PUBLIC GOODS WITH CONSUMPTION INDIVISIBILITY

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Urban Systems Group

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Urban System Group,

Electronics Research Laboratory

University of California

Berkeley, California 94720

ABSTRACT

It is argued that an important property of many public goods is that of consumption indivisibility. By means of a simple example it is shown that such goods can be discussed in the context of Samuelson's framework.

Address for Correspondence:

Prof. P. Varaiya
M. I. T. 35-235
Cambridge, Mass. 02139

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INTRODUCTION

In this note we demonstrate that the notion of "impure" public goods is not essential to the theory of public goods. This is accomplished by a trivial extension of Samuelson's framework for "pure" public goods to include quality variation. Secondly, we show that instead of (or in addition to) the technical dichotomy of public goods into goods which are freely disposable[1] and those which are not, a practically more useful distinction is to isolate those public goods where there is indivisibility of consumption, i.e., where an individual has only the option of consuming (all of) the public good or of not consuming it at all. In many cases this "indivisibility" property can also be treated by a trivial extension of Samuelson's analysis. The optimal allocation of resources and an efficient pricing mechanism for this type of good is discussed in the context of a two-stage planning problem.

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1 Oakland (1972, p. 341) says that this property implies that "exclusion is possible and costless" which is of course a misuse of the term exclusion. We retain the term free disposal since it is well established in the Arrow-Debreu theory.

PUBLIC GOODS WITH QUALITY VARIATION

We are considering public goods which are services flowing over time from the investment of some durable or non-durable resources. To fix ideas consider two examples: a highway, and a classroom lecture. The highway produces a service namely, the ability for an automobile to be driven from some point A to a point B. The classroom lecture produces the service which is the opportunity (for a listener) to acquire a certain knowledge or learning experience. In both instances the service produced and received is of a certain quality, and this quality depends jointly upon the resource inputs as well as upon the number and kinds of individuals partaking the service. Thus the quality of travel (congestion) on the highway depends upon its construction and upon the number and kinds of automobiles using the highway, and the classroom experience depends upon the lecturer, the size of the classroom, etc. (the inputs) and the audience-consumers.

We restrict our attention to a time interval sufficiently small so that the resource inputs and the set of consumers do not change in this interval, and hence the quality of service is also constant. Since each consumer of the service receives the same quality of service it is a public good in the sense of Samuelson. It is crucial to note that we are not defining the "output" of the public good called highway as the number of trips or the "capacity" nor is the output of the lecture defined to be the classroom size.[2]

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[2] This constitutes an important conceptual difference from the otherwise formally equivalent approach of Oakland. He defines the output of the highway as its capacity, say X, and then he writes the consumption constraint as $x_i \leq X$, for all i, where x_i is the ith individual's consumption of the highway. In most examples that one can imagine such a constraint is not very meaningful since it can never be

Our second observation concerning these examples is that an individual has only the option of consuming the good or not consuming it at all (he either takes the trip from A to B using the highway or he does not). In other words, he cannot vary the amount of public good consumed. At first sight this appears to be equivalent to saying that the public good does not have the property of "free disposal". However this is not so because absence of free disposal means that <u>every</u> individual in the economy is consuming (all of) the public good (all of the time) which is clearly not the case here. We may call our examples instances of a class of public goods for which there is <u>indivisibility</u> in consumption.

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binding. Secondly, Oakland uses this notion of capacity chiefly as a device to introduce "congestion" or quality variation which in our framework can be done directly. In our framework capacity would merely be (an index of) the level of resource inputs used to produce the service.

A FORMAL MODEL

We elaborate the idea introduced above in the context of a simple planning problem. There are three goods in the economy: money, another abstract good called "capacity" which is an index for the level of resource inputs used to produce a service which is a public good. The public good has the following properties: the quality of the service can be measured by a scalar, q, which is a function of the input capacity, x, and the number of individuals n partaking the service, $q = Q(n,x)$; secondly, there is indivisibility in consumption of the service and finally, exclusion is possible and costless, i.e., any individual can be effortlessly prevented from partaking the service.

The public good is to provide service for a target community of N individuals. The planner's decisions occur over two stages. In the first stage, he must decide the input capacity based upon an uncertain knowledge of the demand of the population. In the second stage, after the service is being produced, the planner obtains complete information about the demand and he must decide how many individuals are to be excluded from access to the service.[3]

In the first stage the planner's information about the demand is described as follows. The population of N individuals is partitioned into J homogeneous classes. There are $\pi_j(\theta)N$ individuals in the jth class[4] each member of which has the identical cardinal utility function

[3] This two-stage problem corresponds to the common situation where the demand for a (new) good cannot be known accurately prior to the introduction of the good. It is being assumed here that no ex post change in input capacity is possible.

[4] Evidently $\pi_j(\theta) \geq 0, \quad \Sigma\, \pi_j(\theta) = 1$

$$V_j(\theta, m, \delta q) = m + U_j(\theta, \delta q), \tag{1}$$

where m = money consumed, q = quality and $\delta = 1$ or 0 depending upon whether or not the individual has access to the public good, $\theta \in \Theta$ is a random variable distributed according to the (planner's subjective) probability distribution $P(d\theta)$. Suppose that the planner chooses input capacity x. In the second stage the demand is revealed, i.e., a particular θ is realized. Suppose the planner decides to give access to $\alpha_j(\theta) \leq \pi_j(\theta)N$ individuals from the jth class, $j = 1, .., J$. If the planner's welfare criterion is the expected value of the average utility, then the optimal choice of x and $\{\alpha_j(\theta)\}$ is a solution to the following problem.

$$\underset{x}{\text{Maximize}} \ \frac{1}{N}(I - F(x)) + \int_\Theta W(\theta, x)\ P(d\theta), \tag{2}$$

where I = community's income flow, $F(x)$ = cost of input capacity x in terms of money flow, and $W(\theta, x)$ = maximum welfare achievable in the second stage, i.e., it is the optimal value of the following programming problem.

$$\underset{\{\alpha_j\}}{\text{Maximize}} \ \frac{1}{N}\left\{\sum_j \alpha_j U_j(\theta, q) + \sum_j (N\pi_j(\theta) - \alpha_j) U_j(\theta, 0)\right\} \tag{3}$$

$$\text{Subject to} \quad q = Q(\textstyle\sum_j \alpha_j, x) \tag{4}$$

$$0 \leq \alpha_j \leq N\pi_j(\theta), \ j = 1, .., J \tag{5}$$

Let $\hat{x}$, $\{\hat{\alpha}_j(\theta)\}$ be an optimal solution. Then the necessary optimality conditions for the second stage, letting $\hat{\lambda}(\theta)$ denote the Lagrange multiplier corresponding to (4), are

$$\frac{1}{N}[U_j(\theta, \hat{q}(\theta)) - U_j(\theta, 0)] = \begin{cases} -\hat{\lambda}(\theta)\frac{\partial Q}{\partial n} & \text{if } 0 < \hat{\lambda}_j(\theta) < \pi_j(\theta)N \\ \geq -\hat{\lambda}(\theta)\frac{\partial Q}{\partial n} & \text{if } \hat{\alpha}_j(\theta) = \pi_j(\theta)N \\ \leq -\hat{\lambda}(\theta)\frac{\partial Q}{\partial n} & \text{if } \hat{\alpha}_j(\theta) = 0 \end{cases} \tag{6}$$

For future reference set

$$\hat{p}(\theta) = -N\,\hat{\lambda}(\theta)\,\frac{\partial Q}{\partial n} \tag{7}$$

The necessary condition for the first stage problem, assuming $\hat{x} > 0$, is

$$0 = -\frac{1}{N}\frac{\partial F}{\partial x} + \int_\Theta \frac{\partial W}{\partial x}(\theta, \hat{x}) \quad P(d\theta)$$

$$= -\frac{1}{N}\frac{\partial F}{\partial x} + \int_\Theta \hat{\lambda}(\theta)\,\frac{\partial Q}{\partial x}(\Sigma\hat{\alpha}_j(\theta),\hat{x})\;P(d\theta)$$

Using (7) this can be rewritten as

$$\int_\Theta \frac{\hat{p}(\theta)}{\hat{m}(\theta)}\quad P(d\theta) = 1, \tag{8}$$

where

$$\hat{m}(\theta) = -\frac{\partial Q/\partial n}{\partial Q/\partial x}\,(\Sigma\hat{\alpha}_j(\theta),\hat{x})\;\;\partial F/\partial x(\hat{x}) \tag{9}$$

We can give the following interpretation to these conditions. Firstly, $\hat{m}(\theta)$ is the marginal cost of maintaining quality $\hat{q}(\theta)$ if one additional individual is given access to the public good, and the quantity $\hat{p}(\theta)$ is the corresponding marginal contribution to welfare. From (1) and (6) it follows that if $\hat{p}(\theta)$ is the "entrance" price to the public good then it will sustain the optimal numbers $\{\hat{\alpha}_j(\theta)\}$. Thus $\hat{p}(\theta)$ is the appropriate congestion price and <u>not</u> $\hat{m}(\theta)$.[5] Thirdly, (8) says that at the optimum capacity the expected value of the marginal benefit to marginal cost ratio is unity.

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[5] This suggests that we need to carefully specify the *ex ante* and and *ex post* situations when we agree that congestion price = marginal cost for efficiency. In the present exercise $\hat{p}(\theta) = \hat{m}(\theta)$ only for uninteresting distributions $P(d\theta)$. Note that $\hat{p}(\theta)$ is uniform over the population.

A CLARIFYING EXTENSION

In conclusion we present a simple extension which illuminates the remarks in footnotes 3 and 5. Specifically, suppose that in the second stage (after the demand is revealed) it is possible to add more input capacity. Call this addition y, $y \geq 0$, and suppose that its money cost is $G(y)$. It is assumed that the marginal cost of this (<u>ex post</u>) capacity is always larger than the marginal cost of x,

$$\frac{\partial F}{\partial x} < \frac{\partial G}{\partial y} \qquad \text{for all } x \geq 0,\ y \geq 0.$$

The first-stage problem now remains unchanged, whereas the second stage problem ((3), (4), (5)) changes as follows,

$$\underset{\{\alpha_j\},y}{\text{Maximize}}\ \frac{1}{N}\{\sum_j \alpha_j U_j(\theta,q) + \sum_j (N\pi_j(\theta)-\alpha_j)\ U(\theta,o) - G(y)\}$$

$$\text{Subject to} \quad q = Q(\sum_j \alpha_j, x+y)$$

$$0 \leq \alpha_j \leq N\pi_j(\theta) \quad , \quad j=1,\ldots,J$$

$$y \geq 0$$

Letting $\{\hat{\alpha}_j(\theta)\},\hat{y}(\theta)$ denote the optimum solution, and <u>assuming</u> that $\hat{y}(\theta) > 0$ the necessary conditions are (6), and

$$\frac{1}{N}\frac{\partial G}{\partial y}(\hat{y}(\theta)) = \hat{\lambda}(\theta)\frac{\partial Q}{\partial y}.$$

Defining $\hat{p}(\theta)$ by (7) again the condition above also implies that

$$\hat{p}(\theta) = -\frac{\frac{\partial Q}{\partial n}}{\frac{\partial Q}{\partial y}} \frac{\partial G}{\partial y} (\hat{y}(\theta)) \tag{10}$$

The right-hand side of (10) is the (*ex post*) marginal cost of maintaining quality level $\hat{q}(\theta)$ when an additional individual is given access. The left-hand side of (10) is the marginal contribution to welfare. Thus (10) says that when *ex post* adjustments are permitted the conjestion price equals the (*ex post*) marginal cost. In the first-stage problem the necessary condition (8) still retains validity.

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