizer play the Fleming randomized MP-stage game; this is our present version of the "skew play" of <u>Value</u>, IV.

With N=MP and with this new understanding of the skew play, everything goes through here just as it did in Value, IV, §§12, 13, up to the formula (IV.13.1)-(IV.13.2) for the ζ-expectation to the maximizing player, which for us now reads

$$E_N(\zeta, x_t, x_t, t)$$

$$= \iint[\iint[\cdots[\iint\varphi(\mathfrak{r}(1))\,d\mathfrak{P}^{N-1}_{\mathfrak{r}(\tau_{N-1})}(u_{N-1})\,d\mathfrak{Q}^{N-1}_{\mathfrak{r}'(\tau_{N-1})}(v_{N-1})]$$

$$\cdots]\;d\mathfrak{P}^{1}_{\mathfrak{r}(\tau_1)}(u_1)\,d\mathfrak{Q}^{1}_{\mathfrak{r}'(\tau_1)}(v_1)]\;d\mathfrak{P}^{0}_{\mathfrak{r}(\tau_0)}(u_0)\,d\mathfrak{Q}^{0}_{\mathfrak{r}'(\tau_0)}(v_0)\;, \tag{4.19}$$

where $\mathfrak{r}(\tau_n)$ and $\mathfrak{r}'(\tau_n)$, $n=0,\ldots,N-1$, denote respectively the after-shock positions of the maximizer's trajectory $\mathfrak{r}$ in the special game, with the M shocks, and the minimizer's trajectory $\mathfrak{r}'$ in the Fleming game, with the N=MP shocks, so that in particular $\mathfrak{r}(\tau_0) = x_t+Z_{0,P-1}$ and $\mathfrak{r}'(\tau_0) = x_t + z_{0,0}$. The formula for the ζ-expectation $E'_N(\zeta, x_t, x'_t, t)$ to the minimizing player is obtained by replacing $\varphi(\mathfrak{r}(1))$ by $\omega(\mathfrak{r}'(1))$ in the right side of (4.19).

The formula for the difference of the ζ-expectations is thus gotten from (4.19) by replacing the $\varphi(r(1))$ in its right side by the expression $\varphi(r(1)) - \varphi(r'(1))$.

Now, just as at (IV.13.3) in <u>Value</u>, the integral in the right hand side of (4.19) is in fact a finite sum. To each term of this sum there corresponds a definite pair of successions $u_{0,0}, \ldots, u_{M-1,P-1}$ and $v_{0,0}, \ldots, v_{M-1,P-1}$ of controls for the maximizing and minimizing players respectively, and corresponding definite terminal points $r(1)$ and $r'(1)$, satisfying (4.17) or (4.18) depending on whether ζ is good or bad. Hence if ζ is good we have

$$|E_N(\zeta, x_t, x_t, t) - E'_N(\zeta, x_t, x_t, t)|$$

$$< (A\theta M^{-1/3} + 2\mu M^{-1/6}) L e^{2A} , \qquad (4.20)$$

and if ζ is bad

$$|E_N(\zeta, x_t, x_t, t) - E'_N(\zeta, x_t, x_t, t)| \leq 2\mu L. \qquad (4.21)$$

Now we denote by $\Pi_N \mathcal{S}^{\varkappa(1)}$ the distribution of the $\zeta = (z_{0,0}, \ldots, z_{M-1,P-1})$ for the Fleming N=MP-stage game. By the analogue to Proposition 2° in Value, IV, §12, we have the formula

$$E_N(x_t, x_t, t) = \int E_N(\zeta, x_t, x_t, t)\, d\Pi_N \mathfrak{g}^{\varkappa^{(1)}}(\zeta) \qquad (4.22)$$

for the overall expectation to the maximizing player, playing the special MP-stage in the ϵ-construction. Here the shocks in that special game at the times $\tau_m = t+(1-t)m/M$, $m=0,\ldots,M-1$ are understood to be given by the formula $Z_{m,P-1} = z_{m,0} + \cdots + z_{m,P-1}$ as we indicated above following (4.1). For the minimizing player, playing the Fleming N=MP-stage game in the ϵ-construction, we have the formula

$$E'_N(x_t, x_t, t) = \int E'_N(\zeta, x_t, x_t, t)\, d\Pi_N \mathfrak{g}^{\varkappa^{(1)}}(\zeta) , \qquad (4.23)$$

the component shocks $z_{0,0}, \ldots, z_{M-1,P-1}$ of ζ this time being applied just before each move. Since, as we saw above in the paragraph following (4.2), the probability that ζ is bad does not exceed $M^{-1/6}$, it follows from (4.20), (4.21), (4.22) and (4.23) that

$$|E_N(x_t,x_t,t) - E'_N(x_t,x_t,t)|$$

$$< (A\theta M^{-1/3} + 2\mu M^{-1/6})Le^{2A} + 2\mu LM^{-1/6}$$

$$< (A\theta M^{-1/3} + 4\mu M^{-1/6})Le^{2A} \ . \qquad (4.24)$$

Now because the maximizer in the ϵ-construction is always maximizing to accuracy no worse than $\epsilon/N = \epsilon/MP$, we have in analogy to Proposition 3° of Value (formula (IV.12.32)) that

$$E_N(x_t,x_t,t) \geq \Phi\lambda^*_{MP}(t,x_t) - \epsilon \ . \qquad (4.25)$$

Similarly, in analogy to Value, (IV.12.33), we have

$$E'_N(x_t,x_t,t) \leq \Phi\lambda^{\theta}_{MP}(t,x_t) + \epsilon \ . \qquad (4.26)$$

Hence from (4.24)

$$\Phi\lambda^*_{MP}(t,x_t) - \Phi\lambda^{\theta}_{MP}(t,x_t) < (A\theta M^{-1/3} + 4\mu M^{-1/6})Le^{2A} + 2\epsilon. \qquad (4.27)$$

Since the left side of (4.27) does not depend on ϵ and since ϵ is arbitrary, it follows that

$$\Phi\lambda^{*}_{MP}(t,x) - \Phi\lambda^{\theta}_{MP}(t,x) \leq (A\theta M^{-1/3} + 4\mu M^{-1/6}) Le^{2A} \tag{4.28}$$

for any $x = x_t \in R^p$. By reversing the rôles of the players in the ϵ-construction we obtain the opposite inequality. Hence

$$|\Phi\lambda^{*}_{MP}(t,x) - \Phi\lambda^{\theta}_{MP}(t,x)| \leq (A\theta M^{-1/3} + 4\mu M^{-1/6}) Le^{2A} \tag{4.29}$$

for all $x \in R^p$. This is the desired comparison of the value of the special MP-stage randomized game with that of the Fleming MP-stage randomized game.

5. Convergence of $\Phi\lambda_N^\theta(t,x)$ as $N \to \infty$. Theorem I

The first thing we have to do is to find an estimate of the difference between $\Phi\lambda_{MP}^*(t,x)$ and $\Phi\lambda_M^\theta(t,x)$. Put

$$\delta_m = \sup_{x \in R^p} |\varphi\lambda_{MP}^*(t,x,\tau_{m,0}) - \varphi\lambda_M^\theta(t,x,\tau_m)| , \qquad (5.1)$$

where $\tau_{m,0} = \tau_m = t + (1-t)m/M$, $m=0,\ldots,M$. We suppose that $0 \le m \le M-1$ and that δ_{m+1} is known to be finite, as is surely the case when $m=M-1$. We seem an estimate for δ_m.

Consider the problem of the P stages of the special MP-stage game starting after the shock at the time $\tau_{m,0}$ at the point $\bar{x}$. The terminal function for these P stages is $\varphi\lambda_{MP}^*(t,x,\tau_{m+1,0})$. Using the Principle of the Transmission of Continuity as it applies to mixed-strategy games without random shocks (*Value*, II, §3) , and recalling that the Lipschitz constant of $\varphi\lambda_{MP}^*(t,x,\tau_{m+1,0})$ in x is Le^A, we may replace the control function over $[\tau_{m,0}, \tau_{m+1,0})$ by $\bar{\mathfrak{f}}(u,v) = \mathfrak{f}(\bar{x}, \tau_{m,0}, u, v)$ with an error no greater than $(\mu A+a)Le^{A}/M^{2}$. Using that Principle once again, we may replace the terminal function $\varphi\lambda^{*}_{MP}(t,x,\tau_{m+1,0})$ for the P stages of the special MP-stage game by $\varphi\lambda^{\theta}_{M}(t,x,\tau_{m+1})$ with a further error not exceeding δ_{m+1}. Since $\Phi\lambda^{\theta}_{M}(t,x,\tau_{m+1})$ has Lipschitz constant Le^{A} in x and $\varphi\lambda^{\theta}_{M}(t,x,\tau_{m+1}) = \int \Phi\lambda^{\theta}_{M}(t,x+z,\tau_{m+1})d\mathcal{g}^{\varkappa}(z)$ with $\varkappa=pM/\theta^{2}(1-t)$, it follows from the result noted in the first paragraph of §3 that $\nabla\varphi\lambda^{\theta}_{M}(t,x,\tau_{m+1})$ has in x the Lipschitz constant $\sqrt{2\varkappa p/\pi}\ Le^{A} = \nu(t,\theta)Le^{A}\cdot M^{\frac{1}{2}}$, where

$$\nu(t,\theta) = p\sqrt{2/\pi}\,/\theta\sqrt{1-t}\quad . \tag{5.2}$$

When we wish later to emphasize the dependence of $\nu(t,\theta)$ on t and θ, we shall write it out; otherwise we shall simply write ν. In this section both t and θ are fixed, and $\nu=\nu(t,\theta)$ may be regarded as fixed.

We now follow the argument of <u>Value</u>, V, §3; the reader should note that the definitions of ν here and there are different. For any $x \in R^{p}$ we have from the Law of the Mean that

$$\varphi\lambda_M^\theta(t,x,\tau_{m+1}) =$$

$$= \varphi\lambda_M^\theta(t,\bar{x},\tau_{m+1}) + \nabla\varphi\lambda_M^\theta(t,x',\tau_{m+1})\cdot(x-\bar{x}) \quad , \qquad (5.3)$$

where x' is on the segment $[\bar{x},x]$. Since any point x accessible during the P stages starting at $\bar{x}$ must satisfy $|x-\bar{x}| \leq \mu/M$, (5.3) implies that for such a point

$$\varphi\lambda_M^\theta(t,x,\tau_{m+1}) =$$

$$= \varphi\lambda_M^\theta(t,\bar{x},\tau_{m+1}) + \nabla\varphi\lambda_M^\theta(t,\bar{x},\tau_{m+1})\cdot(x-\bar{x})$$

$$+ \nu\mu^2 M^{-3/2} Le^A O(1) \quad , \qquad (5.4)$$

where $O(1)$ here and throughout this paper denotes a scalar satisfying $|O(1)| \leq 1$. It now follows by a third application of the Principle of the Transmission of Continuity that the terminal function $\varphi\lambda_M^\theta(t,x,\tau_{m+1})$ may be replaced by the first two terms in the right side of (5.4) with a further error not exceeding $\nu\mu^2 Le^A M^{-3/2}$.

We are now in the situation of the "simplest linear (mixed-strategy) game" treated in <u>Value</u>, III, §8. According to the (quite trivial) analysis presented there, the value of the P-stage game starting at $\overline{x}$ with the control function $\overline{f}(u,v) = f(\overline{x},\tau_{m,0},u,v)$ and the teminal function given by the first two terms of (5.4) is $c+v/M$, where $c=\varphi\lambda_M^{\theta}(t,\overline{x},\tau_{m+1})$ and

$$v = \underset{U\times V}{\text{Value}}\ \nabla\varphi\lambda_M^{\theta}(t,\overline{x},\tau_{m+1})\cdot f(\overline{x},\tau_{m,0},u,v)\ . \qquad (5.5)$$

We have therefore proved the formula

$$\Phi\lambda_{MP}^{*}(t,\overline{x},\tau_{m,0}) = c + v/M$$

$$+ [(\mu A+a)/M^2 + \nu\mu^2/M^{3/2}]Le^{A}O(1) + \delta_{m+1}O(1)\ . \quad (5.6)$$

For the single stage of the Fleming M-stage game starting after the schock at the time $\tau_m=\tau_{m,0}$ from the point $\overline{x}$, we obtain similarly

$$\Phi\lambda_M^{\theta}(t,\overline{x},\tau_m) = c + v/M$$

$$+ [(\mu A+a)/M^2 + \nu\mu^2/M^{3/2}]Le^{A}O(1)\ , \qquad (5.7)$$

the only transition in terminal functions having this time been the linearization. Hence, uniformly in $\bar{x} \in R^p$,

$$|\Phi\lambda^*_{MP}(t,\bar{x},\tau_{m,0}) - \Phi\lambda^{\theta}_{M}(t,\bar{x},\tau_m)|$$

$$\leq \delta_{m+1} + 2[(\mu A+a)/M^2 + \nu\mu^2/M^{3/2}]Le^A \ . \qquad (5.8)$$

Evidently this same inequality persists for the before position functions at the same time $\tau_{m,0} = \tau_m$. Hence δ_m is finite and

$$\delta_m \leq \delta_{m+1} + 2[(\mu A+a)/M^2 + \nu\mu^2/M^{3/2}]Le^A. \quad (5.9)$$

On concatenating the inequalities (5.9) and recalling that $\delta_M = 0$, we find that

$$|\Phi\lambda^*_{MP}(t,x) - \Phi\lambda^{\theta}_{M}(t,x)|$$

$$\leq \delta_0 \leq 2[(\mu A+a)/M + \nu\mu^2/M^{\frac{1}{2}}]Le^A \ , \qquad (5.10)$$

uniformly on $[0,1] \times R^p$. This is the desired relationship between $\Phi\lambda^*_{MP}(t,x)$ and $\Phi\lambda^{\theta}_{M}(t,x)$ mentioned in the first sentence of this section.

We now combine (5.10) with (4.29), letting a further e^A term into the former. The result is:

$$|\Phi\lambda^{\theta}_{MP}(t,x) - \Phi\lambda^{\theta}_{M}(t,x)|/Le^{2A}$$

$$\leq A\theta M^{-1/3} + 4\mu M^{-1/6} + 2(\mu A+a)M^{-1} + 2\nu\mu^2 M^{-\frac{1}{2}} . \quad (5.11)$$

By interchanging the rôles of the M and P in these last two sections, we obtain the corresponding inequality for $|\Phi\lambda^{\theta}_{MP}(t,x) - \Phi\lambda^{\theta}_{P}(t,x)|/Le^{2A}$, with the M's in the right hand side of (5.11) replaced by P's. Hence

$$|\Phi\lambda^{\theta}_{M}(t,x) - \Phi\lambda^{\theta}_{P}(t,x)|/Le^{2A}$$

$$\leq A\theta\,(M^{-1/3} + P^{-1/3}) + 4\mu\,(M^{-1/6} + P^{-1/6})$$

$$+ 2(\mu A+a)(M^{-1} + P^{-1}) + 2\nu\mu^2(M^{-\frac{1}{2}} + P^{-\frac{1}{2}}) . \quad (5.12)$$

It follows that $\{\Phi\lambda^{\theta}_{N}(t,x)\}$ is a Cauchy sequence, uniformly in t and x. We have thus proved the following theorem, which is essentially due, in a slightly weaker version, to Wendell Fleming, as a consequence of his Lemma 1 in <u>Convergence II</u>:

THEOREM I (Fleming). $\Phi\lambda_N^\theta(t,x)$ *converges uniformly for* $(t,x) \in [0,1]\times R^p$ *to a limit* $\Phi\lambda^\theta(t,x)$ *as* $N \to \infty$.

In our proof of this theorem we have not used the hypothesis stated in the second paragraph of §2 about the Lipschitzian character of $\nabla\varphi(x)$. Nor have we mentioned any PDE.

The function $\Phi\lambda^\theta(t,x)$ is called the *Fleming mixed-strategy value of the randomized game*.

We thus have the first part of our objective, which is existence by elementary means of $\Phi\lambda^\theta(t,x)$. We will now set out, in §§7-17, to prove by elementary means that $\Phi\lambda^\theta(t,x)$ in fact does satisfy Fleming's parabolic PDE. But first, in §6, we run off the easy completion of the proof of the existence of his unrandomized mixed-strategy value, promised in §1.

6. <u>Convergence of</u> $\Phi\lambda_N(t,x)$ <u>as</u> $N\to\infty$. <u>Theorem</u> II

Fleming's unrandomized N-stage mixed-strategy value function $\Phi\lambda_N(t,x)$ is obtained from $\Phi\lambda_N^\theta(t,x)$ by putting $\theta=0$, i.e. eliminating the shocks; or see <u>Value</u>, I, §8.

THEOREM II (Fleming). *Let* B *be a bounded set in* R^p. *Then* $\Phi\lambda_N(t,x)$ *converges uniformly for* $(t,x)\in[0,1]\times B$ *to a limit* $\Phi\lambda(t,x)$ *as* $N\to\infty$.

PROOF. Since the position vector for the deterministic game must remain within a μ-neighborhood $\hat{B}$ of B, we may, by altering φ if necessary outside $\hat{B}$, assume that there is a constant K such that $|\varphi(x)|\leq K$ throughout R^p.

Though now we have a shock at the beginning and not at the end as in <u>Value</u>, IV, §13, and our game is played over $[t,1]$ instead of $[0,1]$, Lemma D of <u>Value</u> has the same form as it did there, and we have in analogy with <u>Value</u>, (IV.13.9) the inequality

$$|\Phi\lambda_N^\theta(t,x) - \Phi\lambda_N(t,x)| \leq (1+Ae^A)\lambda sL + 2K/\lambda^2 \quad . \quad (6.1)$$

Here s is the standard deviation of the sum $z_0 + \ldots + z_{N-1}$ of the shocks, and λ is any number larger than 1. Since each of the shocks has standard deviation $\theta(1-t)^{\frac{1}{2}}/N^{\frac{1}{2}}$, we have $s=\theta(1-t)^{\frac{1}{2}}$. Now suppose $\theta \in (0,1)$ and put $\lambda=\theta^{-1/3}$. (6.1) now becomes

$$|\Phi\lambda_N^{\theta}(t,x) - \Phi\lambda_N(t,x)| \leq C\theta^{2/3} , \qquad (6.2)$$

where $C = (1+Ae^{A})L + 2K$. Now suppose $\epsilon>0$. Choose θ so that $C\theta^{2/3} < \epsilon/3$; this is the only point in this paper at which θ is regarded as anything but a fixed constant given a priori. By Theorem I there is an N_θ such that if $N,N' \geq N_\theta$ then

$$|\Phi\lambda_N^{\theta}(t,x) - \Phi\lambda_{N'}^{\theta}(t,x)| < \epsilon/3 \qquad (6.3)$$

for all $(t,x) \in [0,1] \times R^p$. On combining this with (6.2) we see that if $N,N' \geq N_\theta$ then

$$|\Phi\lambda_N(t,x) - \Phi\lambda_{N'}(t,x)| < \epsilon \qquad (6.4)$$

for all $(t,x) \in [0,1] \times B$. Hence $\{\Phi\lambda_N(t,x)\}$ is a uniform Cauchy sequence for those (t,x), and the theorem is proved. □

Except for the minor point that his control function did not involve time, Fleming proved this theorem in Convergence II. The present author has given a different elementary proof in Value, Chapter V, §10. In all three proofs the basic idea was randomization. No proof is known for which this is not so.

We now return, for the rest of this paper, to the randomized game with a fixed positive θ.

7. The Lipschitz constants γ_n and ℓ_n

In this section we begin a detailed study of the structure of the position functions for the N-stage Fleming randomized game.

Let $1 \leq n \leq N$. We denote by γ_n the Lipschitz constant in x of the increment function $\iota_n(t,x)$ defined at (2.9). We denote by ℓ_n the Lipschitz constant in x of the gradient $\nabla\varphi\lambda_N^\theta(t,x,\tau_n)$ of the before position function $\varphi\lambda_N^\theta(t,x,\tau_n)$ at time τ_n. We wish in this section to find a relation between γ_n and ℓ_n.

Put

$$g_{n-1}(x,u_{n-1},v_{n-1}) = f_{n-1}(x,u_{n-1},v_{n-1}) - \varphi\lambda_N^\theta(t,x,\tau_n), \tag{7.1}$$

f_{n-1} being given by (2.5) with n replaced by n-1.

Let x and x' be two points with $|x-x'| = \eta$. Denote by r the trajectory over $[\tau_{n-1},\tau_n)$ of the differential equation $\dot{r}(\tau) = f(r(\tau),\tau,u_{n-1},v_{n-1})$ starting at x. Similarly we define r'. Next we denote by r'' a path starting at x' and running parallel to r. Put $\delta(\tau) = |r''(\tau) - r'(\tau)|$, $\tau\in[\tau_{n-1},\tau_n]$. Then

$\delta(\tau_{n-1}) = 0$ and

$$\dot{\delta}(\tau) \leq A|\mathfrak{r}(\tau) - \mathfrak{r}'(\tau)| = A|\mathfrak{r}''(\tau) - \mathfrak{r}'(\tau) + x - x'| \leq A[\delta(\tau) + \eta]. \tag{7.2}$$

Hence by a standard calculation like that at (4.9)-(4.10), we get

$$\delta(\tau_n) \leq [(1-t)/N]e^{A/N} < \eta(1-t)e^{A}/N , \tag{7.3}$$

i.e.

$$|X_n(x, u_{n-1}, v_{n-1}) + (x'-x) - X_n(x', u_{n-1}, v_{n-1})|$$
$$\leq \eta(1-t)Le^{A}/N . \tag{7.4}$$

Now put $\mathfrak{s} = X_n(x, u_{n-1}, v_{n-1}) - x$ and $\mathfrak{s}' = X_n(x, u_{n-1}, v_{n-1})$. Then we may rewrite (7.4) as

$$|\mathfrak{s} - \mathfrak{s}'| \leq \eta(1-t)e^{A}/N . \tag{7.5}$$

Since, as we noted following (2.8), $\omega\lambda_N^{\theta}(t, x, \tau_n)$ has

Lipschitz constant Le^{A}, we therefore have

$$g_{n-1}(x', u_{n-1}, v_{n-1})$$

$$= \varphi\lambda_N^{\theta}(t, x'+s', \tau_n) - \varphi\lambda_N^{\theta}(t, x', \tau_n)$$

$$= \varphi\lambda_N^{\theta}(t, x'+s, \tau_n) - \varphi\lambda_N^{\theta}(t, x', \tau_n)$$

$$+ [(1-t)Le^{2A}/N]O(1) \quad . \qquad (7.6)$$

Now, using first the integral form of the Law of the Mean and then noting that $|(x'+\omega s) - (x+\omega s)| = \eta$ for any real ω, and that $|s| \leq \upsilon(1-t)/N$, we get

$$\varphi\lambda_N^{\theta}(t, x'+s, \tau_n) - \varphi\lambda_N^{\theta}(t, x', \tau_n)$$

$$= [\int_0^1 \nabla\varphi\lambda_N^{\theta}(t, x'+\omega s, \tau_n)\,d\omega]\cdot s$$

$$= [\int_0^1 \nabla\varphi\lambda_N^{\theta}(t, x+\omega s, \tau_n)\,d\omega]\cdot s$$

$$+ [\eta(1-t)\mu\ell_n/N]O(1) \quad . \qquad (7.7)$$

Hence we finally have the formula

$$g_{n-1}(x', u_{n-1}, v_{n-1}) = [\int_0^1 \nabla\varphi\lambda_N^\theta(t, x+\omega s, \tau_n)\, d\omega] \cdot s$$

$$+ [\eta(1-t)(Le^{2A} + \mu\ell_n)/N]O(1) \ . \qquad (7.8)$$

Now, directly from the integral form of the Law of the Mean, we have

$$g_{n-1}(x, u_{n-1}, v_{n-1}) = [\int_0^1 \nabla\varphi\lambda_N^\theta(t, x+\omega s, \tau_n)\, d\omega] \cdot s \ .$$

$$(7.9)$$

Hence

$$|g_{n-1}(x', u_{n-1}, v_{n-1}) - g_{n-1}(x, u_{n-1}, v_{n-1})|$$

$$\leq \eta(1-t)(Le^{2A} + \mu\ell_n)/N \ . \qquad (7.10)$$

We may now apply the second part of the Principle of the Transmission of Continuity as it applies to games with mixed strategies (Value, II, §3). Because the estimate (7.10) holds for all $u_{n,1}$, v_{n-1}, then

$$\left| \operatorname*{Value}_{U\times V} g_{n-1}(x', u_{n-1}, v_{n-1}) - \operatorname*{Value}_{U\times V} g_{n-1}(x, u_{n-1}, v_{n-1}) \right|$$

$$\leq \eta(1-t)(Le^{2A} + \mu\ell_n)/N \,. \tag{7.11}$$

But from (7.1) and (2.6)

$$\operatorname*{Value}_{U\times V} g_{n-1}(x', u_{n-1}, v_{n-1})$$

$$= \operatorname*{Value}_{U\times V} f_{n-1}(x', u_{n-1}, v_{n-1}) - \varphi\lambda_N^\theta(t, x', \tau_n)$$

$$= \Phi\lambda_N^\theta(t, x', \tau_{n-1}) - \varphi\lambda_N^\theta(t, x', \tau_n)$$

$$= \iota_n(t, x'), \tag{7.12}$$

and similarly at x. Hence (7.11) may be rewritten as

$$|\iota_n(t,x') - \iota_n(t,x)| \leq \eta(1-t)(Le^{2A} + \mu\ell_n)/N \,, \tag{7.13}$$

a formula called by the author the "double-difference formula". It follows that

$$\gamma_n \leq (1-t)(Le^{2A} + \mu\ell_n)/N \,, \tag{7.14}$$

$n=1,\ldots,N$. This is the desired relation between γ_n and ℓ_n.

8. An inequality for the γ_n

We have $\ell_N = \lambda$, so that

$$\gamma_N \leq (1-t)(\mu\lambda + Le^{2A})/N \,. \tag{8.1}$$

Now suppose that $0 \leq n \leq N-1$, and consider the function $\varphi\lambda_N^{\theta}(t,x,\tau_n)$. If the reader has derived (2.11) from a formal induction, he will have proved the formula

$$\varphi\lambda_N^{\theta}(t,x,\tau_n) =$$

$$= \int \varphi(x+z)\,d\mathfrak{g}^{\varkappa}t^{(N-n)}(z) + \sum_{q=1}^{N-n} \int \iota_{n+q}(t,x+z)\,d\mathfrak{g}^{\varkappa}t^{(q)}(z). \tag{8.2}$$

of which (2.11) is the case n=0. The idea of representing $\varphi\lambda_N^\theta(t,x,\tau_n)$ in the form (8.2) is the principal new idea of this paper.

Now fix attention on a term $\psi(x) = \int \iota_{n+q}(x+z)\,dg^{\varkappa}t^{(q)}(z)$ in (8.2). Since $\iota_{n+q}(t,x)$ has Lipschitz constant γ_{n+q} in x, it follows from the first paragraph of §3 and the formula (2.2) for $\varkappa^{(n)}$ that $\nabla\psi(x)$ has Lipschitz coefficient $\sqrt{2\varkappa_t^{(q)}p/\pi}\cdot\gamma_{n+q} = (\nu N^{\frac{1}{2}}/q^{\frac{1}{2}})\gamma_{n+q}$, where ν was defined at (5.2). Hence from (8.2)

$$\ell_n \leq \lambda + \nu N^{\frac{1}{2}} \sum_{q=1}^{N-n} q^{-\frac{1}{2}}\gamma_{n+q}\,, \tag{8.3}$$

n=0,...,N-1. Now put

$$\Lambda(t) = (1-t)(Le^{2A} + \mu\lambda) \tag{8.4}$$

and

$$y(t,\theta) = (1-t)\mu\nu/N^{\frac{1}{2}} = (\mu p/\theta)\sqrt{2(1-t)/\pi N}\,. \tag{8.5}$$

We will write simply Λ or y when we are not dealing explicitly with the dependence on t, or on t and θ. We then combine (8.3) with (7.14), which requires $n \geq 1$. The result is:

$$\gamma_n \leq \Lambda/N + y \sum_{q=1}^{N-n} q^{-\frac{1}{2}} \gamma_{n+q} , \tag{8.6}$$

$n=1,\ldots,N-1$. We see from (8.1) that (8.6) is valid also for $n=N$. This inequality, holding then for $n=1,\ldots,N$, is the desired one.

We shall spend §§9-13 estimating the γ_n from (8.6); the result is stated at (13.9).

9. The determinants D_i

Consider the system of N equations, $n=1,\ldots,N$:

$$\gamma_n^* = \Lambda/N + y \sum_{q=1}^{N-n} q^{-\frac{1}{2}} \gamma_{n+q}^* , \tag{9.1}$$

in the variables $\gamma_1^*,\ldots,\gamma_N^*$, y and Λ being given by (8.4) and (8.5) respectively. Using (8.1) at n=N and working backward using (8.6), one proves trivially that $\gamma_n \leq \gamma_n^*$, n=1,...,N. From now until the end of §13, we shall be seeking to estimate the γ_n^*.

For each n, consider the system of N-n+1 equations

$$\gamma_{n'}^* - y \sum_{q=1}^{N-n'} q^{-\frac{1}{2}} \gamma_{n+q}^* = \frac{\Lambda}{N} , \tag{9.2}$$

n'=n,...,N, in the N-n+1 unknowns $\gamma_n^*,\ldots,\gamma_N^*$. The determinant of this system is unity. By Cramer's rule, γ_n^* is therefore gotten by replacing the first column in that determinant by the entries on the right hand side of (9.2), i.e.

$$\gamma_n^* = \left|\begin{matrix} \frac{\Lambda}{N} & \frac{-y}{1^{1/2}} & \frac{-y}{2^{1/2}} & \cdots & \frac{-y}{(N-n-1)^{1/2}} & \frac{-y}{(N-n)^{1/2}} \\ \frac{\Lambda}{N} & 1 & \frac{-y}{1^{1/2}} & \cdots & \frac{-y}{(N-n-2)^{1/2}} & \frac{-y}{(N-n-1)^{1/2}} \\ \frac{\Lambda}{N} & 0 & 1 & \cdots & \frac{-y}{(N-n-3)^{1/2}} & \frac{-y}{(N-n-2)^{1/2}} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{\Lambda}{N} & 0 & 0 & \cdots & 1 & \frac{-y}{1^{1/2}} \\ \frac{\Lambda}{N} & 0 & 0 & \cdots & 0 & 1 \end{matrix}\right| . \quad (9.3)$$

We will refer to a path across this, or any other determinant, which has a non-zero product, as a <u>non-zero path</u> .

Let $i>1$. Any non-zero path starting at the $\frac{\Lambda}{N}$ in the i^{th} row of (9.3) must go down the diagonal after the i^{th} column has been passed. Hence the cofactor of that $\frac{\Lambda}{N}$ is the determinant made up of rows $1,\ldots,i-1$ and columns $2,\ldots,i$ in (9.3), prefixed in sign as follows. Consider a non-zero path in (9.3) involving that $\frac{\Lambda}{N}$, which passes through rows $i, i_2, \ldots, i_i$ as $j=2,\ldots,i$. The sign of its product is then that of the permutation $i_2,\ldots,i_i$ of $1,\ldots,i-1$, multiplied by $(-1)^{i-1}$. In the cofactor determinant the columns $1,\ldots,i$ receive the new labels $1^*,\ldots,(i-1)^*$. So we may rewrite the permutation $i_2,\ldots,i_i$ as $i_{1^*},\ldots,i_{(i-1)^*}$. Its sign as a permutation of $1,\ldots,i-1$ is however unchanged. Hence the prefix in sign required on the cofactor determinant is $(-1)^{i-1}$. It follows that for $n=1,\ldots,N$

$$\gamma_n^* = \frac{\Lambda}{N} \left[1 + \sum_{i=2}^{N-n+1} (-1)^{i-1} D_i \right], \tag{9.4}$$

where D_i is the $(i-1)\times(i-1)$ determinant given by

$$D_i = \begin{bmatrix} \frac{-y}{1^{\frac{1}{2}}} & \frac{-y}{2^{\frac{1}{2}}} & \frac{-y}{3^{\frac{1}{2}}} & \cdots & \frac{-y}{(i-2)^{\frac{1}{2}}} & \frac{-y}{(i-1)^{\frac{1}{2}}} \\ 1 & \frac{-y}{1^{\frac{1}{2}}} & \frac{-y}{2^{\frac{1}{2}}} & \cdots & \frac{-y}{(i-3)^{\frac{1}{2}}} & \frac{-y}{(i-2)^{\frac{1}{2}}} \\ 0 & 1 & \frac{-y}{1^{\frac{1}{2}}} & \cdots & \frac{-y}{(i-4)^{\frac{1}{2}}} & \frac{-y}{(i-3)^{\frac{1}{2}}} \\ 0 & 0 & 1 & \cdots & \frac{-y}{(i-5)^{\frac{1}{2}}} & \frac{-y}{(i-4)^{\frac{1}{2}}} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\cdot\, 1 & \frac{-y}{1^{\frac{1}{2}}} & \frac{-y}{2^{\frac{1}{2}}} \\ 0 & 0 & 0 & \cdots & 1 & \frac{-y}{1^{\frac{1}{2}}} \end{bmatrix} . \tag{9.5}$$

Our next objective, accomplished in §13, is to prove that the sum in the brackets in (9.4) is absolutely bounded.

10. The coefficients d_i^k

Let $i \geq 2$. We denote by d_i^k, $k=1,\ldots,i-1$, the coefficients of y^k in $(-1)^{i-1}D_i$.

We denote the indices of the rows and columns of D_i by h and j respectively.

We shall refer to the diagonal just below the main diagonal in D_i as the 1-diagonal. Notice that the entries above the 1-diagonal, i.e. with $h \leq j$, are $\dfrac{-y}{(j-h+1)^{\frac{1}{2}}}$.

We first observe that

$$d_i^1 = \frac{1}{(i-1)^{\frac{1}{2}}} \tag{10.1}$$

for any $i \geq 2$. In particular this disposes of the case $i=2$.

Now suppose that $i \geq 3$, and $2 \leq k \leq i-1$. Any non-zero path involving y^k has to leave the 1-diagonal exactly k-1 times with $j \leq 2$. Suppose it leaves it at the columns $j_1, \ldots, j_{k-1}$ with $j_1 < \cdots < j_{k-1} \leq i-2$. There can be only one non-zero path corresponding to the succession $j_1, \ldots, j_{k-1}$. It goes down the 1-diagonal until column j_1 is reached. It must then go to row 1, and acquire the factor $\frac{-y}{j_1^{\frac{1}{2}}}$. If $j_1 > 1$, the first j_1 row entries are then $2, 3, \ldots, j_1, 1$, which form a permutation of sign $(-1)^{j_1 - 1}$ of the succession $1, \ldots, j_1$. Clearly this sign is also correct if $j_1 = 1$. The path now returns to the 1-diagonal until it reaches column j_2. At that point it must go to row j_1+1 and acquire the factor $\frac{-y}{(j_2 - j_1)^{\frac{1}{2}}}$. We also see that the succession of rows corresponding to columns $j_1+1, \ldots, j_2$ forms a permutation of sign $(-1)^{j_2 - j_1 - 1}$ of the succession $j_1+1, \ldots, j_2$; the reader should check this separately in the cases $j_2 > j_1+1$ and $j_2 = j_1+1$. The path proceeds in a similar way across the segments $[j_2+1, j_3], \ldots, [j_{k-2}+1, j_{k-1}]$, acquiring the factors $\frac{-y}{(j_3 - j_2)^{\frac{1}{2}}}, \ldots, \frac{-y}{(j_{k-1} - j_{k-2})^{\frac{1}{2}}}$ and the permutation signs $(-1)^{j_3 - j_2 - 1}, \ldots, (-1)^{j_{k-1} - j_{k-2} - 1}$. The path across the final segment $[j_{k-1} + 1, i-1]$

similarly acquires the factor $\frac{-y}{(i-1-j_{k-1})^{\frac{1}{2}}}$ and the sign $(-1)^{i-1-j_{k-1}-1} = (-1)^{i-j_{k-1}}$. The rows on the path therefore form overall a permutation of sign

$$(-1)^{(j_1-1)+(j_2-j_1-1)+\ldots+(j_{k-1}-j_{k-2}-1)+(i-j_{k-1})}$$

$$= (-1)^{i-k+1} \tag{10.2}$$

of the succession $1,\ldots,i-1$. There are k minus signs in the factors. The contribution of this path to D_i is therefore

$$\frac{(-1)^{i+1}\, y^k}{(j_1)^{\frac{1}{2}}(j_2-j_1)^{\frac{1}{2}} \cdots (j_{k-1}-j_{k-2})^{\frac{1}{2}}(i-1-j_{k-1})^{\frac{1}{2}}} \ . \tag{10.3}$$

It follows that

$$d_i^k = \sum_{1\le j_1<\cdots<j_{k-1}\le i-2} \frac{1}{(j_1)^{\frac{1}{2}}(j_2-j_1)^{\frac{1}{2}}\cdots(j_{k-1}-j_{k-2})^{\frac{1}{2}}(i-1-j_{k-1})^{\frac{1}{2}}} \ . \tag{10.4}$$

We note that the d_i^k are all positive, hence also all the $(-i)^{i-1}D_i$ appearing in (9.4). For our objective noted at the end of §9 it will therefore suffice to estimate $\sum_{i=2}^{N} (-1)^{i-1} D_i$. We have evidently

$$\sum_{i=2}^{N} (-1)^{i-1} D_i = \sum_{i=2}^{N} \sum_{k=1}^{i-1} d_i^k y^k . \tag{10.5}$$

11. A trivial inequality

In what follows we shall frequently need to estimate the quantity

$$\sum_{q=r+1}^{s-1} \frac{1}{(q-r)^{\frac{1}{2}}(s-q)^{\frac{1}{2}}} , \tag{11.1}$$

r and s being integers with $0 \leq r \leq s-2$. Put $g(w) = \dfrac{1}{(w-r)^{\frac{1}{2}}(s-w)^{\frac{1}{2}}}$. Then, by elementary calculus, $\int_{r}^{(r+s)/2} g(w)\,dw = \int_{(r+s)/2}^{s} g(w)\,dw = \pi/2$, independently of r or s satisfying the inequality indicated above.

$g(w)$ is decreasing on $[r,(r+s)/2]$ and increasing on $[(r+s)/2,s]$. Put $q^* = [\frac{r+s}{2}]$. Then evidently

$$\sum_{q=r+1}^{q^*} g(q) \quad < \quad \int_r^{q^*} g(w)\,dw \;\le\; \int_r^{(r+s)/2} g(w)\,dw = \pi/2 \qquad (11.2)$$

and

$$\sum_{q=q^*+1}^{s-1} g(q) \quad < \quad \int_{q^*+1}^{s} g(w)\,dw \;\le\; \int_{(r+s)/2}^{s} g(w)\,dw = \pi/2\,. \qquad (11.3)$$

Hence the quantity (11.1) is less than π, independently of r and s with $0 \le r \le s-2$.

12. Estimation of the d_i^k

If $i \ge 2$ and $k=1$ we have the trivial exact formula (10.1): $d_i^1 = 1/(i-1)^{\frac{1}{2}}$.

If $i \ge 3$ and $k=2$ we have

$$d_i^2 = \sum_{1 \le j_1 \le i-2} \frac{1}{(j_1)^{\frac{1}{2}}(i-1-j_1)^{\frac{1}{2}}} < \pi \qquad (12.1)$$

according to §11.

If $i \geq 4$ and $k=3$ we have

$$d_i^3 = \sum_{1 \leq j_1 < j_2 \leq i-2} \frac{1}{(j_1)^{\frac{1}{2}}(j_2-j_1)^{\frac{1}{2}}(i-1-j_2)^{\frac{1}{2}}} . \quad (12.2)$$

For each fixed j_1 we have, according to §11,

$$\sum_{j_1 < j_2 \leq i-2} \frac{1}{(j_2-j_1)^{\frac{1}{2}}(i-1-j_2)^{\frac{1}{2}}} < \pi . \quad (12.3)$$

Hence

$$d_i^3 < \pi \sum_{j_1=1}^{i-3} \frac{1}{(j_1)^{\frac{1}{2}}} < \pi \int_0^{i-3} \frac{dw}{w^{\frac{1}{2}}} < \pi \int_0^N \frac{dw}{w^{\frac{1}{2}}} = 2\pi N^{\frac{1}{2}} .$$

(12.4)

Next we suppose that $i \geq 5$ and k is even, $k=2q$ with $2 \leq q \leq \frac{i-1}{2}$. We consider successions $j_2, j_4, \ldots, j_{2q-2}$ satisfying

$$2 \leq j_2 < < j_4 < < \cdots < < j_{2q-2} \leq i-3 , \quad (12.5)$$

where the double inequality indicates that the difference is at least 2. Put $e = [i/2]$. The number of such sets of $q-1$ non-adjacent integers in $[2,i-3]$ certainly does not exceed

$$\frac{(i-4)(i-6)\cdots[i-4-2(q-2)]}{(q-1)!}$$

$$\leq \frac{(2e-3)(2e-5)\cdots[2e-3-2(q-2)]}{(q-1)!}$$

$$< \frac{(2e-2)(2e-4)\cdots[2e-2-2(q-2)]}{(q-1)!}$$

$$= 2^{q-1}\,\frac{(e-1)(e-2)\cdots(e-q+1)}{(q-1)!}$$

$$< 2^{q-1}\,\frac{e(e-1)\cdots(e-q+2)}{(q-1)!}$$

$$= 2^{q-1}\binom{e}{q-1}\,. \tag{12.6}$$

Now with the succession $j_2, j_4, \ldots, j_{2q-2}$ satisfying (12.5) fixed, we group the $k=2q$ factors in the denominator in (10.4) into pairs and sum on $j_1, j_3, \ldots, j_{2q-1}$ satisfying $1 \leq j_1 < j_2 < j_3 < j_4 < \cdots$ $\cdots < j_{2q-2} < j_{2q-1} \leq i-2$. The result, according to §11, is less than π^q. It follows that

$$d_i^k < 2^{q-1}\,\pi^q\,\binom{e}{q-1} < (2\pi)^q\binom{e}{q-1}\ . \tag{12.7}$$

Finally suppose that $i \geq 6$ and k is odd, $k=2q+1$ with $2 \leq q \leq \frac{i-2}{2}$. Fix on a $j_1 \geq 1$ and consider successions $j_3,\ldots,j_{2q-1}$ satisfying

$$j_1+2 \leq j_3 < < j_5 < < \cdots < < j_{2q-1} \leq i-3\ . \tag{12.8}$$

The number of such sets of q-1 non-adjacent integers on $[j_1+2, i-3]$ certainly is less than the number on $[2, i-3]$, and we once again have the overestimate $2^{q-1}\binom{e}{q-1}$ as at (12.6). This time, with j_1 and the succession $j_3,\ldots,j_{2q-1}$ satisfying (12.8) fixed, we group the 2q factors following $(j_1)^{\frac{1}{2}}$ in the denominator of (10.4) into pairs and sum on $j_2, j_4, \ldots, j_{2q}$ satisfying $j_1 < j_2 < j_3 < j_4 < \cdots$ $\cdots < j_{2q-1} < j_{2q} \leq i-2$. The result is once again less than π^q. It follows that

$$d_i^k < 2^{q-1}\,\pi^q\,\binom{e}{q-1}\sum_{j_1=1}^{i-3}\frac{1}{(j_1)^{\frac{1}{2}}} < (2\pi)^q N^{\frac{1}{2}}\binom{e}{q-1}\ .$$

(12.9)

We see from (12.1) and (12.4) that the general estimates (12.7) and (1.29) hold also for $k=2$ and $k=3$, the combinatorial term, in both cases having $q-1=0$, being interpreted in the usual way as unity.

13. Estimation of the γ_n

What we have to estimate is the quantity $\sum_{i=2}^{N}\sum_{k=1}^{i-1} d_i^k y^k$ in the right side of (10.5).

Put

$$\omega(t,\theta) = (\omega p/\theta)\sqrt{2(1-t)/\pi} \quad . \tag{13.1}$$

As usual, we write simply ω when the dependence on t or θ need not be made explicit. Then, from (8.5), $y=\omega/N^{\frac{1}{2}}$.

We first consider the terms in $\sum_{i=2}^{N}\sum_{k=1}^{i-1} d_i^k y^k$ with $k=1$. Using (10.1), we find that these add to

$$y \sum_{i=2}^{N} \frac{1}{(i-1)^{\frac{1}{2}}} < 2yN^{\frac{1}{2}} = 2\omega \, . \tag{13.2}$$

What is left is the quantity

$$\sum_{i=3}^{N} \sum_{k=2}^{i-1} d_i^k y^k \, . \tag{13.3}$$

We shall prove that this is bounded.

Suppose first that $i=2e+1$, $e \geq 1$, is odd. Then the inside sum in (13.3) satisfies, according to (12.7) and (12.10), the outside inequality below:

$$\begin{aligned}
\sum_{k=2}^{2e} d_i^k y^k &= \sum_{q=1}^{e} (d_i^{2q} y^{2q} + d_i^{2q+1} y^{2q+1}) \\
&< \sum_{q=1}^{e} (2\pi)^q \binom{e}{q-1} \left(\frac{\omega^{2q}}{N^q} + \frac{\omega^{2q+1}}{N^q}\right) \\
&= (1+\omega) \sum_{q=1}^{e} \binom{e}{q-1} w^q \\
&= w(1+\omega) \sum_{q=0}^{e-1} \binom{e}{q} w^q \\
&< w(1+\omega) \sum_{q=0}^{e} \binom{e}{q} w^q \\
&= (1+\omega) w (1+w)^e \\
&< 2(1+\omega)\pi\omega^2 e^{\pi\omega^2}/N \, ,
\end{aligned} \tag{13.4}$$

where we wrote $w=2\pi\omega^2/N$.

If $i=2e$, $e\geq 2$, we have

$$\sum_{k=2}^{i-1} d_i^k y^k < \sum_{q=1}^{e} (d_i^{2q} y^{2q} + d_i^{2q+1} y^{2q+1}) < 2(1+\omega)\pi\omega^2 e^{\pi\omega^2}/N , \tag{13.5}$$

the calculation after the second term in (13.5) being the same as in (13.4). Hence

$$\sum_{i=2}^{N} (-1)^{i-1} D_i < 2\omega + 2(1+\omega)\pi\omega^2 e^{\pi\omega^2} < \Gamma , \tag{13.6}$$

where

$$\Gamma = \Gamma(\theta) = 2\omega_0 + 2(1+\omega_0)\pi\omega_0 e^{\pi\omega_0^2} , \tag{13.7}$$

with

$$\omega_0 = \omega(0,\theta) = (\mu p/\theta)\sqrt{2/\pi} . \tag{13.8}$$

Γ is (except for its dependence on θ) the absolute bound asked for at the end of §9.

It follows from (9.4), (13.6), and the inequalities $\gamma_n \leq \gamma_n^*$ noted at the beginning of §9 that, for all $n=1,\ldots,N$,

$$\gamma_n \leq \frac{\Lambda}{N}(1+\Gamma), \tag{13.9}$$

where $\Lambda = \Lambda(t)$ is given by (8.4) and $\Gamma = \Gamma(\theta)$ by (13.7). The whole work of §§7-13 has been directed towards this inequality, which is a central result.

14. Spatial Lipschitz and Hölder coefficients on the spatial derivatives of $\Phi\lambda_N^\theta(t,x)$. Theorems III and IV

To obtain the Lipschitz constant ℓ_0 of $\nabla\Phi\lambda_N^\theta(t,x) = \nabla\varphi\lambda_N^\theta(t,x,\tau_0)$ in x we need only apply formula (8.3) with n=0, using (13.9) and recalling the definitions of ν and Λ at (5.2) and (13.4):

$$\ell_0 \leq \lambda + \frac{\nu\Lambda(1+\Gamma)}{N^{\frac{1}{2}}} \sum_{q=1}^{N} q^{-\frac{1}{2}} < \lambda + 2\nu\lambda(1+\Gamma) < \lambda^\theta, \tag{14.1}$$

where

$$\lambda^{\theta} = \lambda + 2p\sqrt{2/\pi}\,(Le^{2A}+\mu\lambda)(1+\Gamma)/\theta \;, \tag{14.2}$$

Γ being given by (13.7)-(13.8). With θ fixed this estimate is uniform in t,x, and N.

Before we turn to the second derivatives, we note a simple point. Suppose a real-valued function $g(w)$ is bounded by K and has Lipschitz constant ι. Then, for any α with $0<\alpha<1$, $g(w)$ is Hölderian in w with exponent α and coefficient $(2K)^{1-\alpha}\iota^{\alpha}$. We leave the trivial proof to the reader.

We now consider the representation (2.10)-(2.11). We have just proved that $\iota_n(t,x)$ has the uniform Lipzchitz constant $\gamma_n \le \frac{\Lambda}{N}(1+\Gamma)$ given by (13.9). By the results of §3, $\nabla\psi_n$, where ψ_n is given by (2.10), has the uniform Lipschitz constant

$$\sqrt{2\varkappa_t^{(n)}p/\pi}\cdot\gamma_n < p(1+\Gamma)(Le^{2A}+\mu\lambda)\sqrt{2/\pi}\,/\,\theta n^{\frac{1}{2}}N^{\frac{1}{2}} \;. \tag{14.3}$$

Here we have recalled $\varkappa_t^{(n)}$ from (2.2) and dropped a factor $(1-t)^{\frac{1}{2}}$ from the numerator. The right hand side of (14.3) therefore serves as a uniform bound for $\left|\frac{\partial^2\psi_n(t,x)}{\partial x_i \partial x_j}\right|$ over all t,x,i,j.

Next, the Lipschitz constant of $\frac{\partial^2\psi_n(t,x)}{\partial x_i \partial x_j}$ in x has, once again from §3, a value not exceeding

$$2\varkappa^{(n)}\gamma_n \leq 2p(1+\Gamma)(Le^{2A}+\mu\lambda)/\theta^2 n \; . \tag{14.4}$$

Hence, by the trivial observation made two paragraphs back, $\frac{\partial^2\psi_n(t,x)}{\partial x_i \partial x_j}$ has relative to $\alpha\in(0,1)$ the Hölder coefficient

$$\frac{C_\alpha^\theta}{N^{(1-\alpha)/2}\, n^{(1+\alpha)/2}} \quad , \tag{14.5}$$

where

$$C_\alpha^\theta = 2^\alpha \; p(1+\Gamma)\;(Le^{2A}+\mu\lambda)(2/\pi)^{(1-\alpha)/2}/\theta^{1+\alpha} \; . \tag{14.6}$$

The reader should observe that C_α^θ does not depend on t.

Now put

$$\Psi_{ij}^{N}(t,x) = \frac{\partial^2}{\partial x_i \partial x_j} \left[\sum_{n=1}^{N} \psi_n(t,x) \right] . \tag{14.7}$$

It follows from (14.5) that the Hölder coefficient of $\Psi_{ij}^{N}(t,x)$ in x has the estimate

$$\frac{C_\alpha^\theta}{N^{(1-\alpha)/2}} \sum_{n=1}^{N} \frac{1}{n^{(1+\alpha)/2}} < \frac{2C_\alpha^\theta}{1-\alpha} , \tag{14.8}$$

which is independent of t. The reader will see here the reason for passing from Lipschitz to Hölder coefficients for the second derivative. The Lipschitz condition of $\Psi_{ij}^{N}(t,x)$ in x as estimated by summing (14.4) is of order of magnitude log N; the estimate (14.8) is independent of N.

All the discussion to this point has had $t<1$. It is convenient at this point to complete the definitions of §2 in a trivial way by putting $\Phi\lambda_N^\theta(1,x) = \varphi\lambda_N^\theta(1,x,\tau_n) = \Phi\lambda_N^\theta(1,x,\tau_n) = \varphi(x)$ for all $n=0,\ldots,N$ and $x \in R^p$. Then of course $\iota_n(1,x) = 0$ for all $n=1,\ldots,N$ and $x \in R^p$, so that also $\Psi_{ij}^{N}(1,x)$ exists and is identically zero. Hence the Hölder constant $2C_\alpha^\theta/(1-\alpha)$ given by (14.8) for $\Psi_{ij}^{N}(t,x)$ is valid for all $(t,x)\in[0,1]\times R^p$.

As to the other term in the representation (2.11), its gradient $\int \nabla\varphi(x+z)\, d\beta^{\varkappa_t^{(N)}}(z)$ has Lipschitz constant λ in x, which serves as a bound for its second spatial partial derivatives. And by the first part of §3 the second partials have Lipschitz constant estimated by

$$\sqrt{2\varkappa^{(N)}p/\pi} \cdot \lambda = p\lambda\sqrt{2/\pi}/\theta(1-t)^{\frac{1}{2}}. \qquad (14.9)$$

Hence they have with respect to $\alpha\in(0,1]$ the Hölder coefficient in x

$$h_{\alpha,t}^{\theta} = 2^{1-\alpha}\lambda p^{\alpha}(2/\pi)^{\alpha/2}/\theta^{\alpha}(1-t)^{\alpha/2}, \qquad (14.10)$$

which does depend on t.

We are now ready to state our first theorem on Lipschitz and Hölder coefficients.

THEOREM III. Under the hypotheses stated in the second paragraph of §2, in particular under the assumption that the terminal function φ and its gradient $\nabla\varphi$ are uniformly Lipschitzian, but with no hypotheses concerning any second derivatives of φ, the spatial gradient $\nabla\Phi\lambda_N^\theta(t,x)$ is uniformly Lipschitzian in x, with the constant λ^θ given by (14.2).

The second spatial partial derivatives of $\Phi\lambda_N^\theta(t,x)$ may be represented, for $(t,x)\in[0,1)\times R^p$, in the form

$$\frac{\partial^2\Phi\lambda_N^\theta(t,x)}{\partial x_i\,\partial x_j} = \Psi_{ij}^N(t,x) + \frac{\partial^2}{\partial x_i\,\partial x_j}\left[\int \varphi(x+z)\,d\mathfrak{g}^{\varkappa_t^{(N)}}(z)\right], \tag{14.11}$$

$\Psi_{ij}^N(t,x)$ being given by (14.7), and with $\varkappa_t^{(N)} = p/\theta^2(1-t)$ not depending on N. The function $\Psi_{ij}^N(t,x)$ is uniformly Hölderian in x relative to any exponent $\alpha\in(0,1)$, with the coefficient $2C_\alpha^\theta/(1-\alpha)$, C_α^θ being given by (14.6), throughout $[0,1]\times R^p$. The second term on the right hand side of (14.11) has the Hölder coefficient $h_{\alpha,t}^\theta$ in x given by (14.10) relative to any $\alpha\in(0,1]$, having the order of growth $(1-t)^{-\alpha/2}$ near $t=1$. Neither Hölder coefficient depends on N or x.

The following is an immediate consequence of the representation (14.11).

THEOREM IV. *Suppose in addition to the hypotheses of the second paragraph of* §2 *that the terminal function* $\mathfrak{m}$ *has second partial derivatives which are uniformly Hölderian, with constant* h_α^2 , *for some* $\alpha \in (0,1]$. *Then the second spatial partial derivatives*

$$\frac{\partial^2 \mathfrak{m}\lambda_N^\theta(t,x)}{\partial x_i \partial x_j}$$

exist throughout $[0,1] \times R^p$ *and are uniformly Hölderian in* x *there, with the same exponent* α *and with constant* $2C_\alpha^\theta/(1-\alpha) + h_\alpha^2$, C_α^θ *being given by* (14.6), *not depending on* N, t, *or* x.

15. Time Hölder coefficients on the spatial derivatives of $\Phi\lambda_N^\theta(t,x)$. Theorem V

These are obtained by a modification of the lengthy calculation relative to x which we have just finished. However this time we seek Hölder coefficients, rather than Lipschitz constants, on the increments $\iota_n(t,x)$. The modification is not quite trivial.

Until (15.27) is reached we will fix on t,t' with $0\leq t<t'<1$. Put

$$\chi_n^{t,t'} = \sup_{x\in R^p} \frac{|\iota_n(t',x) - \iota_n(t,x)|}{(t'-t)^{\frac{1}{2}}} , \quad (15.1)$$

and

$$h_n^{t,t'} = \sup_{x\in R^p} \frac{|\nabla\varphi\lambda_N^\theta(t',x,\tau_n') - \nabla\varphi\lambda_N^\theta(t,x,\tau_n)|}{(t'-t)^{\frac{1}{2}}} , \quad (15.2)$$

$n=0,\ldots,N$. Because $|\iota_n(t,x)| \leq \mu(1-t)Le^A$ and $|\nabla\varphi\lambda_N^\theta(t,x,\tau_n)| \leq Le^A$ for any n, t, and x, both $\chi_n^{t,t'}$ and $h_n^{t,t'}$ are finite.

Now we seek an analogue of the "double-difference formula" (7.13). We sharpen the notation following (2.3) by denoting explicitly by $X_n^t(x,u_{n-1},v_{n-1})$ the point reached at time τ_n, starting from $\xi(\tau_{n-1}) = x$ at time τ_{n-1} and following the differential equation

$$\dot{\xi}(\tau) = \mathfrak{f}(\xi(\tau),\tau,u_{n-1},v_{n-1}) \tag{15.3}$$

across the interval $[\tau_{n-1},\tau_n)$; similarly for t'. Evidently we may redefine $X_n^{t'}(x,u_{n-1},v_{n-1})$ to be the point reached at time τ_n, starting from $\xi'(\tau_{n-1}) = x$ at time τ_{n-1} and following the differential equation

$$\dot{\xi}'(\tau) = \frac{1-t'}{1-t}\,\mathfrak{f}\left(\xi'(\tau),\ \frac{1-\tau}{1-t}\,t' + \frac{\tau-t}{1-t}\ ,\ u_{n-1},v_{n-1}\right) \tag{15.4}$$

across the interval $[\tau_{n-1},\tau_n)$. Put

$$\delta(\tau) = |\mathfrak{r}(\tau) - \mathfrak{r}'(\tau)| , \tag{15.5}$$

$\tau\in[\tau_{n-1},\tau_n]$. Then by an easy calculation we see that

$$\dot{\delta}(\tau) \leq A\delta(\tau) + \frac{(a+\mu)(t'-t)}{1-t} , \tag{15.6}$$

so that

$$|X_n^{t'}(x,u_{n-1},v_{n-1}) - X_n^t(x,u_{n-1},v_{n-1})|$$

$$\leq \delta(\tau_n) \leq (a+\mu)(t'-t)e^{A/N}/N \leq (a+\mu)(t'-t)e^A/N . \tag{15.7}$$

Hence

$$\varphi\lambda_N^\theta(t',X_n^{t'}(x,u_{n-1},v_{n-1}),\tau_n')$$

$$= \varphi\lambda_N^\theta(t',X_n^t(x,u_{n-1},v_{n-1}),\tau_n')$$

$$+ (\mu+a)(t'-t)Le^A\mathcal{O}(1)/N . \tag{15.8}$$

Now put $\mathfrak{s} = X_n^t(x, u_{n-1}, v_{n-1}) - x$. Then

$$\varphi\lambda_N^\theta(t', X_n^t(x, u_{n-1}, v_{n-1}), \tau_n') - \varphi\lambda_N^\theta(t', x, \tau_n')$$

$$= [\int_0^1 \nabla\varphi\lambda_N^\theta(t', x+\omega\mathfrak{s}, \tau_n')\, d\omega] \cdot \mathfrak{s}$$

$$= [\int_0^1 \nabla\varphi\lambda_N^\theta(t, x+\omega\mathfrak{s}, \tau_n)\, d\omega] \cdot \mathfrak{s}$$

$$+ h_n^{t,t'} \mu(1-t)(t'-t)^{\frac{1}{2}} O(1)/N$$

$$= \varphi\lambda_N^\theta(t, X_n^t(x, u_{n-1}, v_{n-1}), \tau_n) - \varphi\lambda_N^\theta(t, x, \tau_n)$$

$$+ h_n^{t,t'} \mu(1-t)(t'-t)^{\frac{1}{2}} O(1)/N \; . \tag{15.9}$$

Now (15.8) remains valid if we replace the factor $t'-t$ by $(1-t)^{\frac{1}{2}}(t'-t)^{\frac{1}{2}}$. It was in order to make this crucial replacement that we chose $t'>t$. On combining the resulting formula with (15.9) we get

$$\varphi\lambda_N^\theta(t',X_n^{t'}(x,u_{n-1},v_{n-1}),\tau_n') - \varphi\lambda_N^\theta(t',x,\tau_n')$$

$$= \varphi\lambda_N^\theta(t,X_n^t(x,u_{n-1},v_{n-1}),\tau_n) - \varphi\lambda_N^\theta(t,x,\tau_n)$$

$$+ [(\mu+a)(1-t)^{\frac{1}{2}}Le^{2A} + h_n^{t,t'}\mu(1-t)](t'-t)^{\frac{1}{2}}O(1)/N \ .$$

(15.10)

On taking the values in (15.10) over $U \times V$ we therefore get

$$|\iota_n(t',x) - \iota_n(t,x)|$$

$$\leq [(\mu+a)(1-t)^{\frac{1}{2}}Le^{2A} + h_n^{t,t'}\mu(1-t)](t'-t)^{\frac{1}{2}}/N \ .$$

(15.11)

Since this holds for all $x \in R^p$, it follows that

$$\chi_n^{t,t'} \leq [(\mu+a)(1-t)^{\frac{1}{2}}Le^{2A} + h_n^{t,t'}\mu(1-t)]/N \ . \quad (15.12)$$

(15.11) is the desired "double-difference formula", and

(15.12) here plays the rôle of (7.14). These formulas hold for $n=1,\ldots,N$.

Next we suppose that $0 \le n \le N-1$ and consider the representation (8.2) for $\varphi\lambda_N^\theta(t',x,\tau_n')$. The first term in the corresponding representation for $\nabla\varphi\lambda_N^\theta(t',x,\tau_n')$ is $\int \nabla\varphi(x+z')\, d\mathcal{g}^\varkappa{}_{t'}^{(N-n)}(z')$. We may rewrite this relative to the time t as $\int \nabla\varphi(x + (\frac{1-t'}{1-t})^{\frac{1}{2}} z)\, d\mathcal{g}^\varkappa{}_t^{(N-n)}(z)$. Now $1 - (\frac{1-t'}{1-t})^{\frac{1}{2}} < (\frac{t'-t}{1-t})^{\frac{1}{2}}$, so that

$$\int \left|\nabla\varphi(x + (\tfrac{1-t'}{1-t})^{\frac{1}{2}} z) - \nabla\varphi(x+z)\right| d\mathcal{g}^\varkappa{}_t^{(N-n)}(z)$$

$$\le \lambda (\tfrac{t'-t}{1-t})^{\frac{1}{2}} \int |z|\, d\mathcal{g}_t^{(N-n)}(z)$$

$$\le \lambda (\tfrac{t'-t}{1-t})^{\frac{1}{2}} \frac{(N-n)^{\frac{1}{2}}\theta (1-t)^{\frac{1}{2}}}{N^{\frac{1}{2}}}$$

$$\le \lambda\theta (t'-t)^{\frac{1}{2}} . \qquad (15.13)$$

$\int \nabla\varphi(x+z)\, d\mathcal{g}^\varkappa{}_t^{(N-n)}(z)$ has therefore the uniform Hölder coefficient $\lambda\theta$ in t relative to the exponent $\alpha = \frac{1}{2}$.

Now we consider, component by component, the terms $\nabla [\int \iota_{n+q}(t',x+z')\, dg^{\varkappa_{t'}^{(q)}}(z')]$, $q=1,\ldots,N-n$, in the representation corresponding to (2.11) for $\nabla\varphi\lambda_N^\theta(t',x,\tau_n')$. In the following calculations we will write $\varkappa=\varkappa_t^{(q)}$ and $\varkappa'=\varkappa_{t'}^{(q)}$ for typographical convenience. The first component of $\nabla [\int \iota_{n+q}(t',x+z')\; dg^{\varkappa_{t'}^{(q)}}(z')]$ is then

$$\frac{\partial}{\partial x_1} [\int \iota_{n+q}(t',x+z')\, dg^{\varkappa'}(z')$$

$$= -\varkappa' (\tfrac{\varkappa'}{2\pi})^{p/2} \int z_1' \iota_{n+q}(t',x+z') e^{-\varkappa'(z_1'^2 + \ldots + z_p'^2)/2} \, dz_1' \cdots dz_p'$$

$$= -\varkappa (\tfrac{\varkappa}{2\pi})^{p/2} \int (\tfrac{1-t}{1-t'})^{\frac{1}{2}} z_1\, \iota_{n+q}(t',x+ (\tfrac{1-t'}{1-t})^{\frac{1}{2}} z) \, e^{-\varkappa(z_1^2 + \ldots + z_p^2)/2}\; dz_1 \cdots dz_p \;. \qquad (15.14)$$

Since $|\iota_{n+q}(t',z')| \le \mu(1-t')Le^A/N$ for any t',z' , we may replace the factor $(\frac{1-t}{1-t'})^{\frac{1}{2}}$ in the last integral by 1 with an error satisfying

$$\text{Error}_1 \leq [\mu(t'-t)^{\frac{1}{2}}(1-t')^{\frac{1}{2}} L e^A / N]$$

$$\cdot\ \varkappa\left(\frac{\varkappa}{2\pi}\right)^{p/2} \int |z_1| e^{-\varkappa(z_1^2+\dots+z_p^2)/2}\, dz_1 \cdots dz_p$$

$$< \quad \mu L e^A \sqrt{2p/\pi}\,(t'-t)^{\frac{1}{2}}/\theta N^{\frac{1}{2}} q^{\frac{1}{2}} \;. \qquad (15.15)$$

Here we once again use the hypothesis that $t<t'$. Recalling (13.9), we may now replace the entry $x + \left(\frac{1-t'}{1-t}\right)^{\frac{1}{2}} z$ in ι_{n+q} by $x+z$ with a further error satisfying

$$\text{Error}_2 \leq \gamma_n \left(\frac{t'-t}{1-t}\right)^{\frac{1}{2}} \varkappa\left(\frac{\varkappa}{2\pi}\right)^{p/2} \int |z_1||z|\, e^{-\varkappa(z_1^2+\dots+z_p^2)/2}$$

$$dz_1 \cdots dz_p$$

$$\leq \gamma_n \left(\frac{t'-t}{1-t}\right)^{\frac{1}{2}} \varkappa\left(\frac{\varkappa}{2\pi}\right)^{p/2} \int [\, z_1^2 + |z_1|\,|z_2^2+\dots+z_p^2|^{\frac{1}{2}}]$$

$$e^{-\varkappa(z_1^2+\dots+z_p^2)/2} \quad dz_1 \cdots dz_p$$

$$\leq \gamma_n \left(\frac{t'-t}{1-t}\right)^{\frac{1}{2}} \left(1+\sqrt{2p/\pi}\right)$$

$$< 3(Le^{2A}+\mu\lambda)(1+\Gamma)\sqrt{2p/\pi}\,(t'-t)^{\frac{1}{2}}/N \;. \quad (15.16)$$

Here we have dropped a factor $(1-t)^{\frac{1}{2}}$, and noted that $1 < 2\sqrt{2p/\pi}$. Finally, we may replace $\iota_{n+q}(t',z)$ in the altered integral by $\iota_{n+q}(t,z)$ with a final error satisfying

$$\text{Error}_3 \leq \chi_n^{t,t'}(t'-t)^{\frac{1}{2}}\varkappa\left(\frac{\varkappa}{2\pi}\right)^{p/2}\int |z_1|\, e^{-\varkappa(z_1^2+\ldots+z_p^2)/2}$$

$$dz_1 \cdots dz_p$$

$$\leq \sqrt{2\varkappa/\pi}\,(t'-t)^{\frac{1}{2}}\chi_n^{t,t'}$$

$$= \sqrt{2p/\pi}\,N^{\frac{1}{2}}\chi_n^{t,t'}(t'-t)^{\frac{1}{2}}/\theta(1-t)^{\frac{1}{2}}q^{\frac{1}{2}} \;. \quad (15.17)$$

We have thus arrived at the derivative

$\frac{\partial}{\partial x_1}\left[\int \iota_{n+q}(t,x+z)\, d\mathcal{g}^{\varkappa}(z)\right]$ at t, with a total error satisfying

$$\text{Error} \le \text{Error}_1 + \text{Error}_2 + \text{Error}_3$$

$$< \sqrt{2p/\pi}\, [\, \frac{\mu L e^{A}}{\theta N^{\frac{1}{2}} q^{\frac{1}{2}}} + \frac{3(Le^{2A}+\mu\lambda)(1+\Gamma)}{N} + \frac{N^{\frac{1}{2}}}{\theta(1-t)^{\frac{1}{2}} q^{\frac{1}{2}}}\, \chi_n^{t,t'} \,]$$

$$\cdot\ (t'-t)^{\frac{1}{2}}\ . \qquad (15.18)$$

Since the error estimate is the same for $\frac{\partial}{\partial x_2}[\cdot],\ldots,\frac{\partial}{\partial x_p}[\cdot]$, the difference of the gradients does not exceed the right side of (15.18) multiplied by $p^{\frac{1}{2}}$. Adding up the resulting estimates from $q=1$ to $q=N-n$ and taking account of (15.13), we get

$$|\nabla\varphi\lambda_N^{\theta}(t',x,\tau_n') - \nabla\varphi\lambda_N^{\theta}(t,x,\tau_n)|$$

$$< \lambda\theta(t'-t)^{\frac{1}{2}} + p\,\sqrt{2/\pi}\,[2\mu Le^{A}/\theta + 3(Le^{2A}+\mu\lambda)(1+\Gamma)$$

$$+ \frac{N^{\frac{1}{2}}}{\theta(1-t)^{\frac{1}{2}}} \sum_{q=1}^{N-n} q^{-\frac{1}{2}} \chi_n^{t,t'}\,]\ (t'-t)^{\frac{1}{2}}\ . \qquad (15.19)$$

Since the right side of (15.19) is independent of x, we have thus proved that

$$h_n^{t,t'} \le \lambda\theta + p\sqrt{2/\pi}\,[\, 2\mu L e^{A}/\theta + 3(Le^{2A} + \mu\lambda)(1+\Gamma)$$

$$+ \frac{N^{\frac{1}{2}}}{\theta(1-t)^{\frac{1}{2}}} \sum_{q=1}^{N-n} q^{-\frac{1}{2}} \chi_n^{t,t'} \,] \,, \qquad (15.20)$$

$n=0,\ldots,N-1$, analogous to (8.3) but somewhat less obvious.

We may now combine (15.20) with (15.12), $n=1,\ldots,N-1$. The result is:

$$\chi_n^{t,t'} \le H(t)/N + y \sum_{q=1}^{N-n} q^{-\frac{1}{2}} \chi_{n+q}^{t,t'} \,, \qquad (15.21)$$

analogous to (8.6), where

$$H(t) = [\,(\mu+a)Le^{2A} + \mu\lambda\theta + 2\mu^2 p\sqrt{2/\pi}\, L\, e^{A}/\theta$$

$$+ 3\mu p\sqrt{2/\pi}\,(Le^{2A}+\mu\lambda)(1+\Gamma)\,](1-t)^{\frac{1}{2}} \,, \qquad (15.22)$$

and y is given by (8.5). Applying (15.12) with $n=N$ and noting that $h_N^{t,t'} = 0$, we may extend the system (15.21) to $n=N$. The system (15.21) now differs from the system (8.6) only in that Λ is replaced by H(t). Hence we have the analogue to (13.9):

$$\chi_n^{t,t'} \leq \frac{H(t)}{N}\,(1+\Gamma)\;, \tag{15.23}$$

n=1,...,N, H(t) being given by (15.22) and Γ by (13.7). We observe that $\chi_n^{t,t'}$ depends on t but not on $t'\in(t,1)$.

From (15.20) and (15.23) with n=0, we find that

$$h_0^{t,t'} \leq h_{\frac{1}{2}}^{\theta}\;, \tag{15.24}$$

where

$$h_{\frac{1}{2}}^{\theta} = \lambda\theta + p\sqrt{2/\pi}\,[2\mu L e^{A}/\theta + 3(Le^{2A}+\mu\lambda)(1+\Gamma) + 2H(0)(1+\Gamma)/\theta], \tag{15.25}$$

H(0) being the constant in brackets in (15.22). The reader will now understand the importance of the "crucial replacement" of $t'-t$ by $(1-t)^{\frac{1}{2}}(t'-t)^{\frac{1}{2}}$ in (15.8), which we made following (15.9) . Since $h^{\theta}_{\frac{1}{2}}$ does not involve t or t', we have thus proved that

$$|\nabla\Phi\lambda^{\theta}_{N}(t',t) - \nabla\Phi\lambda^{\theta}(t,x)| \leq h^{\theta}_{\frac{1}{2}}|t'-t|^{\frac{1}{2}} \qquad (15.26)$$

for all pairs $t,t' \in [0,1)$ with $t \neq t$, and hence for all pairs $t,t' \in [0,1)$. The reader should now carry out the easy verification that for $t<1$

$$\begin{aligned} &|\nabla\Phi\lambda^{\theta}_{N}(t,x) - \nabla\varphi(x)| \\ &\leq (\lambda\theta + \mu p\sqrt{2/\pi}\, Le^{A}/\theta)(1-t)^{\frac{1}{2}} \\ &< h^{\theta}_{\frac{1}{2}}(1-t)^{\frac{1}{2}}, \end{aligned} \qquad (15.27)$$

so that (15.26) holds for all pairs $t,t' \in [0,1]$. Hence $h^{\theta}_{\frac{1}{2}}$ is a uniform Hölder coefficient in t for $\nabla\Phi\lambda^{\theta}_{N}(t,x)$ relative to $\alpha=\frac{1}{2}$ throughout $[0,1] \times R^{p}$.

In order to understand the situation with the second derivatives, we begin by studying the second partials $\frac{\partial^2 \psi_n(t,x)}{\partial x_i \partial x_j}$, which we denote provisionally by $f(t)$. We now need the bound, sharper than (14.3), which we noted following (14.3):

$$|f(t)| \le K(t) \,, \tag{15.28}$$

where

$$K(t) = p\sqrt{2/\pi}\,(Le^{2A}+u\lambda)(1+\Gamma)(1-t)^{\frac{1}{2}}/\theta n^{\frac{1}{2}} N^{\frac{1}{2}}. \tag{15.29}$$

Now let $0 \le t < t' < 1$. In the following calculation we shall write $\varkappa = \varkappa_t^{(N)}$, $\varkappa' = \varkappa_{t'}^{(N)}$. We suppose at first that $i \ne j$. Then

$$f(t') = \varkappa'^2 \left(\frac{\varkappa'}{2\pi}\right)^{p/2} \int z_i' z_j' \; \iota_n(t', x+z')\, e^{-\varkappa'(z_1'^2 + \dots + z_p'^2)/2}$$

$$dz_1 \cdots dz_p$$

$$= \varkappa^2 \left(\frac{\varkappa}{2\pi}\right)^{p/2} \int z_i z_j \cdot \frac{1-t}{1-t'} \cdot \iota_n\left(t', x + \left(\frac{1-t'}{1-t}\right)^{\frac{1}{2}} z\right)$$

$$e^{-\varkappa(z_1^2 + \dots + z_p^2)/2} \; dz_1 \cdots dz_p \,. \tag{15.30}$$

We may replace the factor $\frac{1-t}{1-t'}$ in the second integral in (15.30) by 1 with an error satisfying

$$\begin{aligned} E_1 &\leq (2\kappa/\pi)(t'-t)\mu/N \\ &< 2\mu p(t'-t)/\theta^2(1-t)n \\ &\leq [2\mu p/\theta^2(1-t)^{\frac{1}{2}}n](t'-t)^{\frac{1}{2}} . \end{aligned} \quad (15.31)$$

Here we have once again made a "crucial replacement" similar to the one we made in (15.8). We may then replace the entry $x + (\frac{1-t'}{1-t})^{\frac{1}{2}}z$ in ι_n by x+z with a further error satisfying

$$\begin{aligned} E_2 &\leq (2\kappa/\pi)(\frac{t'-t}{1-t})^{\frac{1}{2}}\gamma_n \\ &< [2p(Le^{2A}+\mu\lambda)(1+\Gamma)/\theta^2(1-t)^{\frac{1}{2}}n](t'-t)^{\frac{1}{2}} . \end{aligned} \quad (15.32)$$

Here we used the estimate (13.9) for γ_n. Finally we may replace $\iota_n(t',x+z)$ by $\iota_n(t,x+z)$ with a further error satisfying

$$E_3 \leq (2\varkappa/\pi)\chi_n^{t,t'}(t'-t)^{\frac{1}{2}}$$

$$< [2pH(0)(1+\Gamma)/\theta^2(1-t)^{\frac{1}{2}}n](t'-t)^{\frac{1}{2}} . \qquad (15.33)$$

Here we have used the estimate (15.23), valid for $t<t'<1$. In all three estimates we dropped the factor $1/\pi$. We have thus arrived at the formula for $f(t)$, with a total error not exceeding $E_1+E_2+E_3$. Thus we have proved that if $0\leq t\leq t'\leq 1$ then

$$|f(t') - f(t)| \leq M(t)(t'-t)^{\frac{1}{2}} , \qquad (15.34)$$

where

$$M(t) = 2p \frac{\mu + (Le^{2A}+\mu\lambda)(1+\Gamma) + H(0)(1+\Gamma)}{\theta^2(1-t)^{\frac{1}{2}} n} . \qquad (15.35)$$

Now we have a second estimate for $f(t')-f(t)$: since $K(t')\leq K(t)$, then

$$|f(t') - f(t)| \leq 2K(t) . \qquad (15.36)$$

On multiplying (15.34) and (15.36) together and taking the square root, we get

$$|f(t') - f(t)| \leq [2M(t)K(t)]^{\frac{1}{2}}(t'-t)^{\frac{1}{4}} . \qquad (15.37)$$

But $[2M(t)K(t)]^{\frac{1}{2}} = [2M(0)K(0)]^{\frac{1}{2}}$ dies not depend on t or t'. Hence for any pair $t,t' \in [0,1)$ we have

$$\left|\frac{\partial^2\psi_n(t',x)}{\partial x_i \partial x_j} - \frac{\partial^2\psi_n(t,x)}{\partial x_i \partial x_j}\right|$$

$$= |f(t') - f(t)| \leq [2M(0)K(0)]^{\frac{1}{2}}|t'-t|^{\frac{1}{4}} ; \qquad (15.38)$$

$\dfrac{\partial^2\psi_n(t,x)}{\partial x_i \partial x_j}$ is uniformly Hölderian on $[0,1)$ with exponent $\frac{1}{4}$ and coefficient $[2M(0)K(0)]^{\frac{1}{2}}$ given by (15.35) and (15.29).

If i=j the factor $2\varkappa/\pi$ appearing at the outset of each of the estimates (15.21), (15.22), (15.23) is replaced by $2\varkappa$. Since we dropped the factor $1/\pi$ there, these estimates remain valid with some < signs replaced by = or $\leq$. The effect of this trivial difference is entirely wiped out before (15.37) is reached, and (15.38) holds as stated for all pairs i,j.

We now write $[2M(0)K(0)]^{\frac{1}{2}}$ in the form

$$[2M(0)K(0)]^{\frac{1}{2}} = \frac{\Theta}{N^{1/4}n^{3/4}}, \qquad (15.39)$$

where

$$\Theta = \frac{2p(2/\pi)^{\frac{1}{4}}(Le^{2A}+\mu\lambda)^{\frac{1}{2}}(1+\Gamma)^{\frac{1}{2}}[\mu+(Le^{2A}+\mu\lambda)(1+\Gamma)+H(0)(1+\Gamma)]^{\frac{1}{2}}}{\theta^{3/2}}, \qquad (15.40)$$

H(0) being given by (15.22), is an absolute constant. On adding up the inequalities (15.38) we see that the function $\Psi^N_{ij}(t,x)$ defined at (14.7) satisfies the inequality

$$|\Psi^N_{ij}(t',x) - \Psi^N_{ij}(t,x)| \leq 4\Theta|t'-t|^{\frac{1}{4}} \qquad (15.41)$$

for all pairs $t,t'\in[0,1)$.

Now we need to study the behavior of $\Psi^N_{ij}(t,x)$ near t=1. We have first the inequality (15.28) for $f(t) = \dfrac{\partial^2 \psi_n(t,x)}{\partial x_i \partial x_j}$. In addition we have the inequality

$$|f'(t)| \leq 2\kappa\mu(1-t)/N = 2p\mu/\theta^2 n \ , \qquad (15.42)$$

gotten by differentiating in (2.10). On multiplying these inequalities and taking the square root, we get

$$|f(t)| \leq \frac{2p\mu^{\frac{1}{2}}(2/\pi)^{\frac{1}{4}}(Le^{2A}+\mu\lambda)^{\frac{1}{2}}(1+\Gamma)^{\frac{1}{2}}}{\theta^{3/2} n^{3/4} N^{1/4}}$$

$$< \frac{\Theta}{N^{1/4} n^{3/4}} \ , \qquad (15.43)$$

Θ being given by (15.40). Hence

$$|\Psi^N_{ij}(t,x)| < 4\Theta(1-t)^{\frac{1}{4}} \ . \qquad (15.44)$$

Since $\Psi^N_{ij}(1,x) \equiv 0$, this last inequality implies that (15.41) holds for all pairs $t,t' \in [0,1]$ without exception.

As to the other term $\frac{\partial^2}{\partial x_i \partial x_j}[\int \varphi(x+z)\,d\mathcal{g}^{\varkappa}{}_t^{(N)}(z)]$ in the representation based on (2.11) for $\frac{\partial^2}{\partial x_i \partial x_j}[\Phi\lambda_N^\theta(t,x)]$, it is in general not even continuous as $t \to 1$. So no general uniform Hölder condition can be hoped for. The reader should however prove for himself the fact that that term is uniformly Hölderian, with exponent $\alpha=\frac{1}{2}$ and coefficient $\sqrt{2p/\pi}\,(L/\theta + 3\lambda)/(1-\bar{t})$, on the layer $[0,\bar{t}]$, where $\bar{t}<1$.

If however $\frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j}$ exists and is Hölderian in x with exponent α and coefficient h_α^2, one sees trivially that $\frac{\partial^2}{\partial x_i \partial x_j}[\int \varphi(x+z)\,d\mathcal{g}^{\varkappa}{}_t^{(N)}(z)]$ has the uniform Hölder coefficient $h_\alpha^2 \theta^\alpha I_\alpha$ in t relative to the exponent α, where

$$I_\alpha = \left(\frac{p}{2\pi}\right)^{p/2} \int |w|^\alpha e^{-p(w_1^2+\ldots+w_p^2)/2}\, dw_1 \cdots dw_p\ , \qquad (15.45)$$

the expectation of $|w|^\alpha$ relative to the Gaussian spherical distribution of standard deviation 1, depends only on α.

On assembling all the above facts, we have the following theorem.

THEOREM V. Under the hypotheses of the second paragraph of §2, that is under the same hypotheses as in Theorem III, the spatial gradient $\nabla\Phi\lambda_N^\theta(t,x)$ is uniformly Hölderian in t on $[0,1]\times R^p$ with exponent $\alpha=\frac{1}{2}$ and constant $h_{\frac{1}{2}}^\theta$ given by (15.25).

In the representation (14.11) for $\frac{\partial^2\Phi\lambda_N^\theta(t,x)}{\partial x_i \partial x_j}$,
the function $\Psi_{ij}^N(t,x)$ is uniformly Hölderian in t on $[0,1]\times$ with the coefficient 4Θ relative to the exponent $\alpha=\frac{1}{4}$, Θ being given by (15.40). The second term on the right hand side of (14.11) is uniformly
layer $[0,\bar{t}]$, with coefficient $\sqrt{2p/\pi}\,(L/\theta + 3\lambda)/(1-\bar{t})$ and exponent $\alpha = \frac{1}{2}$, provided that $\bar{t}<1$.

If in addition the second partial derivatives $\frac{\partial^2\varphi(x)}{\partial x_i \partial x_j}$ of the terminal function φ are uniformly

Hölderian with coefficient h_α^2 relative to the exponent $\alpha\in(0,1]$, i.e. if the hypotheses of Theorem IV are fulfilled, then the second spatial partial derivatives $\frac{\partial^2\Phi\lambda_N^\theta(t,x)}{\partial x_i\partial x_j}$ are uniformly Hölderian in t on $[0,1]\times R^p$, with coefficient $h_\alpha^2\theta^\alpha I_\alpha + 4\Theta$ and exponent $\beta = \min\{\alpha/2, 1/4\}$, I_α being given by (15.45).

16. Convergence of $\nabla\Phi\lambda_N^\theta(t,x)$ and $\frac{\partial^2\Phi\lambda_N^\theta(t,x)}{\partial x_i\partial x_j}$ to $\nabla\Phi\lambda^\theta(t,x)$ and $\frac{\partial^2\Phi\lambda^\theta(t,x)}{\partial x_i\partial x_j}$. Lipschitz and Hölder coefficients on the latter. Theorems VI and VII

Fix on any $\bar{t}\in[0,1)$, and on any bounded open set G in R^p.

By Theorems III and V, the family $\{\nabla\Phi\lambda_N^\theta(t,x)\}$ is uniformly equicontinuous throughout $[0,1]\times R^p$.

Furthermore, its members are equibounded there by Le^A. Consider any subsequence of $\{\nabla\Phi\lambda_N^\theta(t,x)\}$. A subsequence of that subsequence must converge uniformly on $[0,1]\times G$, and it is a well-known elementary result that the limit must be $\nabla\Phi\lambda^\theta(t,x)$. Hence the whole sequence $\{\nabla\Phi\lambda_N^\theta(t,x)\}$ converges uniformly to $\nabla\Phi\lambda^\theta(t,x)$ on $[0,1]\times G$. In particular it follows that $\nabla\Phi\lambda^\theta(t,x)$ must have on $[0,1]\times G$ the same uniform Lipschitz and Hölder properties as those indicated for the individual $\nabla\Phi\lambda_N^\theta(t,x)$ in Theorems III and V. Since these did not involve G, they must hold throughout $[0,1]\times R^p$.

By Theorems III and V, the family $\{\frac{\partial^2\Phi\lambda_N^\theta(t,x)}{\partial x_i\,\partial x_j}\}$ is uniformly equicontinuous throughout the layer $[0,\bar{t}]\times R^p$. Also, $\{\frac{\partial^2\Phi\lambda_N^\theta(t,x)}{\partial x_i\,\partial x_j}\}$ is obviously equibounded by the Lipschitz constant λ^θ of $\nabla\Phi\lambda_N^\theta(t,x)$ estimated by (14.2). Hence the whole sequence $\{\frac{\partial^2\Phi\lambda_N^\theta(t,x)}{\partial x_i\,\partial x_j}\}$ converges uniformly on $[0,\bar{t}]\times G$ to a function which has to be $\frac{\partial^2\Phi\lambda^\theta(t,x)}{\partial x_i\,\partial x_j}$. That function has the same uniform Hölder coefficients on $[0,\bar{t}]\times G$ as those indicated for the $\frac{\partial^2 \Phi\lambda_N^\theta(t,x)}{\partial x_i \partial x_j}$ in Theorems III and V. Since these do not involve G, they hold on $[0,\bar{t}]^\circ R^p$. Since $\bar{t}<1$ is arbitrary, these derivatives exist, and they satisfy the indicated Hölder conditions, on $[0,1)\times R^p$.

In the statement of Theorem III, we noted that the term $\frac{\partial^2}{\partial x_i \partial x_j}[\int \varphi(x+z)d\mathfrak{g}^{\varkappa}{}_t^{(N)}(z)]$ in the representation (14.11) does not depend on N. It follows that the sequence $\{\Psi_{ij}^N(t,x)\}$ converges on $[0,1)\times R^p$. Since $\Psi_{ij}^N(1,x)\equiv 0$, that sequence converges on $[0,1]\times R^p$. We denote the limit function by $\Psi_{ij}(t,x)$ and note that $\Psi_{ij}(1,x)\equiv 0$. By (15.41), $\Psi_{ij}(t,x)$ is uniformly Hölderian on $[0,1]\times R^p$ with coefficient 4^Θ and exponent $\alpha = \frac{1}{4}$. We have thus proved the following theorem.

THEOREM VI. <u>Suppose the hypotheses of the second paragraph of</u> §2 <u>satisfied. In particular, the terminal function</u> φ <u>and its gradient</u> $\nabla\varphi$ <u>are uniformly Lipschitzian</u>

throughout R^p, but nothing is said about any second derivatives. Then the Fleming randomized value function $\Phi\lambda^\theta(t,x)$, whose existence is asserted by Theorem I, has a gradient $\nabla\Phi\lambda^\theta(t,x)$, defined on $[0,1]\times R^p$, uniformly Lipschitzian in x there with the constant λ^θ given by (14.2), and uniformly Hölderian in t there with the coefficient $h^\theta_{\frac{1}{2}}$ given by (15.25) relative to the exponent $\alpha=\frac{1}{2}$.

The second spatial derivatives of $\Phi\lambda^\theta(t,x)$ may be represented, for $(t,x)\in[0,1)\times R^p$, in the form

$$\frac{\partial^2\Phi\lambda^\theta(t,x)}{\partial x_i\,\partial x_j}$$

$$= \Psi_{ij}(t,x) + \frac{\partial^2}{\partial x_i\,\partial x_j}\left[\int \omega(x+z)\,d\mathfrak{g}^{\varkappa_t}(z)\right], \quad (16.1)$$

$\Psi_{ij}(t,x)$, defined on $[0,1]\times R^p$, being the uniform limit of the functions $\Psi^N_{ij}(t,x)$ defined at (14.7), and with $\varkappa_t = p/\theta^2(1-t)$. The function $\Psi_{ij}(t,x)$ is uniformly Hölderian in x on $[0,1]\times R^p$ relative to any exponent $\alpha\in(0,1)$, with coefficient $2C^\theta_\alpha/(1-\alpha)$, C^θ_α being given by (14.6), and uniformly Hölderian in t on $[0,1]\times R^p$ with coefficient 4Θ relative to the exponent $\alpha=\frac{1}{4}$, Θ being given by (15.40).

The second term in the right side of (16.1) is Hölderian in x for fixed $t<1$ with the constant $h^{\theta}_{\alpha,t}$ given by (14.10) relative to any $\alpha\in(0,1)$, and Hölderian in t on the layer $[0,\bar{t}]$ with $\bar{t}<1$, with the coefficient $\sqrt{2p/\pi}\,(L/\theta + 3\lambda)/(1-\bar{t})$ and exponent $\alpha=\frac{1}{2}$.

Finally, let B be any bounded set in R^p, and $\bar{t}\in[0,1)$. Then the sequence $\{\nabla\Phi\lambda^{\theta}_{N}(t,x)\}$ converges uniformly to $\nabla\Phi\lambda^{\theta}(t,x)$ on $[0,1]\times B$, and the sequence $\{\frac{\partial^2\Phi\lambda^{\theta}_{N}(t,x)}{\partial x_i\,\partial x_j}\}$ converges uniformly to $\frac{\partial^2\Phi\lambda^{\theta}(t,x)}{\partial x_i\,\partial x_j}$ on $[0,\bar{t}]\times B$.

Using the representation (16.1), we deduce trivially the following theorem, analogous to Theorem IV and the last paragraph of Theorem V.

THEOREM VII. *Suppose in addition to the hypotheses of the second paragraph of* §2 *that the terminal function* φ *has second partial derivatives which are uniformly Hölderian, with constant* h_α^2, *for some* $\alpha \in (0,1]$. *Then the second spatial partial derivatives* $\dfrac{\partial^2 \Phi\lambda^\theta(t,x)}{\partial x_i \partial x_j}$ *exist throughout* $[0,1] \times R^p$, *and are uniformly Hölderian there, with constant* $2C_\alpha^\theta/(1-\alpha) + h_\alpha^2$ *in* x *relative to* α , *and constant* $h_\alpha^2 \theta^\alpha I_\alpha + 4\Theta$ *and exponent* $\beta = \min\{\alpha/2, 1/4\}$ *in* t, *where* C_α^θ *is given by* (14.6), Θ *by* (15.40), *and* I_α *by* (15.45) .

17. The parabolic partial differential equation. Theorem VIII

At the end of this section Fleming's parabolic partial differential equation finally makes its appearance. We show (Theorem VIII) that $\Phi\lambda^{\theta}(t,x)$ satisfies it; in §20 we prove this solution unique in a certain class $\mathfrak{K}$.

We fix on any pair $(\bar{t},\bar{x})$ with $\bar{t}\in[0,1)$. We consider $t<\bar{t}$ such that $(\bar{t}-t)/(1-\bar{t})$ is rational; that t might be negative is of no account. If $(\bar{t}-t)/(1-\bar{t}) = P/Q$ with P and Q integers, then we have *)

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*) The reader may notice some superficial similarities with the derivation of the Hamilton-Jacobi equation for the Ω-problem in Value, V, §9.

$$\frac{\bar{t}-t}{P} = \frac{1-\bar{t}}{Q} = \frac{1-t}{P+Q} , \qquad (17.1)$$

so that for any integer $M \geq 1$

$$\frac{\bar{t}-t}{PM} = \frac{1-\bar{t}}{QM} = \frac{1-t}{PM+QM} . \tag{17.2}$$

We consider the (PM+QM)-stage Fleming game, played over the interval $(t,1)$. It has shocks of standard deviation $\theta\left(\frac{1-t}{PM+QM}\right)^{\frac{1}{2}}$. Because of (17.2) *),

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*) This was the original reason that the factor $1-t$ was introduced at (2.2).

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it may therefore be regarded as a PM-stage game, played over $[\bar{t},t)$ and having shocks of standard deviation $\theta\left(\frac{\bar{t}-t}{PM}\right)^{\frac{1}{2}}$, followed by a QM-stage game played over $[\bar{t},1)$ and having shocks of standard deviation $\theta\left(\frac{1-\bar{t}}{QM}\right)^{\frac{1}{2}}$. We therefore fix attention on that PM-stage game, which has the terminal function $\Phi\lambda^{\theta}_{QM}(\bar{t},x)$. We will call it the "Fleming PM-stage game", in order to distinguish it from the "special PM-stage game" defined below.

In the *special* PM-*stage game* (not to be confused with the game in §4), played, along with the Fleming PM-stage game, over the interval $[\bar{t},t))$, there is only one shock Z, of standard deviation $\theta(\bar{t}-t)^{\frac{1}{4}}$. It is applied at time $\bar{t}$, before the shock of the QM-stage game at that time. Thus the special PM-stage game may be assumed to be deterministic, with the terminal function

$$\psi(x) = \int \Phi\lambda^{\theta}_{QM}(\bar{t},x+Z)\,d\mathcal{g}^{\varkappa}t\ (Z)\ , \qquad (17.3)$$

where $\varkappa_t = p/\theta^2(\bar{t}-t)$, $\mathcal{g}^{\varkappa}$ having been defined for general $\varkappa$ at (2.1).

We wish to compare the values of the Fleming and special PM-stage games. In comparing these values we once again (as in §4) use the same underlying sample space, by representing a shock for the special PM-stage game in the form

$$Z = z_0 + \ldots + z_{PM-1} \qquad (17.4)$$

of a sum of shocks from the Fleming PM-stage game.

We say that a sample point $(z_0,\ldots,z_{PM-1})$ for the latter game is <u>good</u> if $|z_0 + \ldots + z_r| \le \theta(\bar{t}-t)^{1/3}$, $r=0,\ldots,PM-1$. Otherwise it is <u>bad</u>. According to the Kolmogorov inequality, the probability that a shock is bad does not exceed $(\bar{t}-t)^{1/3}$. Using this estimate and the $\mathfrak{G}$-construction, we find that the values of the PM-stage Fleming and special games, starting at time t, do not differ by more than

$$(A\theta + 2\mu)L e^{A}(\bar{t}-t)^{4/3} \;; \qquad (17.3)$$

the details can by now safely be left to the reader.

Unitl we reach (17.15), we shall be concerned only with the special PM-stage game with terminal function $\psi(x)$ given by (17.3). Our first objective is to find a suitable formula for $\psi(x)$, which we give at (17.11).

First we write out Taylor's series, to the second power, for $\Phi\lambda^{\theta}_{QM}(\bar{t},x+Z)$, around $\bar{x}$:

$$\Phi\lambda^{\theta}_{QM}(\bar{t},x+Z) = \Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x}) + \nabla\Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x})\cdot(x+Z-\bar{x})$$

$$+ \frac{1}{2}\sum_{ij}\frac{\partial^2\Phi\lambda^{\theta}_{PQ}(\bar{t},x')}{\partial x_i\partial x_j}(x_i+Z_i-\bar{x}_i)(x_j+Z_j-\bar{x}_j) \;, \quad (17.6)$$

where $x'\in[\bar{x},x+Z]$.

For the remainder of this section we fix on an arbitrary $\alpha\in(0,1)$. By Theorem III, $\dfrac{\partial^2\Phi\lambda^{\theta}_{QM}(\bar{t},x)}{\partial x_i\partial x_j}$ is uniformly Hölderian, with exponent α, in x, with coefficient $H^{\theta}_{\alpha,\bar{t}} = 2C^{\theta}_{\alpha}/(1-\alpha) + h^{\theta}_{\alpha,\bar{t}}$ given by (14.8) and (14.10) . Using this and the inequality of convexity $|\xi+\eta|^q \le 2^{q-1}(|\xi|^q+|\eta|^q)$, holding for any pair of vectors $\xi,\eta\in R^p$ and real number $q\ge 1$, we get

$$\Phi\lambda^{\theta}_{QM}(\bar{t},x+z) = \Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x}) + a^{QM}\cdot(x+Z-\bar{x})$$

$$+ \frac{1}{2}\sum c^{QM}_{ij}(x_i+Z_i-\bar{x}_i)(x_j+Z_j-\bar{x}_j)$$

$$+ (p/2)2^{\alpha}H^{\theta}_{\alpha,\bar{t}}(|x-\bar{x}|^{2+\alpha} + |Z|^{2+\alpha})O(1) \;, \quad (17.7)$$

where

$$a^{QM} = \nabla\Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x}) \;, \quad c^{QM}_{ij} = \frac{\partial^2\Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x})}{\partial x_i\partial x_j} \;. \quad (17.8)$$

In the special PM-stage game we are only interested in x reachable from $\bar{x}$ in time $\bar{t}-t$, so that $|x-\bar{x}| \leq \mu(\bar{t}-t)$. We may therefore rewrite the second line in (17.7) as

$$\frac{1}{2} \sum c_{ij}^{QM}(Z_i+x_i-\bar{x}_i)(Z_j+x_j-\bar{x}_j)$$

$$= \frac{1}{2} \sum c_{ij}^{QM} Z_i Z_j + \sum c_{ij}^{QM} Z_i (x_j-\bar{x}_j) + \sum c_{ij}^{QM}(x_i-\bar{x}_i)(x_j-\bar{x}_j)$$

$$= \frac{1}{2} \sum c_{ij}^{QM} Z_i Z_j + \sum c_{ij}^{QM} Z_i (x_j-\bar{x}_j)$$

$$+ (p/2)\lambda^{\theta}\mu^2(\bar{t}-t)^2 O(1) , \qquad (17.9)$$

where we recall that $|c_{ij}^{QM}| \leq \lambda^{\theta}$, λ^{θ} being the uniform Lipschitz constant for $\nabla\Phi\lambda_N^{\theta}(t,x)$ given by (14.2).

As for the third line in (17.7), we first have $|x-\bar{x}|^{2+\alpha} \leq \mu^{2+\alpha}(\bar{t}-t)^{2+\alpha}$. Since we are going to carry out the integration indicated in (17.3), we need a formula for $\int |Z|^{2+\alpha} d\mathfrak{g}^{\varkappa}t(Z)$. For this we put $Z = \theta(\bar{t}-t)^{\frac{1}{2}}\omega$. Then

$$\int |Z|^{2+\alpha}\, d\mathcal{g}^{\varkappa}t\,(z) = \theta^{2+\alpha}(\bar{t}-t)^{1+\alpha/2}\, I_{2+\alpha}\,, \qquad (17.10)$$

where $I_{2+\alpha}$ is the contant depending only on α gotten by replacing α in (15.45) by $2+\alpha$.

We are now ready to carry out the integration indicated in (17.3), using (17.7)-(17.10). In doing so we observe that the middle term in the first expression in (17.9) vanishes, and the terms in the sum $\sum c_{ij}^{QM} Z_i Z_j$ vanish when $i \neq j$. The variance of each Z_i is $\theta^2(\bar{t}-t)/p$. We thus get, for $|x-\bar{x}| \leq \mu(\bar{t}-t)$,

$$\begin{aligned}\psi(x) = \hbar\lambda_{QM}^{\theta}(\bar{t},\bar{x}) &+ a^{QM}\cdot(x-\bar{x}) + \frac{(\bar{t}-t)\theta^2}{2p}\sum c_{ii}^{QM} \\ &+ (p/2)\lambda^{\theta}\mu^2(\bar{t}-t)^2 O(1) \\ &+ (p/2)2^{\alpha}H_{\alpha,\bar{t}}^{\theta}[\mu^{2+\alpha}(\bar{t}-t)^{2+\alpha} \\ &\qquad + \theta^{2+\alpha}(\bar{t}-t)^{1+\alpha/2} I_{2+\alpha}]O(1)\,,\end{aligned}$$

(17.11)

the desired expression for $\psi(x)$. This formula bears some resemblance to Fleming's (2.7) on page 201 of Convergence II. It is here that the Laplacian term $\sum c_{ii}^{QM}$ first makes its appearance, in the integration of $\frac{1}{2}\sum c_{ij}^{QM} z_i z_j$.

Thus, with the error indicated in (17.11), the terminal function $\psi(x)$ for the special PM-stage game may be replaced by

$$\psi^*(x) = C + \alpha^{QM}\cdot(x-\bar{x}) \quad , \tag{17.12}$$

where

$$C = \Phi\lambda_{QM}^{\theta}(\bar{t},\bar{x}) + \frac{(\bar{t}-t)\theta^2}{2p}\sum c_{ii}^{QM} \quad . \tag{17.13}$$

The control function $\mathfrak{f}(x,t,u,v)$ may now be replaced by $\bar{\mathfrak{f}}(u,v) = \mathfrak{f}(\bar{x},\bar{t},u,v)$ with a further error no greater than

$$(A\mu+a)(\bar{t}-t)^2|\alpha^{QM}| \leq (A\mu+a)(\bar{t}-t)^2 Le^A \quad . \tag{17.14}$$

We are now once again dealing with the "simplest linear game" of <u>Value</u>, III, §8 (recall e.g. this paper, at (5.5)) . The value of this game is $C + (\bar{t}-t)v^{QM}$, where

$$v^{QM} = \underset{U\times V}{\text{Value}}\ \alpha^{QM}\cdot \mathfrak{f}(\bar{x},\bar{t},u,v) \ . \tag{17.15}$$

We now return to the Fleming PM-stage game starting at $\bar{x}$ at time t, whose value is $\Phi\lambda^{\theta}_{PM+QM}(t,\bar{x})$. In passing from the Fleming game to the special game we incurred the error (17.5). In passing from the terminal function $\psi(x)$ given by (17.3) to the terminal function $\psi^*(x)$ given by (17.12)-(17.13) we incurred the additional error given in the second and third lines of (17.11). Finally, in passing to the control function $\mathfrak{f}(\bar{x},\bar{t},u,v)$ we incurred the error estimated by (17.14) . Putting all this together, we get

$$\begin{aligned}\Phi\lambda^{\theta}_{PM+QM}(t,\bar{x}) &= \Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x}) + (\bar{t}-t)v^{QM} + \frac{(\bar{t}-t)\theta^2}{2p}\sum c^{QM}_{ii} \\ &+ (A\theta+2\mu)Le^{A}(\bar{t}-t)^{4/3}\,O(1) + (p/2)\lambda^{\theta}\mu^2(\bar{t}-t)^2\,O(1) \\ &+ (p/2)2^{\alpha}H^{\theta}_{\alpha,\bar{t}}\,[\mu^{2+\alpha}(\bar{t}-t)^{2+\alpha} + \theta^{2+\alpha}(\bar{t}-t)^{1+\alpha/2}\,I_{2+\alpha}]\,O(1) \\ &+ (A\mu+a)(\bar{t}-t)^2 Le^{A}\,O(1) \ ,\end{aligned} \tag{17.16}$$

where $I_{2+\alpha}$ is gotten from (15.45) by replacing α by $2+\alpha$. By Theorem I, $\Phi\lambda^{\theta}_{PM+QM}(t,\bar{x}) \to \Phi\lambda^{\theta}(t,\bar{x})$ and $\Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x}) \to \Phi\lambda^{\theta}(\bar{t},\bar{x})$ as $M \to \infty$. By the last sentence of Theorem VI, $a^{QM} = \nabla\Phi\lambda^{\theta}_{QM}(\bar{t},\bar{x}) \to \nabla\Phi\lambda^{\theta}(\bar{t},\bar{x})$ as $M \to \infty$, so that v^{QM} as given by (17.15) converges to $v = v(\bar{t},\bar{x})$, where

$$v(\bar{t},\bar{x}) = \underset{U \times V}{\text{Value}}\ \nabla\Phi\lambda^{\theta}(\bar{t},\bar{x}) \cdot f(\bar{x},\bar{t},u,v). \qquad (17.17)$$

Finally, by the last sentence of Theorem VI, c^{QM}_{ii} as given by (17.8) converges to

$$c_{ii} = \frac{\partial^2 \Phi\lambda^{\theta}(\bar{t},\bar{x})}{\partial x_i^{\,2}} \qquad (17.18)$$

as $M \to \infty$. Equation (17.16) therefore holds with the subscripts PM+QM, QM deleted. Since all its terms (except the $O(1)$ terms, which indicate inequalities) are continuous in t, it holds not only for the special $t<\bar{t}$ for which the fraction $(\bar{t}-t)/(1-\bar{t})$ is rational but for all $t<\bar{t}$. We now transpose $\Phi\lambda^{\theta}(\bar{t},\bar{x})$ in (17.16) to the left, multiply through by -1, divide by $\bar{t}-t$, and pass to the limit as $t \to \bar{t}-$. The result is:

$$\Phi\lambda^{\theta}_{t-}(\bar{t},\bar{x}) = -v - \sum c_{ii} , \qquad (17.19)$$

the subscript t- indicating the left partial derivative , and v and c_{ii} being given by (17.17) and (17.18) respectively. Finally, it is a particular consequence of Theorem VI that the right hand side of (17.19) is continuous in $\bar{t}$ for each fixed $\bar{x}$. Hence so is $\Phi\lambda^{\theta}_{t-}(\bar{t},\bar{x})$. It follows from an elementary argument that the right partial derivative $\Phi\lambda^{\theta}_{t+}(\bar{t},\bar{x})$ exists as well for any $\bar{t}<1$, and equals $\Phi\lambda^{\theta}_{t-}(\bar{t},\bar{x})$. Hence the partial derivative $\Phi\lambda^{\theta}_{t}(\bar{t},\bar{x})$ exists for any $\bar{t}< 1$, and is given by

$$\Phi\lambda^{\theta}_{t}(\bar{t},\bar{x}) = -v - \sum c_{ii} , \qquad (17.20)$$

v and c_{ii} being given by (17.17) and (17.18) respectively. This is Fleming's parabolic equation.

The local Hölder properties of $\Phi\lambda_t^\theta(t,x)$ for $t<1$ may easily be read off from the properties of $\nabla\varphi\lambda^\theta(t,x)$ and $\dfrac{\partial^2\Phi\lambda^\theta(t,x)}{\partial x_i \partial x_j}$ as stated in the general and special cases in Theorems VI and VII. We shall not present these in detail, only noting that in the special case $\Phi\lambda_{t-}^\theta(1,x)$ exists and satisfies (17.20).

We have thus proved the following theorem.

THEOREM VIII (Fleming's parabolic equation). *Under the hypotheses of the second paragraph of* §2, *the value function* $\Phi\lambda^\theta(t,x)$ *has, for any* $t<1$ *and any* $x\in R^p$, *a partial derivative* $\Phi\lambda_t^\theta(t,x)$ *with respect to* t, *which is continuous in* t *and* x. *This derivative is given explicitly by the formula*

$$\Phi\lambda_t^\theta(t,x) = -v(t,x) - (\theta^2/2p)\ \Delta\Phi\lambda^\theta(t,x)\ , \qquad (17.21)$$

where $v(t,x)$ *is given by* (17.17) *and the Laplacian* $\Delta\Phi\lambda^\theta(t,x) = \sum c_{ii}$ *in terms of the* c_{ii} *given by* (17.18).

If in addition the terminal function φ _has second partial derivatives which are uniformly Hölderian with exponent_ $\alpha \in (0,1]$, _the partial derivative_ $\Phi_t(t,x)$ _exists throughout_ $[0,1] \times R^p$ _and is uniformly Hölderian in_ t _with exponent_ $\beta = \text{Min}\{\alpha/2, 1/4\}$ _and in_ x _with exponent_ α.

For the second paragraph of this theorem we need only note that $v(t,x)$ is in any case uniformly Hölderian in t with exponent $\frac{1}{2}$ and uniformly Lipschitzian in x.

Thus we have accomplished the main objective of this paper, to prove, without recourse to existing PDE theory and in fact in a completely elementary manner, that Fleming's value function $\Phi\lambda^{\theta}(t,x)$ exists and satisfies his parabolic PDE. We have done this without any assumption on the derivatives of the terminal function beyond the first.

In the next section we clear up an apparent deviation, noted preceding (2.1), between our definition of Fleming's value and Fleming's own definition of that value.

18. Fleming's, and other, randomizations

The randomization introduced in this paper, at (2.1) and (2.2), is different from the one used in Fleming's original definition of his value in Convergence II. For Fleming (top of page 199 in Convergence II) the i^{th} component z_i of the shock is either (in our notation) $+\theta(1-t)^{\frac{1}{2}}/p^{\frac{1}{2}}N^{\frac{1}{2}}$ or $-\theta(1-t)^{\frac{1}{2}}/p^{\frac{1}{2}}N^{\frac{1}{2}}$, each with probability ½. Thus Fleming's sample space for his shocks has only finitely many points; this was essential for his proof of his Lemma 2.

On the other hand, the Gaussian shocks are essential for our method, in the first place because they yield smooth before position functions, and for other reasons as well.

It is the object of this section to show that both randomizations lead, in the limit, to the same value.

In fact we shall prove a slightly stronger proposition. We shall call a distribution $\mathcal{B}$ of the shocks for the N-stage game at each of the times $\tau_0,\ldots,\tau_{N-1}$ <u>admissible</u> provided it is the product of orthogonally and independently distributed one-dimensional distributions of mean zero and standard deviation $\theta(1-t)^{\frac{1}{2}}/p^{\frac{1}{2}}N^{\frac{1}{2}}$. With this definition we have the following theorem.

THEOREM IX. *Under the hypotheses of the second paragraph of* §2, *it does not matter which admissible distribution of shocks is used in defining the value of the* N-*stage randomized Fleming mixed-strategy game; the value function always converges uniformly as* $N\to\infty$ *on* $[0,1]\times R^p$ *to the function* $\Phi\lambda^\theta(t,x)$ *defined in* §§2-5 *using the Gaussian distribution*.

PROOF. Consider an admissible distribution $\mathcal{B}^*$. We shall denote the corresponding position functions and value function by putting a star in place of θ.

Write

$$\delta_n = \sup_{x\in R^p} |\varphi\lambda_N^*(t,x,\tau_n) - \varphi\lambda_N^\theta(t,x,\tau_n) , \qquad (18.1)$$

$n=0,\ldots,N$. Then $\delta_N=0$. Suppose that $0\le n\le N-2$ and that we have a finite estimate for δ_{n+1} ; we are leaving the estimation of δ_{N-1} until later, at (18.16). By the Principle of the Transmission of Continuity, we have

$$\sup_{x\in R^p} |\Phi\lambda_N^*(t,x,\tau_n) - \Phi\lambda_N^\theta(t,x,\tau_n)| \le \delta_{n+1} , \qquad (18.2)$$

so that for any $x\in R^p$

$$|\varphi\lambda_N^*(t,x,\tau_n) - \varphi\lambda_N^\theta(t,x,\tau_n)|$$

$$\le |\int \Phi\lambda_N^\theta(t,x+z,\tau_n)d\mathcal{B}^*(z) - \int \Phi\lambda_N^\theta(t,x+z,\tau_n)d\mathcal{B}^{\varkappa}t^{(1)}(z)|$$

$$+ \delta_{n+1} . \qquad (18.3)$$

Now $\Phi\lambda_N^\theta(t,x,\tau_n)$ has the representation

$$\Phi\lambda_N^\theta(t,x,z)$$

$$= \sum_{q=2}^{N-n} \int \iota_{n+q}(t,x+z)\,dg^{\varkappa}{}_t^{(q-1)}(z) + \int \varphi(x+z)\,dg^{\varkappa}{}_t^{(N-n-1)}(z)$$

$$+ \iota_{n+1}(t,x) \ , \qquad (18.4)$$

analogous to (8.2) . Using essentially the same arguments as those leading to Theorem III, we see that the first term in this representation has second spatial derivatives having the Hölder coefficient $2C_\alpha^\theta/(1-\alpha)$, C_α^θ being given by (14.6), and that the second term has second spatial derivatives having the Hölder coefficient $[N^{\frac{1}{2}}/(N-n-1)^{\frac{1}{2}}]h_{\alpha,t}^\theta$, where $h_{\alpha,t}^\theta$, given by (14.9), is fixed for t fixed. Thus the sum $\psi(x)$ of the first two terms in (18.4) is C^∞, and the second derivative has Hölder coefficient

$$X_{\alpha,n} = \frac{2C_\alpha^\theta}{1-\alpha} + \left(\frac{N}{N-n-1}\right)^{\frac{1}{2}} h_{\alpha,t}^\theta \ . \qquad (18.5)$$

We may therefore write

$$\psi(x+z) = \psi(x) + \nabla\psi(x)\cdot z + \frac{1}{2}\sum \frac{\partial^2\psi(x)}{\partial x_i\,\partial x_j} z_i z_j$$

$$+ (p/2)X_{\alpha,n}|z|^{2+\alpha}O(1) \ . \qquad (18.6)$$

Hence

$$\int \psi(x+z)\,d\beta^*(z) = \psi(x) + \frac{\theta^2(1-t)}{2p} \sum \frac{\partial^2\psi(x)}{\partial x_i^{\,2}}$$

$$+ [(p/2)X_{\alpha,n}\theta^{2+\alpha}(1-t)^{1+\alpha/2} I^*_{2+\alpha}/N^{1+\alpha/2}]\, O(1) \quad , \tag{18.7}$$

where

$$I^*_{2+\alpha} = \int |w|^{2+\alpha}\, dW^*(w) \; , \tag{18.8}$$

W^* being the distribution of standard deviation 1 and homothetic to β^* gotten by putting $z = \theta(1-t)^{\frac{1}{2}}w/N^{\frac{1}{2}}$, depends only on α. We have the similar formula for $\int \psi(x+z)\,d\beta^{\varkappa}t^{(1)}(z)$, in terms of the $I_{2+\alpha}$ gotten by replacing the α in (15.45) by $2+\alpha$. Hence

$$\left| \int \psi(x+z)\,d\beta^*(z) - \int \psi(x+z)\,d\beta^{\varkappa}t^{(1)}(z) \right|$$

$$\leq (p/2)X_{\alpha,n}(I^*_{2+\alpha} + I_{2+\alpha})/N^{1+\alpha/2} \; . \tag{18.9}$$

The remsining term in the representation (18.4), $\iota_n(t,x)$, is in general not even differentiable. But we have the estimate (13.9) for its Lipschitz constant γ_n. Hence

$$\iota_n(t,x+z) = \iota_n(t,x) + |z|\Lambda(1+\Gamma)\mathcal{O}(1)/N , \qquad (18.10)$$

so that

$$\int \iota_n(t,x+z)\, d\mathcal{g}^*(z) = \iota_n(t,x) + \Lambda(1+\Gamma)\theta(1-t)^{\frac{1}{2}}\mathcal{O}(1)/N , \qquad (18.11)$$

and similarly for $\int \iota_n(t,x+z)\, d\mathcal{g}^{\varkappa}{}_t^{(1)}(z)$; hence

$$\left| \int \iota_n(t,x+z)\, d\mathcal{g}^*(z) - \iota_n(t,x+z)\, d\mathcal{g}^{\varkappa}{}_t^{(1)}(z) \right| \leq 2\Lambda(1+\Gamma)\theta/N^{3/2} . \qquad (18.12)$$

On combining (18.3), (18.9), and (18.12), we get

$$\delta_n \leq \delta_{n+1} + (p/2)\theta^{2+\alpha}(I^*_{2+\alpha} + I_{2+\alpha})X_{\alpha,n}/N^{1+\alpha/2} + 2\Lambda(1+\Gamma)\theta/N^{3/2}, \qquad (18.13)$$

$n=0,\ldots,N-2$. On noting that

$$\frac{1}{N}\sum_{n=0}^{N-2} X_{\alpha,n} < \frac{2C_{\alpha}^{\theta}}{1-\alpha} + 2h_{\alpha,n}^{\theta} \quad , \tag{18.14}$$

we find that

$$\delta_0 \le \delta_{N-1} + p\theta^{2+\alpha}(I_{2+\alpha}^{*} + I_{2+\alpha})\left(\frac{C_{\alpha}^{\theta}}{1-\alpha} + h_{\alpha,t}^{\theta}\right)/N^{\alpha/2}$$

$$+ 2(N-1)\Lambda(1+\Gamma)\theta/N^{3/2} \quad . \tag{18.15}$$

One now easily fills in the missing step at $n=N-1$ and gets

$$\delta_{N-1} \le 2L\theta/N^{\frac{1}{2}} + 2\Lambda(1+\Gamma)\theta/N^{3/2} \quad , \tag{18.16}$$

so that

$$|\Phi\lambda_N^{*}(t,x) - \Phi\lambda_N^{\theta}(t,x)|$$

$$\le p\theta^{2+\alpha}(I_{2+\alpha}^{*} + I_{2+\alpha})\left(\frac{C_{\alpha}^{\theta}}{1-\alpha} + h_{\alpha,t}^{\theta}\right)/N^{\alpha/2} + 2L\theta/N^{\frac{1}{2}}$$

$$+ 2\Lambda(1+\Gamma)\theta/N^{\frac{1}{2}} \quad . \tag{18.17}$$

Now for any sample point $\zeta^* = (z_0^*, \ldots, z_{N-1}^*)$ of the alternate randomization and any succession of controls $u_0, v_0, \ldots, u_{N-1}, v_{N-1}$, we have

$$|x_N - x| \leq \mu(1-t) + |z_0^* + \ldots + z_{N-1}^*| , \tag{18.18}$$

$x=x_0$ being the starting point, so that

$$|\varphi(x_N) - \varphi(x)| \leq \mu(1-t)L + |z_0^* + \ldots + z_{N-1}^*| L . \tag{18.19}$$

Using an ε-construction, one immediately sees that

$$|\Phi\lambda_N^*(t,x) - \varphi(x)| \leq \mu(1-t)L + \theta(1-t)^{\frac{1}{2}}L . \tag{18.20}$$

We have the similar inequality for the Gaussian randomization, so that

$$|\Phi\lambda_N^*(t,x) - \Phi\lambda_N^\theta(t,x)| \leq 2(\mu+\theta)(1-t)^{\frac{1}{2}}L . \tag{18.21}$$

Let $\varepsilon>0$. We choose $\bar{t}\in[0,1)$ so that $2(\mu+\theta)(1-\bar{t})^{\frac{1}{2}} < \varepsilon$. Then we choose N_0 so large that the right side of (18.17), calculated at N_0 and $\overline{t}$, is less than ϵ. Since $h^{\theta}_{\alpha,t} < h^{\theta}_{\alpha,\overline{t}}$ if $t<\overline{t}$, then $|\Phi\lambda^*_N(t,x) - \Phi\lambda^{\theta}_N(t,x)| < \epsilon$ for all $N \geq N_0$ and all $t \in [0,1]$. $\Phi\lambda^*_N(t,x)$ therefore converges uniformly to $\Phi\lambda^{\theta}(t,x)$ as $N \to \infty$ on $[0,1] \times R^p$, and the theorem is proved. □

The proof of this theorem, given the machinery now at our command, was quite simple. However the theorem itself, using the delicate estimate (13.9) for γ_n and the equally delicate Hölder estimates on the second derivatives, is far from trivial. It answers affirmatively the conjecture made by Fleming in Convergence II, top of page 199, where he said "The central limit theorem suggests that the form of the distribution of the [shocks] is unimportant for large [N] ." However the question does not appear to this writer to have anything to do with the central limit theorem.

19. The integral equation. Theorem X

It is possible to find an integral generalization of the parabolic equation, as follows.

THEOREM X. *Suppose that* $\Phi(t,x)$ *is uniformly Lipschitzian in* $[0,1] \times R^p$ *and has bounded generalized spatial partial derivatives* $\frac{\partial^2}{\partial x_i^2}[\Phi(t,x)]$ *such that the equation*

$$\Phi_t(t,x) + \frac{\theta^2}{2p} \Delta\Phi(t,x) + v^*(t,x) = 0 \qquad (19.1)$$

holds almost everywhere, where

$$v^*(t,x) = \operatorname*{Value}_{U \times V} \nabla\Phi(t,x) \cdot \mathfrak{f}(x,t,u,v) \quad . \qquad (19.2)$$

Suppose further that $\Phi(1,x) = \varphi(x)$ *throughout* R^p.

Then for all $(t,x)\in[0,1]\times R^p$ *we have*

$$\Phi(t,x) = \int_{R^p} \varphi(x+z)\, d\mathfrak{g}^{p/\theta^2(1-t)}(z) + \int_t^1 d\tau \int_{R^p} v^*(\tau,x+z)\, d\mathfrak{g}^{p/\theta^2(\tau-t)}(z), \tag{19.3}$$

where $\mathfrak{g}^{\varkappa}$ *was defined at* (2.1).

PROOF. Put $f(t) = \Phi(t,x)$, and

$$f(\tau) = \left(\frac{\varkappa}{2\pi}\right)^{p/2} \int_{R^p} \Phi(\tau,x+z)\, e^{-\varkappa(z_1^2+\dots+z_p^2)/2}\, dz_1 \cdots dz_p \tag{19.4}$$

for $\tau\in(t,1]$, where $\varkappa = p/\theta^2(\tau-t)$. Then $f(\tau)$ is continuous on $[t,1]$ and

$$\dot{f}(\tau) = (\tfrac{\varkappa}{2\pi})^{p/2} \int_{R^p} \Phi_t(\tau,x+z)e^{-\varkappa(z_1^2+\ldots+z_p^2)/2}$$

$$dz_1 \cdots dz_p$$

$$+ (\tfrac{\varkappa}{2\pi})^{p/2}[p/2\theta^2(\tau-t)^2]$$

$$\int_{R^p} (z_1^2+\ldots+z_p^2)\Phi(\tau,x+z)e^{-\varkappa(z_1^2+\ldots+z_p^2)/2}$$

$$dz_1 \cdots dz_p$$

$$- (p/2)(\tfrac{\varkappa}{2\pi})^{p/2}(2\pi/\varkappa)(1/2\pi)[p/\theta^2(\tau-t)^2]$$

$$\int_{R^p} \Phi(\tau,x+z)e^{-\varkappa(z_1^2+\ldots+z_p^2)/2}$$

$$dz_1 \cdots dz_p$$

$$= (\tfrac{\varkappa}{2\pi})^{p/2} \int_{R^p} \Phi_t(\tau,x+z)e^{-\varkappa(z_1^2+\ldots+z_p^2)/2}$$

$$dz_1 \cdots dz_p$$

$$+ (\tfrac{\varkappa}{2\pi})^{p/2}(\theta^2/2p) \int_{R^p} [\varkappa^2(z_1^2+\ldots+z_p^2) - p\varkappa]\Phi(\tau,x+z)$$

$$e^{-\varkappa(z_1^2+\ldots+z_p^2)/2} \quad dz_1 \cdots dz_p$$

$$= \left(\frac{\varkappa}{2\pi}\right)^{p/2} \int_{R^p} \Phi_t(\tau, x+z)\, e^{-\varkappa(z_1^2+\dots+z_p^2)/2}$$

$$dz_1 \cdots dz_p$$

$$+ \left(\frac{\varkappa}{2\pi}\right)^{p/2} (\theta^2/2p)$$

$$\Delta \Big[\int_{R^p} \Phi(\tau, x+z)\, e^{-\varkappa(z_1^2+\dots+z_p^2)/2}$$

$$dz_1 \cdots dz_p \Big]$$

$$= \left(\frac{\varkappa}{2\pi}\right)^{p/2} \int_{R^p} [\Phi_t(\tau, x+z) + (\theta^2/2p)\Delta\Phi(\tau, x+z)]$$

$$e^{-\varkappa(z_1^2+\dots+z_p^2)/2}\, dz_1 \cdots dz_p \;. \qquad (19.5)$$

Since Φ is uniformly Lipschitzian in t and the derivatives $\partial^2\Phi/\partial x_i{}^2$ bounded, it follows that $\dot{f}(\tau)$ is bounded on $(t, 1]$, and hence f uniformly Lipschitzian there. Hence

$$f(1) - f(t) = \int_t^1 \dot{f}(\tau)\, d\tau \;. \qquad (19.6)$$

Formula (19.3) is now an immediate consequence of (19.5) and (19.1). The theorem is proved. □

20. Uniqueness. Theorem XI

We shall say that a function $\Phi(t,x)$ defined on $[0,1]\times R^p$ is in the class $\mathfrak{K}$ provided it is uniformly Lipschitzian in both variables, and satisfies equation (19.3) almost everywhere in $[0,1]\times R^p$.

We have seen from Theorem X that the class $\mathfrak{K}$ includes all uniformly Lipschitzian functions $\Phi(t,x)$, taking on the prescribed boundary values $\Phi(1,x)=\varphi(x)$, and having bounded generalized second partial derivatives $\frac{\partial^2}{\partial x_i^2}[\Phi(t,x)]$ such that the parabolic equation (19.1) holds almost everywhere. In particular, from Theorem VIII, the class $\mathfrak{K}$ includes the Fleming value function $\Phi\lambda^\theta(t,x)$.

THEOREM XI. *Under the hypotheses of the second paragraph of* §2, *the class* $\mathfrak{K}$ *consists of the single element* $\Phi\lambda^\theta(t,x)$.

PROOF. The result is trivial if $t=1$, so we fix on a $t<1$. We need first to estimate the spatial derivatives of the right hand integral in (19.3). If we do this through the Gaussian distributions we will arrive at an improper integral. We therefore seek to estimate the difference quotients of the integral $\int_t^{t_1} d\tau \int_{R^p} v^*(\tau, x+z)\, dg^{p/\theta^2(\tau-t)}(z)$, where $t_1 \in (t,1]$, directly.

Suppose that $\tilde{x}_i \neq x_i$. Put $x = (x_1, \ldots, x_p)$ and $\tilde{x}_i = (x_1, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_p)$, so that x and $\tilde{x}$ differ only in the i^{th} coordinate. We are interested in the difference

$$\int_t^{t_1} d\tau \int_{R^p} [v^*(\tau, \tilde{x}+z) - v^*(\tau, x+z)]\, dg^{p/\theta^2(\tau-t)}(z) . \tag{20.2}$$

v^* having been defined for $\Phi \in \mathfrak{R}$ by (19.2). Now we fix on a $\tau \in (t, t_1]$ and once again write $\varkappa = p/\theta^2(\tau-t)$. We write the inside integral in (20.2) in the form

$$\int_{R^p} [v^*(\tau,\tilde{x}+z) - v^*(\tau,x+z)]\, d\mathfrak{g}^{\varkappa}(z)$$

$$= \left(\frac{\varkappa}{2\pi}\right)^{p/2} \int_{R^{p-1}} e^{-\varkappa(z_1^2+\ldots+z_{i-1}^2 + z_{i+1}^2 + \ldots+z_p^2)/2}$$

$$dz_1 \cdots dz_{i-1}\, dz_{i+1} \cdots dz_p \quad \times$$

$$\times \int_{R^1} v^*(\tau,z_1,\ldots,z_{i-1},y_i,z_{i+1},\ldots,z_p)$$

$$[e^{-\varkappa(y_i-\tilde{x}_i)^2/2} - e^{-\varkappa(y_i-x_i)^2/2}]\, dy_i \;.$$

$$(20.3)$$

For the quantity in square brackets we use the integral form of the law of the mean:

$$e^{-\varkappa(y_i-\tilde{x}_i)^2/2} - e^{-\varkappa(y_i-x_i)^2/2}$$

$$= \varkappa \int_{x_i}^{\tilde{x}_j} (y_i - \xi_i)\, e^{-\varkappa(y_i-\xi_i)^2/2}\, d\xi_i \;. \quad (20.4)$$

Using this we estimate the inside integral in (20.3) as not larger in modulus than

$$\int_{x_i}^{\tilde{x}_i} |d\xi_i| \int_{R^1} \varkappa|y_i-\xi_i|\mu\Lambda^* e^{-\varkappa(y_i-\xi_i)^2/2}\, dy_i$$

$$= 2\mu\Lambda^*|\tilde{x}_i-x_i| \ . \qquad (20.5)$$

Hence the modulus of the overall integral in (20.3) does not exceed

$$\sqrt{2\varkappa/\pi}\ \mu\Lambda^*\ |\tilde{x}_1-x_1| = (\mu\Lambda^*/\theta)\sqrt{2p/\pi}\,(\tau-t)^{-\frac{1}{2}}|\tilde{x}_1-x_1| \ .$$

$$(20.6)$$

It follows that the modulus of the difference (20.2) does not exceed

$$(2\mu\Lambda^*/\theta)\sqrt{2p/\pi}\,(t_1-t)^{\frac{1}{2}}|\tilde{x}_1-x_1| \ . \qquad (20.7)$$

Thus the desired difference quotient does not exceed $(2\mu\Lambda^*/\theta)\sqrt{2p/\pi}\,(t_1-t)^{\frac{1}{2}}$.

Now we consider the other part of the integral in the right hand side of (20.3), with t_1 still on $(t,1]$. We get

$$\frac{\partial}{\partial x_i}\left[\int_{t_1}^{1} d\tau \int_{R^p} v^*(\tau,x+z)\,dg^{p/\theta^2(\tau-t)}(z)\right]$$

$$= \int_{t_1}^{1} \frac{p\,d\tau}{\theta^2(\tau-t)} \int_{R^p} z_i v^*(\tau,x+z)\,dg^{p/\theta^2(\tau-t)}(z)$$

$$= \int_{t}^{1} \frac{p\,d\tau}{\theta^2(\tau-t)} \int_{R^p} z_i v^*(\tau,x+z)\,dg^{p/\theta^2(\tau-t)}(z)$$

$$+ \; (2\mu\Lambda^*/\theta)\sqrt{2p/\pi}\,(t_1-t)^{\frac{1}{2}}\,O(1). \qquad (20.8)$$

On combining this with the estimate (20.7) and taking account of the fact that t_1 is arbitrary on $(t,1]$, we see that the derivative of the whole second integral on the right side of (19.3) exists and that in fact

$$\frac{\partial}{\partial x_i}\left[\int_{t}^{1} d\tau \int_{R^p} v^*(\tau,x+z)\,dg^{p/\theta^2(\tau-t)}(z)\right]$$

$$= \int_{t}^{1} \frac{p\,d\tau}{\theta^2(\tau-t)} \int_{R^p} z_i v^*(\tau,x+z)\,dg^{p/\theta^2(\tau-t)}(z)\,.$$

(20.9)

It follows in particular that the gradient $\nabla\Phi(t,x)$ exists everywhere on $[0,1]\times R^p$.

The above formulas have been written down for an arbitrary element Φ of $\mathfrak{K}$. Now suppose also that $\Phi'\in\mathfrak{K}$, and denote by $v^{*\prime}$ the value function corresponding to Φ' under (19.2). Put

$$\delta(t) = \sup_{x\in R^p} |\nabla\Phi(t,x) - \nabla\Phi'(t,x)| . \tag{20.10}$$

Evidently $\delta(t)\leq 2\Lambda^*$.

Using formulas (19.3) and (20.9), we find immediately that

$$\left|\frac{\partial}{\partial x_i}[\Phi(t,x)] - \frac{\partial}{\partial x_i}[\Phi'(t,x)]\right|$$

$$\leq \int_t^1 \frac{p\delta(\tau)d\tau}{\theta^2(\tau-t)} \int_{R^p} |z_i|\, dg^{p/\theta^2(\tau-t)}(z)$$

$$= \frac{\mu\sqrt{2p/\pi}}{\theta} \int_t^1 \frac{\delta(\tau)}{(\tau-t)^{\frac{1}{2}}}\, d\tau , \tag{20.11}$$

so that

$$\delta(t) \leq \frac{\mu p\sqrt{2/\pi}}{\theta} \int_t^1 \frac{\delta(\tau)}{(\tau-t)^{\frac{1}{2}}}\, d\tau \leq (\mu p/\theta) \int_t^1 \frac{\delta(\tau)}{(\tau-t)^{\frac{1}{2}}}\, d\tau \,. \tag{20.12}$$

Since $\delta(t) \leq 2\Lambda^*$, we find from (20.12) that

$$\delta(t) \leq K_1(1-t)^{\frac{1}{2}} \,, \tag{20.13}$$

where $K_1 = 4\mu p\Lambda^*/\theta$. Suppose that $m \geq 1$ and that we have proved that

$$\delta(t) \leq K_m(1-t)^{m/2} \tag{20.14}$$

for some K_m. On substituting this into (20.12) we get

$$\delta(t) \leq (\mu p/\theta)K_m \int_t^1 \frac{(1-\tau)^{m/2}}{(\tau-t)^{\frac{1}{2}}}\, d\tau = (2\mu p/\theta)K_m \int_0^c (c^2 - s^2)^{m/2}\, ds$$

$$= K_{m+1}(1-t)^{(m+1)/2} \,, \tag{20.15}$$

where

$$K_{m+1} = (2\mu p/\theta)K_m \int_0^{\pi/2} (\cos\omega)^{m+1}\, d\omega \qquad (20.16)$$

and we wrote $c = (1-t)^{\frac{1}{2}}$ so as to recognize the integral. If we now choose m_0 so that $(2\mu p/\theta) \int_0^{\pi/2} (\cos\omega)^{m+1}\, d\omega < 1$ for all $m \geq m_0$, then $K_{m+1} < K_m$ for all $m \geq m_0$. It follows that $\delta(t) \leq K_{m_0} (1-t)^{m/2}$ for all $m \geq m_0$, so that $\delta(t) = 0$ for all $t \in (0,1]$. Then obviously $\delta(0)=0$ as well. Hence $v^*(t,x) = v^{*\prime}(t,x)$ for all pairs t,x, so that from (19.3) $\Phi(t,x) = \Phi'(t,x)$ for all pairs t,x, and the theorem is proved. □

21. Other values. Theorem XII

One may also define N-stage games with shocks of standard deviation $\theta(1-t)^{\frac{1}{2}}/N^{\frac{1}{2}}$ corresponding to the unrandomized games having values $\Phi\lambda_N^+(t,x)$, $F_N^+(t,x)$, $VRD_N^+(t,x)$, $VREKD_N^+(t,x)$ and $EK_N^+(t,x)$ as defined in Value, Chapter I *). We denote these values by

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*) Except that here we have a general starting time $t\in[0,1]$, as we did in §§8-10 of Value, Chapter V.

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the addition of a superscript θ. The existence proof for the limit $\Phi\lambda^{+\theta}(t,x)$ of the sequence $\{\Phi\lambda_N^{+\theta}(t,x)\}$ follows the general lines of §§4,5 , except that we use a piecewise-constant ϵ-construction as for the corresponding problem in the early sections of Chapter IV of Value, and the v in (5.5) is replaced by a Min Max . One proves by a trivial carryover of the arguments of Value, V, §7 that the limits $F^{+\theta}(t,x)$ and $VRD^{+\theta}(t,x)$ exist as well and equal $\Phi\lambda^{+\theta}(t,x)$. In the corresponding parabolic equation the v(t,x) of (17.17) is replaced by

$$\bar{v}(t,x) = \underset{v \in V}{\mathrm{Min}}\ \underset{u \in U}{\mathrm{Max}}\ \nabla\Phi\lambda^{+\theta}(t,x) \cdot f(x,t,u,v) \quad . \qquad (21.1)$$

One then proves as in Value, IV, §11 that the limits $VREKD^{+\theta}(t,x)$ and $EK^{+\theta}(t,x)$ exist and equal $\Phi\lambda^{\theta}(t,x)$. We summarize these facts in the following theorem.

THEOREM XII. *The upper randomized values* $\Phi\lambda^{+\theta}(t,x)$, $F^{+\theta}(t,x)$, *and* $VRD^{+\theta}(t,x)$ *exist and are equal. They satisfy a parabolic equation analogous to* (17.21) *except that* $v(t,x)$ *is replaced by the* $\bar{v}(t,x)$ *of* (21.1). *All the facts stated in Theorems* VI-XI *for* $\Phi\lambda^{\theta}(t,x)$ *carry over to* $\Phi\lambda^{+\theta}(t,x)$.

The upper randomized values $VREKD^{+\theta}(t,x)$ *and* $EK^{+\theta}(t,x)$ *exist and equal* $\Phi\lambda^{\theta}(t,x)$.

There is also a parabolic equation for the Ω-problem. The formulation is rather more clumsy than the others and we leave it to the interested reader.

§22. Fleming's trick and a special class of parabolic equations with Laplacian operator. Theorem XIII

One may combine the methods of this paper with a simplified version of a clever device introduced by Fleming in §4 of [6] to obtain an elementary proof of an existence and uniqueness theorem for a parabolic equation in appearance somewhat more general than (17.21). The equation in question is

$$\Phi_t(t,x) + \Delta\Phi(t,x) + F(t,x,\nabla\Phi(t,x)) = 0, \qquad (22.1)$$

where $F(t,x,\alpha)$ is locally Lipschitzian in t and x with constant $C(1+|\alpha|)$, uniformly Lipschitzian in α with constant ι, and satisfies

$$|F(t,x,0)| \le K \qquad (22.2)$$

for some K and all $(t,x)\in[0,1]\times R^p$. These conditions are rather more restrictive than Fleming's conditions (4.3') in [6]. The terminal conditions are:

$$\Phi(1,x) = \varphi(x) \tag{22.3}$$

for all $x \in R^p$, where φ, as in §2, is Lipschitzian with constant L and has a gradient $\nabla\varphi$ which is Lipschitzian with constant λ.

As in §19, we consider the integral equation

$$\Phi(t,x) = \int_{R^p} \varphi(x+z)\, d\mathcal{g}^{1/2(1-t)}(z) + \int_t^1 d\tau \int_{R^p} F(\tau,x,\nabla\Phi(\tau,x))\, d\mathcal{g}^{1/2(\tau-t)}(z) \; . \tag{22.4}$$

Any uniformly Lipschitzian function Φ on $[0,1] \times R^p$ having bounded generalized second spatial partial derivatives such that (22.1)-(22.3) is satisfied almost everywhere also satisfies (22.4). We may therefore regard (22.4) as the generalized form of (22.1)-(22.3) .

Our version of Fleming's trick is as follows. We play the game in R^{p+1}. As U we take the set of p-vectors $u = (u^1, \dots, u^p) \in R^p$ satisfying $|u| \leq (L+1)e^A$, where $A = 3(p+1)^{\frac{1}{2}}C/2$. As V we take the unit sphere $(v^1)^2 + \dots + (v^p)^2 = 1$. We put

$$f^i(x,t,u,v) = \frac{F(t,x,u)u^i}{1+|u|^2} + (\omega+\iota)v^i \;, \; i=1,\dots,p \;;$$

$$f^{p+1}(x,t,u,v) = \frac{F(t,x,u)}{1+|u|^2} - (\omega+\iota)u\cdot v \;, \tag{22.5}$$

where $\omega = (3/2)\text{Max}\{K, \iota\}$. As the payoff we take the function $\psi(x, x^{p+1}) = \varphi(x) + x^{p+1}$, where $x = (x^1, \dots, x^p) \in R^p$.

Since $(s+s^2)/(1+s^2) < 3/2$ for all real s, we see that f is bounded by $\mu = (p+1)^{\frac{1}{2}} \cdot (3/2)\text{Max}\{K, \iota\} + (p+1)^{\frac{1}{2}}(\omega+\iota) = (p+1)^{\frac{1}{2}}(\omega+\iota)$. Similarly, f is uniformly Lipschitzian in t and x with the constant $A = 3(p+1)^{\frac{1}{2}}C/2$ defined above. The terminal function ψ is uniformly

Lipschitzian in R^{p+1} with constant L+1, and its gradient $(\nabla\varphi(x),1)$ is uniformly Lipschitzian there with constant λ. Here ∇ denotes the gradient operator in R^p. We take $\theta = \sqrt{2p}$. The hypotheses of the second paragraph of §2 concerning the game defined by U, V, f, ψ, and θ are then all satisfied.

We take the game to be the Fleming lower (minorant) randomized game, in which the operation of taking the value at each stage is replaced by a Max-Min operation. Following §21, we denote the value of the N-stage game by $\Phi\lambda_N^{-\theta}(t,x,x^{p+1})$. As we noted in §21, the entire apparatus of this paper carries over to this game. The limiting value function $\Phi\lambda^{-\theta}(t,x,x^{p+1})$ then exists, has the same Lipschitz and Hölder properties as the mixed-strategy value $\Phi\lambda^{\theta}(t,x)$, and satisfies the parabolic partial differential equation (17.21) in which v has been replaced by the $\underline{v}$ gotten by replacing the Min-Max in (21.1) by Max-Min. All we have now to do is to calculate $\underline{v}$ in the case at hand.

Evidently we may decompose $\Phi\lambda^{-\theta}(t,x,x^{p+1})$ into the form

$$\Phi\lambda^{-\theta}(t,x,x^{p+1}) = \phi(t,x) + x^{p+1} , \qquad (22.6)$$

so that

$$\nabla_{R^{p+1}} \Phi\lambda^{-\theta}(t,x,x^{p+1}) = (\nabla\Phi(t,x),1) , \qquad (22.7)$$

the unsubscripted ∇ always denoting the gradient in R^p. If we write $a = \nabla\Phi(t,x)$ we get

$$\underline{v}(t,x,x^{p+1}) = \underset{u \in U}{\mathrm{Max}} \; \underset{v \in V}{\mathrm{Min}} \left[\frac{F(t,x,u)(1+u\cdot a)}{1+|u|^2} - (\omega+\iota)(u-a)\cdot v \right] . \qquad (22.8)$$

For each fixed $u \neq a$, if we choose $v = (u-a)/|u-a|$ we get

$$\frac{F(t,x,u)(1+u\cdot a)}{1+|u|^2} - (\omega+\iota)(u-a)\cdot v$$

$$= F(t,x,u) + \frac{F(t,x,u)u\cdot(a-u)}{1+|u|^2} - (\omega+\iota)|u-a|$$

$$\leq F(t,x,a) + \iota|u-a| + \omega|u-a| - (\omega+\iota)|u-a|$$

$$= F(x,t,a) \ . \tag{22.9}$$

It we choose $u=a$, the quantity in brackets in (22.8) is equal to $F(t,x,a)$ independently of v. Hence

$$\underline{v}(t,x,x^{p+1}) = F(t,x,\nabla\Phi(t,x)) \ , \tag{22.10}$$

so that Φ satisfies (22.1) . Φ therefore satisfies (22.4), and is, by the carryover of Theorem XI, the only uniformly Lipschitzian solution of (22.4). We have thus proved the following theorem.

THEOREM XIII. Suppose F and φ satisfy the conditions stated in the first paragraph of this section. Then the integral equation (22.4), which is the generalized form of the parabolic equation (22.1)-(22.3), has, among the class of all uniformly Lipschitzian Φ defined on $[0,1]\times R^p$, a unique solution. $\nabla\Phi$ is uniformly Hölderian in t with exponent $\frac{1}{2}$, and uniformly Lipschitzian in x. The second term on the right of (22.4) has second spatial derivatives which are uniformly Hölderian in t with exponent $\frac{1}{4}$ and uniformly Hölderian in x relative to any exponent $\alpha \in (0,1)$.

If in addition φ has second partial derivatives satisfying a uniform Hölder condition with exponent $\alpha \in (0,1]$, then Φ_t and the second spatial partial derivatives $\partial^2\Phi/\partial x_i \partial x_j$ are uniformly Hölderian in t with exponent $\beta = \mathrm{Min}\{\alpha/2, 1/4\}$ and uniformly Hölderian in x with exponent α.

The first paragraph in this theorem, with its light conditions on φ, is apparently new. The second paragraph is a very special case of Theorem XIV of [10], which appears to make the stronger assertion that $\beta = \alpha/2$ *) .

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*) Interim note in draft: I have written Professor Oleĭnik questioning her on this point.

If F is not Lipschitzian in t, Φ as given by (22.6) still exists and satisfies (22.1)-(22.3), but $\nabla\Phi$ and the $\partial^2\Phi/\partial x_i \partial x_j$ lose their Hölder properties relative to t, and there is no uniqueness assertion.

Fleming's trick in [6] applied to an F involving not only $\nabla\Phi$ but also Φ. That does not make the trick itself any harder, but it leads to a differential game of a type radically different from that treated in this paper, in which the position function is at each stage multiplied by a function (near unity) depending on the strategies of the players (see formula (3.3) in [6]) . The problem of a direct approach to such games on the lines of the present paper appears to be very difficult.

GLOSSARY

We have tried in this paper to adhere to a uniform notation, consistent with that of Value, and to avoid using special symbols for different things. We have ourselves, considering that this paper has 236 displayed formulas and uses a good fraction of the available alphabets, had difficulty in recalling notations. So we have provided this Glossary.

Gothic alphabet

$\mathfrak{a}$	p-vector, denoting a gradient; used at (17.8) ff.
$\mathfrak{f}$	control function; see second paragraph of §2
$\mathfrak{K}$	class of solutions of integral equation; see §19
$\mathfrak{B}$	global strategy for maximizing player in Fleming's mixed-stragegy game; see just preceding (4.19)
$\mathfrak{D}$	same, for minimizing player

$\mathfrak{s}$ "stick", used at (7.5)-(7.9) and at (15.9)

$\mathfrak{r}$ function whose values are position vectors

$\mathfrak{g}$ $\mathfrak{g}^{\varkappa}$ is spherically symmetric Gaussian distribution in R^p with standard deviation $\sqrt{p/\varkappa}$; defined at (2.1)

<u>Greek alphabet</u>

α Hölder exponent, exclusively

β Hölder exponent, defined first in Theorem VII

γ γ_n is Lipschitz coefficient of $\iota_n(t,x)$; see §7

Γ a constant defined at (13.7); exclusive

Δ the conventional spatial Laplacian in R^p

δ generic; used in estimating deviations

ε used only in referring to the ε-construction; see §4 and Chapter IV of <u>Value</u>

ζ a sample point in the product space of shocks; see §4

η generic; used in estimations

θ $\theta(1-t)^{\frac{1}{2}}/N^{\frac{1}{2}}$ is the standard deviation of one shock in the Fleming game; see the second paragraph of §2

Θ — a constant, defined at (15.40)

ι — ι_n is the fundamental incremental function, defined at (2.9)

$\varkappa$ — parameter in Gaussian distribution; used variously at (2.1) and in §§19,20

λ — Lipschitz constant of gradient $\nabla\varphi$ of terminal function; defined in second paragraph of §2

λ^θ — Lipschitz constant of $\nabla\Phi\lambda^\theta(t,x)$ in x

Λ — defined at (8.4); exclusive

Λ^* — Lipschitz constant used in §§19,20

μ — bound for $|\varphi|$; see second paragraph of §2

ν — defined at (5.2); exclusive

ξ — generic, preceding (17.7); also, ξ_i is a parameter defined at (20.4)

Π — joint distribution of shocks; see following (4.21)

π — 3.1415926535...

τ — time parameter, used throughout

$\Phi\lambda^\theta$ — Fleming's mixed-strategy randomized value function; also after position function; see §2

$\varphi\lambda^\theta$ — before position function; see (2.4) and (2.7)

Φ	a solution of the parabolic equation; see §§19,20
φ	terminal function, exclusively; see §2
χ	generic; sometimes a Hölder coefficient, as at (15.1), sometimes a function, as at (3.1)
X	$X_{\alpha,n}$, at (18.5), is a Hölder coefficient. Here we intend capital greek chi, not latin capital x
ψ	used generically in several places. $\psi_n(t,x)$ is used specifically in the representation (2.10)-(2.11)
Ψ	the functions Ψ_{ij} are defined at (14.7)
ω	generic; there is a special definition of $\omega(t,\theta)$ at (13.1)

Latin alphabet

A	Lipschitz constant of $\mathfrak{f}$ in x
a	Lipschitz constant of $\mathfrak{f}$ in t
B	bounded set in R^p
c	generic; used at (5.6) for a constant, in §17 for partial derivatives, and at (20.15) to denote $(1-t)^{\frac{1}{2}}$
C	generic; the important constant C_α^θ is defined at (14.6)

D — the determinants D_i are defined at (9.5)

d — the coefficients d_i^k of y^k in $(-1)^{i-1}D_i$ are defined in the first paragraph of §10

E — expectation, in §4, and error, in §15

e — $e=[i/2]$, defined following (12.5); to be distinguished from e

e — exponential function (exclusively)

f — generic function, used several times

F — see §22

G — bounded open region in R^p

g — generic function, used several times

h — Hölder coefficient, modified variously by subscripts and superscripts as follows:

$h^\theta_{\alpha,t}$ is defined at (14.10)

$h_n^{t,t'}$ is defined at (15.2)

$h^\theta_{\frac{1}{2}}$ is defined at (15.25)

h^2_α is defined in the statement of Theorem VII

H — $H(t)$ is a function defined at (15.22). $H^\theta_{\alpha,t}$ is defined preceding (17.7)

I — I_α is defined at (15.45) ; $I_{2+\alpha}$ is gotten from I_α by replacing α by $2+\alpha$

i	sometimes row of matrix, as in §9, other times index of coordinate in R^p
j	sometimes column of matrix, as in §9, other times index of coordinate in R^p
k	integer index, introduced in §10
K	generic, always a bound
ℓ	ℓ_n is Lipschitz constant of $\nabla\varphi\lambda^\theta(t,x,\tau_n)$ in x
ι	generic for a Lipschitz constant; see §3 or §14
L	Lipschitz constant of the terminal function φ
m	generic; used in many ways
M	number of groups of stages; see §4. Also: $M(t)$ is a function, defined at (15.35)
n	index of a stage in an N-stage game
N	number of stages in a game
O	$O(1)$ is any scalar or vector with $\|O(1)\|\le 1$
P	generic integer
p	dimension of the playing space
Q	generic integer
q	generic integer, as at (8.6) or (11.1)
r	generic integer

R^p	Euclidean p-dimensional space
s	generic; used in many ways, e.g. a standard deviation in §6
t	starting time of game
U	control space for maximizing player
$\mathcal{u}$	a control for maximizing player
V	control space for minimizing player
$\mathcal{v}$	a control for minimizing player
v	value of a game, used frequently; to be distinguished from $\mathcal{v}$
w	generic; usually variable of integration
x	position vector in R^p
X	$X_{n+1}(x_n, \mathcal{u}_n, \mathcal{v}_n)$ is defined following (2.3). Do not confuse with greek letter Χ, q.v.
y	parameter, defined at (8.5)
z	shock; see §2 ff.
Z	sum of shocks; see in particular (4.1)

<u>Special symbols</u>

∇	gradient operator, in R^p

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