A QUADRATICALLY CONVERGENT PRIMAL-DUAL ALGORITHM WITH GLOBAL CONVERGENCE PROPERTIES FOR SOLVING OPTIMIZATION PROBLEMS WITH EQUALITY CONSTRAINTS
(Revised)

by

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ABSTRACT

This paper presents a globally convergent multiplier method which utilizes an explicit formula for the multiplier. The algorithm solves finite dimensional optimization problems with equality constraints. A unique feature of the algorithm is that it automatically calculates a value for the penalty coefficient, which, under certain assumptions, leads to global convergence.

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1. Introduction

In the 1930's and 1940's (see for example [9]) it was customary to convexify the Lagrangian of equality constrained variational problems by adding a penalty term. Later in 1958, Arrow and Solow [1], in considering gradient methods for solving the problem P, $\min\{f(x)|g(x) = 0\}$ with $f : \mathbb{R}^n \to \mathbb{R}^1$, $g : \mathbb{R}^n \to \mathbb{R}^m$ $(m < n)$, proposed to solve P by finding a saddle point of the convexified Lagrangian, $\bar{\psi}_c(x,y) = f(x) + \langle y,g(x)\rangle + \frac{1}{2}c \|g(x)\|^2$, for c sufficiently large. In 1969, Hestenes [10] and Powell [16] and in 1970, Haarhoff and Buys [8] proposed related methods for solving P, based on the convexified Lagrangian. Within a short period of time, these methods became known as methods of multipliers and generated a great deal of interest [2], [3], [4], [5], [6], [7], [13], [14] in the form of specific refinements and extensions. There are two basic types of multiplier methods: Those that compute estimates to the Lagrange multipliers $\bar{y}$ after each inner iteration (e.g. as described in [10], [16], etc.), and those that estimate the multipliers $y(x)$ continuously, as first described in [5]. The basic deficiency common to all these schemes is that they do not incorporate a method for selecting a finite value of c adequate to ensure convergence of the sequences constructed.

In this paper we present a multiplier method which uses an explicit formula for $y(x)$. Our algorithm appears to be the first in the multiplier methods family, to incorporate an automatic scheme for determining a finite value of c compatible with convergence, and it converges quadratically under mild assumptions. The algorithm is partly based on ideas due to Fletcher [7], which we gratefully acknowledge.

2. Preliminary Results

We shall consider the following minimization problem:

$$\min\{f(x) \mid g(x) = 0\} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^1$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, with $m \leq n$. We shall denote the components of a vector by superscripts, the elements of a sequence by subscripts and we shall make use of the following hypotheses.

Assumption 1: The functions f and g are three times continuously differentiable. ⌑

Assumption 2: For every $x \in \mathbb{R}^n$ the vectors $\nabla g^i(x)$, $i = 1,2,\ldots,m$, are linearly independent. ⌑

We recall that the Lagrangian $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^1$, for problem (1) is defined by

$$\ell(x,y) = f(x) + \langle y, g(x) \rangle \tag{2}$$

where $\langle \cdot,\cdot \rangle$ denotes the Euclidean scalar product.

Given any $x \in \mathbb{R}^n$, it is possible to define $y(x) \in \mathbb{R}^m$ as the unique minimizer of $\|\nabla_x \ell(x,y)\|^2$ over $\mathbb{R}^m$, i.e.,

$$y(x) = \arg\min\{\|\nabla_x \ell(x,y)\|^2 \mid y \in \mathbb{R}^m\} \tag{3}$$

where $\| \ \|$ denotes the Euclidean norm and we use the notation $\nabla_x \ell(x,y) \triangleq [\frac{\partial}{\partial x} \ell(x,y)]^T$. The symbol ∇ will always denote a column vector of first partial derivatives. We shall use a subscript on ∇ only when confusion is possible. We shall make use of the following two properties of the multiplier function $y(\cdot)$

Proposition 1: For every $x \in \mathbb{R}^n$,

$$\frac{\partial g(x)}{\partial x} \nabla_x \ell(x,y(x)) = 0 \tag{4}$$

Proof: It follows immediately from (3) that

$$y(x) = -\left(\frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T\right)^{-1} \frac{\partial g(x)}{\partial x} \nabla f(x) \tag{5}$$

and hence

$$\frac{\partial g(x)}{\partial x} \nabla_x \ell(x,y(x)) = \frac{\partial g(x)}{\partial x}\left(\nabla f(x) + \frac{\partial g(x)}{\partial x}^T y(x)\right) = 0 \tag{6}$$

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Proposition 2: The function y is twice continuously differentiable and its Jacobian matrix is given by

$$\frac{\partial y(x)}{\partial x} = -\left(\frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T\right)^{-1} \left[\frac{\partial g(x)}{\partial x} \frac{\partial^2 \ell(x,y(x))}{\partial x^2} + \sum_{j=1}^{m} e_j \nabla_x \ell(x,y(x))^T \frac{\partial^2 g^j(x)}{\partial x^2}\right] \tag{7}$$

where e_j denotes the jth column of the m×m identity matrix.

Proof: That y is twice continuously differentiable follows directly from Assumption 1 and (5). To obtain formula (7), we make use of a dyadic expansion of (4) as follows.

$$\frac{\partial g(x)}{\partial x} \nabla_x \ell(x,y(x)) = \sum_{j=1}^{m} e_j \frac{\partial g^j(x)}{\partial x} \nabla_x \ell(x,y(x)) = 0 \tag{8}$$

Hence, differentiating (8), we obtain,

$$\sum_{j=1}^{m} e_j\left(\nabla_x \ell(x,y(x))^T \frac{\partial^2 g^j(x)}{\partial x^2} + \frac{\partial g^j(x)}{\partial x}\left[\frac{\partial^2 \ell(x,y(x))}{\partial x^2} + \frac{\partial^2 \ell(x,y(x))}{\partial y \partial x} \frac{\partial y(x)}{\partial x}\right]\right) = 0 \tag{9}$$

Since $\frac{\partial^2 \ell(x,y(x))}{\partial y \partial x} = \frac{\partial g(x)^T}{\partial x}$, (7) follows immediately from (9). ⌑

To obtain a penalty function type method which does not require infinite penalization for the infraction of constraints in (1) to yield a solution of (1), Hestenes [9] and Powell [13] have independently proposed the use of the following, parametrized, augmented Lagrangian $\bar{\psi}_c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^1$, with

$$\bar{\psi}_c(x,y) \triangleq \ell(x,y) + \tfrac{1}{2}c\|g(x)\|^2 \tag{10}$$

where $c \geq 0$ is a parameter. We shall make use of a simple modification of (10), also used by Fletcher [7], viz., $\psi_c : \mathbb{R}^n \to \mathbb{R}^1$, defined for $c \geq 0$, by

$$\psi_c(x) = \ell(x,y(x)) + \tfrac{1}{2}c\|g(x)\|^2 \tag{11}$$

We shall depend on the fact that there is an important relationship between stationary points of ψ_c and of ℓ, as we now show.

Proposition 3 If a pair $(\bar{x},\bar{y})$ is a stationary point for $\ell(\cdot,\cdot)$, then $\bar{y} = y(\bar{x})$ and, for any $c \geq 0$, $\bar{x}$ is a stationary point for $\psi_c(\cdot)$.

Proof: Since $(\bar{x},\bar{y})$ satisfy $\nabla_x \ell(\bar{x},\bar{y}) = 0$, $\nabla_y \ell(\bar{x},\bar{y}) = 0$, it follows from (3) that $\bar{y} = y(\bar{x})$ and from (2) that $g(\bar{x}) = 0$. Hence, for any $c \geq 0$, $\nabla \psi_c(\bar{x}) = \nabla_x \ell(\bar{x},\bar{y}) = 0$. ⌑

The following result shows to what extent is the converse of proposition 3 true.

Proposition 4 For every compact subset S of $\mathbb{R}^n$, there exists a $c_S \geq 0$ such that for all $c \geq c_S$, if $\bar{x} \in S$ is a stationary point of $\psi_c(\cdot)$, then $(\bar{x},y(\bar{x}))$ is a stationary point of $\ell(\cdot,\cdot)$.

is a stationary point of $\ell(\cdot,\cdot)$.

Proof: Let S be a compact subset of $\mathbb{R}^n$ and suppose that for some $c \geq 0$, $\bar{x} \in S$ satisfies $\nabla\psi_c(\bar{x}) = 0$. Hence, making use of (4),

$$0 = \frac{\partial g(\bar{x})}{\partial x} \nabla\psi_c(\bar{x}) = \frac{\partial g(\bar{x})}{\partial x}\left(\nabla_x \ell(\bar{x},y(\bar{x})) + \frac{\partial y(\bar{x})}{\partial x}^T g(\bar{x}) + c\,\frac{\partial g(\bar{x})}{\partial x}^T g(\bar{x})\right)$$

$$= \left[c\,\frac{\partial g(\bar{x})}{\partial x}\,\frac{\partial g(\bar{x})}{\partial x}^T + \frac{\partial g(\bar{x})}{\partial x}\,\frac{\partial y(\bar{x})}{\partial x}^T\right] g(\bar{x}) \qquad (12)$$

Consequently, we must have

$$\left[\, c\,I + \left(\frac{\partial g(\bar{x})}{\partial x}\,\frac{\partial g(\bar{x})}{\partial x}^T\right)^{-1} \frac{\partial g(\bar{x})}{\partial x}\,\frac{\partial y(\bar{x})}{\partial x}^T\right] g(\bar{x}) = 0 \qquad (13)$$

Since S is compact and all the matrices in (13) and continuous, there exists a $c_S \geq 0$ such that for all $c \geq c_S$ and any $\bar{x} \in S$, the matrix in (13) multiplying $g(\bar{x})$ is nonsingular. Hence, if $c \geq c_S$, $g(\bar{x}) = 0$ and therefore, $g(\bar{x}) = \nabla_y \ell(\bar{x},y(\bar{x})) = 0$ and $\nabla_x \ell(\bar{x},y(\bar{x})) = \nabla\psi_c(\bar{x}) = 0$, which completes our proof. ⌑

Since the algorithm in the next section is designed to find local minima of ψ_c, for adequately large values of c, it is interesting to establish the relationship between the local minima of ψ_c and f.

Corollary 1: For every compact subset S of $\mathbb{R}^n$ there exists a $c_S \geq 0$ such that for all $c \geq c_S$, if $\bar{x} \in S$ is a local minimum of $\psi_c(\cdot)$, then $\bar{x}$ is a local minimum of $f(\cdot)$ on $\Omega = \{x | g(x) = 0\}$.

Proof: By proposition 4, there exists a $c_S \geq 0$ such that if $c \geq c_S$ and $\bar{x}$ is a local minimum of $\psi_c(\cdot)$, then $g(\bar{x}) = 0$. But $\psi_c(x) = f(x)$ for all $x \in \Omega$, and hence the corollary follows. ⌑

3. The Algorithm: Convergence

The algorithm which we are about to describe is related to the Gauss-Newton method and makes use of an approximate Hessian of $\psi_c(\cdot)$, defined as follows. For any $c \geq 0$ and $x \in \mathbb{R}^n$, let $H_c(x)$ be an $n \times n$ matrix defined by

$$H_c(x) = \frac{\partial^2 \ell(x,y(x))}{\partial x^2} + \frac{\partial g(x)^T}{\partial x} \frac{\partial y(x)}{\partial x} + \frac{\partial y(x)^T}{\partial x} \frac{\partial g(x)}{\partial x} + c \frac{\partial g(x)^T}{\partial x} \frac{\partial g(x)}{\partial x} \tag{14}$$

<u>Proposition 5</u>: For all $\bar{x}$ satisfying $g(\bar{x}) = 0$ and any $c \geq 0$,

$$H_c(\bar{x}) = \frac{\partial^2 \psi_c(\bar{x})}{\partial x^2} \tag{15}$$

<u>Proof</u>: Eq. (15) follows directly by calculation from the fact that for any $c \geq 0$ and $x \in \mathbb{R}^n$,

$$\frac{\partial^2 \psi_c(x)}{\partial x^2} = H_c(x) + \sum_{j=1}^{m} g^j(x) \left(\frac{\partial^2 y^j(x)}{\partial x^2} + c \frac{\partial^2 g^j(x)}{\partial x^2} \right) \quad \square \tag{16}$$

The algorithm below makes use of a preselected monotonically increasing sequence $\{c_j\}_{j=0}^{\infty}$, with $c_{j+1} > c_j \geq 0$, $j = 0,1,2,\ldots$, and $c_j \to \infty$ as $j \to \infty$. For example, one could use sequences defined by $c_j = c_0 + j\rho$, $j = 0,1,2,\ldots$, $\rho > 0$, or $c_j = c_0 \nu^j$, $j = 0,1,2,\ldots$, with $\nu > 1$ and $c_0 > 0$.

<u>Algorithm</u>: Parameters: $\alpha \in (0,\frac{1}{2})$, $\beta \in (0,1)$, $0 < \varepsilon_0 << 1$, $0 < \varepsilon_1 << 1$, $\gamma \geq 1$, $\{c_j\}_{j=0}^{\infty}$. Initial guess: z_0.

<u>Step 0</u>: Set $j = 0$.

Step 1: Set $i = 0$, and set $x_0 = z_j$.

Step 2: If $\nabla\psi_{c_j}(x_i) \neq 0$, go to step 4; else go to step 3.

Step 3: If $g(x_i) = 0$, stop; else go to step 10.

Step 4: If $\left\langle \frac{\partial g(x_i)}{\partial x}^T \left(\frac{\partial g(x_i)}{\partial x} \frac{\partial g(x_i)}{\partial x}^T \right)^{-1} g(x_i), \nabla\psi_{c_j}(x_i) \right\rangle \geq$ $\|g(x_i)\|^2$, go to step 5; else go to step 10.

Comment: The test in step 4, roughly, is on the angle between $\nabla\psi_{c_j}(x_i)$ and the Newton direction $v(x_i) \triangleq \frac{\partial g(x_i)}{\partial x}^T \left(\frac{\partial g(x_i)}{\partial x} \frac{\partial g(x_i)}{\partial x}^T \right)^{-1} g(x_i)$, for solving $g(x) = 0$, defined by $v(x_i) = \arg\min\{\|v\|^2 \,|\, g(x_i) + \frac{\partial g(x_i)}{\partial x} v = 0\}$.

Step 5: If $|\det H_{c_j}(x_i)| \geq \varepsilon_0$, go to step 6; else go to step 7.

Step 6: If

$$\langle \nabla\psi_{c_j}(x_i), H_{c_j}(x_i)^{-1} \nabla\psi_{c_j}(x_i) \rangle \geq$$

$$\min\{\varepsilon_1, \|\nabla\psi_{c_j}(x_i)\|^\gamma\} \|\nabla\psi_{c_j}(x_i)\| \, \|H_{c_j}(x_i)^{-1} \nabla\psi_{c_j}(x_i)\|, \tag{17}$$

set $h(x_i) = -H_{c_j}(x_i)^{-1} \nabla\psi_{c_j}(x_i)$ and go to step 8; else go to step 7.

Step 7: Set $h(x_i) = -\nabla\psi_{c_j}(x_i)$.

Step 8: Compute the smallest nonnegative integer $\ell_i \geq 0$ such that

$$\psi_{c_j}(x_i + \beta^{\ell_i} h(x_i)) - \psi_{c_j}(x_i) \leq \beta^{\ell_i} \alpha \langle \nabla\psi_{c_j}(x_i), h(x_i) \rangle \tag{18}$$

Step 9: Set $x_{i+1} = x_i + \beta^{\ell_i} h(x_i)$, set $i = i+1$, and go to step 2.

Step 10: Set $z_{j+1} = x_i$, set $j = j + 1$ and go to step 1. ⌑

Lemma 1: Suppose that x_i is such that $\nabla\psi_{c_j}(x_i) \neq 0$, and $h(x_i)$ is as constructed by the algorithm, in step 6 or in step 7, then there exists an $\ell_i < \infty$ such that (18) holds.

Proof: Note that

$$\langle \nabla\psi_{c_j}(x_i), h(x_i) \rangle \leq \delta_j(x_i)\|\nabla\psi_{c_j}(x_i)\| \, \|h(x_i)\| < 0 \tag{19}$$

where

$$\delta_j(x_i) \triangleq \max\{-1, -\min\{\varepsilon_1, \|\nabla\psi_{c_j}(x_i)\|^{\gamma}\}\} \tag{20}$$

The lemma now follows from the Mean Value Theorem. ⌑

Lemma 2: Suppose that the algorithm has constructed an infinite sequence $\{x_i\}_{i=0}^{\infty}$ (i.e., step 10 is reached a finite number of times only), then any accumulation point x^* of $\{x_i\}_{i=0}^{\infty}$ satisfies $\nabla\psi_{c_{j^*}}(x^*) = 0$, and $g(x^*) = 0$, where j^* is the last value of j.

Proof: Suppose that x^* is an accumulation point of $\{x_i\}_{i=0}^{\infty}$. Then, because of the test in step 5, given any $\varepsilon^* > 0$ there exists an $M^* \in (0,\infty)$ such that $\|h(x_i)\| \leq M^* \|\nabla\psi_{c_{j^*}}(x_i)\|$ for all $\|x_i - x^*\| \leq \varepsilon^*$. Making use of this fact, of (19) and some of the results in section 2.1 of [12], we conclude that $\nabla\psi_{c_{j^*}}(x^*) = 0$ must hold.

Next, denoting by $K \subset \{0,1,2,\ldots\}$ the indices of the convergent subsequence, since j never grows beyond j^*, we must have, according to step 4, for $i=0,1,2,\ldots$

$$\|g(x_i)\| \leq \left\langle \frac{\partial g(x_i)}{\partial x}^T \left(\frac{\partial g(x_i)}{\partial x} \frac{\partial g(x_i)}{\partial x}^T \right)^{-1} g(x_i), \nabla\psi_{c_{j^*}}(x_i) \right\rangle \tag{21}$$

But $\nabla\psi_{c_{j^*}}(x_i) \to 0$ as $i \to \infty$, $i \in K$, and $g(\cdot)$, $\frac{\partial g}{\partial x}(\cdot)$ are continuous, and $\frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T$ is nonsingular for all x. Hence we must have $g(x_i) \to 0$ as $i \to \infty$, $i \in K$ and hence, $g(x^*) = 0$. ⌑

Lemma 3: Suppose that the algorithm has constructed an infinite sequence of points $\{z_j\}_{j=0}^{\infty}$. Then $\{z_j\}_{j=0}^{\infty}$ has no accumulation points.

Proof: Suppose that the algorithm has constructed an infinite sequence $\{z_j\}_{j=0}^{\infty}$ which has an accumulation point z^*. We shall show that this leads to a contradiction. Let $K \subset \{0,1,2,\ldots\}$ be such that $z_j \to z^*$ as $j \to \infty$, $j \in K$. Obviously, the set $S = \{z_j\}_{j \in K}$ is compact. Now consider the test in step 4, whose failure results in the construction of a new z_j. Let $\theta : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$ be defined by

$$\theta(c,x) = \langle \frac{\partial g(x)}{\partial x}^T \left(\frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T\right)^{-1} g(x), \nabla\psi_c(x) \rangle - \|g(x)\|^2 \tag{22}$$

Then, making use of the right hand side of (12), we find that

$$\theta(c,x) = \langle \left(\frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T\right)^{-1} g(x), \frac{\partial g(x)}{\partial x} \nabla\psi_c(x) \rangle - \|g(x)\|^2$$

$$= \langle \left(\frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T\right)^{-1} g(x), \left(c \frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T + \frac{\partial g(x)}{\partial x} \frac{\partial y(x)}{\partial x}^T\right) g(x) \rangle - \|g(x)\|^2$$

$$= \langle g(x), \left[(c-1)I + \left(\frac{\partial g(x)}{\partial x} \frac{\partial g(x)}{\partial x}^T\right)^{-1} \frac{\partial g(x)}{\partial x} \frac{\partial y(x)}{\partial x}^T\right] g(x) \rangle \tag{23}$$

Since S is compact and θ is continuous, it now follows that there exists a $c^* \geq 0$, such that $\theta(c,x) \geq 0$ for all $c \geq c^*$ and for all $x \in S$. Let j^* be such that $c_j \geq c^*$ for all $j \geq j^* - 1$, then the algorithm could not have constructed any point z_j, j with $j \geq j^*$, $j \in K$, on account of a failure to satisfy the test in step 4. Hence, it must have constructed the points z_j, $j \in K$, $j \geq j^*$ because of the transfer in step 3, i.e., we must have $\nabla\psi_{c_{j-1}}(z_j) = 0$, $g(z_j) \neq 0$ for all $j \in K$, $j \geq j^*$. But according to proposition 4, there exists a c_S such that if $\nabla\psi_c(z) = 0$ then $g(z) = 0$ for all $z \in S$, for all $c \geq c_S$. Let $j^{**} \geq j^*$ be such that $c_j \geq c_S$ for all $j \geq j^{**} - 1$, then we see that z_j, $j \in K$, $j \geq j^{**}$ could not have been constructed. Thus we have a contradiction. ⌑

We can now collect our results and state them as a theorem.

<u>Theorem 1</u>: (a) If the algorithm constructs a finite sequence $\{x_i\}_{i=0}^{\nu}$ and stops, then $\nabla_x \ell(x_\nu, y(x_\nu)) = 0$ and $g(x_\nu) = 0$. (b) If the algorithm constructs an infinite sequence $\{x_i\}_{i=0}^{\infty}$, then any accumulation point x^* of $\{x_i\}_{i=0}^{\infty}$ satisfies $\nabla_x \ell(x^*, y(x^*)) = 0$, $g(x^*) = 0$. (c) If the algorithm constructs an infinite sequence $\{z_i\}_{j=0}^{\infty}$ then this sequence has no accumulation points. ⌑

4. Rate of Convergence of the Algorithm

We shall now investigate the rate of convergence of the algorithm described in Section 3, when it converges to a local nonsingular minimum point of the problem (1). We recall that a point $\bar{x}$ is a local nonsingular minimum point of (1) if (i) $g(\bar{x}) = 0$, (ii) there exists a multiplier $\bar{y} \in \mathbb{R}^m$ such that

$$\nabla_x \ell(\bar{x},\bar{y}) = 0 \tag{24}$$

and

$$\langle x, \frac{\partial^2 \ell(\bar{x},\bar{y})}{\partial x^2} x \rangle > 0 \text{ for all } x \in \{x \mid \frac{\partial g(\bar{x})}{\partial x} x = 0,\ x \neq 0\} \tag{25}$$

Proposition 6: If $\bar{x}$ is a nonsingular local minimum point of (1), then there exists a $\bar{c} > 0$ such that for all $c \geq \bar{c}$, $\bar{x}$ is a strong local minimum point of $\psi_c(\cdot)$, i.e. $\nabla\psi_c(\bar{x}) = 0$, $\dfrac{\partial^2\psi_c(\bar{x})}{\partial x^2} > 0$.

Proof: By propositions 3 and 5, for any $c \geq 0$, $\Delta\psi_c(\bar{x}) = 0$ and $\dfrac{\partial^2\psi_c(\bar{x})}{\partial x^2} = H_c(\bar{x})$. From (14), summing the first three matrices and calling the result $A(\bar{x})$, we have

$$H_c(\bar{x}) = A(\bar{x}) + c\,\frac{\partial g(\bar{x})}{\partial x}^T \frac{\partial g(\bar{x})}{\partial x} \tag{26}$$

Now, because of (25), $\langle x, A(\bar{x})\, x \rangle > 0$ for all $x \neq 0$ such that $\dfrac{\partial g(\bar{x})}{\partial x}\, x = 0$ and hence there exists a $\bar{c} > 0$ such that $H_c(\bar{x})$, and consequently $\dfrac{\partial^2\psi_c(\bar{x})}{\partial x^2}$, is positive definite for all $c \geq \bar{c}$. ¤

The converse of this result follows trivially from (26) and proposition 4.

Proposition 7: For every compact subset S of $\mathbb{R}^n$ there exists a $\bar{c} > 0$ such that for any $c \geq \bar{c}$, if $\bar{x} \in S$ is a strong local minimum of $\psi_c(\cdot)$, then $\bar{x}$ is a nonsingular local minimum of (1). ¤

Corollary 2: Suppose that the algorithm has constructed a sequence $\{x_i\}_{i=0}^{\infty}$ which converges to x^* a strong local minimum of $\psi_{c_{j^*}}(\cdot)$, where j^* is the last value of j. Then x^* is a nonsingular local minimum of (1).

Proof: By theorem 1, $g(x^*) = 0$, i.e., x^* is feasible and $\nabla_x \ell(x^*, y(x^*)) = 0$. Since $H_{c_{j^*}}(x^*)$ is positive definite, we must have $\dfrac{\partial^2 \ell(x^*, y(x^*))}{\partial x^2}$ positive definite on the linear manifold $\{x \,|\, \dfrac{\partial g(x^*)}{\partial x}\, x = 0\}$, hence we are done. ¤

Lemma 4: Suppose that the algorithm has constructed a sequence $\{x_i\}_{i=0}^{\infty}$ which converges to x^*, a strong local minimum of $\psi_{c_{j^*}}(\cdot)$ and suppose that

$\varepsilon_0 < \frac{1}{2} |\det H_{c_{j*}}(x^*)|$, then there exists an i^* such that for all $i \geq i^*$, $h(x_i) = - H_{c_{j*}}(x_i)^{-1}\nabla\psi_{c_{j*}}(x_i)$ and $\ell_i = 0$.

Proof: Since $H_{c_{j*}}(x^*) > 0$ by assumption, and $H_{c_{j*}}(\cdot)$ is continuous, there exists a closed ball $B(x^*)$ with center at x^* such that $H_{c_{j*}}(x) > 0$ and $H_{c_{j*}}(\cdot)$ is continuous, for all $x \in B(x^*)$. Since $x_i \to x^*$, there exists an i_1 such that for all $i \geq i_1$, $x_i \in B(x^*)$ and $|\det H_{c_{j*}}(x_i)| \geq \varepsilon_0$. Next, because $H_{c_{j*}}(x)^{-1} > 0$ for all $x \in B(x^*)$ and $B(x^*)$ is compact and because $\nabla\psi_{c_{j*}}(x_i) \to 0$, there exists an $m \in (0,\infty)$ and an $i_2 \geq i_1$, such that for all $i \geq i_2$, (since $\gamma \geqq 1$)

$$\begin{aligned}
&\langle \nabla\psi_{c_{j*}}(x_i), H_{c_{j*}}(x_i)^{-1} \nabla\psi_{c_{j*}}(x_i)\rangle - \\
&\qquad \min\{\varepsilon_1, \|\nabla\psi_{c_{j*}}(x_i)\|^{\gamma}\} \|\nabla\psi_{c_{j*}}(x_i)\| \|H_{c_{j*}}(x_i)^{-1}\nabla\psi_{c_{j*}}(x_i)\| \\
&\qquad \geqq \|\nabla\psi_{c_{j*}}(x_i)\|^2 (m - \|\nabla\psi_{c_{j*}}(x_i)\|^{\gamma-1}\|H_{c_{j*}}(x_i)^{-1}\nabla\psi_{c_{j*}}(x_i)\|) \\
&\qquad \geqq 0
\end{aligned} \tag{27}$$

Hence, for all $i \geq i_2$, the test (17) in step 6 is satisfied and the algorithm sets $h(x_i) = - H_{c_{j*}}(x_i)^{-1}\nabla\psi_{c_{j*}}(x_i)$.

Next, we consider the test (18) in step 8 for $i \geq i_2$. For $x \in B(x^*)$ let

$$\phi(x) = \psi_{c_{j*}}(x+h(x)) - \psi_{c_{j*}}(x) - \alpha \langle \nabla\psi_{c_{j*}}(x), h(x)\rangle \tag{28}$$

where $\alpha \in (0,\frac{1}{2})$ is as in the algorithm. Then making use of the second order Taylor formula, for all $i \geq i_2$, we obtain

$$\phi(x_i) = (1-\alpha)\langle \nabla\psi_{c_{j*}}(x_i), h(x_i)\rangle + \int_0^1 (1-t)\langle h(x_i), \frac{\partial^2\psi_{c_{j*}}(x_i+th(x_i))}{\partial x^2} h(x_i)\rangle dt$$

$$= (\tfrac{1}{2}-\alpha)\ \langle \nabla\psi_{c_{j*}}(x_i),\ h(x_i)\rangle$$

$$+ \int_0^1 (1-t)\ \langle h(x_i), \left[\frac{\partial^2\psi_{c_{j*}}(x_i+th(x_i))}{\partial x^2} - H_{c_{j*}}(x_i)\right] h(x_i)\rangle\, dt \qquad (29)$$

But $H_{c_{j*}}(\cdot)^{-1}$ is continuous on $B(x^*)$ and hence there exists an $M \in [m,\infty)$ such that

$$\phi(x_i) \le \left[-(\tfrac{1}{2}-\alpha)m + \tfrac{1}{2}M^2 \sup_{t\in[0,1]} \left\|\frac{\partial^2\psi_{c_{j*}}(x_i+th(x_i))}{\partial x^2} - H_{c_{j*}}(x_i)\right\|\right] \|\nabla\psi_{c_{j*}}(x_i)\|^2 \qquad (30)$$

Since $\sup_{t\in[0,1]} \left\|\frac{\partial^2\psi_{c_{j*}}(x_i+th(x_i))}{\partial x^2} - H_{c_{j*}}(x_i)\right\| \to 0$ as $i \to \infty$, there exists an $i^* \ge i_2$ such that $\phi(x_i) \le 0$ for all $i \ge i^*$, i.e., the test (18) is passed with $\ell_i = 0$ for all $i \ge i^*$. ⌑

<u>Theorem 2</u>: Suppose that the algorithm has constructed a sequence $\{x_i\}_{i=0}^{\infty}$ which converges to x^*, a strong local minimum of $\psi_{c_{j*}}(\cdot)$, and that $\varepsilon_0 < \frac{1}{2}|\det H_{c_{j*}}(x^*)|$. Then

$$\frac{\|x_{i+1} - x^*\|}{\|x_i - x^*\|} \to 0 \qquad \text{as } i \to \infty \qquad (31)$$

Furthermore, if the functions f and g are three times Lipschitz continuously differentiable on a convex neighborhood of x^* then there exist an i^* and a $\nu \in (0,\infty)$ such that

$$\|x_{i+1} - x^*\| \le \nu\, \|x_i - x^*\|^2 \text{ for all } i \ge i^*. \qquad (32)$$

Proof: Let i^* be as in Lemma 4. Then for all $i \geq i^*$, $x_{i+1} = x_i - H_{c_{j^*}}(x_i)^{-1} \nabla\psi_{c_j}(x_i)$, and hence,

$$\|x_{i+1} - x^*\| = \|(x_i - x^*) - H_{c_{j^*}}(x_i)^{-1}(\nabla\psi_{c_{j^*}}(x_i) - \nabla\psi_{c_{j^*}}(x^*))\|$$

$$\leq \|H_{c_{j^*}}(x_i)^{-1}\| \int_0^1 \|H_{c_{j^*}}(x_i) - \frac{\partial^2\psi_{c_{j^*}}(x^*+t(x_i-x^*))}{\partial x^2}\| dt \; \|x_i - x^*\| \tag{33}$$

But for all $i \geq i^*$, $\|H_{c_{j^*}}(x_i)^{-1}\| \leq M < \infty$, where M is as in lemma 5, and hence

$$\frac{\|x_{i+1} - x^*\|}{\|x_i - x^*\|} \leq M \sup_{t\in[0,1]} \|H_{c_{j^*}}(x_i) - \frac{\partial^2\psi_{c_{j^*}}(x^*+t(x_i-x^*))}{\partial x^2}\| \tag{34}$$

Since $\sup_{t\in[0,1]} \|H_{c_{j^*}}(x_i) - \frac{\partial^2\psi_{c_j}(x^*-t(x_i-x^*))}{\partial x^2}\| \to 0$ as $i \to \infty$, (31) follows from (34).

Next, suppose that f and g are three times Lipschitz continuously differentiable on a convex neighborhood B of x^*. Without loss of generality, we may assume that $x_i \in B$ for all $i \geq i^*$. Hence, (34) yields

$$\frac{\|x_{i+1} - x^*\|}{\|x_i - x^*\|} \leq M \sup_{t\in[0,1]} \{\|H_{c_{j^*}}(x_i) - H_{c_{j^*}}(x^*)\| + \|\frac{\partial^2\psi_{c_{j^*}}(x^*+t(x_i-x^*))}{\partial x^2} - H_{c_{j^*}}(x^*)\|\}$$

$$\leq 2ML \|x_i - x^*\| \text{ for all } i \geq i^*, \tag{35}$$

where L is a Lipschitz constant which is valid both for $H_{c_{j*}}(\cdot)$ and for $\frac{\partial^2 \psi_{c_{j*}}(\cdot)}{\partial x^2}$, and where we have made use of the fact that $\frac{\partial^2 \psi_{c_{j*}}(x^*)}{\partial x^2} = H_{c_{j*}}(x^*)$.

Setting $\nu = 2ML$ in (35), we obtain (32). ¤

Conclusion:

We have shown in this paper that there is at least one scheme for automatically finding an adequate penalty for use in a multiplier method. This scheme ensures convergence, but it does not guarantee quadratic convergence, since it does not ensure that the matrices $H_{e_{j*}}(x_i)$ do in fact become positive definite. One could, of one so desired, add a test to check if the matrices $H_{c_{j*}}(x_i)$ are positive definite and increase c_j to make them positive definite. However, it is not clear that the extra work is justified, since in practice it has been found that the c_{j*} resulting from a straightforward use of the algorithm is, usually, large enough to ensure quadratic convergence.

References

[1] K. J. Arrow and R. M. Solow: "Gradiwnt Methods for Constrained Maxima with Weakened Assumptions," in Studies in Linear and Nonlinear Programming, K. J. Arrow, L. Hurwicz & H. Uzawa (ed.), Stanford University Press, Stanford, California, 1958.

[2] D. P. Bertsekas: "Combined Primal-Dual and Penalty Methods for Constrained Minimization," EES Dept. Working Paper, Stanford University, Jan. 1973, to appear in SIAM J. on Control.

[3] D. P. Bertsekas: "On Penalty and Multiplier Methods for Constrained Minimization," EES Dept. Working Paper, Stanford University, Aug. 1973, submitted to SIAM J. on Control.

[4] J. D. Buys: "Dual Algorithms for Constrained Optimization Problems," Doctorate Dissertation, University of Leiden, June 1972.

[5] R. Fletcher: "A Class of Methods for Nonlinear Programming with Termination and Convergence Properties," in Integer and Nonlinear Programming, J. Abadie (ed.), North-Holland, 1970.

[6] R. Fletcher and S. Lill: "A Class of Methods for Nonlinear Programming II: Computational Experience," in Nonlinear Programming, J. B. Rosen, O. L. Mangasarian and K. Ritter (ed.), Academic Press, 1971.

[7] R. Fletcher: "A Class of Methods for Nonlinear Programming III: Rate of Convergence," Technical Paper T.P. 445, Theoretical Physics Division, U.K.A.E.A. Research Group, Atomic Energy Research Establishment, HARWELL, May 1971.

[8] P. C. Haarhoff and J. D. Buys: "A New Method for the Optimization of a Nonlinear Function Subject to Nonlinear Constraints," Computer Journal, Vol. 13, pp. 178-184, 1970.

[9] M. R. Hestenes: "Multiplier and Gradient Methods," Journal of Optimization Theory and Application, Vol. 4, No. 5, pp. 303-320, 1969.

[10] A. Miele, P. E. Moseley and E. E. Cragg: "Numerical Experiments on Hestenes' Method of Multipliers for Mathematical Programming Problems," Rice University, Aero-Astronautics Report No. 85, 1971.

[11] A. Miele, P. E. Moseley and E. E. Cragg: "A Modification of the Method of Multipliers for Mathematical Programming Problems," in Techniques of Optimization, A. V. Balakrishnan, (ed.), Academic Press, New York, 1972.

[12] E. Polak: Computational Methods in Optimization: A Unified Approach, Academic Press, New York, 1971.

[13] M. J. D. Powell: "A Method for Nonlinear Constraints in Minimization Problems," in Optimization, R. Fletcher (ed.), Academic Press, New York, 1969.