MARTINGALE APPROACH TO WAITING LINE PROBLEMS

by

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ABSTRACT

In this thesis we present a new approach to waiting line problems, using the theory of martingales and stochastic integration. We show that this approach allows us to formulate and analyse more general problems than the classical approach.

Some of the advantages of the martingale approach are as follows: (1) the queueing problem for random and time-varying arrival rates and service rates can be formulated in great generality without the assumption of Markov property for the queue process. (2) In many cases where the classical approach yields solutions, the same solution can be obtained more directly. For example, Laplace transform need not to be invoked. (3) Queueing with feedback can be formulated, and thereby permitting optimal control problems to be considered. (4) Recursive equations for estimators of various kinds can be obtained.

The advantages cited above are exploited in this dissertation in connection with single-server first-come-first-served queues to yield a variety of results, the most important ones being on control and estimation. Extension to queues of more general types is in principle not difficult, but it is not considered in this dissertation.

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CHAPTER 1

INTRODUCTION

Waiting line (or queueing) problems arise in a large number of applications. These include: telephone congestion, air and road traffic, operation of dams, analysis of particle counters, and design of waiting room facilities. The basic problem in common involves a process of arrivals, and a process by which they are served. The interaction between arrivals and serving produces the queueing process whose properties constitute the main objective of the theory of queues.

The standard model in queue, which we shall follow, involves: a stochastic process representing the arrivals, a queue discipline which determines how service is offered, and a process representing the successive service durations.

The traditional goal of queueing has been one of analysis. The emphasis has been to derive properties of the queue (e.g. average queue length) under a given assumption on the queue discipline and the probability laws governing the arrival and service times. Possibly a more important goal is optimization. Of the various problems in optimization, the ones involving feedback control are probably the most important and also the most difficult. A satisfactory formulation of queues with feedback requires a modeling of the dynamics of the queueing process itself as opposed to the dynamics of the probability law, the latter being the starting point in most analyses involving queues.

The basic approach of this thesis, which we believe to be new, is to model the queue as the solution of a stochastic integral equation driven by the arrival and service processes, which are in turn modeled by semi-martingales. This allows us to bring to bear on the problem the powerful calculus made available by recent developments in martingale theory. Our approach will be to deal with the queue process itself rather than with its probability law.

Although our approach is in no way restricted to single-server single-queue systems, the detailed analysis in this thesis will be largely confined to such cases. On the other hand a very general class of arrival and service-time distributions will be allowed, including cases where they depend on the queue (feedback).

In chapter 2 we will present the basic tools needed for our goal; it will be a review of martigale theory, stochastic integration, transformation of measure, stochastic calculus, etc. The modeling of the waiting-line process will be done in chapter 3. Chapter 4 will deal with the derivation of some properties of the queue. The control problem will be treated in chapter 5 and problems related to estimation and identification in chapter 6. We end in chapter 7 with discussion and conclusions.

CHAPTER 2

PRELIMINARIES

2.1. Introduction

In this chapter we shall summarize the basic mathematical material needed in our work, and establish the notations used in the remaining chapters. The main topics of the chapter will be definition and properties of general stochastic processes, martingales, Poisson process and related topics, stochastic integrals, martingale representation results, absolute continuity of measures and transformation of martingales.

2.2. Stochastic processes

Probability spaces

1. A probability space, denoted by $(\Omega,\mathcal{F},\mathcal{P})$, is a triple consisting of a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω and a probability measure $\mathcal{P}$ defined on $\mathcal{F}$. $\mathcal{F}$ will be called complete with respect to $\mathcal{P}$, iff $B \subset A$ and $\mathcal{P}(A) = 0$ imply $B \in \mathcal{F}$. A probability space is said to be complete, iff $\mathcal{F}$ is complete with respect to $\mathcal{P}$. In our work we will deal exclusively with complete probability spaces.

2. T will denote the index set, which will be a subset of the real line.

3. $(\mathcal{F}_t, t \in T)$ is a family of σ-fields satisfying:
   a. $\mathcal{F}_t \subset \mathcal{F}$ for all $t \in T$
   b. $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $t, s \in T$, $s \leq t$
   c. $\bigcap\limits_{s>t} \mathcal{F}_s = \mathcal{F}_t$ for all $t \in T$ (right-continuity)
   d. $\mathcal{F}_o$ contains all the null sets of $\mathcal{F}$.

4. We will denote $\mathcal{F}_\infty = \bigvee\limits_{t\in T} \mathcal{F}_t$

Stochastic processes

5. Given a probability space $(\Omega,\mathcal{F},\mathcal{P})$ and an index set T, a stochastic process $X = (X_t, t \in T)$ is a family of random variables on $(\Omega,\mathcal{F})$ indexed by T, that is, X_t is a random variable on $(\Omega,\mathcal{F})$ for any t in T.

6. $(X_t, t \in T)$ is said to be adapted to the family of σ-fields $(\mathcal{F}_t, t \in T)$ if X_t is $\mathcal{F}_t$-measurable for all $t \in T$.

7. If the stochastic process X has sample functions that are right continuous with left-limits, we define:

$$\Delta X_t \triangleq X_t - X_{t^-} = \text{jump of X at time t}$$

Stopping times

8. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space and let $(\mathcal{F}_t, t \in T)$ be an increasing family of sub-σ-fields of $\mathcal{F}$. A positive random variable τ defined on Ω is said to be a stopping time if we have $(\tau \leq t) \in \mathcal{F}_t$ for every $t \in T$.

9. Let τ be a stopping time relatively to $(\mathcal{F}_t, t \in T)$ then

$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty | A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in T\}$

10. Given a stochastic process $(x_t, \mathcal{F}_t, t \in T)$, τ a stopping time with respect to $(\mathcal{F}_t, t \in T)$, we define:

$$X_{t\wedge\tau} \triangleq X_t I_{(\tau<t)} + X_\tau I_{(\tau\leq t)}$$

This process is called the process X stopped at τ.

11. A stopping time τ is predictable, if there is a sequence of stopping times τ_n such that:

$$\tau_n \uparrow \tau$$

$$\tau_n < \tau \quad \text{on } \{\tau>0\}, \text{ for all } n$$

12. A stopping time τ is said to be totally inaccesible, if τ is not almost sure infinite and if, for every sequence $(\tau_n) \in \mathcal{J}_T$,

$$\mathcal{P}\{\omega: \lim_n \tau_n(\omega) = \tau(\omega),\ \tau_n(\omega) < \tau(\omega) < \infty \quad \text{for every } n \in N\} = 0$$

13. The σ-field $\mathcal{A}$ on $\Omega \times T$ generated by all left continuous adapted process on $T \times \Omega$ is called the predictable σ-field.

14. A stochastic process $(X_t, \mathcal{F}_t, t \in T)$ is said to be predictable, if it is measurable with respect to $(T\times\Omega, \mathcal{A})$

Increasing processes

15. A process $A = (A_t, t \in T)$ is said to be an increasing process, if its trajectories are right-continuous increasing functions and $A_o = 0$

16. $BV^+ = \{A/A$ is an increasing process$\}$

17. $BV = \{A/A = A_1-A_2/A_1,A_2 \in BV^+\}$

18. $IV^+ = \{A/A \in BV^+$ and A is integrable$\}$

19. $IV = \{A/A = A_1-A_2/A_1,A_2 \in IV^+\}$

20. $LIV^+ = \{A/A \in BV^+$ and $\exists\ \{\tau_n\}, \lim_n \tau_n = \infty$ a.s., $\forall n : A_{t\wedge\tau_n} \in IV^+$

Remark: For the definitions and notation of this section we follow [Meyer,1966] and [Dellacherie,1972]

2.3. Martingales

In this section we shall introduce some definitions and results concerning martingales. More details can be found in [Meyer, 1966], [Kunita-Watanabe,1967] and [Doléans-Dade,Meyer, 1970].

1. A martingale $M = (M_t, \mathcal{F}_t, t\geq 0)$ is a stochastic process, such that:
   a. M_t is $\mathcal{F}_t$-measurable, integrable
   b. $E(M_t/\mathcal{F}_s) = M_s$ a.s. for all $t,s,t \geq s$
2. A process M is a submartingale, if (b) in (2.3.1) is replaced by:

b. $E(M_t/\mathcal{F}_s) \geq M_s$ a.s. for all $t,s,t \geq s$

3. A process M is a supermartingale, if (b) in (2.3.1) is replaced by

b. $E(M_t/\mathcal{F}_s) \leq M_s$ a.s. for all $t,s,t \geq s$

4. If $(M_t,\mathcal{F}_t,t\geq 0)$ is a martingale and h is a convex function such that $h(M_t)$ is integrable, then $(h(M_t),\mathcal{F}_t,t\geq 0)$ is a submartingale.

5. If a function h is convex, increasing and $(M_t,\mathcal{F}_t,t\geq 0)$ is a submartingale, then $(h(M_t),\mathcal{F}_t,t\geq 0)$ is a submartingale (if $h(M_t)$ is integrable)

Remark: If M is a submartingale, then -M is a supermartingale. Therefore, any property concerning submartingales can be extended easily to supermartingales and vice-versa.

Let us define some classes of martingales:

6. $\mathcal{M}_1 \triangleq \{M/M$ is a right continuous with left limits martingale with respect to some family $(\mathcal{F}_t, t \in T)$, $M_o = 0$ and M is uniformly integrable$\}$

7. $\mathcal{M}_2 \triangleq \{M \in \mathcal{M}_1/\sup_{t\in T} E(M_t^2) < \infty\}$ = class of L_2-bounded martingales

8. $\mathcal{M}_2^c \triangleq \{M \in \mathcal{M}_2/M$ is sample continuous$\}$

9. $\mathcal{M}_{loc} = \{M/M$ is a right continuous, adapted stochastic process, $M_0 \equiv 0$, and there is an increasing sequence of stopping times $\{\tau_n\}$, $\lim_{n\to\infty} \tau_n = \infty$ a.s., such that for all n, on the set $\{\tau_n>0\}$, $M_{t\wedge\tau_n} \in \mathcal{M}_1\}$ = class of local martingales

10. $\mathcal{M}_{loc}^c \triangleq \{M \in \mathcal{M}_{loc}/M$ is sample continuous$\}$

11. $\mathcal{M}_{2loc} \triangleq \{M \in \mathcal{M}_{loc}/$there exists $\{\tau_n\}$, $\lim_{n\to\infty} \tau_n = \infty$ a.s., such that for all $n, M_{t\wedge\tau_n} \in \mathcal{M}_2\}$

These classes of martingales will be used in the decomposition results and in defining stochastic integrals.

Decomposition of supermartingales

12. A uniformly integrable supermartingale $(X_t, \mathcal{F}_t, t \geq 0)$ is of class D, if $\{X_\tau$, for all stopping times $\tau\}$ is a uniformly integrable family.

13. A uniformly integrable supermartingale $(X_t, \mathcal{F}_t, t \geq 0)$ is of class D_a, if $\{X_\tau$, τ a stopping time, $\tau \leq a\}$ is a uniformly integrable family.

14. A uniformly integrable supermartingale $(X_t, \mathcal{F}_t, t \geq 0)$ is locally class D, if it is class D_a for all positive real a.

15. Theorem [Meyer,1966]: A supermartingale $(X_t, \mathcal{F}_t, t \geq 0)$ is locally class D, iff

$$X_t = M_t - A_t$$

where M is a martingale, A is a predictable process, right continuous increasing paths, $A_0 = 0$, $EA_t < \infty$. This decomposition is unique.

Decomposition of martingales

16. Two local martingales M and N are said to be orthogonal, if $MN = (M_t N_t, \mathcal{F}_t, t \geq 0) \in \mathcal{M}_{loc}$. In the same way $M, N \in \mathcal{M}_2$ are orthogonal iff $MN \in \mathcal{M}_1$. Similarly, $M, N \in \mathcal{M}_{2loc}$ are orthogonal, iff $MN \in M_{loc}$.

17. Theorem: [Doléans-Dade,Meyer 1970]

If $M \in \mathcal{M}_{loc}$, then there is a unique decomposition of M in the form:

$$M = M^c + M^d$$

where M^c and M^d belong to $\mathcal{M}_{loc}$, $M^c \in \mathcal{M}^c_{loc}$ and M^d is a local martingale, orthogonal to all local martingales belonging to $\mathcal{M}^c_{loc}$.

18. Theorem: [Doléans-Dade,Meyer,1970].

If $M \in \mathcal{M}_2$, then M can be decomposed uniquely in the form:

$$M = M^{(1)} + M^{(2)} + M^{(3)}$$

where $M^{(1)} \in \mathcal{M}_2^{loc}$, $M^{(2)}$ is a sum of martingales, each having just one jump at a predictable stopping time; each of these martingales orthogonal to any martingales having no common discontinuities with it; $M^{(3)}$ is a sum of martingales having only one jump; the jump is at a totally unpredictable stopping time; each martingale is orthogonal to any martingale having no common discontinuity with it.

Increasing processes associated with martingales

19. If $M \in \mathcal{M}_2$, there is a unique, predictable, increasing process called $\langle M,M\rangle_t$, such that $M_t^2 - \langle M,M\rangle_t$ is an $\mathcal{M}_1$-martingale [Meyer,1966, VIII 23]

20. For $M,N \in \mathcal{M}_2$ we define

$$\langle M,M\rangle_t = \frac{1}{2}\,[\langle M+N,M+N\rangle_t - \langle M,M\rangle_t - \langle N,N\rangle_t]$$

21. If $M \in \mathcal{M}_{loc}$ we define:

$$[M,M]_t = \langle M^c,M^c\rangle_t + \sum_{s\leq t} (\Delta M_s)^2$$

22. Similarly, if $M,\ N \in \mathcal{M}_{loc}$:

$$[M,N]_t = \frac{1}{2}\,([M+N,M+N]_t - [M,M]_t - [N,N]_t)$$

23. Theorem: [Doléans-Dade,Meyer,1970]

If $M,N \in \mathcal{M}_{loc}$, then $MN - [M,N]$ belongs to $\mathcal{M}_{loc}$.

24. Theorem: [Doléans-Dade,Meyer,1970]

In the case $M \in \mathcal{M}_2$, then both $[M,M]$ and $\langle M,M\rangle$ are well defined. Therefore, $[M,M] - \langle M,M\rangle \in \mathcal{M}_1$.

25. We can define $\langle N,N\rangle$ for any $M,N \in \mathcal{M}_{loc}$ in the following way: $\langle M,N\rangle$ is a predictable process, adapted, of bounded variation, and such that $[M,N] - \langle M,N\rangle$ is in $\mathcal{M}_{loc}$.

Semi-martingales

26. A process $(X_t, \mathcal{F}_t, t \geq 0)$ is called a semi-martingale, if it can be written in the following form:

$$X_t = X_0 + A_t + M_t$$

where $A \in IV$ and $M \in \mathcal{M}_{loc}$

This decomposition is not unique, but X_0 is unique and also the continuous part M^c of M. We define $\langle X^c, X^c \rangle = \langle M^c, M^c \rangle$

2.4. Counting processes

1. A real valued stochastic process $(N_t, t \in T)$ is a counting process, if:

   a. $N_0 = 0$

   b. N is constant, except for positive unit jumps at random time

   c. N has right continuous sample functions almost surely.

2. A stochastic process $(M_t, t \in T)$ is a Poisson process with constant rate λ, iff:

   a. N is a counting process.

   b. N has independent increments.

   c. $N_t - N_s$ has a Poisson distribution with parameter $\lambda|t-s|$, $\lambda > 0$.

for all $t,s \in T$.

If $\lambda=1$, we call N a standard Poisson process.

3. The Poisson process has the following properties:

   a. N is a strong Markov process.

   b. N is quasi-left continuous.

   c. $N_t - \lambda t$ is a martingale.

These properties of the Poisson process are a consequence of being a process with stationary independent increments.

4. Another way of defining the Poisson process is by saying that the Poisson process is a right-continuous jump Markov process with the infinitesimal conditions:

$$\lim_{h\to 0} \frac{1}{h} \mathcal{P}\{N_{t+h} - N_t = 1/\mathcal{F}_{N_t}\} \overset{a.s}{=} \lambda$$

$$\lim_{h\to 0} \frac{1}{h} \mathcal{P}\{N_{t+h} - N_t > 1/\mathcal{F}_{N_t}\} \overset{a.s}{=} 0$$

where $\mathcal{F}_{N_t} = \sigma\{N_s, 0\le s\le t\}$

5. With the use of martingales, we can also define the Poisson process as being the process $(N_t, \mathcal{F}_t, t\ge 0)$ which satisfies:

a. it is adapted.

b. it is a counting process.

c. $(N_t - t, \mathcal{F}_t, t\ge 0) \in \mathcal{M}_{loc}$.

Extensions of Poisson Processes

6. In engineering we often deal with processes $N = (N_t, t\ge 0)$ called generalized Poisson processes or point processes with random rate $\lambda = (\lambda_t, t\ge 0)$, where λ is a measurable, non-negative process. Such a process can be defined as follows:

a. $N_0 \equiv 0$, N is a counting process

b. $\lim_{h\to 0} \frac{1}{h} E(I_{\{N_{t+h} - N_t = 1\}}/N_s, 0\le s\le t) = \lambda t$ a.s.

c. $\lim_{h\to 0} \frac{1}{h} E(I_{\{N_{t+h} - N_t > 1\}}/N_s, 0\le s\le t) = 0$ a.s.

In the section on trasformation of measures we will see that this process really exists.

2.5. Stochastic Integration

We are interested in defining integrals of the form $\int_0^t C_s dX_s$, where

$(X_t, \mathcal{F}_t, t \geq 0)$ is a stochastic process and $(C_t, t \geq 0)$ is a process adapted to $(\mathcal{F}_t, t \geq 0)$. In the case of a discrete martingale $(X_n, \mathcal{F}_n, n=1,2,3\ldots)$ we can define random variables $Y_n = \sum_{n=1}^{n} u_k d_k$, where $d_1 = X_1$, $d_2 = X_2 - X_1, \ldots,$ $d_k = X_k - X_{k-1}$, and $(u_n, n=1,2,\ldots)$ is a family of random variables such that u_n is $\mathcal{F}_{n-1}$-measurable; $(Y_n, \mathcal{F}_n, n=1,2,\ldots)$ is also a martingale. We should be able to obtain a similar result for continuous time martingales, under certain conditions. The first person to define such integrals for such a process was Ito [1944], that constructed an integral for the case where X was the Brownian motion and C belong to a class of functions. Later works extended the definition to other class of martingales. Among them, we should mention [Kunita,Watanabe,1967], where it is treated the case of square integrable martingales, [Millar,1968], and [Doléans-Dade,Meyer 1970], where the results are extended to local martingales.

We are going to state briefly the results of this last work.

1. If $A \in IV$, we define $L^1(A)$, the set of all the adapted, predictable processes C, such that:

$$E\int_0^\infty |C_s||dA_s| < \infty$$

2. If $M \in \mathcal{M}_2$, we define:

$$L^2(M) = \{C/C \text{ is adapted, predictable, } E\int_0^\infty |C_s|^2 d\langle M,M\rangle_s < \infty\}$$

3. Theorem: [Doléans-Dade,Meyer,1970]

Let us have $M \in \mathcal{M}_2$ and $C \in L^2(M)$, then $C \in L^2(M,N)$ for all $N \in \mathcal{M}_2$, and there is a unique element C.M of $\mathcal{M}_2$, such that, for all $N \in \mathcal{M}_2$, we have:

$$\langle C.M,N\rangle_t = \int_0^t C_s d\langle M,N\rangle_s$$

We say that this element is the stochastic integral of the process C with respect to the martingale M, and will denote $(C,M)_t = \int_0^t C_s dM_s$

4. Theorem: [Doléans-Dade,Meyer,1970]

If $M \in \mathcal{M}_2$, then the space $L^2(M)$ is such that

$$L^2(M) = \{C/C \text{ adapted, predictable, } \int_0^\infty C_s^2 \, d[M,M]_s < \infty\}$$

Also, if $C \in L^2(M)$, the stochastic integral C.M is the unique element of $\mathcal{M}_2$ such that

$$[C.M,N]_t = \int_0^t C_s \, d[M,N]_s \text{ a.s. for all } N \in \mathcal{M}_2, \text{ all } t$$

5. A process C is called locally bounded, if there is an increasing sequence of stopping times $\{\tau_n\}$, $\lim_{n\to\infty} \tau_n = \infty$ a.s. such that for all n, $|C_{t\wedge\tau_n} I_{(\tau_n<\infty)}| \le M_n < \infty$, where $\{M_n\}$ are real, positive constants.

6. LB = {C/C is adapted, predictable and locally bounded}

7. Theorem: [Doleans-Dade,Meyer,1970]

If $M \in \mathcal{M}_{loc}$, $C \in LB$, then there exists an unique element $C.M \in \mathcal{M}_{loc}$, called the stochastic integral, such that, for all $N \in \mathcal{M}_{loc}$, we have:

$$[C.M,N]_t = \int_0^t C_s \, d[M,N]_s \quad \text{for all } t$$

For a process $M \in \mathcal{M}_{2loc}$ we can, by using stopping-time arguments, define

8. $L_{2loc}(M)$ = {C/C adapted, predictable and there exists an increasing sequence of stopping times $\{\tau_n\}$, $\lim_{n\to\infty} \tau_n = \infty$, such that for all n, $E[\int_0^{\tau_n} |C_s|^2 d\langle M,M\rangle_s] < \infty$}

9. If $M \in \mathcal{M}_{loc}$ and $C \in L_{2loc}(M)$, there exists a unique element $C.M \in \mathcal{M}_{2loc}$, such that for all $N \in \mathcal{M}_{2loc}$:

$$\langle C.M,N\rangle_t = \int_0^t C_s \, d\langle M,N\rangle_s \quad \text{for all } t \geq 0$$

The Stochastic integral and the Lebesque-Stieltjes integral

In some cases, the stochastic integral coincides with the Stieltjes integral

10. Theorem: [Doléans-Dade,Meyer,1970]

If $M \in \mathcal{M}_1 \cap BV$, $C \in L_1(M)$, then the Stieltjes integral $\int_0^t C_s dM_s \in \mathcal{M}_1$.

11. Theorem: [Doléans-Dade,Meyer,1970]

If $M \in \mathcal{M}_2 \in IV$ and $C \in L^2(M) \cap L^1(M)$, then the stochastic integral $(C.M)_t$ is almost surely equal to the Stieltjes integral $\int_0^t C_s dM_s$.

The differentiation rule

12. If X is a semi-martingale with values in R^n and $f : R^n \to R$ is a twice continuously differentiable function with first partial derivatives $\frac{\partial f}{\partial x^{(i)}}$, $i = 1,2,\ldots,n$ and second partial derivatives $\frac{\partial^2 f}{\partial x^{(i)} \partial x^{(j)}}$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,n$, then $f(X)$ is a semi-martingale of the form

$$f(X_t) = f(X_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial X_s^{(i)}} (X_{s-}) dX_s^{(i)}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \int_0^t \frac{\partial^2 f(X_{s-})}{\partial X_s^{(i)} X_s^{(j)}} \, d\langle X^{(i)c}, X^{(j)c}\rangle_s$$

$$+\sum_{s\leq t}\left[f(X_s)-f(X_{s-})-\sum_{i=1}^{n}\frac{\partial f}{\partial X_s^{(i)}}(X_{s-})\Delta X_s^{(i)}\right]$$

[Doléans-Dade,Meyer,1970]

13. We can extended this formula for the case $f(t,X_t)$, where this function satisfies the conditions of (2.5.12) and also is once continuously differentiable in t. In this case:

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial s}(s,X_{s-})ds + \sum_{i=1}^{n}\int_0^t \frac{\partial f}{\partial X_s^{(i)}}(s,X_{s-})dX_s^{(i)}$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t \frac{\partial^2 f(s,X_{s-})}{\partial X_s^{(i)}\partial X_s^{(j)}}\, d\langle X^{(i)c},X^{(j)c}\rangle_s$$
$$+\sum_{s\leq t}\left[f(s,X_s)-f(s,X_{s-})-\sum_{i=1}^{n}\frac{\partial f(s,X_{s-})}{\partial X_s^{(i)}}\Delta X_s^{(i)}\right]$$

Two important cases of application of this formula are:

14. If X, Y are semi-martingales in R, then:

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s + [X,Y]_t$$

15. If X = Y, we have in the above formula

$$X_t^2 = 2\int_0^t X_{s-}dX_s + [X,X]_t$$

2.6. Martingale representation theorems

We have seen that a stochastic integral is a martingale. Now our problem is given a martingale M with respect to a family of σ-fields $(\mathcal{F}_t)$, generated by another martingale N, when does this martingale M has a representation as a stochastic integral with respect to the martingale N? Ito [1951b] was the first one to answer this question.

He considered square integrable functionals, on the σ-field generated by a Brownian motion process, and obtained a representation as a stochastic integral with respect to the Brownian motion. Wong [1916b] has given a extension of the results to local martingales. Kunita and Watanabe [1961] gave us results for the case where $M \in \mathcal{M}_2$ and $\mathcal{F}_t$ is generated by a Hunt process. Another work, dealing with martingales generated by Brownian motion, was written by Clark [1970]. A recent work by Boel, Varaiya and Wong [1973] permits us represent local martingales with respect to σ-fields, generated by a certain kind of jump processes in stochastic integral form.

1. Let $(W_t, t \geq 0)$ be a standard Brownian motion, and let $(\mathcal{F}_{W_t}, t \geq 0)$ be the family of σ-fields generated by it.

2. Theorem: [Ito,1951b;Kunita,Watanabe,1967]

   If $(M_t, \mathcal{F}_{W_t}, t \geq 0) \in \mathcal{M}_2$, then M has the unique representation

$$M_t = \int_0^t C_s dW_s, \text{ for all } t \geq 0, \text{ where } C \in L_2(W).$$

3. Theorem [Clark,1970;Wong,1971b]

   If $(M_t, \mathcal{F}_{W_t}, t \geq 0) \in \mathcal{M}_{1oc}$ then M has a unique representation

$$M_t = \int_0^t C_s dW_s \text{ for all } t, \text{ where } C \in L_{2loc}(W).$$

4. Let $N_t, t \geq 0$ be a standard Poisson process and let $(\mathcal{F}_{N_t}, t \geq 0)$ be the family of σ-fields generated by it.

5. Theorem: [Kunita,Watanabe,1967]

   If $(M_t, \mathcal{F}_{N_t}, t \geq 0) \in \mathcal{M}_2$, then M has the unique representation

$$M_t = \int_0^t C_s (dN_s - ds) \text{ a.s., for all } t \geq 0, \text{ where } C \in L_2(N_t - t)$$

Now we will state some results of the work by Boel, Varaiya and Wong [1973,a]

6. A <u>Blackwell Space</u> $(Z, \mathcal{Z})$ is a measurable space, such that $\mathcal{Z}$ is

a separable σ-field, and every measurable function $f : Z \to R$ maps Z onto an analytic subset of R.

7. Let $(X_t, \mathcal{F}_t, t \geq 0)$ be a process with values in $(Z, \mathcal{Z})$. Suppose that, with probability 1, all the sample paths of X are piecewise constant, right-continuous, and have only a finite number of discontinuities in every finite interval. If $(Z, \mathcal{Z})$ is a Blackwell space and the jump times of the process are totally inaccesible, then X is <u>fundamental process</u>.

8. Let us suppose that $(X_t, \mathcal{F}_t, t \in T)$ is a counting process, and there exists a process $\lambda = (\lambda_t, \mathcal{F}_{X_t}, t \in T)$, $\lambda_t \geq 0$, for all $t \in T$, $(\int_0^t \lambda_s ds, t \in T) \in$ LIV, such that $(X_t - \int_0^t \lambda_s ds, \mathcal{F}_{X_t}, t \in T)$ belongs to $\mathcal{M}_{loc}$. Then:

9. Theorem: [Borel,Varaiya,Wong,1973]

If $M \in \mathcal{M}_{loc}$ is a local martingale with respect to the family $\mathcal{F}_{X_t}$, generated by $(X_t, t \in T)$, then $M_t = \int_0^t C_s(dX_s - \lambda_s ds)$, for all t where C is a predictable process, such that $\int_0^t |C_s|\lambda_s ds < \infty$ for all $t \in R$.

2.7. <u>Transformation of measure and translation of martingales</u>

The purpose of this section is to examine absolutely continuous transformation of measures and their relation with martingales. The Radon-Nikodym derivative of an absolutely continuous transformation has a natural interpretation as a martingale, and this connection gives rise to some important representation results for Radon-Nikodym derivatives. A second important connection arises in examining how martingales are transformed under a charge in probability law. Results of this type originated in the form of the theorem of Cameron and Martin [1944] on translations of Wiener processes, and were generalized by Gursanov [1960]. Their expression in terms of local martingales was obtained by Van Schuppen and Wong [1973].

1. The exponential formula: [Doléans-Dade,1970]

If $(X_t, \mathcal{F}_t, t \in T)$ is a real valued semi-martingale, $X_0 \equiv 0$, then there exists a unique semi-martingale $(X_t, \mathcal{F}_t, t \in T)$ satisfying $Z_t = 1 + \int_0^t Z_{s-}\, dX_s$, and Z is given by

$$Z_t = \exp(X_t - \frac{1}{2}\langle X^c, X^c\rangle_t) \prod_{s \leq t} (1+\Delta X_s) e^{-\Delta X_s} \tag{2}$$

We will denote (2.6.2) by $Z_t = \varepsilon(X_t)$

2. Theorem: [Van Schuppen,1973] If $(Z_t, \mathcal{F}_t, t \in T)$ is a semi-martingale, and Z_t, $Z_{t-} > 0$ a.s. for all $t \in T$, and $Z_0 \equiv 1$, then there is a semi-martingale $(X_t, \mathcal{F}_t, t \in T)$, $X_0 \equiv 0$, $\Delta X_t > -1$ a.s. for all $t \in T$, such that $Z_t = \varepsilon(X_t)$.

3. Theorem: [Van Schuppen, 1973]

If a. $(X_t, \mathcal{F}_t, t \in [0,1]) \in \mathcal{M}_{loc}$, $X_0 \equiv 0$, $\Delta X_t > -1$ a.s. for all $t \in [0,1]$, $\langle X^c, X^c\rangle_1 < \infty$ a.s.

b. $(\langle X,X\rangle_t, \mathcal{F}_t, t \in [0,1])$ exists and is a process of locally integrable variation, and satisfies $d\langle X,X\rangle_t = \psi_t\, dt$ where $(\psi_t, \mathcal{F}_t, t \in T)$ $\in L_{1loc}(t)$ satisfies $|\psi_t| \leq k(t)$ a.s. for all $t \in [0,1]$ for some positive valued function $k : T \to R$, such that $\int_0^1 k(s)ds < \infty$

then $E(\varepsilon(X_1)) = 1$

Two important corollaries of this theorem are:

4. If W is Brownian notion, $\Phi \in LB$ and if $|\Phi_t^2| \leq k(t)$ a.s. for all $t \in T = [0,1]$, where k is a positive function, such that $\int_0^1 k(s)ds < \infty$, then $E(\varepsilon(\int_0^1 \Phi s\, dWs)) = 1$

5. If N is a standard Poisson process, $(\Phi_t, \mathcal{F}_{N_t}, t \in T = [0,1]) \in LB$ and $|\psi_t|^2 \leq k(t)$ a.s. for all $t \in T = [0,1]$, where k(t) is a positive

function, such that $\int_0^1 k(s)ds < \infty$, then $E(\varepsilon(\int_0^1 \psi_s(dN_s - ds)) = 1$

Absolute Continuity of Measures and Translation of Martingales

6. Let us have a measurable space $(\Omega, \mathcal{F})$ and two probability measures and $\mathcal{P}_o$ on this space. $\mathcal{P}$ is said to be absolutely continuous with respect to $\mathcal{P}_o$, if for all $A \in \mathcal{F}$ such that $\mathcal{P}_o(A) = 0$ we have that $\mathcal{P}(A) = 0$ (notation $\mathcal{P} << \mathcal{P}_o$). $\mathcal{P}$ and $\mathcal{P}_o$ are said to be equivalent iff $\mathcal{P} << \mathcal{P}_o$ and $\mathcal{P}_o << \mathcal{P}$ (notation $\mathcal{P} \sim \mathcal{P}_o$)

7. If $\mathcal{P} << \mathcal{P}_o$ by the Radon-Nikodym theorem, we have $P(A) = \int_A \Lambda(w)\mathcal{P}_o(dw)$ for all $A \in \mathcal{F}$

where $\Lambda(.)$ is a non-negative, $\mathcal{F}$-measurable function.

If $\{\mathcal{F}_t, t \in T\}$ is an increasing family of sub σ-fields of $\mathcal{F}$, we have that $\Lambda_t = E_o(\Lambda/\mathcal{F}_t)$ is a martingale with respect to $(\mathcal{F}_t)$

8. Theorem [Van Schuppen, 1973]

a. Given a probability space $(\Omega, \mathcal{F}, \mathcal{P}_o)$. Let $(S_t, \mathcal{F}_t, t \in [0,1]) \in \mathcal{M}_{loc}$ be such that $X_0 \equiv 0$, $\langle X^c, X^c \rangle < \infty$ a.s., $\Delta X_t < -1$ a.s. for all $t \in T$, and $E_o[\varepsilon(X_1)] = 1$, then $(\varepsilon(X_t), \mathcal{F}_t, t \in [0,1]) \in \mathcal{M}_1$. The formula $\frac{d\mathcal{P}}{d\mathcal{P}_o} = \varepsilon(X_1)$ introduces a new probability measure $\mathcal{P}$ on $(\Omega, \mathcal{F})$ and $\mathcal{P}$ is equivalent to $\mathcal{P}_o$.

b. Let $(\Omega, \mathcal{F})$ be a measurable space with two equivalent probability measures $\mathcal{P}$ and $\mathcal{P}_o$ defined on it. Let $(\mathcal{F}_t, t \in T)$ be a family of sub σ-fields satisfying the conditions in (2.2.3), then there is a process $(\hat{X}_t, \mathcal{F}_t, t \in T) \in \mathcal{M}_{loc}$, $\hat{X}_0 \equiv 0$, $\Delta\hat{X}_t > -1$ a.s. for all $t \in [0,1]$, $\langle \hat{X}^c, \hat{X}^c \rangle < \infty$ a.s., such that $\Lambda_t = \varepsilon(\hat{X}_t)$ for all $t \in [0,1]$.

Therefore, under certain conditions, a local martingale introduces a new probability measure. Conversely the estimate of the Radon-Nikodym derivative of two equivalent measures given some family of σ-fields is characterized by a local martingale $\hat{X}$.

Translation of martingales

9. Theorem [Van Schuppen,Wong,1973]

a. Let $(\Omega,\mathcal{F},\mathcal{P}_o)$ be a probability space,

b. Define a transformation of measure by $\frac{d\mathcal{P}}{d\mathcal{P}_o} = \varepsilon(X_1)$, where $(X_t,\mathcal{F}_t,t \in [0,1])$ is a real-valued $\mathcal{P}_o$ local martingale, such that $\langle X^c,X^c\rangle_1 < \infty$ a.s., $\Delta x_t < -1$ for all $t \in [0,1]$, and $E_o[\varepsilon(X_1)] = 1$.

c. Let $(Y_t,\mathcal{F}_t,t \in [0,1])$ be a local martingale under $\mathcal{P}_o$ with values in R^n

d. Suppose there exists a predictable process, denoted by $(\langle Y,X\rangle_t,\mathcal{F}_t,t \in [0,1]) \in IV$, such that $([Y,X]_t - \langle Y,X\rangle_t,\mathcal{F}_t,t \in [X,1]) \in\mathcal{M}_{loc}$ under $\mathcal{P}_o$.

Then $\mathcal{P}$ is a probability measure on $(\Omega,\mathcal{F})$ and the process M, defined by $M_t \triangleq Y_t - \langle Y,X\rangle_t$, satisfies $(M_t,\mathcal{F}_t,t \in [0,1]) \in\mathcal{M}_{loc}$ under $\mathcal{P}$. If, in addition, $\langle Y,X\rangle$ is sample continuous, then $[M,M] = [Y,Y]$.

Two important consequences of this theorem are as follows:

10. If a. $(\Omega,\mathcal{F},\mathcal{P}_o)$ is a probability space,

b. $(W_t,\mathcal{F}_t,t \in T)$ is sample continuous Brownian motion in R^n,

c. $(\Phi_t,\mathcal{F}_t,t \in T) \in L_{2loc}(W)$ in R^n, $\int_0^1 |\Phi_s|^2 ds < \infty$ a.s. $\mathcal{P}_o$ and satisfies $E_o[\varepsilon(\int_0^1 \Phi_s dW_s)] = 1$, and

d. We introduce a new measure $\frac{d\mathcal{P}}{d\mathcal{P}_o} = \varepsilon(\int_0^1 \Phi_s dWs)$

Then $\mathcal{P}$ is a probability measure. Further, if

$$M_t \triangleq W_t - \int_0^t \Phi_s ds$$

Then $(M_t,\mathcal{F}_t,t \in T) \in\mathcal{M}^c_{loc}$ with $\langle M,M\rangle_t = \langle W,W\rangle_t = t$

Hence M is a sample continuous Brownian motion under $\mathcal{P}$.

11. Let $(\Omega,\mathcal{F},\mathcal{P}_o)$ be a probability space [Gursanov,1960]

Let $(N_t,\mathcal{F}_t,t \in T)$ be a real valued standard Poisson process $(N_t-t,\mathcal{F}_t,t \in T) \in \mathcal{M}_{1oc}(\mathcal{P}_o)$

Let $\lambda = (\lambda_t,\mathcal{F}_t,t \in T) \in L_{1loc}(t)$, $\lambda_t > 0$ a.s. for all $t \in T$, and satisfy $E_o[\varepsilon(\int_0^1 (\lambda_s-1)\ (dN_s-ds)] = 1$

Suppose we introduce a new measure

$$\frac{d\mathcal{P}}{d\mathcal{P}_o} = \varepsilon(\int_0^1 (\lambda_s-1)\ (dN_s-ds))$$

Then $\mathcal{P}$ is a probability measure, and if

$$M_t = N_t - \int_0^t \lambda_s ds,$$

Then $(M_t,\mathcal{F}_t,t \in T) \in \mathcal{M}_{2loc}(\mathcal{P})$ with $[M,M]_t = N_t$ and $\langle M,M\rangle_t = \int_0^t \lambda_s ds$. [Brémaud,1972]

CHAPTER 3

THE MARTINGALE APPROACH FOR WAITING TIME PROBLEMS

3.1. Introduction

In this chapter we begin the study of waiting time problems using the martingale approach. As it was said before, we will mainly study the one-server queue with service in order of arrival. In section 3.2 we will show how the queue size can be represented as a stochastic integral for a very simple kind of queue whose existance is already known in the classical literature. In section 3.3 we will see that this approach can be extended for a wide class of problems by the use of transformation of measures. In section 3.4 we will show how to use the martingale approach for the case when we have queues with interarrival times independent, identically distributed and service times independent, identically distributed. The martingale approach to other kind of queues will be discussed very briefly in 3.5, and we will end with a summary and discussion in section 3.6, where we will show the advantages of our approach.

3.2. Representation of the queue length as a stochastic integral for single-server, first-come-first-served queues

Let us suppose that customers are arriving at a counter where there is only one server and the customers are served in the order of their arrival. Assume that the times between successive arrivals are mutually independent, identically distributed random variables with distribution function $F(x) = 1 - e^{x}$, $x \geq 0$, i.e., exponential distribution with parameter 1. The service times are also assumed to be mutually independent, identically distributed random variables with distribution function $G(y) = 1 - e^{-y}$, $y \geq 0$. Let us define the process $Z^{(1)} = (Z_t^{(1)}, t \geq 0)$ as

$Z_t^{(1)}$ = number of customers in the system, that is, the number of customers waiting plus the one that is being served. This process $Z^{(1)}$ will be called the queue length. By our description it is easy to see that $Z^{(1)}$ is a counting process.

Remark: We will always consider that, at time t = 0, no one is in the system, that is, $Z_0^{(1)} = 0$

The process $Z^{(1)}$, defined above, consists in a special case of the processes called birth-and-death processes. They belong to the class of jump Markov processes and, therefore, are strong Markov processes [Breiman,1968]. The process $Z^{(1)}$ has state space $\{0,1,2,...\}$ and the state 0 is reflexive. By the theory of the jump Markov processes we know that the transition probabilities $p_t(i,j)$ of Z must satisfy certain infinitesimal conditions:

a. $p_{t+h}(i,i+1) = h + o(h)$

b. $p_{t+h}(i,i-1) = h + o(h)$ for $i \geq 1$ (1)

c. $p_{t+h}(i,i) = 1 - 2h + o(h)$

d. 0 is reflexive

where $p_t(i,j)$ = probability that $Z_{t+h}^{(1)} = j$ at time t+h, given that at time t, we have $Z_t^{(1)} = i$.

From the infinitesimal conditions it is possible to obtain the discrete density function of the process $Z_t^{(1)}$. Denoting by $P_n^{(1)}(t)$ the probability that $Z_t^{(1)} = m$ we find the following differential equations:

$$\frac{dP_n^{(1)}(t)}{dt} = -2P_n^{(1)}(t) + P_{n-1}^{(1)}(t) + P_{n+1}^{(1)}(t) \qquad n \geq 1$$

$$\frac{dP_o^{(1)}(t)}{dt} = - P_o^{(1)}(t) + P_1^{(1)}(t) \tag{2}$$

for all $t \geq 0$, initial conditions $P_o(0) = 1$, $P_n(0) = 0$, $n \geq 1$.

Defining $P^{(1)}(a,t) = \sum_{n=0}^{\infty} P_n^{(1)}(t)a^n = E\, a^{Z_t^{(1)}}$, $a < 1$, we have using the equations above:

$$\text{a.} \quad \frac{\partial P^{(1)}(a,t)}{\partial t} = (1-a)\ [(1-a)P^{(1)}(a,t) - P_o^{(1)}(t)] \tag{3}$$

Knowing $P^{(1)}(u,t)$ is possible to calculate $P_n^{(1)}(t)$ for all n.

Above the value of $P_o^{(1)}(t)$ is not known. To get it we use the fact that $\sum_{n=0}^{\infty} P_n^{(1)}(t) = 1$. There are many ways of doing that (see e.g., [Takács,1961], [Saaty,1961]).

The same problem has an alternative approach as follows:

Suppose that $(X_t, \mathcal{F}_t, t \geq 0)$, $(Y_t, \mathcal{F}_t, t \geq 0)$ are independent standard Poisson processes with $\mathcal{F}_t \supseteq \sigma((X_s, Y_s),\ 0 \leq s \leq t)$

Define a counting process Z such that:

$$\Delta Z_t = \Delta X_t - 1(Z_{t-})\Delta Y_t \quad \text{for all } t \geq 0 \tag{4}$$

where $1(z) = 0 \quad \text{if } z \leq 0$

$\quad 1(z) = 1 \quad \text{if } z \geq 1$

We can write:

$$Z_t = \sum_{s \leq t} (\Delta X_s - 1(Z_{s-})\Delta Y_s) = X_t - \sum_{s \leq t} 1(Z_{s-})\Delta Y_s \tag{5}$$

The last term can be written as an integral

$$\sum_{s \leq t} 1(Z_{s-})\Delta Y_s = \int_0^t 1(Z_{s-})dY_s \tag{6}$$

since Y is a process of integrable variation, $1(Z_{s-})$ is adapted and predictable and $E[\int_0^t 1(Z_{s-})dY_t] \leq t$. Therefore:

$$Z_t = X_t - \int_0^t 1(Z_{s-})dY_s \tag{7}$$

Equation (7) describes our queue length as a function of the arrival and service processes. Comparing the two approaches we see that in the first one we are interested mainly with the distribution of the process. Now with equation (7) we should be able to work with the process itself and apply the theory of martingales with its stochastic calculus to the problems in queueing. We don't need to worry about distributions in a very early stage which constitutes, in our point of view, a more intuitive way.

To show how we can get the same results of the classical theory we will calculate the discrete density function of the process Z. Rewriting (7) we have

$$Z_t = (X_t - t) - \int_0^t 1(Z_{s-})(dY_s - ds) + \int_0^t (1 - 1(Z_{s-}))ds \tag{8}$$

and $(X_t - t) \in \mathcal{M}_{loc}$

$\int_0^t 1(Z_{s-})(dY_s - ds) \in \mathcal{M}_{loc}$ since $1(Z_{t-})$ is adapted, predictable, $|1(Z_{t-})| \leq 1$ for all $t \geq 0$, and $(Y_t - t) \in \mathcal{M}_{loc}$

$\int_0^t (1 - 1(Z_{s-}))ds \in IV$

Then, by (2.3.26), Z is a semi-martingale. Applying the differentiation rule (2.5.12) to the function a^{Z_t}, $a \leq 1$, we get:

$$a^{Z_t} = 1 + \int_0^t (\log a) a^{Z_{s-}} dZ_s + \sum_{s \le t} (a^{Z_s} - a^{Z_{s-}}) - (\log a) a^{Z_{s-}} \Delta Z_s$$

$$= 1 + \int_0^t (\log a)\ a^{Z_{s-}} dt_s + \sum_{s \le t} (a^{Z_s} - a^{Z_{s-}}) - \int_0^t (\log a) a^{Z_{s-}} dZ_s$$

$$= 1 + \sum_{s \le t} (a-1) a^{Z_{s-}} \Delta X_s + \sum_{s \le t} (1/a-1) a^{Z_{s-}} 1(Z_{s-}) \Delta Y_s$$

$$= 1 + \int_0^t (a-1) a^{Z_{s-}} (dX_s - ds) + \int_0^t (1/a-1) Z^{s-} 1(Z_{s-}) (dY_s - ds)$$

$$+ \int_0^t (a-1) a^{Z_s} ds + \int_0^t (1/a-1) a^{Z_s} 1(Z_s) ds$$

Taking the expected value, we have:

$$E_o a^{Z_t} = 1 + \int_0^t (a-1)\ E_o a^{Z_s} ds + \int_0^t (1/a-1) E_o a^{Z_s} 1(Z_s) ds \tag{9}$$

This is equivalent to:

$$a\ \frac{\partial P(a,t)}{\partial t} = (1-a)[(1-a)P(a,t) - P_o(t)] \tag{10}$$

Initial condition $P(a,0) = 1$

Comparing the expressions of $P^{(1)}(a,t)$ and $P(a,t)$, we can see that the two functions are the same.

3.3. Extension to a class of inter-arrival and service-time distributions

Equation (1) which relates the queue-length process Z to the arrival and service processes clearly does not depend on the fact that X and Y are Poisson processes with rate 1. Using the results on transformation of martingales under a change of probability law, we can study the queue-length process for a general class of processes X and Y. The overall plan is as follows: Let $\mathcal{P}_o$ denote a probability measure which respect to which X and Y are independent, standard Poisson processes. Let $\mathcal{P}$ be any probability measure absolutely continuous with respect to $\mathcal{P}_o$. Clearly, expectation and conditional expectations with respect to can be computed by integrating with respect to $\mathcal{P}_o$ using the density $\frac{d\mathcal{P}}{d\mathcal{P}_o}$.

Let us assume the following

a. $(\Omega, \mathcal{F}, \mathcal{P}_o)$ is a probability space

b. $(X_t, \mathcal{F}_t, t \in [0,1])$, $(Y_t, \mathcal{F}_t, t \in [0,1])$ are real valued, independent standard Poisson processes, hence $\mathcal{F}_t \supseteq \sigma((X_s, Y_s), 0 \le s \le t)$, $(X_t - t, \mathcal{F}_t, t \in [0,1]) \in \mathcal{M}_{loc}(\mathcal{P}_o)$, $(Y_t - t, \mathcal{F}_t, t \in [0,1]) \in \mathcal{M}_{loc}(\mathcal{P}_o)$

c. $\lambda = (\lambda_t, \mathcal{F}_t, t \in [0,1]) \in L_{1loc}(t)$, $\lambda_t \ge 0$ a.s. for all $t \in [0,1]$, $\mu = (\mu_t, \mathcal{F}_t, t \in [0,1]) \in L_{1loc}(t)$, $\mu_t \ge 0$ a.s. for all $t \in [0,1]$ and satisfies $E_o[\varepsilon(\int_0^1 (\lambda_{s-} - 1)(dX_s - ds) + \int_0^t (\mu_{s-} - 1)(dY_s - ds))] = 1$

d. We introduce a new measure

$$\frac{d\mathcal{P}}{d\mathcal{P}_o} = \varepsilon(\int_0^1 (\lambda_{s-} - 1)(dX_s - ds) + \int_0^1 (\mu_{s-} - 1)(dY_s - ds)) \tag{11}$$

Then $\mathcal{P}$ is a probability measure, and using theorem (2.7.9), we have that

$$M_t^{(1)} = X_t - t - \langle \int_0^t (\lambda_{s-} - 1)(dX_s - ds) + \int_0^t (\mu_{s-} - 1)(dY_s - ds), X_t - t \rangle_t$$

$$= X_t - \int_0^t \lambda_s ds$$

$$M_t^{(2)} = Y_t - t - \langle \int_0^t (\lambda_{s-} - 1)(dX_s - ds) + \int_0^t (\mu_{s-} - 1)(dY_s - ds), Y_t - t \rangle_t$$

$$= Y_t - \int_0^t \mu_s ds$$

are such that $(M_t^{(1)}, \mathcal{F}_t, t \in [0,1])$, $(M_t^{(1)}, \mathcal{F}_t, t \in [0,1]) \in \mathcal{M}_{loc}$ under $\mathcal{P}$.

Then, if we define in $(\Omega, \mathcal{F}, \mathcal{P}_o)$ the process Z as we have done in (3.2), we will get in $(\Omega, \mathcal{F}, \mathcal{P})$ a process Z that corresponds to a queue length process. To this process we can relate to other processes, λ and μ; λ is related to X that corresponds to the arrival process in the queue, therefore, λ will be called the arrival rate; μ is related to Y that corresponds to the service process, therefore, μ will be called the service rate. Then the process Z under $\mathcal{P}$ will be a queue length with arrival rate λ and service rate μ.

Remark: For now on E_o will denote expectation with respect to the measure $\mathcal{P}_o$ and E will denote expectation with respect to $\mathcal{P}$.

So far, we have considered that the condition $E_o[\varepsilon[\int_0^1 (\lambda_{s-}-1)(dX_s-ds) + \int_0^1 (\mu_{s-}-1)(dY_s-ds)] = 1$ is satisfied for the given λ and μ. To our knowledge there are no necessary and sufficient conditions for λ and μ to be satisfied in order to the expectation to be 1. The best we can get are sufficient conditions. We are going to discuss some of these.

We have:

$$\langle \int_0^t (\lambda_{s-}-1)(dX_s-ds) + \int_0^t (\mu_{s-}-1)(dY_s-ds), \int_0^t (\lambda_{s-}-1)(dX_s-ds)$$
$$+ \int_0^t (\mu_{s-}-1)(dY_s-ds) \rangle_t = \int_0^t [(\lambda_s-1)^2 + (\mu_s-1)^2]ds \tag{12}$$

By (2.7.3), a sufficient condition for the expectation of the exponential formula being 1 should be:

$$(\lambda_t-1)^2 + (\mu_t-1)^2 \leq k(t) \quad \text{a.s. for all } t \in [0,1]$$

where $k(t)$ is a positive-valued function on $[0,1]$ such that $\int_0^1 k(s)ds < \infty$.

For $|\lambda_t| \leq$ constant and $|\mu_t| \leq$ constant a.s. for all $t \in [0,1]$ we have the desired condition. Also for the case $\lambda_t = f(t)$ and $\mu_t = g(t)$ for all t, where f and g are bounded, non-random functions defined on the interval $[0,1]$, the condition is satisfied.

One case that does not satisfy the condition above is the case where $\lambda_t = \lambda =$ constant for all t and $\mu_t = c(t)Z_{t-} + d(t)$, where $c(t)$ and $d(t)$ are non-negative functions on $[0,1]$, bounded respectively by constants c and d. However, this case can be handled by a special argument as follows. Since $R_t = \varepsilon(\int_0^t (\lambda_{s-}-1)(dX_s-ds) + \int_0^t (\mu_{s-}-1)(dY_s-ds)) \in \mathcal{M}_{loc}$, there is a sequence of stopping times (τ_n), $\lim_{n\to\infty} \tau_n = 1$, such that $R_{t\wedge\tau_n} \in \mathcal{M}_1$. But:

$$R_{t\wedge\tau_n} = \prod_{t_i\leq\tau_n} \lambda_{t_{i-}} \prod_{s_j\leq\tau_n} \mu_{s_{j-}} \exp[-\int_0^t (\lambda_s+\mu_s-2)ds] \tag{13}$$

where (t_i) are the times that X jumps

(s_i) are the times that Y jumps

But

$$R_{t\wedge\tau_n} \leq \lambda^{X_1}(cX_1+d)^{Y_1} \exp(2) \tag{14}$$

and

$$E_o\lambda^{X_1} (cX_1+d)^{Y_1} \exp 2 = \exp[\lambda e^c+d] \tag{15}$$

Therefore, $(R_{t\wedge\tau_n}, n \geq 1)$ is uniformly integrable. This implies that $\lim_{n\to\infty} R_{t\wedge\tau_n} = 1$ since $E_o R_{t\wedge\tau_n} = 1$ for all $n \geq 1$.

Then

$$E[\varepsilon(\int_0^1 (\lambda_{s-}-1)(dX_s-ds) + \int_0^1 (\mu_{s-}-1)(dY_s-ds))] = 1$$

We should mention that it is not always necessary to prove that $E_o\varepsilon = 1$ for the method to be useful. It is undoubtedly true for many cases which cannot be proved.

3.4. Waiting time process and renewal process

In this section we are going to show how to apply the martingale approach to a particular kind of queue. The usual model for queues assumes that the interarrival times and the service times are independent and identically distributed with distribution functions F(.) and G(.) respectively. The corresponding densities will be denoted by f(.) and g(.).

Theorem: [Brêmaud,1972]

Assume the same conditions (a), (b), (d) of section (3.3). In condition (i) we assume

$$\lambda_t = \frac{f(t-\theta_t)}{1-F(t-\theta_t)} \tag{16}$$

$$\mu_t = \frac{g(t-\xi_t)}{1-G(t-\xi_t)} \tag{17}$$

where θ_t is the last jump of X before t and ξ_t is the last jump of Y before t. In addition, assume that

$$E_o[\varepsilon(\int_0^1 (\lambda_{s-}-1)(dX_s-ds) + \int_0^t (\mu_{s-}-1)(dY_s-ds)] = 1$$

Then, under $\mathcal{P}$, X is a renewal process with renewal distribution function F(.) and Y is a renewal process with renewal distribution G(.)

Proof: Let $\tau_n = \inf\{t/X_t = n\}$. Then τ_n is a stopping time and so is $T_n + s$, for any $s \in R^+$.

$$\mathcal{P}\{X(T_n+s) - X(T_n) = 0/\mathcal{F}_{T_n}\} = E(1_{\{X(T_n+s)-X(T_n)=0\}}/\mathcal{F}_{T_n})$$

$$= E_o(1_{\{X(T_n+s)-X(T_n)=0\}} \frac{\Lambda_{T_n+s}}{\Lambda_{T_n}} /\mathcal{F}_{T_n})$$

$$= E_o[\exp(-\int_{T_n}^{T_n+s} \frac{f(u-T_n)}{1-F(u-T_n)} ds)/\mathcal{F}_{T_n}] = 1-F(s)$$

$$\text{where } \Lambda_{T_n+s} = \varepsilon(\int_0^{T_n+s} (\lambda_{s-}-1)(dX_s-ds) + \int_0^{T_n+s} (\mu_{s-}-1)(dY_s-ds)$$

The same is true for Y. The, if we construct our process Z as in (3.3), we will have Z as the queue length with the desired properties.

Remark: For the condition $E_o[\int_0^1 (\lambda_{s-}-1)(dX_s-ds) + (\mu_{s-}-1)(dY_s-ds)] = 1$ we need to impose some conditions on λ and μ as we said before. In this case, if $\frac{f(x)}{1-F(x)}$ is bounded for all $n \geq 0$ and $\frac{g(y)}{1-G(y)}$ is bounded for all $y \geq 0$, we have the desired property.

3.5. The martingale approach for other kinds of queues

Queues with service disciplines other than "one-server first-come-first-served" can also be modeled by stochastic integral equations. For example:

a. Two queues in series:

$dZt = dX_t - 1(Z_{t-})dY_t$ = first queue length (18)

$dW_t = 1(Z_{t-})dY_t - 1(W_{t-})dP_t$ = second queue length

X,Y,P being standard Poisson processes

b. Two queues with the first one having priority over the second (same server)

$$dZ_t = dX_t - 1(Z_{t-})dY_t = \text{first queue length} \tag{19}$$

$$dW_t = dP_t - (1-1(Z_{t-}))\, 1(W_{t-})dY_t = \text{second queue length}$$

X,Y,P being standard Poisson process

Analyses for these cases are similar. However, we will not treat them in detail.

3.6. Summary and Discussion

We have seen above how, beginning with a very simple case of the queue defined on the probability space $\mathcal{P}_o$, we are able to find a expression for the queue length relating the arrival and service processes with it. Then, doing a transformation of measures, we could define, under certain conditions, a new probability measure $\mathcal{P}$, on which our original queue length has associated with it two processes λ and μ which we called the arrival and service rate respectively. In obtaining the queue length Z on $\mathcal{P}$ we did not have to define infinitesimal conditions, that is, there was no need of any Markov assumptions concerning the process, as in the classical approach. Besides that, as it was said before in section (3.2), we have the possibility of working with the process itself, and not with its distributions. These two main advantages will become more apparent in the next chapters where we will deal with problems in optimization, estimation and filtering of queueing processes.

CHAPTER 4

CALCULATION OF CHARACTERISTICS OF A WAITING LINE

4.1. Introduction

We have seen in Chapter 3 how a queueing process with arrival rate $\lambda = (\lambda_t,\ t \in T)$ and service rate $\mu = (\mu_t,\ t \in T)$ can be constructed; λ and μ are non-negative stochastic processes. This was done starting with a probability measure $\mathcal{P}_o$ on our probability space $(\Omega, \mathcal{F})$ under which the queueing process is a very simple one. We then define a transformation of measure. Under the new probability measure $\mathcal{P}$, the queueing process corresponds to one with an arrival rate λ and a service rate μ. Since $\mathcal{P}$ is absolutely continuous with respect to $\mathcal{P}_o$, expectation with respect to $\mathcal{P}$ can be computed working with $\mathcal{P}_o$, exploiting the fact that, under $\mathcal{P}_o$, the queue has rather simple properties.

As we have seen in Chapter 3, we defined:

$$\Lambda_t = E_o\left(\frac{d\mathcal{P}}{d\mathcal{P}_o} \,\Big|\, \mathcal{F}_t\right) = \varepsilon\left(\int_o^t (\lambda_{s-}-1)\ (dX_s - ds) + \int_o^t (\mu_{s-}-1)\ (dY_s - ds)\right) = \tag{1}$$

$$= \prod_{t_i \le t} \lambda_{t_{i-}} \prod_{s_i \le t} \mu_{s_{i-}} \exp\left[-\int_o^t (\lambda_s + \mu_s - 2)\ ds\right]$$

where: (t_i) are the times that X jumps

(s_i) are the times that Y jumps

$\mathcal{F}_t \supseteq \sigma(X_s, Y_s),\ 0 \le s \le t)$

If M is an $\mathcal{F}_t$-measurable random variable, it can be deduced (see [Loève, 1963] or [Wong, 1971a]):

$$E(M|\mathcal{F}_t) = \frac{E_o(\Lambda_t\, E_o(M|\mathcal{F}_t)|\mathcal{F}_t)}{\Lambda_t} \tag{2}$$

A particular case of (2) happens when we want the mean:

$$EM = E_o\Lambda_t\, M \tag{3}$$

The formula (3) allows us to compute expectation with respect to $\mathcal{P}$ working with $\mathcal{P}_o$.

Our major objective in this chapter will be to obtain the characteristics for the case where the interarrival times and service times are independent and identically distributed with distributions $F(x) = \int_0^x f(u)\, du$ and $G(x) = \int_0^x g(u)\, du$. In section (4.2) we will get the expressions for the queue length distribution. In section (4.3) we will calculate the busy-period distribution. In section (4.4) we will deal with the distribution of the nth customer and in section (4.5) we will see how moments of the queue size and an expression for the average waiting time can be computed using stochastic calculus. We will end in section (4.7) with a comparison with the classical method.

4.2 Queue length distribution.

We are interested in calculating the probability that at time t our queue length Z_t is equal to m, a nonnegative integer. Using (3), we can write:

$$\mathcal{P}\{Z_{t=m}\} = E\, I_{\{Z_{t=m}\}} = E_o\Lambda_t\, I_{\{Z_{t=m}\}} = E_o\Lambda_t(I_{\{Z_{t\leq m}\}} - I_{\{Z_{t\leq m-1}\}})$$

Let us suppose that $(\tau_i,\ i\geq 1)$ are defined in the following way:

$$\tau_1 = \{\inf t:\ X_t = 1$$
$$\tau_2 = \{\inf t:\ X_{\tau_1+t} = 2\}$$
$$\vdots$$
$$\tau_n = \{\inf t:\ X_{\tau_{n-1}+t} = n\}$$
$$\vdots$$

In the same way, $(\sigma_j,\ j\geq 1)$ are defined:

$$\sigma_1 = \{\inf t:\ Y_t = 1\}$$
$$\sigma_2 = \{\inf t:\ Y_{\sigma_1+t} = 2\}$$
$$\vdots$$
$$\sigma_n = \{\inf t:\ Y_{\sigma_{n-1}+t} = n\}$$
$$\vdots$$

Then:

$$\mathcal{P}\{Z_{t=m}\} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} E_o(\lambda(\tau_1)\ \lambda(\tau_1+\tau_2)\ \ldots \lambda(\tau_1+\tau_2+\ldots+\tau_k) \cdot$$
$$\cdot\ \mu(\sigma_1)\ \mu(\sigma_1+\sigma_2)\ \ldots\ \mu(\sigma_1+\sigma_2+\ldots+\sigma_\ell)\ \exp[-\int_0^t (\lambda_s+\mu_s-2)ds] \cdot$$
$$\cdot\ [I_{\{\tau_1+\tau_2+\ldots+\tau_k\leq t,\ \tau_1+\tau_2+\ldots+\tau_{k+1} > t\}} \cdot I_{\{\sigma_1+\sigma_2+\ldots+\sigma_2\leq t,\sigma_1+\sigma_2+\ldots}$$
$$+\ \sigma_{\ell+1} > t\} \cdot [I_{\{Z_t\leq m\}} - I_{\{Z_t\leq m-1\}}])$$

But

$$I_{\{\tau_1+\tau_2+\ldots+\tau_k\leq t,\ \tau_1+\tau_2+\ldots+\tau_{k+1}>t\}}\ I_{\{\sigma_1+\sigma_2+\ldots+\sigma_\ell\leq t\ ,\sigma_1+\sigma_2+\ldots+\sigma_{\ell+1}>t\}} \cdot$$

$$\cdot\ I_{\{Z_t\leq m\}} = I_{\{\tau_1+\tau_2+\ldots+\tau_k\leq t_1\tau_1+\tau_2+\ldots+\tau_{k+1}>t,\ \sigma_1+\sigma_2+\ldots+\sigma_\ell\geq t,}$$

$$\sigma_1+\sigma_2+\ldots+\sigma_{\ell+1}>t\} \cdot I_{\{\sigma_1+\sigma_2+\ldots\sigma_\ell\geq\sigma_1+\sigma_2+\ldots+\sigma_{k-m}}\ ,$$

$$\sigma_1+\sigma_2+\ldots+\sigma_{\ell-1}\geq\tau_1+\tau_2+\ldots+\tau_{k-m-1},\ldots,\sigma_1+\sigma_2+\ldots+\sigma_{\ell-k+m+1}\geq\tau_1\}$$

$$= Q^{(m)}_{k\ell}(t) \tag{4}$$

In the same way, we find:

$$I_{\{\tau_1+\tau_2+\ldots+\tau_k\leq t_1\tau_1+\tau_2+\ldots+\tau_{k+1}>t,\ \sigma_1+\sigma_2+\ldots+\sigma_\ell\leq t,\ \sigma_1+\sigma_2+\ldots+\sigma_{\ell+1}\geq t\}}$$

$$I_{\{Z_t=m-1\}} = I_{\{\tau_1+\tau_2+\ldots+\tau_k\leq t,\ \tau_1+\tau_2+\ldots+\tau_{k+1}>t,\ \sigma_1+\sigma_2+\ldots+\sigma_\ell\leq t,}$$

$$_{\sigma_1+\sigma_2+\ldots+\sigma_{\ell+1}\geq t\}} \cdot I_{\{\sigma_1+\sigma_2+\ldots+\sigma_\ell\geq\tau_1+\tau_2+\ldots+\tau_{k-m+1},}$$

$$_{\sigma_1+\sigma_2+\ldots+\sigma_{\ell-1}\geq\tau_1+\tau_2+\ldots+\tau_{k-m},\ldots,\sigma_1+\sigma_2+\ldots+\sigma_{\ell-k+}\geq\tau_1\}} =$$

$$= Q^{(m-1)}_{k\ell}(t) \tag{5}$$

Since we must have $Z_t = m$, our expression for $\mathcal{P}\{Z_t=m\}$ reduces to

$$\mathcal{P}\{Z_t=m\} = \sum_{k=m}^{\infty}\ \sum_{\ell=k-m}^{\infty} E_o(\lambda(\tau_1)\lambda(\tau_1+\tau_2)\ldots\lambda(\tau_1+\tau_1+\ldots+\tau_k)\ \cdot$$

$$\cdot\ \mu(\sigma_1)\mu(\sigma_1+\sigma_2)\ldots\mu(\sigma_1+\sigma_2+\ldots+\sigma_\ell)\ \cdot\ [Q^{(m)}_{k\ell}X(t) - Q^{(m-1)}_{k\ell}]\ \cdot$$

$$\cdot\ \exp\left[-\int_0^t (\lambda_s+\mu_s-2)\,ds\right]) \tag{6}$$

The expression (6) is valid for any λ and any μ, satisfying the hypotheses in (3.3). Under our assumptions on the interarrival and service times, we have:

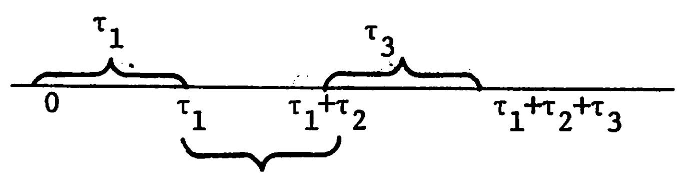
$$\mathcal{P}\{Z_t=m\} = \sum_{k=m}^{\infty} \sum_{\ell=k-m}^{\infty} E_o(f(\tau_1)\ f(\tau_2)\ldots f(\tau_k)\ \cdot$$

$$\cdot\ [1-F(t-(\tau_1+\tau_2+\tau_k))]\ g(\sigma_1)g(\sigma_2)\ldots g(\sigma_\ell)\ \cdot\ [1-G(t-(\sigma_1+\sigma_2+\ldots+\sigma_\ell))]\ \cdot$$

$$\cdot\ [Q_{k\ell}^{(m)}(t) - Q_{k\ell}^{(m-1)}(t)]\ \cdot\ \exp\,(-2t)) \tag{7}$$

Under $\mathcal{P}_o$, $(\tau_i, i\geq 1)$ and $(\sigma_j, j\geq 1)$ use two independent families of independent and identically random variables with an exponential distribution having parameter 1. The density function of τ_1, $\tau_1+\tau_2$, $\tau_1+\tau_2+\tau_3,\ldots,\tau_1+\tau_2+\ldots+\tau_k$, τ_{k+1} is given by:

$$p(u_1,u_2,\ldots,u_k,t_{k+1}) = e^{-u_k} e^{-t_{k+1}}\ dt_{k+1} du_1\ du_2\ \ldots\ d\mu_k \tag{8}$$



The density function of $\sigma_1,\sigma_1+\sigma_2,\sigma_1+\sigma_2+\sigma_3,\ldots,\sigma_1+\sigma_2+\sigma_3+\ldots+\sigma_\ell,\sigma_{\ell+1}$, is given by:

$$p(v_1,v_2,\ldots,v_\ell,s_{\ell+1}) = e^{-v_\ell} e^{-t_{\ell+1}} ds_{\ell+1}\ dv_1\ \ldots\ dv_\ell\ ds_{\ell+1} \tag{9}$$

Therefore:

$$\mathcal{P}\{Z_t=m\} - \sum_{k=m}^{\infty} \sum_{\ell-k-m} \int_0^t \int_0^{u_k} \int_0^{u_{k-1}} \cdots \int_0^{u_2} \int_{u_{k-n-1}}^{t} \int_{u_{k-m-1}}^{v_\ell} \cdots$$

$$\int_{u_1}^{v_{\ell-k+m+2}} \int_0^{v_{\ell-k+m+1}} \cdots \int_0^{v_2} \int_{t-u_k}^{\infty} \int_{t-v_\ell}^{\infty} e^{-u_k} e^{-v_\ell - t_{k+1}} e^{-s_{\ell+1}}$$

$$f(u_1)f(u_2-u_1)f(u_3-u_2)\ldots f(u_k-u_{k-1}) \cdot$$

$$\cdot\ [1-F(t-u_k)]\ g(v_1)g(v_2-v_1)g(v_1)g(v_2-v_1)g(v_3-v_2)\ldots g(v_\ell-v_{\ell-1}) \cdot$$

$$\cdot\ [1-G\ (t-v_\ell)]\ \exp(2t)\ ds_{\ell+1}dt_{k+1}dv_1\ldots dv_{\ell-k+m}dv_{\ell-k+m+1}\ldots dv_{\ell-1} \cdot$$

$$\cdot\ dv_\ell\ du_1\ \ldots\ du_{k-2}\ du_{k-1}\ du_k] -$$

$$-\sum_{k=m}^{\infty}\ \sum_{\ell=k-m}^{\infty} [\int_0^t\int_0^{u_k}\int_0^{u_{k-1}}\cdots\int_0^{u_2}\int_{u_{k-m+1}}^{t}\int_{u_{k-m}}^{v_\ell}\cdots\int_{u_1}^{v_{\ell-k+m+1}}\int_0^{v_{\ell-k+m}}\cdots$$

$$\cdots\int_0^{v_2}\int_{t-u_k}^{\infty}\int_{t-v_\ell}^{\infty} e^{-u_k}e^{-v_\ell}e^{-t_{k+1}}e^{-s_\ell}\ f(u_1)f(u_2-u_1)f(u_3-u_2)\ldots f(u_k-u_{k-1}) \cdot$$

$$\cdot\ [1-F(t-u_k)]\ g(v_1)g(v_2-v_1)g(v_3-v_2)\ldots g(v_\ell-v_{\ell-1})[1-G(t-v_\ell)]\ell^{2t}ds_{\ell+1} \cdot$$

$$\cdot\ dt_{k+1}dv_1\ldots dv_{\ell-k+m-1}dv_{\ell-k+m}\ldots dv_{\ell-1}dv_\ell du_1\ldots du_{k-2}du_{k-1}du_k] \qquad (10)$$

The expression (10) allows us to calculate the queue size distribution for queues with interarrival times and service times that are independent and identically distributed with distributions $f(x) = \int_o^n f(u)\ du$ and $G(y) = \int_o^y g(u)\ du$ respectively. We did not use generating functions, Laplace transforms and complex number theory to get the desired results, as it is done in the classical literature; the probability $\mathcal{P}\{Zt = m\}$, is obtained directly as a function of t. Besides that, we obtained a general answer for the problem, instead of a solution for particular cases (see, for instance, [Takács, 1962] and [Saaty, 1961]. Therefore the answer obtained in (10) is more general and, in our point of view, more straightforward.

For the case where the distribution of X is exponential with parameter λ = constant, and the distribution of Y is exponential with parameter μ = constant, we have:

$$\lambda_s = \frac{\lambda\, e^{-\lambda(s-\theta_s)}}{e^{-\lambda(s-\theta_s)}} = \lambda$$

$$\mu_s = \frac{\mu\, e^{-\mu(s-\xi_s)}}{e^{-\lambda(s-\xi_s)}} = \mu$$

and

$$\mathcal{P}\{Z_t=m\} = \sum_{k=m}^{\infty} \sum_{\ell=k-m}^{\infty} \lambda^k \mu^\ell e^{-(\lambda+\mu)t} (E_o\, Q_{k\ell}^{(m)} - E_o\, Q_{k\ell}^{(m-1)}) =$$

$$\sum_{k=}^{\infty} \sum_{\ell=K-}^{\infty} (\lambda^k \mu^\ell e^{-(\lambda+\mu)t} \int_0^t \int_0^{u_k} \int_0^{u_{k-1}} \cdots \int_0^{u_2} \int_{u_{k-m}}^{t} \int_{u_{k-1}}^{v_\ell} \cdots$$

$$\int_{u_1}^{v_{\ell-k+m+2}} \int_0^{v_{\ell-k+m+1}} \cdots \int_0^{v_2} \int_{t-u_k}^{\infty} \int_{t-v_\ell}^{\infty} e^{-u_k} e^{-v_\ell} \,.$$

$$\cdot\; e^{-t_{k+1}} e^{-s_{\ell+1}} e^{2t} ds_{\ell+1} dt_{k+1} dv_1 \ldots dv_{\ell-k+m} dv_{\ell-k+m+1} \ldots dv_{\ell-1} dv\; du_1 \;\cdot$$

$$\ldots du_{k-2} du_{k-1} d\mu_k \text{-} \lambda^k \mu^\ell e^{-(\lambda=\mu)t} \int_0^t \int_0^{u_k} \int_0^{u_{k-1}} \cdots \int_0^{u_2} \int_{u_{k-m+1}}^{t} \int_{u_{k-m}}^{v_\ell} \cdots$$

$$\cdots \int_{u_1}^{v_{\ell-k+m+1}} \int_0^{v_{\ell-K+m}} \cdots \int_0^{v_2} \int_{t-u_k}^{\infty} \int_{t-v_\ell}^{\infty} e^{-u_k} e^{-v_\ell} e^{-t_{k+1}} e^{-s_\ell} e^{2t} \cdot$$

$$ds_{\ell+1} dt_{m+1} dv_1 \ldots dv_{\ell-k+m-1} dv_{\ell-k+m} \ldots dv_{\ell-1} dv_\ell du_1 \ldots du_{k-2} du_{k-1} du_k) \quad (11)$$

For the particular case $m = 0$ we get, with the expression (11):

$$\mathcal{P}\{Z_t = 0\} = \exp\,[-(\lambda+\mu)t] \cdot \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} \lambda^k \mu^\ell \frac{t^{\ell+k}}{(\ell+1)!k!} \quad (12)$$

Rearranging the terms in the summation, we can find the classical answer:

$$\mathcal{P}\{Z_t = 0\} = \exp\,[-(\lambda+\mu)t] \cdot [I_0(2\sqrt{\lambda\mu}\ t) +$$

$$+ (\sqrt{\tfrac{\mu}{\lambda}})\ I_1(2\sqrt{\lambda\mu}\ t) + (1 - \tfrac{\lambda}{\mu}) \sum_{k=2}^{\infty} (\sqrt{\tfrac{\mu}{\lambda}})^k\ I_k(2\sqrt{\lambda\mu}\ t)] \quad (13)$$

Comparing the method used for obtaining $\mathcal{P}\{Zt = 0\}$, we see that there is no need to apply Rouché's theorem, as it is done in [Saaty, 1961]; or consider that we have a finite queue, then calculate the answer for this case and, by limits, to obtain the result for infinite queue, as in [Takács, 1962].

4.3. Busy period distribution

A busy period is an interval of time during which there are customers in the system. A busy period begins with the arrival of a customer when the queue length is zero, and ends at the time the queue length is zero again. Then, since we are primarily interested in the case of independent, identically distributed service times, we can treat the problem as a queue which starts with $Z_0 = 1$ and the busy period is the time where, for the first time, there is noone waiting in the system.

Remark: If a queue starts with customers waiting at time 0, the initial busy period will have different distribution than the subsequent busy periods, which are the ones we will deal with primarily. To get the distribution in that case will involve the same reasoning, with some modifications, that we will apply.

We are interested in calculating

$$\mathcal{P}\{T_b \leq x\} = EI_{\{T_b \leq x\}} = E_o \Lambda_x \, I_{\{T_b \leq x\}} = E_o \Lambda_{T_b} \, I_{\{T_b \leq x\}}$$

where T_b is the busy period.

Let us define:

$$\lambda_t^{(1)} = \begin{cases} \lambda_t & \text{if } Z_u \neq 0,\ 0 \leq u \leq t \\ 1 & \text{otherwise} \end{cases}$$

$$\mu_t^{(1)} = \begin{cases} \mu_t & \text{if } Z_u \neq 0,\ 0 \leq u \leq t \\ 1 & \text{if not} \end{cases}$$

Then we can express the Radon-Nikodym derivative by:

$$\Lambda_t^{(1)} = \varepsilon\left(\int_0^t (\lambda_{s-}^{(1)} - 1)\ (dX_s - ds) + \int_0^t (\mu_{s-}^{(1)} - 1)\ (dY_s - ds)\right. \qquad (14)$$

and $\Lambda_X^{(1)} = \Lambda_{T_b}$

Therefore:

$$\mathcal{P}\{T_b \le x\} = E_o \Lambda_X^{(1)} \, I_{\{T_b \le x\}} \tag{15}$$

and

$$\mathcal{P}\{T_b \le x\} = E_o\Big(\sum_{k=0}^{\infty} \lambda(\tau_1)\lambda(\tau_1+\tau_2)\ldots\lambda(\tau_1+\tau_2+\ldots+\tau_k)\mu(\sigma_1) \cdot$$

$$\cdot\ \mu(\sigma_1+\sigma_2)\ldots\mu(\sigma_1+\sigma_2+\ldots+\sigma_{k+1}) \exp\Big[-\int_0^x (\lambda_s+\mu_s-2)\, ds\Big] \cdot$$

$$\cdot\ I_{\{\sigma_1+\sigma_2+\ldots+\sigma_{k+1}\le x\}} \tag{16}$$

Specializing to the case where the interarrival times and the service times are both i.i.d., we have for (16)

$$\mathcal{P}\{T_b \le x\} = \sum_{k=0}^{\infty} (f(\tau_1)f(\tau_2)\ldots f(\tau_k)[1-F(\sigma_1+\sigma_2+\ldots\sigma_{k+1}-\tau_1-\tau_2-\ldots-\tau_k)]$$

$$g(\sigma_1)g(\sigma_2)\ldots g(\sigma_k)g(\sigma_{k+1})\ e^{2(\sigma_1+\sigma_2+\ldots+\sigma_k)} \cdot I_{\{\sigma_1+\sigma_2+\ldots+\sigma_{k+1}\le x\}} =$$

$$= \sum_{k=0}^{\infty} \int_0^x \int_0^{v_{k+1}} \int_0^{u_k} \int^{u_{k-1}} \cdots \int_0^{u_2} \int_0^{v_{k+1}} \int_0^{v_k} \cdots \int_0^{v_2} \int_0^{\infty} e^{-u_{k+1}} e^{-v_{k+1}} \cdot f(u_1) \cdot$$

$$\cdot\ f(u_2-u_1)\ldots f(u_k-u_{k-1})[1-F(v_{k+1}-u_k)]\ g(v_1)g(v_2)\ldots g(v_{k+1}-v_k)$$

$$e^{2v_{k+1}} du_{k+1} \cdot dv_1\ldots dv_{k-1}dv_2du_1\ldots du_{k-2}du_{k-1}dv_{k+1} =$$

$$= \sum_{k=0}^{\infty} \int_0^x \int_0^{v_{k+1}} \int_0^{u_k} \int_0^{u_{k-1}} \cdots \int_0^{u_2} \int_0^{v_{k+1}} \int_0^{v_k} \cdots \int_0^{v_2} f(u_1) f(u_2-u_1) \ldots$$

$$\ldots f(u_k-u_{k-1}) \cdot [1-F(v_{k+1}-u_k)]\, g(v_1) g(v_2) \ldots g(v_{k+1}-v_k)$$

$$dv_1 \ldots dv_{k-1} du_1 \ldots du_{k-2} du_{k-1} du_k dv_{k+1} \tag{17}$$

Let us use the expression to calculate the busy period distribution for the case $\lambda_s = \lambda =$ constant and $\mu_s = \frac{g(s-\xi_s)}{1-G(s-\xi_s)}$.

$$\mathcal{P}\{T_b \leq x\} = \sum_{k=0}^{\infty} \int_0^x \int_0^{v_{k+1}} \int_0^{u_k} \int_0^{u_{k-1}} \cdots \int_0^{u_2} \int_0^{v_{k+1}} \cdots \int_0^{v_2}$$

$$\lambda^k e^{-\lambda v_{k+1}} g(v_1) \cdot g(v_2) \ldots g(v_{k+1}-v_k) dv_1 \ldots dv_{k-1} du_1 \ldots du_{k-2} du_{k-1}$$

$$du_k dv_{k+1} = \sum_{k=0}^{\infty} \int_0^x \int_0^{v_{k+1}} \int_0^{u_k} \int_0^{u_{k-1}} \cdots \int_0^{u_2} \lambda^k e^{-\lambda v_{k+1}} dG_{k+1}(v_{k+1})$$

$$du_1 \ldots du_{k-2} du_{k-1} du_k dv_{k+1} = \sum_{k=0}^{\infty} \int_0^x e^{-\lambda v_{k+1}} \frac{(\lambda u)^k}{k!} dG_{k+1}(v_{k+1}) \tag{18}$$

Remark: $G_{k+1}(x)$ denotes the <u>k</u>th iterated convolution of $G(x)$ with itself.

The expression (18) can be found in [Takács, 1962, p. 58].

We can also specialize formula (17) for the case

$\lambda_s = \frac{f(s-\theta_s)}{1-F(s-\theta_s)}$ and $\mu_s = \mu =$ constant. We obtain

$$\mathcal{P}\{T_b \le x\} = \sum_{k=0}^{\infty} \int_0^x \int_0^{v_{k+1}} \int_0^{u_k} \int_0^{u_{k-1}} \cdots \int_0^{v_2} \int_0^{v_{k+1}} \int_0^{v_k} \cdots \int_0^{v_2} f(u_1) \cdot$$

$$\cdot\ f(u_2-u_1)\ldots f(u_k-u_{k-1})[1-F(v_{k+1}-u_k)]\ \mu^{k+1} e^{-\mu v_{k+1}} dv_1\ldots dv_{k-1} \cdot$$

$$\cdot\ du_1\ldots du_{k-2}du_{k-1}du_k dv_{k+1} = \mu \sum_{k=0}^{\infty} \int_0^x e^{-\mu v_{k+1}} \frac{(\mu v_{k+1})^k}{(k-1)!}$$

$$[1-F_{k+1}(v_{k+1})]\ dv_{k+1} \tag{19}$$

This corresponds to the result given by Takács [1962] in page 124. Since (18) and (19) correspond to known results, we know that (17) is the correct distribution. Moreover, we have obtained a more general solution.

4.4 <u>Calculation of the distribution of the nth customer waiting time</u>

Let us define the waiting time α_n by

$\alpha_n = \inf \{t \geq 0: Z_t = n\}$

α_n is the arrival time of the <u>n</u>th customer. Then:

$\alpha_{n,0} = \inf \{t \geq 0: \ Z_{\alpha_{n+t}} = 0\}$ = <u>n</u>th customer waiting time.

We are interested in calculating $\{\alpha_{n,0} \leq n\}$, the waiting time distribution of the <u>n</u>th customer. Let us define:

$$\lambda_t^{(1)} = \begin{cases} \lambda_t & \text{if the number of arrivals up to time t is less than n} \\ 1 & \text{otherwise} \end{cases}$$

$$\mu_t^{(1)} = \begin{cases} \mu_t & \text{if } t \le \alpha_n + \alpha_{n,0} \\ 1 & \text{otherwise} \end{cases}$$

If we define $\Lambda_t^{(1)}$ using $\Lambda_t^{(1)}$, $\mu_t^{(1)}$, we get:

$$\mathcal{P}\{\alpha_{n,0} \le x\} = E_o \Lambda_t^{(1)} \; I_{\{\alpha_{n,0} \le n\}} =$$

$$= E_o(\sum_{\ell=n}^{\infty} \lambda(\tau_1)\lambda(\tau_1+\tau_2)\ldots\lambda(\tau_1+\tau_2+\ldots+\tau_n)\; \mu(\sigma_1) \cdot$$

$$\cdot\; \mu(\sigma_1+\sigma_2)\ldots\mu(\sigma_1+\sigma_2+\ldots+\sigma_\ell) \cdot \exp\,[-\int_0^{\tau_1+\tau_2+\ldots+\tau_n} \lambda_s\, ds -$$

$$- \int_0^{\sigma_1+\sigma_2+\ldots+\sigma_\ell} \mu_s\, ds + (\tau_1+\tau_2+\ldots+\tau_n)+(\sigma_1+\sigma_2+\ldots+\sigma_\ell)] \cdot$$

$$I_{\{\sigma_1+\sigma_2+\ldots+\sigma_\ell \le \tau_1+\tau_2+\ldots+\tau_n+n\}} \cdot I_{\{\tau_1+\tau_2+\ldots+\tau_n \le \sigma_1+\sigma_2+\ldots+\sigma_\ell,}$$

$$\tau_1+\tau_2+\ldots+\tau_{n-1} \le \sigma_1+\sigma_2+\ldots+\sigma_{\ell-1},\; \ldots,\; \tau_1 \le \sigma_1+\sigma_2+\ldots+\sigma_{\ell-n+1}\} \tag{20}$$

For the case where the interrival times and the service times are both i.i.d., we have:

$$\mathcal{P}\{\alpha_{n,0} \le n\} = \sum_{\ell=n}^{\infty} E_o(f(\tau_1)f(\tau_2)\ldots f(\tau_n)g(\sigma_1)g(\sigma_2)\ldots g(\sigma_2)$$

$$\exp[(\tau_1+\tau_2+\ldots+\tau_n)+(\sigma_1+\sigma_2+\ldots+\sigma_\ell)]\; I_{\{\sigma_1+\sigma_2+\ldots+\sigma_\ell \le \tau_1+\tau_2+\ldots+\tau_n+x\}} \cdot$$

$$\cdot\; I_{\{\tau_1+\tau_2+\ldots+\tau_n \le \sigma_1+\sigma_2+\ldots+\sigma_\ell,\; \tau_1+\tau_2+\ldots+\tau_{n-1} \le \sigma_1+\sigma_2+\ldots+\sigma_{\ell-1},\; \ldots,}$$

$$\tau_1 \le \sigma_1 + \ldots + \sigma_{\ell-n+1}\} = \sum_{\ell=n}^{\infty} \int_0^{\infty} \int_0^{u_n} \cdots \int_0^{u_2} \int_{u_n}^{u_{n+X}} \int_{u_{n-1}}^{u_n} \cdots \int_{u_1}^{v_{\ell-n+1}} \int_0^{v_{\ell-n}} \cdots$$

$$\cdots \int_0^{v_2} f(u_1)f(u_2-u_1)\ldots f(u_{k-1}) \cdot g(v_1)g(v_2-v_1)\ldots g(v_\ell - v_{\ell-1})$$

$$dv_1 \ldots dv_{\ell-n-1} dv_{\ell-n} \ldots dv_{\ell-1} dv \quad du_1 \ldots du_{n-1}\, du_n \tag{21}$$

We have tried to compare (21) with the results in the classical literature, but we have found it very difficult to do that because the results are in Laplace transform form.

4.5 Calculation of the virtual waiting time

The virtual waiting time at time t is the time that a customer would wait if he joined the queue at the instant t. Then, the virtual waiting time is the difference between the time that all customers, waiting at time t, are served and the time t.

Let us define:

$$\lambda_s^{(1)} = \begin{cases} \lambda_s & \text{if } s \le t \\ 1 & \text{if } s > t \end{cases}$$

$$\mu_s^{(1)} = \begin{cases} \mu_s & \text{if } s < t \text{ on } s \ge t \text{ and } Z_u \ne 0 \text{ for } t \le u \le s \\ 1 & \text{otherwise} \end{cases}$$

With $\lambda_s^{(1)}$ and $\mu_s^{(1)}$ define $\Lambda_t^{(1)}$ as usual. Then, if we called the virtual waiting time at time t, η_t, we have:

$$\mathcal{P}\{N_t \le x\} = E_o \Lambda^{(1)}_{t+} I_{\{\eta_t \le x\}} =$$

$$= E_o\Big(\sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} \lambda(\tau_1)\lambda(\tau_1+\tau_2)\ldots\lambda(\tau_1+\tau_2+\ldots+\tau_k)\ \mu(\sigma_1)\mu(\sigma_1+\sigma_2)\ldots$$

$$\mu(\sigma_1+\sigma_2+\ldots+\sigma_\ell)\exp[-\int_0^t (\lambda_s-1)\,ds - \int_0^{\sigma_1+\sigma_2+\ldots+\sigma_\ell} (\mu_s-1)\,ds] \cdot$$

$$\cdot\ I_{\{\tau_1+\tau_2+\ldots+\tau_k\le\sigma_1+\sigma_2+\ldots+\sigma_\ell,\ \tau_1+\tau_2+\ldots+\tau_{k-1}\le\sigma_1+\sigma_2+\ldots+\sigma_{\ell-1},\ldots}$$

$$\ldots,\ \tau_1\le\sigma_1+\sigma_2+\ldots\sigma_{\ell-k+1}\}} \cdot I_{\{\alpha_1+\alpha_2+\ldots+\alpha_{\ell-1}\le\tau_1+\tau_2+\ldots\tau_k,}$$

$$\sigma_1+\sigma_2+\ldots+\sigma_\ell\le\tau_1+\tau_2+\ldots+\tau_\ell+x\} \tag{22}$$

For the case where the interarrival times and the service times are both i.i.d., we obtain:

$$\mathcal{P}\{\eta_t\le x\} = \sum_{k=0}^{\infty}\sum_{\ell=k}^{\infty}\int_0^t\int_0^{u_k}\cdots\int_0^{u_2}\int_{u_k}^{u_{k+n}}\cdots\int_{u_1}^{v_{\ell-k+1}}\int_0^{v_{\ell-k}}\cdots\int_0^{v_2} f(u_1)\ \cdot$$

$$\cdot\ f(u_2-u_1)\ldots f(u_k-u_{k-1})[1-F(t-u_k)]\ g(v_1)g(v_2-v_1)\ldots g(v_\ell-v_{\ell-1})$$

$$dv_1\ldots dv_{\ell-k-1}dv_{\ell-k}\ldots dv_\ell\ du_1\ldots du_{k-1}\ du_k$$

For the case $\lambda_s = \lambda$ = constant, $\mu_s = \mu$ = constant, we have:

$$\mathcal{P}\{\eta_t<n\} = \sum_{k=0}^{\infty}\sum_{\ell=k}^{\infty}\int_0^t\int_0^{u_k}\cdots\int_0^{u_k+x}\cdots\int_{u_1}^{v_{\ell-n+1}}\int_0^{v_{\ell-k}}\cdots\int_0^{v_\ell}\lambda^k e^{-\lambda t}$$

$$\mu^{\ell} \quad {}^{-v_\ell} dv_1 \ldots dv_{\ell-1-1}\, dv_{\ell-n} \ldots dv_\ell\, du_1 \ldots du_{k-1}\, du_2 \tag{23}$$

Rearranging the expression (22) we have:

$$\mathcal{P}\{\eta_t \le x\} = \mathcal{P}\{Z_t = 0\} + \sum_{k=1}^{\infty} \mathcal{P}\{Z_t = k\} \int_0^k e^{-\mu y} \frac{(\mu y)^{k-1}}{(k-1)!} \mu dy \tag{24}$$

This is the result obtained by Takács [1962, p. 38]. For other types of queues, the expression for the virtual waiting time becomes to much complicated and it is difficult to compare with expressions already obtained.

4.6 Calculation of the queue length moments and the average waiting time

Knowing the distribution of the queue length, we can calculate the moments, but, in some cases, it is possible to get expressions for them simpler to calculate.

The moment of order n, n = 1,2,3,..., will be given by

$$EZ_t^n = E_o\, \Lambda_t\, Z_t^n \tag{25}$$

Applying the differentiation rule to $\Lambda_t Z_t^n$, we get:

$$\Lambda_t Z_t^n = \int_0^t n\Lambda_{s-}\, Z_s^{n-1}\, dZ_s + \int_0^t Z_{s-}^n d\Lambda_s + \sum_{s\le t} [\Lambda_s Z_s^n - \Lambda_{s-} Z_{s-}^n -$$

$$- N\, \Lambda_{s-}\, Z_s^{n-1} \Delta Z_s - \Lambda_{s-}\, Z_{s-}^n\, \Delta\Lambda_s] =$$

$$= \int_0^t Z_{s-}^n\, d\Lambda_s + \sum_{s\le t} [(\Lambda_{s-} + \Delta\Lambda_s)(Z_{s-} + \Delta Z_s)^n - \Lambda_{s-} Z_{s-}^n - \Lambda_{s-} Z_{s-}^n \Delta\Lambda_s] =$$

$$= \int_0^t Z_{s-}^n d\Lambda_s + \sum_{s\leq t} [(\Lambda_{s-}+\Delta\Lambda_s)\ (\binom{n}{1}\ Z_{s-}^{n-1}(\Delta Z_s) + \binom{n}{2}\ Z_{s-}^{n-2}(\Delta Z_s)^2 + \ldots +$$

$$+ (\Delta Z_s)^n] =$$

$$= \int_0^t Z_{s-}^n d\Lambda_s + \sum_{s\leq t}(\lambda_{s-}\Lambda_{s-}\Delta X_s + \mu_{s-}\Lambda_{s-}\Delta Y_s)\ [\binom{n}{1}\ Z_s^{n-1}(\Delta Z_s) + \binom{n}{2} Z_x^{n-2}(\Delta Z_s)^2 +$$

$$+ \ldots + (\Delta Z_s)^n] =$$

$$= \int_0^t Z_{s-}^n d\Lambda_s + \int_0^t \lambda_{s-}\Lambda_{s-}[\binom{n}{1}\ Z_{s-}^{n-1} + \binom{n}{2}\ Z_{s-}^{n-2} + \ldots + 1]\ dX_s +$$

$$+ \int_0^t \mu_{s-}\Lambda_{s-}[-\binom{n}{1}\ Z_{s-}^{n-1} + \binom{n}{2} Z_{s-}^{n-2} + \ldots + (-1)^n]\ 1(Z_{s-})\ dY_s$$

Taking the expected value, we obtain:

$$E_o[\Lambda_t Z_t^n] = \int_0^t [\binom{n}{1} E_o\Lambda_s\lambda_s Z_s^{n-1} + \binom{n}{2}\ E_o\Lambda_s\lambda_s Z_s^{n-2} + \ldots + E_o\Lambda_s\lambda_s]\ ds +$$

$$+ \int_0^t [-\binom{n}{1}\ E_o\Lambda_s\mu_s Z_s^{n-1} + \binom{n}{2}\ E_o\Lambda_s\mu_s Z_s^{n-2} + \ldots +$$

$$+ \binom{n}{n-1}\ (-1)^{n-1}\ E_o\Lambda_s\mu_s Z_s]\ ds + \int_0^t (-1)^n\ E_o\Lambda_s\mu_s n(n_s)\ ds \qquad (26)$$

Using the formula we can calculate the mean:

$$EZ_t = \int_0^t E_o\Lambda_s\lambda_s \, ds - \int_0^t E_o\Lambda_s\mu_s \, 1(Z_s) \, ds =$$

$$= \int_0^t E(\lambda_s-\mu_s) \, ds + \int_0^t E \, \mu_s \, I_{\{Z_s=0\}} \, ds \tag{27}$$

For the case $\lambda_t = \lambda$ = constant and $\mu_t = \mu$ = constant, we have

$$E_o\Lambda_t Z_t^n = \int_0^t [\binom{n}{1} \lambda \, E \, Z_s^{n-1} + \binom{n}{2} \lambda \, E \, Z_s^{n-2} + \dots + \lambda] \, ds +$$

$$+ \int_0^t [-\binom{n}{1} \mu \, E \, Z_s^{n-1} + \binom{n}{2} \mu \, E \, Z_s^{n-2} + \dots + \binom{n}{n-1}(-1)^{n-1} \mu \, E \, Z_s] \, ds +$$

$$+ \int_0^t (-1)^n \mu \, E \, 1(Z_s) \, ds \tag{28}$$

$$EZ_t = (\lambda-\mu)t + \mu \int_0^t \mathcal{P}\{Z_s = 0\} \, ds \tag{29}$$

In this case, for calculating the mean, we only have to know the value of $\mathcal{P}\{Z_t = 0\}$.

Another interesting case arises when $\lambda_t = \lambda$ = constant and $\mu_t = c \, Z_{t-}$, where c is a constant. Then we find, after differentiating the expression corresponding to $E \, Z_t^n$:

$$\frac{d(EZ_t^n)}{dt} = \lambda\binom{n}{1} EZ_t^{n-1} + \lambda\binom{n}{2} EZ_t^{n-2} + \ldots + \lambda -$$

$$- c\binom{n}{1} EZ_t^n + c\binom{n}{2} EZ_t^{n-1} + \ldots + c\binom{n}{n-1}(-1)^{n-1}EZ_t^2 + c(-1)^n EZ_t \qquad (30)$$

The mean is:

$$\frac{d(EZ_t)}{dt} = \lambda - c(EZ_t) \qquad (31)$$

The solution of equation (31) is $EZ_t = \frac{\lambda}{c}(1-e^{-ct})$

The average waiting time is given by the expected value of the virtual waiting time. In the case of i.i.d. service times, the virtual waiting time is given by:

$$d\eta_t = \alpha_t \, dX_t - 1(Z_{t-}) \, dt \qquad (32)$$

where $(\alpha_t, t \geq 0)$ is the waiting time of a customer arriving at time t. The average waiting time will be given by:

$$E\eta_t = E_o \Lambda_t \eta_t \qquad (33)$$

Using stochastic calculus in $\Lambda_t \eta_t$, we get:

$$\Lambda_t \eta_t = \Lambda_0 \eta_0 + \int_0^t \Lambda_{s-} d\eta_s + \int_0^t \eta_{s-} d\Lambda_{s-} + \sum_{s \leq t} \Delta\eta_s \, \Delta\Lambda_s =$$

$$= \Lambda_0 \eta_0 + \int_0^t \Lambda_{s-} d\eta_s + \int_0^t \eta_{s-} \Lambda d \quad s + \sum_{s \leq t} (\alpha_s \Delta X_s)[(\lambda_{s-}-1)\Lambda_s \Delta X_s +$$

$$+ (\mu_{s-}-1)\, \Lambda_{s-}\, \Delta Y_s] =$$

$$= \Lambda_0 \eta_0 + \int_0^t \Lambda_{s-} d\eta_s + \int_0^t \eta_{s-}\, d\Lambda_s + \int_0^t (\lambda_{s-}-1)\, \alpha_s \Lambda_{s-} dX_s \tag{34}$$

Therefore:

$$E_o \Lambda_t \eta_t = E_o \Lambda_0 \eta_0 + \int_0^t E_o \Lambda_s \alpha_s\, ds - \int_0^t E_o \Lambda_s\, 1(\eta_s)\, ds +$$

$$+ \int_0^t E_o (\lambda_s - 1)\, \alpha_s \Lambda_s\, ds =$$

$$= E_o \Lambda_0 \eta_0 - \int_0^t E_o \Lambda_s\, 1(Z_s)\, ds + \int_0^t E_o \lambda_s \alpha_s \Lambda_s\, ds$$

Since $\mathcal{P}\{\eta_t = 0\} = \mathcal{P}\{Z_t = 0\}$, we get

$$E_o \Lambda_t \eta_t = E_o \Lambda_0 \eta_0 - t + \int_0^t \mathcal{P}\{Z_s = 0\}\, ds + \int_0^t E_o \Lambda_s \alpha_s \lambda_s\, ds$$

If, under $\mathcal{P}$, $(\alpha_t, t \geq 0)$ and $(\lambda_s,\ 0 \leq s \leq t)$ are independent, we have,

$$E_o \Lambda_t \eta_t = E_o \Lambda_0 \eta_0 - t + \int_0^t \mathcal{P}\{Z_s = 0\}\, ds + \int_0^t (E\alpha_s)(E\lambda_s)\, ds$$

Calling $E\, \alpha_t = \alpha$, we have:

$$\text{Average waiting time} = E\eta_t = E\eta_0 - t + \int_0^t \mathcal{P}\{Z_s = 0\}\, ds + \int_0^t \alpha E(\lambda_s)\, ds \tag{35}$$

If we take $\lambda_s = \lambda$ = constant, we will find the expression, given by Takács [1962]:

$$E\eta_t = E\eta_0 + \int_0^t \mathcal{P}\{Z_s = 0\}\, ds - t\,(1-\alpha\lambda) \tag{36}$$

4.7 Comparison with the classical method

The methods presented in the section (4.2), (4.3) and (4.4) were intended to solve the case where the interarrival and service times are both i.i.d.. The main reason for doing that was that it is the case treated in the classical literature and we would like to compare the two approaches. The use of the transformation of measures $\frac{d\mathcal{P}}{d\mathcal{P}_o}$ can be used in problems other than this; the complexity of the calculations, however, increases.

Comparing the two methods, we see that in the case of the martingale approach, given the transformation of measure, we only need to do some combinatorics and then, taking advantage of the behavior of the processes X and Y in $\mathcal{P}_o$, calculate the probabilities. Instead in the classical approach we have to solve differential equations using Laplace transforms and complex number theory to obtain the desired results. Besides that the methods vary with the case being treated. In the calculations of the moments and the average waiting time, the expressions are obtained directly by use of stochastic calculus and without the necessity of calculating generating functions.

New results are obtained for the queue length, busy period, _n_th customer waiting time and virtual waiting time distributions in the case where the interarrival times are i.i.d. with distribution $F(x)$ and the service times are also i.i.d. with distribution $G(y)$ (formulas (10), (17), (21), (22)). These results were checked with particular cases given in the classical literature.

CHAPTER 5

OPTIMAL CONTROL FOR WAITING-TIME PROCESSES

5.1. Introduction

Suppose that we have a waiting time process where it is possible to vary, in some way, our service mechanism. Therefore we can consider trying to adjust the service in order to improve the performance of the system. This performance is usually measured by the number of customers in the system and the cost of service. Our decision in changing the service arrangement is usually based in our observations of the behaviour of the queue length up to the time we do the adjustment.

The main objective of this chapter will be to derive a Hamilton-Jacobi equation for the control of the waiting time processes. Our work will be based mainly on the paper by Davis and Varaiya [1973] in which the Hamilton-Jacobi equation is obtained for the Brownian notion case.

In section (5.3) we will derive the Hamilton-Jacobi equations. In sections (5.4) and (5.5) we will apply these equations to obtain the optimal control for linear and for quadratic objective functions. It is worth mentioning that the results obtained for the Hamilton-Jacobi equations can be extended to any queueing process where the arrival process X is a jump Markov process and the transformation of measure $\frac{d\mathcal{P}}{d\mathcal{P}_o}$ exists.

5.2. Preliminaries

1. Let us start with a probability space $(\Omega,\mathcal{F},\mathcal{P}_o)$. Under $\mathcal{P}_o$, $(X_t,\mathcal{F}_t,t \in [0,1]$, $(Y_t,\mathcal{F}_t,t \in [0,1])$ are real valued, independent standard Poisson processes.

Let us define the process Z as in equation (7) in section (3.2).

Define:

$$\rho_u^t(\mu) = \varepsilon\left(\int_s^t (\lambda-1)(dX_s-ds) + \int_s^t (\mu_{s-}-1)(dY_s-ds)\right) \quad (1)$$

2. The class of Markov controls, denoted by $\mathcal{M} = \mathcal{M}_0^1$, where $\mathcal{M}_s^t$ is the class of functions satisfying the following conditions:

a. μ: $[0,1] \times [0,1,2,..] \to \Sigma \subset R$ is non-negative, jointly measurable.

b. For each t, $\mu(t,.)$ is adapted to $\sigma(Z_{t-})$

c. $E_o[\rho_s^t(\mu)/\mathcal{F}_s] = 1$ a.s. $\mathcal{P}_o$ for all $t,s \in [0,1]$

Remark: If condition (c) is satisfied we know that (see (3.3)) the processes:

$$M_t^{(1)} = X_t - \lambda t$$

$$M_t^{(2)} = Y_t - \int_0^t \mu_s ds$$

are such that $(M_t^{(1)}, \mathcal{F}_t, t \in [0,1])$, $(M_t^{(1)}, \mathcal{F}_t, t \in [0,1]) \in \mathcal{M}_{loc}$ under $\mathcal{P}$, the measure defined by ρ_0^1. Therefore the process Z will have arrival rate λ = constant and service rate $\mu = (\mu_t, t \in [0,1])$ under $\mathcal{P}$.

3. Let c: $[0,1] \times [0,1,2,...] \times \Sigma \to R^+$ be a non-negative real value function satisfying:

a. c: $[0,1] \times [0,1,2,...] \times \Sigma \to R^+$ is jointly measurable

b. For fixed (t,μ), $c(t,.,\mu)$ is adapted to $\mathcal{F}_t$.

Then the cost ascribed to a Markov control μ will be

$$J(\mu) = E_\mu \int_0^T c(s,t_s,\mu(s,Zs))ds = E_o[\rho_0^1(\mu)\int_0^1 c_s(\mu)ds]$$

Our objective will be to find an admissible control μ^* such that $J(\mu)$ is minimized.

This way of formulating the problem is different from the approach usually taken for optimization of Markov processes. (See for instance, [Fleming,1969]). There, a probability space $(\Omega,\mathcal{F},\mathcal{P})$ is given a priori, and different processes have different sample paths defined by having control-dependent coefficients in some differential equation. This kind of formulation causes a number of theoretical problems, one of them being the necessity of smooth controls, what usually does not happen for the optimal control. The approach used in our work has the following features: (i) the control is closed loop; (ii) admissible controls need not to be smooth, (iii) the transformation of measures $\rho_0^1(\mu)$ gives a measure corresponding to the queue with arrival rate λ = constant and service rate $\mu = (\mu_t, t \in [0,t])$. More details about this approach can be found in [Benes,1970].

5.3. The Hamilton-Jacobi equation for waiting-time systems

1. Suppose a control $u \in \mathcal{M}_o^t$ is used on $[0,t]$ and $v \in \mathcal{M}_t^1$ is used on $[t,1]$. Then the expected remaining cost at time t, given the value of Z at time t-, is:

$$\psi_{uv} = E_{uv}[\int_t^1 c_s ds/Z_{t-}] = \frac{E_o[\rho_o^t(u)\rho_t^1(v)\int_t^1 c_s^{(v)} ds/Z_{t-}]}{E_o[\rho_o^t/Z_{t-}]} \tag{2}$$

Since, under $\mathcal{P}_o$, Z is a Markov process and u_t is Z_t-measurable, we have:

$$E_o[\rho_o^t(u)\rho_0^1(v)\int_t^1 c_s ds/Z_{t-}] - E_o[\rho_o^t(u)/Z_t]E_o[\rho_t^1(v)\int_t^1 c_s ds/Z_{t-}] \tag{3}$$

Then:

$$\psi_{uv}(t) = E_o[\rho_t^1(v)\int_t^1 c_s ds/Z_{t-}] \tag{4}$$

Therefore, we can write

$$\psi_u(t) = E_o[\rho_t^1(u)\int_t^1 c_s^{(u)} ds/Z_{t-}] \tag{5}$$

Let us define

$$U_t = U(t,Z_t) = \underset{u\in\mathcal{M}_t^1}{\text{infimum}}\ \psi_u(t) \tag{6}$$

In [Davis,Varaiya,1971] it is proved that this minimum exists in L_1 for each t.

2. Markov principle of optimality [Davis,Varaiya,1971].

Let $\mu \in \mathcal{M}$. Then, for each t,h:

$$V_t \leq E_\mu[\int_t^{t+h} c_s^{(\mu)} ds/Z_{t-}] + E_\mu[U_{t+h}/Z_{t-}]$$

3. Lemma: There exist measurable functions ΛU: $[0,1] \times [0,1,2,\ldots] \to R$, $U_X : [0,1] \times [0,1,2,\ldots] \to R$ and $U_Y : [0,1] \times [0,1,2,\ldots] \to R$, such that:

$$E_o\int_0^1 |\Lambda U(t,Z_t)|dt < \infty$$

$$\int_0^1 |U_X(t,Z_t)|^2 dt < \infty$$

$$\int_0^1 |U_Y(t,Z_t)|^2 dt < \infty$$

$$U(t,Z_t) = J_M + \int_0^t \Lambda U(s,Z_{s-})ds + \int_0^t U_X(s,Z_{s-})dX_s$$

$$+ \int_0^t U_Y(s,Z_{s-})1(Z_{s-})dY_s \tag{7}$$

where $J_M = \inf_{\mu \in \mathcal{M}} J(\mu)$, the minimum Markov cost.

4. Theorem: $\mu^* \in \mathcal{M}$ is optimal if and only if there exist a constant J^* and processes $(\eta_t, t \in [0,1])$, $(\xi_t^{(1)}, t \in [0,1])$, $(\xi_t^{(1)}, t \in [0,1])$, taking values in R, adapted to $\sigma(Z_{t-})$, and satisfying the following conditions

a. $\int_0^1 |\xi_t^{(1)}|^2 dt < \infty$ a.s.

$\int_0^1 |\xi_t^{(1)}|^2 dt < \infty$ a.s.

b. $X(1) = 0$ a.s. where $X(t) = J^* + \int_0^t \eta_s ds + \int_0^t \xi_s^{(1)} dX_s$

$+ \int_0^t \xi_s^{(2)} \; 1(Z_{s-})dY_s$

c. $\eta_t + \xi_t^{(1)}\lambda + \xi_t^{(2)}\mu(t,Z_{t-})1(Z_{t-}) + c_t^{(\mu)} \geq$

$= \eta_t + \xi_t^{(1)}\lambda + \xi_t^{(2)}\mu^*(t,Z_{t-})1(Z_{t-}) + c_t^{(\mu^*)}$

Then $X(t) = U_t$ a.s. and $J_M = J(\mu^*)$, the cost of the optimal Markov policy.

Remark: (5.3.3) and (5.3.4) are proved almost exactly as similar results in [Davis,Varaiya,1971]. The difference is that, here we are dealing with Z, a counting process, and, in the mentioned work, the process Z is a diffusion process. Therefore, in the proofs of (5.3.3) and (5.3.4) we have to use the results given in (2.6.9), corresponding to the representation of processes that are martingales with respect to a family generated by a jump process.

5. Suppose that the function $U(t,z)$ has continuous first partial derivative in t and continuous first and second derivatives in then

$$U_X(t,z) = U(t,z+1) - U(t,z)$$

$$U_Y(t,z) = U(t,z-1) - U(t,z) \tag{8}$$

$$\Lambda U(t,z) = \frac{\partial U(t,z)}{\partial t}$$

Proof: Under the measure $\mathcal{P}_\mu$ we have:

$$X_t - \int_0^t \lambda ds = X-\lambda t \in \mathcal{M}_{loc}(\mathcal{P}_\mu)$$

$$Y_t - \int_0^t \mu(s,Z_s)ds \in \mathcal{M}_{loc}(\mathcal{P}_\mu)$$

Therefore:

$$Z_t = X_t - \int_0^t 1(Z_{s-})dY_s = t - \int_0^t \mu(s,Z_s)1(Z_s)ds + X_t - \lambda t$$

$$- \int_0^t 1(Z_{s-})(dY_s - \mu(s,Z_s)ds)$$

is a semi-martingale under $\mathcal{P}_\mu$

Then we can apply the differentiation rule to $U(t,Z_t)$:

$$U(t,Z_t) = J_n + \int_0^t \frac{\partial U(s,Z_s)}{\partial s}\, ds + \int_0^t \frac{\partial U(s,Z_{s-})}{\partial Z_s}\, dZ_s$$

$$+ \sum_{s\leq t} [U(s,Z_s)-U(s,Z_{s-}) - \frac{\partial U(s,Z_{s-})}{\partial Z_s} \quad \Delta Z_s]$$

$$= J_M + \int_0^t \frac{\partial U(s,Z_s)}{\partial s}\, ds + \int_0^t [U(s,Z_{s-}+1)-U(s,Z_{s-})]dX_s$$

$$+ \int_0^t [U(s,Z_{s-}-1)-U(s,Z_{s-})]1(Z_{s-})dY_s$$

Comparing with (5.3.3), we find:

$$U_X(t,z) = U(t,z+1) - U(t,z)$$

$$U_Y(t,z) = U(t,z-1) - U(t,z)$$

$$\Lambda U(t,z) = \frac{\partial U(t,z)}{\partial t}$$

7. <u>Hamilton-Jacobi equation for waiting time processes control</u>

If the function $U(t,z)$ satisfies the conditions given in (5.3.6), then the condition (c) in (5.3.4) reduces to:

$$\frac{\partial U(t,z)}{\partial t} + \lambda[U(t,z+1) - U(t,z)] + \min_{v\in\Sigma} \{[U(t,z-1) - U(t,z)]1(z)v$$

$$+ c(t,z,v)\} = 0$$

for all $(t,z) \in [0,1] \times [0,1,2,...]$ and the optimal policy μ^* is characterized by the property that

$$[U(t,z-1)-U(t,z)]\ 1(z)v + c(t,z,v)$$

is minimized by $v = \mu^*(t,z)$

Proof: The conditions of (5.3.4) are satisfied taking:

$$\eta_t = \Lambda(t,Z_t)$$

$$\xi_t^{(1)} = U_X(t,Z_t)$$

$$\xi_t^{(2)} = U_Y(t,Z_t)$$

Since by (5.3.6), we have

$$U_X(t,Z_t) = U(t,Z_t+1) - U(t,Z_t)$$

$$U_Y(t,Z_t) = U(t,Z_t-1) - U(t,Z_t)$$

$$\Lambda(t,Z_t) = \frac{\partial U(t,Z_t)}{\partial t}$$

The condition (c) in (5.3.4) reduces to:

$$\frac{\partial U(t,Z_t)}{\partial t} + \lambda[U(t,Z_{t-}+1) - U(t,Z_{t-})] + [U(t,Z_{t-}-1) - U(t,Z_{t-})]1(Z_{t-})$$

$$\cdot \mu^*(t,Z_{t-}) + c(t,Z_{t-},\mu^*(t,Z_{t-})) = 0 \quad \text{a.s.}$$

Since $\mu(t,Z_{t-})$ can take any value in Σ, we have:

$$\frac{\partial U(t,z)}{\partial t} + \lambda[U(t,z+1)-U(t,z)]$$

$$+ \min_{v\in\Sigma} \{[U(t,z-1)-U(t,z)]1(z)v + c(t,z,v)\} = 0$$

for all $(t,z) \in [0,1] \times [0,1,2,\ldots]$

8. If the conditions of (5.3.4) are satisfied, we have:

$$\inf_{\mu\in\mathcal{M}} J(\mu) = \inf_{\mu\in\mathcal{N}} J(\mu)$$

where $\mathcal{N}$ is the class of non-antecipative controls, that is, the admissible controls are functionals of the past of Z (measurable with respect to $\mathcal{Z}_t = \sigma(Z_s, 0 \leq s \leq t)$)

Proof: By (5.3.7) we have:

$$\Lambda U(t,z) + \lambda U_X(t,z) + U_Y(t,z)1(z)v + c(t,z,v) \geq 0$$

for all $(t,x,v) \in [0,1] \times [0,1,2,\ldots] \times \Sigma$

Let $\mu \in \mathcal{N}$, then the processes:

$$M_t^{(1)} = X_t - \int_0^t \lambda ds$$

$$M_t^{(2)} = Y_t - \int_0^t \mu_s ds$$

belong to $\mathcal{M}_{loc}(\mathcal{P}_\mu)$.

But, by (5.3.3)

$$U_t = J_M + \int_0^t \Lambda U(s,Z_s)ds + \int_0^t U_X(s,Z_{s-})dX_s + \int_0^t U_Y(z,Z_{s-})1(Z_{s-})dY_s$$

and $U(1) = 0$. Then

$$J_M = -E\left(\int_0^1 \Lambda U(s,Z_s)ds + \int_0^t U_X^{(1)}(s,Z_{s-})dX_s + \int_0^t U_Y(s,Z_{s-})1(Z_{s-})dY_s\right)$$

$$= -E_\mu\left(\int_0^1 [\Lambda U(s,Z_s) + \lambda U_X(s,Z_s) + U_Y(s,Z_s)\mu(s,Z_s)1(Z_s)]ds\right)$$

$$\leq E_\mu \int_0^1 c^{(\mu)} ds = J(\mu)$$

Since was arbitrary:

$$J_M \leq \inf_{\mu \in \mathcal{N}} J(\mu)$$

Now $J_M \geq \inf_{\mu \in \mathcal{N}} J(\mu)$ since $\mathcal{M} \subset \mathcal{N}$. Therefore:

$$J_M = \inf_{\mu \in \mathcal{N}} J(\mu)$$

5.4. Optimal Control for Linear Cost

In this and in the next section we will apply the Hamilton-Jacobi equation for special kinds of cost functions. First we shall suppose that we can only vary our service rate from 0 to c = constant, that is, $\Sigma = \{v : 0 \leq v \leq c\}$. Our cost will be given by:

$$J(\mu) = \int_0^T c(s,Z_s,\mu(s,Z_s))ds = \int_0^T (eZ_s+\mu(s,Z_s))ds \tag{9}$$

where T = fixed time.

Therefore, our optimal control $\mu^*(t,Z_t)$ must be such to minimize (by 5.3.7):

$$v1(z)[U(t,z-1)-U(t,z)] + ez + v \tag{10}$$

for all $(t,z) \in [0,1,] \times [0,1,2,\ldots]$

Then:

$$\mu^*(t,z) = \begin{cases} 0 & \text{if} \quad z = 0 \\ 0 & \text{if} \quad z > 0 \quad \text{and } U(t,z) - U(t,z-1) -1 \leq 0 \\ c & \text{if} \quad z > 0 \quad \text{and } U(t,z) - U(t,z-1) -1 > 0 \end{cases} \tag{11}$$

Remark: $\mu^*(t,z)$, in this case, belongs to $\mathcal{M}_0^1$ since $\mu^*(t,z) \leq c$, for all (t,z).

We have to calculate $(U(t,z) - U(t,z-1)-1)$ in order to know, for $z \geq 1$, what is the value of the optimal control. $U(t,z)$ must satisfy:

$$\frac{\partial U(t,0)}{\partial t} + \lambda[U(t,1) - U(t,0)] = 0$$

$$\frac{\partial U(t,z)}{\partial t} + \lambda[U(t,z+1) - U(t,z)] + c1(U(t,z) - U(t,z-1)-1)[U(t,z-1) - U(t,z)]$$

$$+ ez + c\ 1(U(t,z) - U(t,z-1)-1) = 0 \quad \text{for } z \geq 1 \qquad (12)$$

$U(T,z) = 0$ for all $z \geq 0$

Since $U(T,z) = 0$ $\forall z \geq 0$, we must have, near the terminal time T, that $U(t,z) - U(t,z-1) -1 < 0$. Therefore, our optimal service rate will be $\mu^*(t,z) = 0$, $\forall z \geq 0$, for t very near to T. Then calling:

$$U(t,z) - U(t,z-1) -1 = P(t,z) \qquad (13)$$

we have

$$\frac{\partial P(t,z)}{\partial t} + \lambda P(t,z+1) - \lambda P(t,z) + e = 0 \quad \text{for all } z \geq 0 \qquad (14)$$

$$P(T,z) = -1 \quad \text{for all } z \geq 0$$

The solution to this equation is given by

$$P(t,z) = e(T-t)-1$$

Therefore, for $t \in [T-1/e,T]$, we have:

$$\mu^*(t,z) = 0 \qquad \forall z \geq 0$$

since $P(t,z)$ is positive

For $t \leq T -1/e$ we have

$$\frac{\partial P(t,1)}{\partial t} + \lambda P(t,2) - (\lambda+c)P(t,1) + e = 0$$

$$\frac{\partial P(t,z)}{\partial t} + \lambda P(t,z+1) - (\lambda+c)P(t,z) + c\ P(t,z-1) + e = 0 \quad \forall z \geq 2$$

$$P(T,z) = 0 \qquad \forall z \geq 1 \tag{15}$$

For solving the above equations we need to use the method of generating functions (see [Pinney,1958]), we define:

$$Q(t) = \sum_{z=1}^{\infty} a^z P(t,z) \qquad a < 1 \tag{16}$$

Then we get by using (15)

$$\frac{\partial Q(t)}{\partial t} + \frac{\lambda}{a}\ Q(t) - \frac{\lambda}{a} P(t,1) - (\lambda+c)Q(t) + caQ(t) + e\ \frac{a}{1-a} = 0 \tag{17}$$

$Q(T) = 0$

Making the transformation $u = T-t$, we have

$$\frac{\partial Q(u)}{\partial u} - \frac{\lambda}{a} Q(u) + \frac{\lambda}{a} P(u,1) + (\lambda+c)Q(u) - caQ(u) - \frac{ea}{1-a} = 0$$

$$Q(0) = 0 \tag{18}$$

Taking the Laplace transform of Q(u), we obtain:

$$\mathcal{L}[Q(u)] = \frac{-\frac{\lambda}{a}\mathcal{L}[P\ (u,1)] + \frac{ea}{1-a}\ \frac{1}{s}}{S - \lambda/a + (\lambda+c) - ac} \tag{19}$$

Applying Rouché's theorem to the Laplace transform (see [Saaty,1961]), we get:

$$\mathcal{L}[P\ (u,1)] = \frac{e}{\lambda} \cdot \frac{\lambda^2}{c^2} [3\ \frac{\lambda}{c}\ (\sqrt{\frac{c}{\lambda}})^3\ u^{-1} I_3(2\sqrt{\lambda c}\ u)$$

$$+ 4\ \frac{\lambda^2}{c^2}\ (\sqrt{\frac{c}{\lambda}})^4\ u^{-1} I_4(2\sqrt{\lambda c}\ \ u) + \ \ldots] \tag{20}$$

Since $I_n(2\sqrt{\lambda c}\ u) \geq 0$ for all $u \geq 0$, any n, we have:

$$P(T-t,1) \geq 0 \quad \forall t \in [0,T-1/e]$$

Substituting in the equation (19) the value of $\mathcal{L}[P(u,1)]$, we obtain

$$\mathcal{L}[Q(u)] = \frac{e}{cs}\left[\frac{1}{\theta_1}\left[a + \frac{\lambda/c}{\theta_1} + a^2 + \frac{\lambda/c}{\theta_1} + \frac{\lambda^2/c^2}{\theta_1^2} + a^3 + \frac{\lambda/c\ a^2}{\theta_1} + \frac{\lambda^2/c^2}{\theta_1^2} + \frac{\lambda^3/c^3}{\theta_1^3} + \ldots\right]\right. \cdot \sum_{n=0}^{\infty}\left(\frac{a}{\theta_1}\right)^n \tag{21}$$

Therefore $P(T-t,z) \geq 0 \quad \forall t \in [0,T-1/e],\ \forall z \geq 1$

We can now summarize:

$$\mu^*(t,z) = \begin{cases} 0 & \text{for} \quad z = 0,\ \forall t \in [0,T] \\ 0 & \text{for} \quad z \geq 1,\ t \in [T-1/e,T] \\ c & \text{for} \quad z > 1,\ t \in [0,T-1/e] \end{cases} \tag{22}$$

We have obtained a bang-bang control. It was obvious from the beginning that the value of the service rate would be zero when no one is in the system, but, now, we have found out that for some period of time near the terminal time, we should also have no service at all. The reason for this is the fact that, for a small period of time, the probability of one person being served is very small, and we will increase our cost by trying to serve.

5.5. Optimal control for quadratic cost

Let us suppose that we have $\Sigma = R$ and our cost function is given by:

$$J(\mu) = \int_0^T c(s,Z_s,\mu(s,Z_s))ds = \int_0^T [eZ_s^2 + \frac{\mu^2(s,Z_s)}{\lambda^2}]ds \quad (23)$$

Our control must be such that to minimize the following expression:

$$v1(z)[U(t,z-1) - U(t,z)] + ez^2 + \frac{v^2}{\lambda^2} \quad (24)$$

for all $(t,z) \in [0,1] \times [0.1,2,...]$

Then

$$\mu^*(t,z) = \frac{\lambda^2}{2}[U(t,z)-U(t,z-1)]1(z) \quad (25)$$

and $U(t,z)$ must satisfy the following equations:

$$\frac{\partial U(t,o)}{\partial t} + \lambda[U(t,1)-U(t,0)] = 0$$

$$\frac{\partial U(t,z)}{\partial t} + \lambda[U(t,z+1)-U(t,z)] - \frac{\lambda^2}{4}[U(t,z)-U(t,z-1)]^2 + ez^2 = 0$$

for $z \geq 1$

$$U(T,z) = 0 \quad \forall z \geq 0 \quad (26)$$

We couldn't find a closed form solution for these equations. Trying to solve using Picard's method, we realize that the optimal control is almost linear with the queue length, that is, $\mu^*(t,z) \simeq a(t)z + b(t)$, for $z > 0$. Now, if we change our cost function by adding a cost $f(t)/\lambda^2$, whenever there is a queue length different from 0, we should be able to obtain a linear control by finding a convenient f. Supposing that this cost is small in relation to $eZ^2 + \frac{\mu^2(t,Z_t)}{\mu^2}$, we are in position to state that the optimal control for our original problem is indeed approximately linear.

Therefore, our objective function is:

$$J_1(\mu) = \int_0^T [eZ_s^2 + \frac{\mu^2(s,Z_s)}{\lambda^2} + \frac{f(s)}{\lambda^2} 1(Z_s)]ds \tag{27}$$

The expression to be minimized is:

$$v1(z) + [U_1(t,z-1)-U_1(t,z)] + ez^2 + \frac{v^2}{\lambda^2} + 1(z) \frac{f(t)}{\lambda^2} \tag{28}$$

for all $(t,z) \in [0,1] \times [0,1,2,\ldots]$

Then:

$$\mu^*(t,z) = \frac{\lambda^2}{2} [U_1(t,z)-U_1(t,z-1)]1(z) \tag{29}$$

and $U_1(t,z)$ must satisfy the following equations:

$$\frac{\partial U_1(t,z)}{\partial t} + \lambda[U_1(t,1)-U_1(t,0)] = 0$$

$$\frac{\partial U_1(t,_0)}{\partial t} + \lambda[U_1(t,z+1)-U_1(t,z)] - \frac{\lambda^2}{4} [U_1(t,z)-U_1(t,z-1)]^2$$

$$+ ez^2 + 1(z) \frac{f(t)}{\lambda^2} = 0$$

$$U_1(T,z) = 0 \qquad \forall z \geq 0 \tag{30}$$

calling $R(t,z) = \lambda^2[U_1(t,z)-U_1(t,z-1)], z \geq 1$, we have:

$$\frac{\partial R(t,1)}{\partial t} + \lambda R(t,2) - \lambda R(t,1) - \frac{1}{4} R^2(t,1) + \lambda^2 e + f(t) = 0$$

$$\frac{\partial R(t,Z)}{\partial t} + \lambda R(t,z+1) - \lambda R(t,z) - \frac{1}{4} R^2(t,z) + \frac{1}{4} R^2(t,z-1)$$

$$+ \lambda^2 e(z-1) = 0 \tag{31}$$

$$R(T,z) = 0 \qquad \text{for all } z \geq 1$$

Let us try a solution of the form $R(t,z) = a(t)z + b(t)$, $z \geq 1$ for these equations. Then $a(t)$, $b(t)$, $f(t)$ must satisfy:

$$\dot{a}(t) + \frac{1}{2} a^2(t) + 2e\lambda^2 = 0 \qquad a(T) = 0 \tag{32}$$

$$\dot{a}(t) + \dot{b}(t) + \lambda a(t) - \frac{1}{4} a^2(t) - \frac{a(t)b(t)}{2} - \frac{b^2(t)}{4} + e\lambda^2$$
$$+ f(t) = 0 \tag{33}$$

$$f(t) = \frac{b^2(t)}{4} \tag{34}$$

Solving the equations (32), (33) and (34) we get:

$$a(t) = 2\lambda\sqrt{e} \tanh[\lambda\sqrt{e}\ (T-t)] \tag{35}$$

$$b(t) = 2\lambda - 2\lambda \operatorname{sech}[\lambda\sqrt{e}\ (T-t)] + \sqrt{e}\ \lambda \operatorname{sech}[\lambda\sqrt{e}\ (T-t)].$$
$$\cdot \tan^{-1}[\sinh(\lambda\sqrt{e}\ (T-t))] \tag{36}$$

$$f(t) = \frac{1}{4} (2\lambda-2\lambda \operatorname{sech}[\lambda\sqrt{e}\ (T-t)] + \sqrt{e}\ \lambda \operatorname{sech}[\lambda\sqrt{e}\ (T-t)]$$
$$\cdot \tan^{-1}[\sinh(\lambda\sqrt{e}\ (T-t))])^2 \tag{37}$$

Our optimal control is given by:

$$\mu^*(t,z) = \begin{cases} \lambda\sqrt{e}\ z \tan[\lambda\sqrt{e}\ (T-t)] + \sqrt{f(t)} & \text{if } z > 0 \\ 0 & \text{if } z = 0 \end{cases} \tag{38}$$

The optimal control obtained belongs to the class $\mathcal{M}$ of Markov controls since satisfies the conditions (a), (b) obviously and conditions (c) is a consequence of a result proved in section

(3.3)

Let us study the function f(t) and make comparisons with $eZ_t^2 + [\mu^*(t,z)]^2$. Since $\mu^*(t,z_1) \geq \mu^*(t,z_2)$ when $z_1 \geq z_2$, we only need to compare f(t) with $e + [\mu^*(t,1)]^2$. Therefore, taking $\lambda = 1$, we find the following results for different e's:

e = 0.25

T-t	f(t)	$(\mu^*)^2$+e
0.2	0.0009	0.27
0.4	0.0049	0.32
0.6	0.0144	0.42
0.8	0.0324	0.56
1.0	0.0625	0.81
1.2	0.090	0.94
1.4	0.140	1.19
1.6	0.190	1.25

e = 1

T-t	f(t)	$(\mu^*)^2$+e
0.2	0.024	1.10
0.4	0.063	1.40
0.6	0.16	1.98
0.8	0.28	2.42
1.0	0.40	2.93
1.2	0.52	3.40
1.4	0.62	3.82
1.6	0.71	4.10

e = 4

T-t	f(t)	$(\mu^*)^2$+e
0.1	0.04	4.18
0.2	0.21	4.71
0.3	0.53	5.61
0.4	0.98	5.01
0.5	1.32	7.65
0.6	2.04	9.11
0.7	2.69	10.35
0.8	3.17	11.29
0.9	3.57	12.01
1.0	4.16	13.00

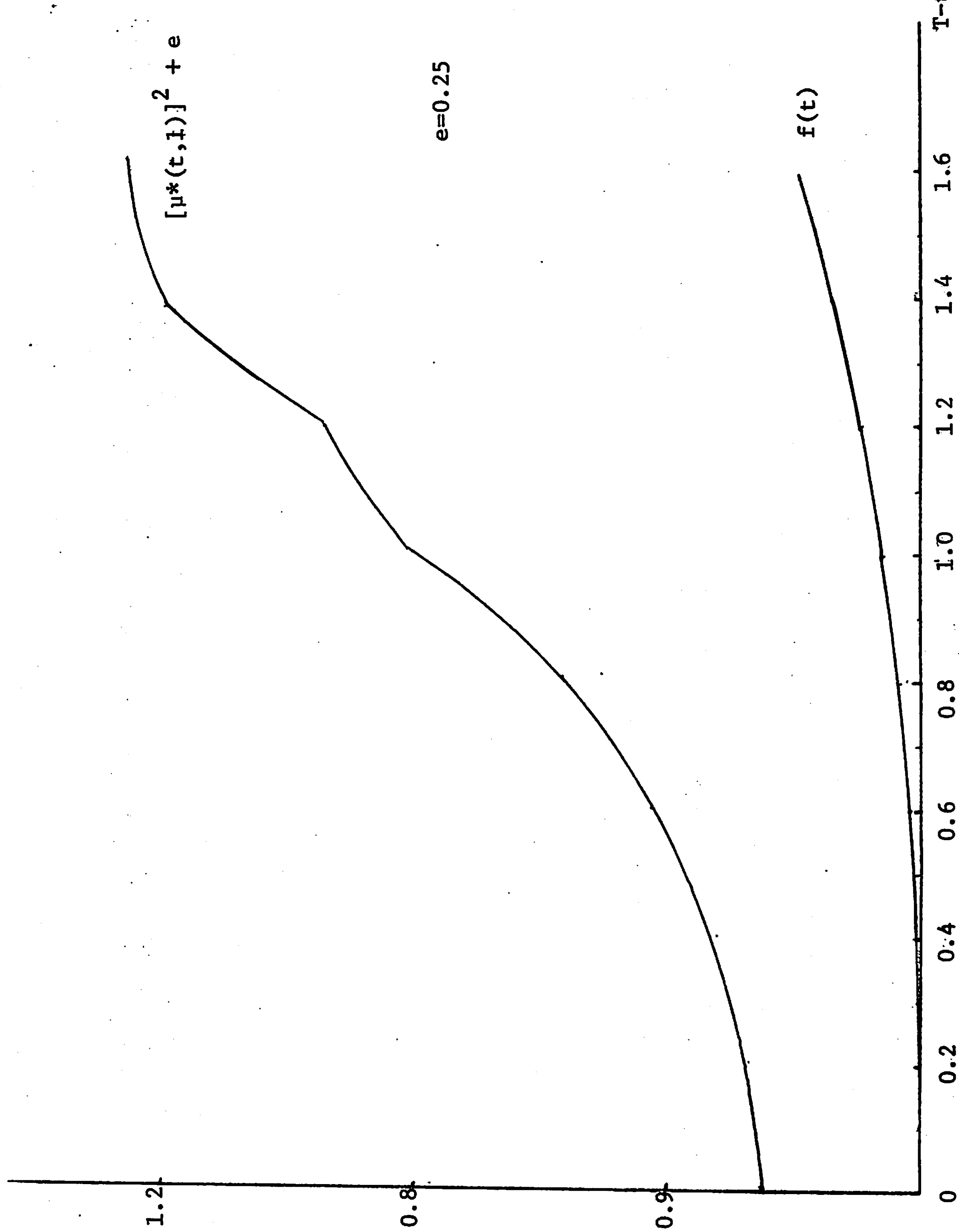
By the data provided above, we can conclude that:

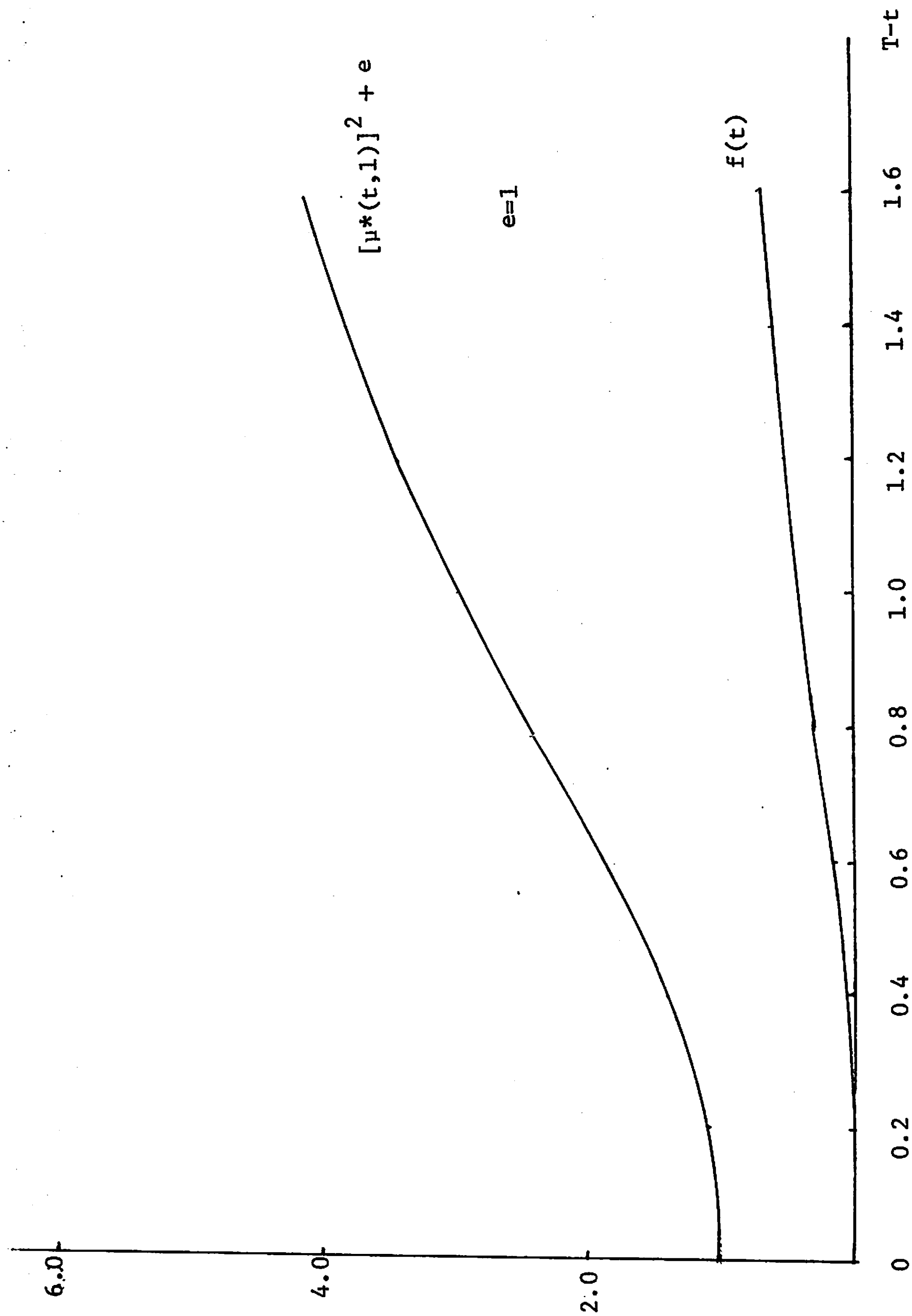
(a) for $\lambda = 1$, $e = 0.25$ and $T \leq 1.0$, the function f(t) is very small with comparison to $eZ_t^2 + [\frac{\mu^*(t,Z_t)}{\lambda}]^2$. Then, if we use as objective function $eZ_t^2 + [\frac{\mu(t,Z_t)}{\lambda}]^2$ instead of $eZ_t^2 + [\frac{\mu(t,Z_t)}{\lambda}]^2 + f(t)1(Z_t)$, our optimal control will be almost the same, that is

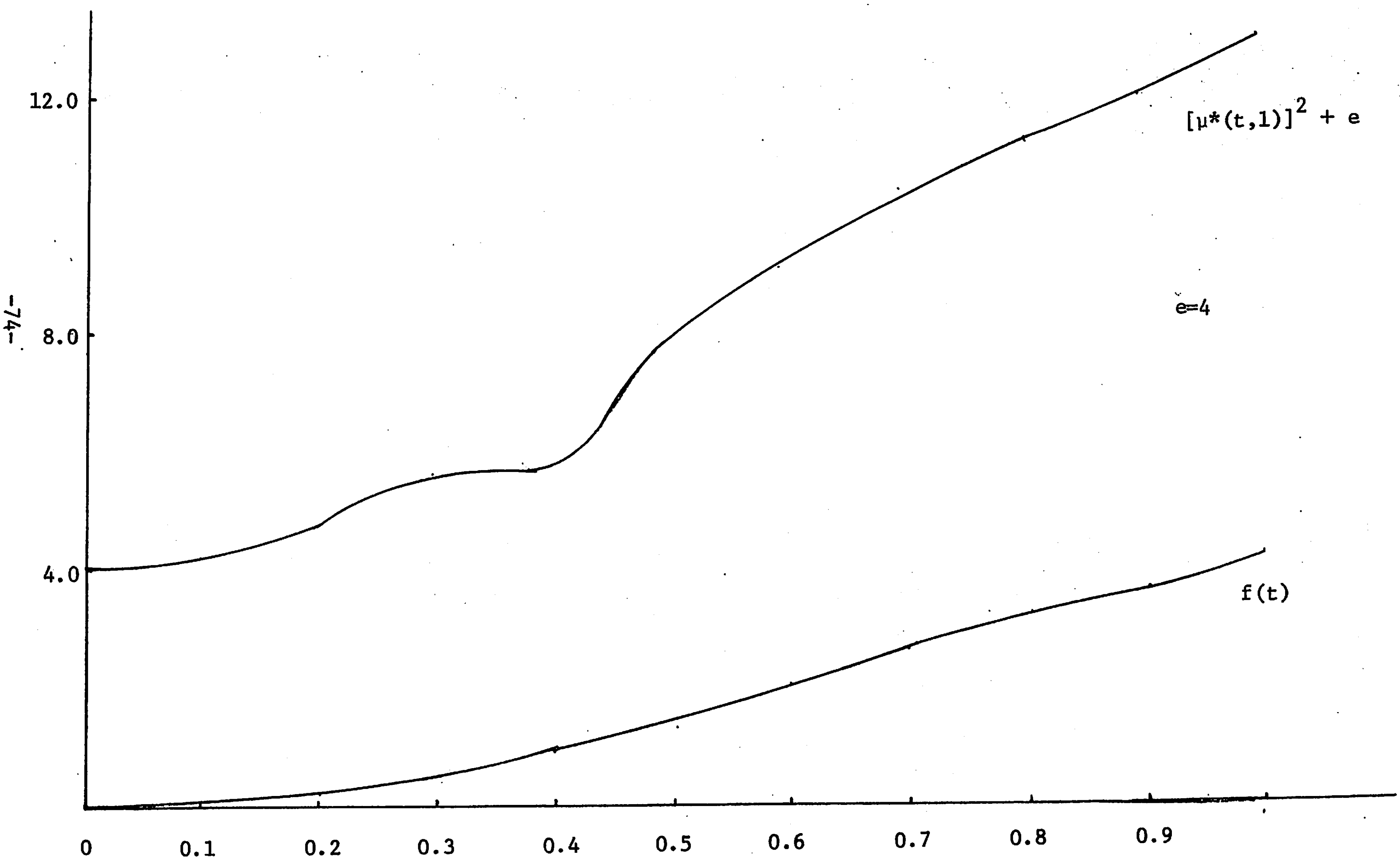
$$\mu^*(t,z) = \begin{cases} \sqrt{e}\; z \tanh[\sqrt{e}\,(T-t)] + \sqrt{f(t)} & z > 0 \\ 0 & \text{if } z = 0 \end{cases}$$

(b) The same is true, if $\lambda = 1$, $e = 1$, $T \leq 0.6$ and $\lambda = 1$, $e = 4$, $T \leq 0.3$.

This implies that the optimal control for the quadratic case will be almost linear depending on the values of T, λ and e being used.


[μ*(t,1)]² + e
e=0.25
f(t)
T-t
0
0.2
0:4
0.6
0.8
1.0
1.2
1.4
1.6
1.2
0.8
0.9


$[\mu^*(t,1)]^2 + e$
e=1
f(t)
T-t
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
2.0
4.0
6.0



12.0
8.0
4.0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
[μ*(t,1)]^2 + e
e=4
f(t)

CHAPTER 6

IDENTIFICATION AND ESTIMATION IN WAITING LINE PROBLEMS

6.1. Introduction

In the chapter we shall deal with some problems in identification and estimation related to queueing processes.

Concerning identification, we will find the maximum likelihood estimators for unknown parameters related to the arrival and servicee rates, based on the observation of the past of the queueing process. The likelihood ratio for queueing processes will be derived as it was done in [Van Schuppen,Wong.1973]

In estimation, we will be mainly interested in estimating the random processes λ and μ, the arrival and service rates, at time t, based in the past of the queue length and knowing the dynamical equations that govern both λ and μ. In this case, we will apply the unnormalized conditional density method, used by Wong [1971a] for Brownian motion cases, and by Boel, Varaiya and Wong [1973b] for jump processes. Another problem to be consider is the estimation of the queue length given the input flow or given the output flow. For this last problem, martingale methods apply directly. A case in which approximated filtering is possible will be shown.

6.2. Maximum likelihood estimators:

In section (3.3) we introduced a transformation of measures $\frac{d\mathcal{P}}{d\mathcal{P}_o}$ defined by:

$$\frac{d\mathcal{P}}{d\mathcal{P}_o} = \varepsilon\left(\int_0^1 (\lambda_{s-}-1)(dX_s-ds) + \int_0^1 (\mu_{s-}-1)(dY_s-ds)\right) \tag{1}$$

Also:

$$E_o\left\{\frac{d\mathcal{P}}{d\mathcal{P}_o}/\mathcal{F}_t\right\} = \varepsilon\left(\int_0^t (\lambda_{s-}-1)(dX_s-ds) + \int_0^t (\mu_{s-}-1)(dY_s-ds)\right) \quad (2)$$

We are interested in finding:

$$E_o\left\{\frac{d\mathcal{P}}{d\mathcal{P}_o}/\mathcal{Z}_t\right\} = L_t \quad (3)$$

where $\mathcal{Z}_t = \sigma\{Z_s, 0\leq s\leq t\}$

By the martingale representation theorem, presented in (2.6.9), we have:

$$L_t = 1 + \int_0^t \alpha_{s-}^{(+)}(dX_s-ds) + \int_0^t \alpha_{s-}^{(-)}\, 1(Z_{s-})(dY_s-ds)$$

where $\alpha^{(+)}$ and $\alpha^{(-)}$ are predictable processes, satisfying:

$$\int_0^t |\alpha_s^{(+)}|ds < \infty \qquad \text{for all } t \in T$$

$$\int_0^t |\alpha_s^{(-)}|ds < \infty \qquad \text{for all } t \in T$$

Then:

$$\Lambda_t X_t = \int_0^t \Lambda_{s-}dX_s + \int_0^t X_{s-}d\Lambda_s + \int_0^t (\lambda_{s-}-1)\Lambda_{s-}dX_s$$

$$L_t X_t = \int_0^t L_{s-}dX_s + \int_0^t X_{s-}dL_s + \int_0^t \alpha_t^{(+)}dX_s$$

Since $E_o\{\Lambda_t X_t/\mathcal{Z}_t\} = L_t\Lambda_t$, we must have:

$$E_o(d(\Lambda_t X_t)/\mathcal{Z}_t) = E_o(d(L_t X_t)/\mathcal{Z}_t)$$

implying:

$$E_o(\Lambda_t dX_t + X_{t-} d\Lambda_t + (\lambda_{t-}-1)\Lambda_{s-} dX_t / \mathcal{Z}_t)$$
$$= E_o(L_{t-} dX_t + X_{t-} dL_t + \alpha_{t-}^{(+)} dX_t / \mathcal{Z}_t)$$

and

$$\alpha_t^{(+)} = E_o[(\lambda_t - 1)\Lambda_t / \mathcal{Z}_t] \tag{4}$$

In the same way, we have:

$$\alpha_t^{(-)} = E_o[(\mu_t - 1)\Lambda_t / \mathcal{Z}_t] \tag{5}$$

Therefore:

$$L_t = 1 + \int_0^t E_o[(\lambda_s - 1)\Lambda_s / \mathcal{Z}_s](dX_s - ds) + \int_0^t E_o[(\mu_s - 1)\Lambda_s / \mathcal{Z}_s] 1(Z_{s-}) \cdot (dY_s - ds) \tag{6}$$

But

$$\hat{\lambda}_s - 1 = E[(\lambda_s - 1)/\mathcal{Z}_s] = \frac{E_o[(\lambda_s - 1)\Lambda_s / \mathcal{Z}_s]}{E_o[\Lambda_s / \mathcal{Z}_s]} = \frac{E_o[(\lambda_s - 1)\Lambda_s / \mathcal{Z}_s]}{L_s}$$

$$\hat{\lambda}_s - 1 = E[(\mu_s - 1)/\mathcal{Z}_s] = \frac{E_o[(\mu_s - 1)\Lambda_s / \mathcal{Z}_s]}{E_o[\Lambda_s / \mathcal{Z}_s]} = \frac{E_o[(\mu_s - 1)\Lambda_s / \mathcal{Z}_s]}{L_s}$$

Then:

$$L_t = 1 + \int_0^t (\hat{\lambda}_s - 1) L_{s-} (dX_s - ds) + \int_0^t (\hat{\mu}_s - 1) L_{s-} 1(Z_{s-})(dY_s - ds) \tag{7}$$

Therefore L_t is given by the exponential formula:

$$L_t = \prod_{t_i \le t} \hat{\lambda}_{t_i} \prod_{s_j \le t} \hat{\mu}_{s_j -} \exp[-\int_0^t ((\hat{\lambda}_s - 1) + (\hat{\mu}_s - 1) 1(Z_s)) ds] \tag{8}$$

Where (t_i) are the times of the positive jumps of Z, and (s_j), the times of the negative jumps of Z.

Let us suppose that $\lambda_t = \lambda$ = unknown constant and $\mu_t = \mu$ unknown constant, and we want to find the maximum likelihood estimators $e(\lambda)$ and $e(\mu)$ respectively. In this case:

$$L_t = \lambda^{N_t} \mu^{M_t} \exp[-(\lambda-1)t-(\mu-1)S_t] \tag{9}$$

where N_t = number of positive jumps up to time t

M_t = number of negative jumps up to time t

S_t = busy period up to time t

The maximum likelihood estimates for λ and μ will maximize L_t. Differentiating with respect to λ and μ, we get:

$$e(\lambda) = \frac{N_t}{t} \tag{10}$$

$$e(\mu) = \frac{M_t}{S_t} \tag{11}$$

Suppose now that λ is known and $\mu_t = \mu_1 + \mu_2 I_{\{Z_t=1\}}$, that is, we have two servers in parallel with service rates μ_1 and μ_2, constant and unknown. Then

$$L_t = \lambda^{N_t} \mu_1^{M_t} (\mu_1+\mu_2)^{P_t} \exp\{-(\lambda-1)t-(\mu_1-1)s_1-\mu_2 s_2\} \tag{12}$$

where: M_t = number of jumps from 1 to 0 up to time t.

P_t = all other negative jumps up to time t.

s_1 = busy period up to time t.

s_2 = period of time up to time t, when there are more than one costumer waiting.

The differentiation of the likelihood ratio gives us:

$$M_t(e(\mu_1) + e(\mu_2)) + P_t e(\mu_1) - s_1 e(\mu_1)(e(\mu_1) + e(\mu_2)) = 0$$

$$P_t - s_2(e(\mu_1) + e(\mu_2)) = 0$$

Therefore:

$$e(\mu_1) = \frac{M_t P_t}{P_t(s_1-s_2)} = \frac{M_t}{s_1-s_2} \tag{13}$$

$$e(\mu_2) = \frac{P_t}{s_2} - \frac{M_t}{s_1-s_2} \tag{14}$$

Another case happens when λ is constant and $\mu_t = cZ_{t-}$, where c is an unknown constant. Then:

$$L_t = \lambda^{N_t} \prod_{s_j \leq t} cZ_{s_j^-} \exp\left[-\int_0^t (\lambda-1)ds - \int_0^t (cZ_s-1)1(Z_s)ds\right] \tag{15}$$

and the maximum likelihood estimator for c is given by:

$$e(c) = (M_t \prod_{s_j \leq t} Z_{s_j^-}) \cdot \frac{1}{\int_0^t Z_s ds} \tag{16}$$

6.3. <u>Estimation of the arrival rate and service rate of a queue using the unnormalized conditional density method</u>

We are interested in estimating the pair $\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}$ given $\mathcal{Z}_t = \sigma(Z_s, 0 \leq s \leq t)$. For this purpose, we begin as in section (3.3), defining a transformation of measures $\frac{d\mathcal{P}}{d\mathcal{P}_o}$, using as $\mathcal{F}_t = \left(\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}, 0 \leq s \leq t\right) \vee \sigma((X_s, Y_s), 0 \leq s \leq t)$. To the hypothesis already imposed in (3.13) we have to add the following:

(a) $\left(\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}, \mathcal{F}_t, \mathcal{P}_o\right)$ is a Markov process whose sample paths are right continuous and have left-limits, and the jump times of $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ are totally in acessible.

(b) The processes (X,Y) and $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ are independent under $\mathcal{P}_o$

Let us define:

$$\mathcal{A}_t = \sigma((X_s,Y_s),0\leq s\leq t) \tag{17}$$

If a function $g(\lambda_t,\mu_t)$ belongs to G = {space of all bounded, measurable functions, defined on $(A\times B)$, where $A \times B$ is the range of $\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}$ }, we define:

$$\Pi_t(g) = E_o g(\lambda_t,\mu_t)\Lambda_t/\mathcal{Y}_t] \tag{18}$$

where

$$\Lambda_t = \varepsilon\left(\int_0^t (\lambda_{s-}-1)(dX_s-ds) + \int_0^t (\mu_{s-}-1)(dY_s-ds)\right)$$

and therefore, satisfies:

$$\Lambda_t = 1 + \int_0^t \Lambda_{s-}(\lambda_{s-}-1)(dX_s-ds) + \int_0^t \Lambda_{s-}(\mu_{s-}-1)(dY_s-ds)$$

Then:

$$E_o[g(\lambda_t,\mu_t)\Lambda_t/\mathcal{A}_t] = E_o[g(\lambda_t,\mu_t)/\mathcal{A}_t] + E_o[\int_0^t g(\lambda_t,\mu_t)\Lambda_{s-}(\lambda_{s-}-1)(dX_s-ds)$$

$$+ \int_0^t g(\lambda_t,\mu_t)\Lambda_{s-}(\mu_{s-}-1)(dY_s-ds)/\mathcal{A}_t] \tag{19}$$

Since (X_t,Y_t) and $\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}$ are independent under $\mathcal{P}_o$, we obtain:

$$E_o[g(\lambda_t,\mu_t)/\mathcal{A}_t] = E_o g(\lambda_t,\mu_t) \tag{20}$$

Also

$$E_o[g(\lambda_t,\mu_t)\Lambda_{s-}(\lambda_{s-}-1)/\mathcal{A}_t] = E_o[E_o(g(\lambda_t,\mu_t)\Lambda_{s-}(\lambda_{s-}-1)/\mathcal{A}_t \vee \mathcal{G}_s)/\mathcal{A}_t]$$

where $\mathcal{G}_s=\sigma\left(\begin{pmatrix}\lambda_p\\ \mu_p\end{pmatrix}, 0\le p\le s\right)$. Since $\mathcal{A}_t$ and $\mathcal{G}_s$ are independent,

$$= E_o[E_o(g(\lambda_t,\mu_t)/\mathcal{G}_s)\Lambda_{s-}(\lambda_{s-}-1)/\mathcal{A}_t]$$

Using the Markov property of $\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}$, we have:

$$= E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_{s-}(\lambda_{s-}-1)/\mathcal{A}_t\right]$$

$$= E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_{s-}(\lambda_{s-}-1)/\mathcal{A}_s \vee \sigma((X_u,Y_u,s\le u\le t)\right]$$

But (X,Y) has independent increments

$$= E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_{s-}(\lambda_{s-}-1)/\mathcal{A}_s\right]$$

In the same way, we can get

$$E_o[g(\lambda_t,\mu_t)\Lambda_{s-}(\mu_{s-}-1)/\mathcal{A}_t] = E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_{s-}(\mu_{s-}-1)/\mathcal{A}_s\right]$$

Therefore:

$$E_o[g(\lambda_t,\mu_t)\Lambda_t/\mathcal{A}_t] = E_o g(\lambda_t,\mu_t) + \int_0^t E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_s(\lambda_s-1)/\mathcal{A}_s\right]$$
$$\cdot\ (dX_s-ds) + \int_0^t E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_s(\mu_s-1)/\mathcal{A}_s\right](dY_s-ds)$$

Now:

$$E_0[g(\lambda_t,\mu_t)\Lambda_t/\mathcal{Z}_t] = E_o[E_o(g(\lambda_t,\mu_t)\Lambda_t/\mathcal{A}_t)/\mathcal{Z}_t] = E_o g(\lambda_t,\mu_t)$$
$$+ E_o\left[\int_0^t E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_s(\lambda_s-1)/\mathcal{A}_s\right](dX_s-ds)\right.$$
$$\left.+\int_0^t E_o\left[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_s(\mu_s-1)/\mathcal{A}_s\right](dY_s-ds)/\mathcal{Z}_t\right]$$

Let us define

$$P_t = \int_0^t E_o[E_o\left(g(\lambda_t,\mu_t)/\binom{\lambda_s}{\mu_s}\right)\Lambda_s(\lambda_s-1)/\mathcal{A}_s](dX_s-ds)$$

$$\hat{P}_t = E_o(P_t/\mathcal{Z}_t)$$

$$Q_t = \int_0^t E_o[E_o\left(g(\lambda_t,\mu_t)/\binom{\lambda_s}{\mu_s}\right)\lambda_s(\mu_s-1)/\mathcal{A}_s](dY_s-ds)$$

$$\hat{Q}_t = E_o(Q_t/\mathcal{Z}_t)$$

By the martingale representation theorem, we should have:

$$\hat{P}_t = \int_0^t \psi^+_{s-}(dX_s-ds) + \int_0^t \psi^-_{s-}1(Z_{s-})(dY_s-ds)$$

$$\hat{Q}_t = \int_0^t \xi^+_{s-}(dX_s-ds) + \int_0^t \xi^-_{s-}1(Z_{s-})(dY_s-ds)$$

Defining $Z^+_t = X_t-t$ and $Z^-_t = \int_0^t 1(Z_{s-})(dY_s-ds)$, we obtain:

$$Z^+_tP_t = \int_0^t Z^+_{s-}dP_s + \int_0^t P_{s-}dZ^+_s + \int_0^t E_o[E_o\left(g(\lambda_t,\mu_t)/\binom{\lambda_s}{\mu_s}\right)\Lambda_s(\lambda_s-1)/\mathcal{A}_s]dX_s$$

$$Z^+_t\hat{P}_t = \int_0^t Z^+_{s-}d\hat{P}_s + \int_0^t \hat{P}_{s-}dZ^+_s + \int_0^t \psi^+_{s-}dX_s$$

Since $E_o[Z^+_tP_t/\mathcal{Z}_t] = Z^+_t\hat{P}_t$, we must have

$$E_o(d(Z^+_sP_s)/\mathcal{Z}_s) = E_o(d(Z^+_s\hat{P}_s)/\mathcal{Z}_s) \tag{21}$$

Therefore:

$$E_o[Z^+_sdP_s + P_{s-}dZ^{(+)}_s + E_o[E_o\left((\lambda_t,\mu_t)/\binom{\lambda_s}{\mu_s}\right)\Lambda_s(\lambda_s-1)/\mathcal{A}_s]dX_s/\mathcal{A}_s]$$

$$= E_o[Z^+_sd\hat{P}_s + \hat{P}_{s-}dZ^{(t)}_s+\psi^+_{s-}dX_s/\mathcal{Z}_s]$$

Using equality (21), we find:

$$\psi_s^+ = E_o(E_o[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_s(\lambda_s-1)/\mathcal{A}_s]/\mathcal{Z}_s)$$

$$= E_o[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_s(\lambda_s-1)/\mathcal{Z}_s] \tag{22}$$

In the same way, we obtain:

$$\psi_s^- = 0 \tag{23}$$

$$\xi_s^+ = 0 \tag{24}$$

$$\xi_s^- = E_o[E_o\left(g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right)\Lambda_s(\lambda_s-1)/\mathcal{Z}_s] \tag{25}$$

We can write:

$$\pi_t g = \pi_0 g + \int_0^t \pi_s[(H_{t,s}g)(\lambda_s-1)](dX_s-ds)$$

$$+ \int_0^t \pi_s[(H_{t,s}g)(\mu_s-1)]1(Z_{s-})(dY_s-ds) \tag{26}$$

where the operator π_t is defined in (18), and

$$H_{t,s}(g) = E_o[g(\lambda_t,\mu_t)/\begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}] \tag{27}$$

Hence:

$$E[g(\lambda_t,\mu_t)/\mathcal{Z}_t] = \frac{E_o[g(\lambda_t,\mu_t)\Lambda_t/\mathcal{Z}_t]}{E_o[\Lambda_t/\mathcal{Z}_t]}$$

$$= \frac{\pi_o g + \int_0^t \pi_s[(H_{t,s}g)(\lambda_s-1)](dX_s-ds)+\int_0^t \pi_s[(H_{t,s}g)(\mu_s-1)]1(Z_{s-})(dY_s-ds)}{1 + \int_0^t \pi_s[(\lambda_s-1)](dX_s-ds)+\int_0^t \pi_s[(\mu_s-1)]1(Z_{s-})(dY_s-ds)} \tag{28}$$

Let us take a set C, belonging to A × B, and let us choose $g(\lambda_t,\mu_t) = I_C(\lambda_t,\mu_t)$. If we suppose $\mathcal{P}_o\{(\lambda_t,\mu_t) \in C\} = \mathcal{P}\{(\lambda_t,\mu_t) \in C\} = 0$, we have:

$$E_o[I_C\Lambda_t/\mathcal{Z}_t] = 0$$

Therefore, there exists a measurable function $U_t : A \times B \to R$ such that, for any $C \in A \times B$, we have:

$$E_o[\Lambda_t I_C/\mathcal{Z}_t] = \int_C U_t(a,b)P_t(da,db) \tag{29}$$

where P_t is the marginal distribution of $\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}$ under $\mathcal{P}_o$ and $\mathcal{P}_1$. Then, for any $h \in \mathcal{G}$, we have:

$$E_o[\Lambda_t h(\lambda_t,\mu_t)/\mathcal{Z}_t] = \int_{A\times B} U_t(a,b)h(a,b)P_t(da,db)$$

and:

$$E_o(g(\lambda_t,\mu_t)\Lambda_t/\mathcal{Z}_t) = \pi_t g = \int_{A\times B} g(a,b)P_t(da,db)$$

$$+ \int_0^t \int_{A\times B} (a-1)[\int_{A\times B} g(a',b')P(da',db',t/a,b,s)]U_s(a,b)P_s(da,db)(dX_s-ds)$$

$$+ \int_0^t (b-1)[\int_{A\times B} g(a',b')P(da',db',t/a,b,s)]U_s(a,s)P_s(da,db)1(Z_{s-})(dY_s-ds) \tag{30}$$

where $P(da',db',t/a,b,s)$ = transition probability of the Markov process $\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}$.

Rearranging the order of integration and using:

$$P(da',db',t/a,b,s)P_s(da,db) = P_t(da',db')$$
$$= P(da,db,s/a',b',t)P_t(da,db),$$

we have

$$= \int_{A\times B} g(a,b)P_t(da,db)$$

$$+ \int_{A\times B} g(a',b') \int_0^t \int_{A\times B} (a-1)U_s(a,b)P(da,\ db,s/a',b',t)(dX_s-ds)P_t(da',db')$$

$$+ \int_{A\times B} g(a',b') \int_0^t (b-1)U_s(a,b)P(da,db,s/a',b',t)1(Z_{s-})(dY_s-ds)P_t(da',db') \tag{31}$$

Since g is arbitrary:

$$U_t(a,b) = 1 + \int_0^t \int_{A\times B} (a-1)U_s(a',b')P(da',db',s/a,b,t)(dX_s-ds)$$
$$+ \int_0^t \int_{A\times B} (b-1)U_s(a',b')P(da',db',s/a,b,t)1(Z_{s-})(dY_s-ds) \tag{32}$$

The equation (32) is similar to the results obtained in [Wong,1971a] for the case of Brownian motion processes. (32) represents the "equation of motion" for a system with infinitely dimensional state space. In some special cases, (32) can be implemented. We will show one of them in example (a) below.

With some additional assumptions is possible to obtain a more convenient form for the "equation of evolution" (32). Let us suppose that the operators $H_{t,s}$ have the following properties:

(a) $\lim_{s\uparrow t} H_{t,s} = I$, I = identity matrix (33)

(b) there exist operators $\mathcal{L}_t$, $t \geq 0$ on $\mathcal{G}$ such that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} (H_{t+\varepsilon,s} - H_{t,s})(g) = H_{t,s}\mathcal{L}_t(g) \tag{34}$$

The operators $\mathcal{L}_t$ are often referred to as infinitesimal generators.

Using the above assumptions, we obtain:

$$E_o g(\lambda_t,\mu_t) = \int_{A\times B} (H_{t,0}(g))(a,b)P_0(da,db) = E_0(H_{t,s}(g))(a_0,b_0) \tag{35}$$

and

$$E_o(g(\lambda_{t+\varepsilon},\mu_{t+\varepsilon}) - g(\lambda_t,\mu_t)) = \int_{A\times B} (H_{t+\varepsilon,0} - H_{t,0})(g)(a,b)P_0(da,db)$$

$$\cong \varepsilon\int_{A\times B} (H_{t,0}\mathcal{L}_t(g))(a,b)P_0(da,db) = \varepsilon[(\mathcal{L}_t(g))(\lambda_t,\mu_t)] \tag{36}$$

$$(\pi_{t+\varepsilon} - \pi_t)(g) = \varepsilon E_o(\mathcal{L}_t(g)) + \varepsilon\pi_t[(\lambda_t - 1)g(\lambda_t,\mu_t)]$$

$$+ \varepsilon\int_0^t \pi_s[(\lambda_s - 1)H_{t,s}\mathcal{L}_t(g)](dX_s - ds) + \varepsilon\pi_t[(\mu_t - 1)g(\lambda_t,\mu_t)]$$

$$+ \varepsilon\int_0^t \pi_s[(\mu_s - 1)H_{t,s}\mathcal{L}_t(g)]1(Z_{s-})(dY_s - ds) \tag{37}$$

Hence:

$$\pi_t(g) = \pi_0(g) + \int_0^t \pi_s[(\lambda_s - 1)g(\lambda_s,\mu_s)](dX_s - ds)$$

$$+\int_0^t \pi_s[(\mu_s-1)g(\lambda_s,\mu_s)]1(Z_{s-})(dY_s-ds) + \int_0^t \pi_s(\mathcal{L}_s g)ds \tag{38}$$

The equation (38) is similar to the result obtained for jump processes in [Boel,Varaiya,Wong,1973b]. If we repeat the same procedure using the expression for $U_t(a,b)$ given in (38) and defining $V_t(a,b,\mathcal{L}_t) = P_t(a,b)U_t(a,b)$, we should be able to get a similar result to the one given by Wong [1971a]. Doing this, we find

$$V_t(a,b,\mathcal{Z}_t) = P_0(a,b) + \int_0^t (a-1)V_s(a,b,\mathcal{Z}_s)(dX_s-ds)$$

$$+\int_0^t (b-1)V_s(a,b,\mathcal{Z}_s)1(Z_{s-})(dY_s-ds) + \int_0^t \mathcal{L}_s V_s(a,b,\mathcal{Z}_s)ds \tag{39}$$

Hence:

$$P_t(a,b/\mathcal{Z}_t) = \frac{V_t(a,b,\mathcal{Z}_t)}{\int_{A\times B} V_t(a,b,\mathcal{Z}_t)dadb} \tag{40}$$

The advantage of the equations (39) over (32) are the following: (a) the term involving the observation depends only on $V_t(\cdot,\cdot,\mathcal{Z}_t)$ at (a,b) (b) the change due to dt is local in the sense that $\mathcal{L}_t$ is a local operator. These two factors cause considerable savings in computation.

Let us apply the formulas arrived to some problems:

a. Let us suppose our queue has the following characteristics: a stream of Poisson-type customers arrives at a single service station. The arrival rate is not homogeneous; there exist two arrival intensities corresponding to two kinds of customers that alternate. The time interval that the system receives customers of type i, $i = 1,2$, is exponentially distributed possessing the expected value 1. It is assumed that any realization of a time interval associated with uniform arrival rate λ_i is independent of previous history. Service time is exponentially distributed; if the system is at level i, that is, it is servicing customers type i, the service intensity possesses the value μ_i, and, as before, statistical independence between two realizations is assumed. This model can be applied in computers (i.e., one computer that services two kinds of sources, each one feeding the computer with programs) and other areas; for more details see [Yechialy, Naor, 1971].

By the description, we see that λ behaves as a random telegraph signal with levels λ_1 and λ_2; μ behaves in the same way and the only difference is that we have levels μ_1 and μ_2 whenever λ is at level λ_1 and λ_2, respectively. The random telegraph process satisfies all hypothesis needed in our analysis, and we have the following properties:

$$\mathcal{P}\left\{\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix} = \begin{pmatrix}\lambda_1\\ \mu_1\end{pmatrix}\right\} = \mathcal{P}\left\{\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix} = \begin{pmatrix}\lambda_2\\ \mu_2\end{pmatrix}\right\} = \frac{1}{2}$$

$$\mathcal{P}\left\{\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix} = \begin{pmatrix}\lambda_s\\ \mu_s\end{pmatrix}\right\} = \frac{1}{2}\left(1+e^{-|t-s|}\right) \tag{41}$$

Then, using (32), we get:

$$U_t(\lambda_1,\mu_1) = 1 + \frac{1}{2}\int_0^t [1+e^{-(t-s)}](\lambda_1-1)U_s(\lambda_1,\mu_1)(dX_s-ds)$$

$$+\frac{1}{2}\int_0^t [1-e^{-(t-s)}](\lambda_2-1)U_s(\lambda_2,\mu_2)(dX_s-ds)$$

$$+\frac{1}{2}\int_0^t [1+e^{-(t-s)}](\mu_1-1)U_s(\lambda_1,\mu_1)\ 1(Z_{s-})(dY_s-ds)$$

$$+\frac{1}{2}\int_0^t [1-e^{-(t-s)}](\mu_2-1)U_s(\lambda_2,\mu_2)1(Z_{s-})(dY_s-ds) \tag{42}$$

and

$$U_t(\lambda_2,\mu_2) = 1+\frac{1}{2}\int_0^t [1-e^{-(t-s)}](\lambda_1-1)Y_s(\lambda_1,\mu_1)(dX_s-ds)$$

$$+\frac{1}{2}\int_0^t [1+e^{-(t-s)}](\lambda_2-1)U_s(\lambda_2,\mu_2)(dX_s-ds)$$

$$+\frac{1}{2}\int_0^t [1-e^{-(t-s)}](\mu_1-1)U_s(\lambda_1,\mu_1)1(Z_{s-})(dY_s-ds) \tag{43}$$

$$+\frac{1}{2}\int_0^t [1+e^{-(t-s)}](\mu_2-1)U_s(\lambda_2,\mu_2)1(Z_{s-})(dY_s-ds)$$

By (28), we obtain

$$\hat{\lambda}_t = E(\lambda_t/\mathcal{Z}_t) = \frac{\frac{\lambda_2}{2}\ U_t(\lambda_1,\mu_1)+\frac{\lambda_2}{2}\ U_t(\lambda_2,\mu_2)}{1/2U_t(\lambda_1,\mu_1)+1/2U_t(\lambda_2,\mu_2)} \tag{44}$$

$$\hat{\mu}_t = E(\mu_t/\mathcal{Z}_t) = \frac{\frac{\mu_2}{2}U_t(\lambda_1,\mu_1)\quad \frac{\mu_2}{2}\ U_t(\lambda_2,\mu_2)}{1/2U_t(\lambda_1,\mu_1)+1/2U_t(\lambda_2,\mu_2)} \tag{45}$$

Denoting:

$$A_t = \frac{\lambda_1U_t(\lambda_1,\mu_1)+\lambda_2U_t(\lambda_2,\mu_2)}{2} \tag{46}$$

$$B_t = \frac{\mu_1U_t(\lambda_1,\mu_1)+\mu_2U_t(\lambda_2,\mu_2)}{2} \tag{47}$$

$$C_t = \frac{U_t(\lambda_1,\mu_1)+U_t(\lambda_2,\mu_2)}{2} \quad , \tag{48}$$

We obtain, using (42) and (43),

$$C_t = 1 + \int_0^t (A_{s-}-C_{s-})(dX_s-ds) + \int_0^t (B_{s-}-C_{s-})1(Z_{s-})(dY_s-ds) \tag{49}$$

$$\begin{aligned} A_t &= \frac{\lambda_1+\lambda_2}{2} + \frac{1}{2}\int_0^t [(\lambda_1+\lambda_2)A_{s-}-(\lambda_1+\lambda_2)C_{s-} \\ &\quad + e^{-(t-s)}((\lambda_1+\lambda_2-2)A_{s-}+(\lambda_1+\lambda_2-2\lambda_1\lambda_2)C_{s-})](dX_s-ds) \\ &\quad + \frac{1}{2}\int_0^t [(\lambda_1+\lambda_2)B_{s-}-(\lambda_1+\lambda_2)C_{s-}+e^{-(t-s)}((2\mu_1-2)A_{s-}+(\lambda_2-\lambda_1)B_{s-} \\ &\quad + (\lambda_1+\lambda_2-2\lambda_2\mu_1)C_{s-})]1(Z_{s-})(dY_s-ds) \end{aligned} \tag{50}$$

$$\begin{aligned} B_t &= \frac{\mu_1+\mu_2}{2} + \int_0^t [(\mu_1+\mu_2)A_{s-}-(\mu_1+\mu_2)C_{s-}+e^{-(t-s)}((\mu_2-\mu_1)A_{s-} \\ &\quad + (2\lambda_1-2)B_{s-}+(\mu_1+\mu_2-2\mu_2\lambda_1)C_{s-})](dX_s-ds) \\ &\quad + \frac{1}{2}\int_0^t [(\mu_1+\mu_2)B_{s-}-(\mu_1+\mu_2)C_{s-}+e^{-(t-s)}((\mu_1+\mu_2-2)B_{s-}+(\mu_1+\mu_2-2\mu_1\mu_2)C_{s-}] \\ &\quad 1(Z_{s-})(dY_s-ds) \end{aligned} \tag{51}$$

Since, by (44) and (45), $\hat{\lambda}_t = \frac{A_t}{C_t}$ and $\hat{\mu}_t = \frac{B_t}{C_t}$, we can use the differentiation rule to get:

$$\begin{aligned} d\hat{\lambda}_t &= -[(\lambda_1+\lambda_2-2)\hat{\lambda}_t-\lambda_1\lambda_2+\hat{\lambda}_t^2]dt - [\lambda_2\hat{\mu}_t-(\mu_1-2)\hat{\lambda}_t-\lambda_2\mu_1+\hat{\lambda}_t\hat{\mu}_t]1(Z_{t-})dt \\ &\quad - e^{-t}D_t dt + \frac{(1-\hat{\lambda}_{t-})[(\lambda_1+\lambda_2-1)\hat{\lambda}_{t-}-\lambda_1\lambda_2]}{1+(\lambda_1+\lambda_2-1)\hat{\lambda}_{t-}-\lambda_1\lambda_2}\, dX_t \end{aligned}$$

$$+\frac{(1-\hat{\lambda}_{t-})[\lambda_1\hat{\mu}_{t-}+(\mu_1-1)\hat{\lambda}_{t-}-\lambda_2\mu_1]}{1+(\mu_1-1)\hat{\lambda}_{t-}-\lambda_2\mu_1} \quad 1(Z_{t-})dY_t \tag{52}$$

and

$$d\hat{\mu}_t = -[\mu_2\hat{\lambda}_t+(\lambda_1-2)\hat{\mu}_t-\mu_2\lambda_1+\hat{\lambda}_t\hat{\mu}_t]dt - [\hat{\mu}_t^2+(\mu_1+\mu_2-2)-\mu_1\mu_2]1(Z_{t-})dt$$

$$- e^{-t}E_t dt + \frac{(1-\hat{\mu}_{t-})[(\mu_1+\mu_2-1)\hat{\mu}_{t-}-\mu_1\mu_2]}{1+(\mu_1+\mu_2-1)\hat{\mu}_t-\mu_1\mu_2} \quad dX_t$$

$$+\frac{(1-\hat{\mu}_{t-})[\mu_1\hat{\lambda}_{t-}+(\lambda_1-1)\hat{\mu}_{t-}-\lambda_1\lambda_2]}{1+\mu_1\hat{\lambda}_{t-}+(\lambda_1-1)\hat{\mu}_{t-}-\mu_2\lambda_1} \quad 1(Z_{t-})dY_t \tag{53}$$

where

$$D_t = \frac{1}{2C_t}\Big[\int_0^t e^s((\lambda_1+\lambda_2-2)A_{s-}+(\lambda_1+\lambda_2-2\lambda_1\lambda_2)C_{s-})(dX_s-ds)$$

$$+\int_0^t e^s((\lambda_2-\lambda_1)B_{s-}+(2\mu_1-2)A_{s-}+(\lambda_1+\lambda_2-2\lambda_1\lambda_2)C_{s-})1(Z_{s-})(dY_s-ds)\Big] \tag{54}$$

and

$$E_t = \frac{1}{2C_t}\Big[\int_0^t e^s((\mu_2-\mu_1)A_{s-}+(2\lambda_1-2)B_{s-}+(\mu_1+\mu_2-2\mu_2\lambda_1)C_{s-})(dX_s-ds)$$

$$+\int_0^t e^s((\mu_1+\mu_2-2)B_{s-}+(\mu_1+\mu_2-2\mu_1\mu_2)C_{s-})1(Z_{s-})(dY_s-ds)\Big] \tag{55}$$

Again using the differentiation rule in D_t and E_t, we get:

$$dD_t = \frac{-e^t}{2}[(\lambda_1+\lambda_2-2)\hat{\lambda}_t+(\lambda_1+\lambda_2-2\lambda_1\lambda_2)]dt - [D_t\hat{\lambda}_t-D_t]dt$$

$$-\frac{e^t}{2}[(\lambda_2-\lambda_1)\hat{\mu}_t+(2\mu_1-2)\hat{\lambda}_t+(\lambda_1+\lambda_2-2\lambda_1\lambda_2)]1(Z_t)dt - [D_t\hat{\mu}_t-D_t]1(Z_t)dt \tag{56}$$

and

$$dE_t = \frac{-e^t}{2}\,[(\mu_2-\mu_1)\hat{\lambda}_t-(2\lambda_1-2)\hat{\mu}_t+(\mu_1+\mu_2-2\mu_1\mu_2)]dt - [E_t\hat{\lambda}_t-E_t]dt$$

$$- \frac{e^t}{2}\,[(\mu_1+\mu_2-2)\hat{\mu}_t+(\mu_1+\mu_2-2\mu_1\mu_2)]1(Z_t)dt - [E_t\hat{\mu}_t-E_t]1(Z_t)\,dt \tag{57}$$

With equations (52), (53), (56) and (57) we have a set of recursive equations for the pair $\begin{pmatrix}\lambda_t\\ \mu_t\end{pmatrix}$. By recursive we mean that we have equations for the dynamics that depend on $\hat{\lambda}_t$, $\hat{\mu}_t$, D_t and E_t.

(b) Let us suppose that a queue is servicing different kinds of customers; first, it serves customers of type 0 that arrive with arrival rate λ_0 and are served with service rate μ_0; then, customers of type 1 with arrival rate λ_1 and service rate μ_1; and so on. Therefore the queue will serve a infinite number of kinds of customers. We will consider that the time that customers of type i can arrive is exponentially distributed with mean 1. In this case:

$$H_{t,s}(g)(\lambda_e,\mu_e) = E(g(\lambda_t,\mu_t)/\lambda_s = \lambda_e,\ \mu_s = \mu_e)$$

$$= \sum_{k=0}^{\infty} g(\lambda_{e+k},\mu_{e+k})\,\frac{(t-s)^k}{k!}\,e^{-(t-s)} \tag{58}$$

Then:

$$\frac{\partial}{\partial t} H_{t,s}(g)(\lambda_e,\mu_e) = \sum_{k=0}^{\infty} \frac{(t-s)^k}{k!}\,e^{-(t-s)}[g(\lambda_{e+k+1},\mu_{e+k+1})$$

$$- g(\lambda_{e+k},\mu_{e+k})] \tag{59}$$

and we can say:

$$(\mathcal{L}_t(g))(\lambda_e,\mu_e) = (\mathcal{L}g)(\lambda_e,\mu_e) = g(\lambda_{e+1},\mu_{e+1}) - g(\lambda_e,\mu_e) \tag{60}$$

Let us define:

$$\delta_k(\lambda_t,\mu_t) = I_{\{\lambda_t=\lambda_k,\mu_t=\mu_k\}} \tag{61}$$

Using formula (38), we obtain:

$$\pi_t(\delta_e) = \pi_0(\delta_e) + \int_0^t (\lambda_e-1)\pi_s(\delta_e)(dX_s-ds) + \int_0^t (\mu_e-1)\pi_s(\delta_e)1(Z_{s-})(dY_s-ds) + \int_0^t [\pi_s(\delta_{e-1})-\pi_s(\delta_e)]ds \tag{62}$$

Therefore:

$$\pi_t(\delta_0) = 1 + \int_0^t (\lambda_0-1)\pi_s(\delta_0)(dX_s-ds) + \int_0^t (\mu_0-1)\pi_s(\delta_0)1(Z_{s-})(dY_s-ds) + \int_0^t \pi_s(\delta_0)ds \tag{63}$$

$$\pi_t(\delta e) = \int_0^t (\lambda_e-1)\pi_s(\delta_e)(dX_s-ds) + \int_0^t (\mu_e-1)\pi_s(\delta_e)1(Z_{s-})(dY_s-ds) + \int_0^t [\pi_s(\delta_{e-1})-\pi_s(\delta_e)]\, ds \qquad \text{for } \ell \geq 1 \tag{64}$$

Solving (63) and (64), we have:

$$e^t\pi_t(\delta_0) = 1 + \int_0^t (\lambda_0-1)e^s\pi_s(\delta_0)(dX_s-ds) + \int_0^t (\mu_0-1)e^s\pi_s(\delta_0)1(Z_{s-})(dY_s-ds)$$

$$\pi_t(\delta_0) = e^{-t} \prod_{t_i \leq t} \lambda_0 \prod_{s_j \leq t} \mu_0 \exp[-\int_o^t (\lambda_0 - 1)ds - \int_0^t (\mu_0 - 1(Z_s)ds] \tag{65}$$

$$\pi_t(\delta_e) = \int_0^t e^{-(t-s)} \pi_s(\delta_{e-1}) [\prod_{s \leq t_i \leq t} \lambda_e \prod_{s \leq s_j \leq t} \mu_e \exp[-\int_s^t (\lambda_e - 1)du$$

$$-\int_s^t (\mu_e - 1) 1(Z_u) du] ds, \qquad e \geq 1 \tag{66}$$

where (t_i) are the positive jumps of Z

(s_j) are the negative jumps of Z.

In this case we are unable to find a workable expression for $\hat{\lambda}_t$ and $\hat{\mu}_t$ but, at least, we can find (65) and (66) that can completely determinate $\hat{\lambda}_t$ and $\hat{\mu}_t$ at time t.

6.4. Estimation of the queue length given the input flow or the output flow

Sometimes we can not observe the queue length, itself, but; instead, the arrival process (input flow) or the output folw (customers that are leaving the queue). We will be interested in estimating given one of these flows.

Let us first obtain $E(Z_t/\mathcal{J}_t)$, where $\mathcal{J}_t = \sigma(X_s, 0 \leq s \leq t)$, X being the input flow. We consider that the arrival and service rate are constant. Then:

$$\hat{Z}_t = E[Z_t/\mathcal{J}_t] = E[X_t - \int_0^t 1(Z_{s-}) dY_s/\mathcal{J}_t]$$

$$= X_t - E[\int_0^t 1(Z_{s-}) dY_s/\mathcal{J}_t] = X_t - \int_0^t \widehat{1(Z_s)} \mu ds + E(M_t/\mathcal{J}_t) \tag{67}$$

where $M_t = \int_0^t 1(Z_{s-})(dY_s - ds)$

$$\widehat{1(Z_s)} = E(1(Z_s)/\mathcal{Y}_s)$$

By the representation theorem, we have:

$$E(M_t/\mathcal{Y}_t) = \int_0^t \psi_1(dX_s - \lambda ds) \tag{68}$$

To obtain ψ_1, we calculate:

$$Z_t X_t = \int_0^t Z_{s-}dX_s + \int_0^t X_{s-}dZ_s + \int_0^t dX_s$$

$$\hat{Z}_t X_t = \int_0^t \hat{Z}_{s-}dX_s + \int_0^t X_{s-}d\hat{Z}_s + \int_0^t (\psi_1+1)dX_s$$

Since $E(d(Z_t X_t)/\mathcal{Y}_t) = E(d(\hat{Z}_t X_t)/\mathcal{Y}_t)$, we have:

$$\psi_1 = 0 \tag{69}$$

Then

$$\hat{Z}_t = X_t - \int_0^t \widehat{1(Z_s)}\mu ds \tag{70}$$

In the expression (70) we don't know $\widehat{1(Z_s)}$. In order to obtain it, we have to calculate $\widehat{a^{Z_t}}$.

$$a^{Z_t} = 1 + \int_0^t (a-1)a^{Z_{s-}}\lambda ds + \int_0^t (1/a-1)a^{Z_{s-}}1(Z_{s-})\mu ds + N_t$$

$$\text{where } N_t = \int_0^t (a-1)a^{Z_{s-}}(dX_s - \lambda ds) + \int_0^t (1/a-1)a^{Z_{s-}}1(Z_{s-})(dY_s - ds) \tag{71}$$

Using the martingale representation theorem, we get

$$\hat{N}_t = \int_0^t \psi_2(dX_s - \lambda ds)$$

and

$$\widehat{a^{Z_t}} = 1 + \int_0^t (a-1)\widehat{a^{Z_s}}\lambda ds + \int_0^t (1/a-1)\widehat{a^{Z_s}1(Z_s)}\mu ds + \hat{N}_t$$

To get ψ_2, we use the same procedure used before, calculating:

$$a^{Z_t}X_t = \int_0^t X_{s-}da^{Z_s} + \int_0^t a^{Z_{s-}}dX_s + \int_0^t (a-1)a^{Z_{s-}}dX_s$$

$$\widehat{a^{Z_t}}X_t = \int_0^t X_{s-}d\widehat{a^{Z_t}} + \int_0^t \widehat{a^{Z_{s-}}}\, dX_s + \int_0^t \psi_1 dX_s$$

By equating $E(d(a^{Z_t}X_t)/J_t) = E(d(\widehat{a^{Z_t}}X_t)/J_t)$, we obtain:

$$\psi_1 = (a-1)\ \widehat{a^{Z_s}} \tag{72}$$

and

$$\widehat{a^{Z_t}} = 1 + \int_0^t (a-1)\widehat{a^{Z_s}}\lambda ds + \int_0^t (1/a-1)\widehat{a^{Z_s}1(Z_s)}\mu ds + \int_0^t (a-1)\widehat{a^{Z_s}}(dX_s-\lambda ds) \tag{73}$$

Since $a^{Z_t} = I_{\{Z_t=0\}} + a\, I_{\{Z_t=1\}} + a^2 I_{\{Z_t=2\}} + \ldots$, and can use (73) to obtain:

$$d\hat{I}_{\{Z_t=0\}} = \hat{I}_{\{Z_t=1\}}\, dt - \hat{I}_{\{Z_t=0\}}\, dX_t \tag{74}$$

$$d\hat{I}_{\{Z_t=e\}} = \mu[\hat{I}_{\{Z_t=e+1\}} - \hat{I}_{\{Z_t=e\}}]dt + [\hat{I}_{\{Z_t=e-1\}}-\hat{I}_{\{Z_t=e\}}]dX_t \tag{75}$$

Taking $J_t^T = (\hat{I}_{\{Z_t=0\}}, \hat{I}_{\{Z_t=1\}}, \hat{I}_{\{Z_t=2\}}, \ldots)$ we can put (74) and (75) in the form:

$$dJ_t = \mu BJ_t dt + AJ_{t-}\, dX_t \tag{76}$$

where:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \cdots & & & & \cdots \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 & 0 & \cdots \\ 1 & -1 & 0 & \cdots \\ 0 & 1 & -1 & \cdots \\ \cdots & & & \cdots \end{bmatrix}$$

Solving equation (73), we get:

$$J_t = e^{\mu Bt} \prod_{t_i \leq t} (I+e^{-\mu Bt} A e^{\mu Bt}) C \tag{77}$$

where e = initial condition for J_t, I = identity matrix

Remark: Since B is an infinite matrix, we define:

$$e^{\mu Bt} = I + \mu + B + \mu^2 t^2 \frac{B^2}{2!} + \mu^3 t^3 \frac{B^3}{3!} + \ldots$$

Let us obtain now $E(Z_t/\mathcal{K}_t)$ where $K_t = \int_0^1 1(Z_{s-})dY_s$, the output flow, and $\mathcal{K}_t = \sigma(K_s, 0 \leq s \leq t)$. We consider that λ and μ are constant. Then:

$$E(Z_t/\mathcal{K}_t) = E(P_t/\mathcal{K}_t) + \int_0^t \lambda ds - \int_0^t 1(Z_{s-})\, dY_s \tag{78}$$

where $P_t = X_t - \lambda t$

By the martingale representation theorem, we have:

$$\hat{P}_t = \int_0^t \psi_3 (1(Z_{s-})dY_s - \widehat{1(Z_s)}ds)$$

where $\widehat{1(Z_s)} = E(1(Z_s)/\mathcal{K}_s)$

Calculating

$$Z_tK_t = \int_0^t K_{s-}dKs + \int_0^t K_{s-}dZ_s - \int_0^t 1(Z_{s-})dYs$$

$$\hat{Z}_tK_t = \int_0^t \hat{Z}_{s-}dK_s + \int_0^t K_{s-}d\hat{Z}_s + \int_0^t (\psi_3-1)1(Z_{s-})dY_s$$

and equating $E(d(Z_tK_t)(\mathcal{K}_t) = E(d(\hat{Z}_tK_t)/\mathcal{K}_t)$, we have

$$\psi_3 = \frac{\widehat{Z_t1(Z_t)}-\hat{Z}_t\widehat{1(Z_t)}}{\widehat{1(Z_t)}} \tag{79}$$

Therefore:

$$\hat{Z}_t = \int_0^t \lambda ds - \int_0^t 1(Z_{s-})dY_s + \int_0^t \frac{\hat{Z}_{s-}(1-\widehat{1(Z_s)})}{\widehat{1(Z_{s-})}}(1(Z_{s-})dY_s-\widehat{1(Z_t)}ds) \tag{80}$$

We need to know the value of $\widehat{1(Z_s)}$. Calculating $\widehat{a^{Z_t}}$, we have:

$$a^{Z_t} = 1 + \int_0^t (a-1)\widehat{a^{Z_s}}\lambda ds + \int_0^t [\widehat{a^{Z_s}1(Z_s)}-\widehat{a^{Z_s}}\widehat{1(Z_s)}]\mu ds$$
$$+ \int_0^t 1/a\,\frac{\widehat{a^{Z_{s-}}1(Z_{s-})}-\widehat{a^{Z_{s-}}}\widehat{1(Z_{s-})}1(Z_{s-})}{1(Z_{s-})}\,dY_s \tag{81}$$

and finally

$$d\hat{I}_{\{Z_t=0\}} = -\lambda\hat{I}_{\{Z_t=0\}} + \mu[\hat{I}_{\{Z_t=0\}}(1-\hat{I}_{\{Z_t=0\}})]dt$$
$$+ [\hat{I}_{\{Z_{t-}=1\}} - \hat{I}_{\{Z_{t-}=0\}}(1-\hat{I}_{\{Z_{t-}=0\}})]\cdot\frac{1}{1-\hat{I}_{\{Z_{t-}=0\}}}1(Z_{t-})dYt \tag{82}$$

$$d\hat{I}_{\{Z_t=e\}} = [\hat{I}_{\{Z_t=e-1\}} - \hat{I}_{\{Z_t=e\}}]dt$$

$$+ \mu[\hat{I}_{\{Z_t=e\}}(1-\hat{I}_{\{Z_t=0\}})-\hat{I}_{\{Z_t=e\}}]dt$$

$$+ [\hat{I}_{\{Z_{t-}=e+1\}}-I_{\{Z_t=e\}}(1-\hat{I}_{\{Z_{t-}=0\}})]\cdot \frac{1}{1-\hat{I}_{\{Z_{t-}=0\}}} 1(Z_{t-})dY_t \tag{83}$$

In this case we are not able to get a closed form solution, as in the preceding case, and, therefore, the equations (82) and (83) are not too useful.

6.5. Approximate filtering

Let us suppose that our provess is such that both our service rate and arrival rate are functions of a parameter a. It is supposed that a takes values in a specified set A and the distribution of parameter values in A is known. Therefore, we can calculate:

$$P_t(a/\mathcal{Z}_t) = \frac{R_t}{Q_t} \tag{84}$$

where:

$$R_t = P_o(a) + \int_0^t R_{s-}(\lambda_{s-}(a)-1)(dX_s-ds) + \int_0^t R_{s-}(\mu_{s-}(a)-1)\cdot$$
$$\cdot\ 1(Z_{s-})(dY_s-ds) \tag{85}$$

$$Q_t = 1 + \int_0^t Q_{s-}(\hat{\lambda}_s-1)(dX_{s-}ds) + \int_0^t Q_{s-}(\hat{\mu}_s-1)1(Z_{s-})(dY_s-ds) \tag{86}$$

Applying the differentiation rule to $\frac{R_t}{Q_t}$, we obtain:

$$dP_t(a/\mathcal{Z}_t) = [\lambda_t(a)-\hat{\lambda}_t]\hat{\lambda}_t^{-1}P_t(a/\mathcal{Z}_t)(dX_t-\hat{\lambda}_t dt)$$
$$+ P_t(a/\mathcal{Z}_t)[\mu_{t-}(a)-\hat{\mu}_{t-}]\hat{\mu}_{t-}^{-1}1(Z_{t-})(dY_t-\hat{\mu}_t dt) \tag{87}$$

Using the martingale representation theorem, we obtain:

$$\hat{a}_t = Ea + \int_0^t \psi_1(dX_s - \hat{\lambda}_s ds) + \int_0^t \psi_2 1(Z_{s-})(dY_s - \hat{\mu}_s ds) \tag{88}$$

Calculating:

$$aX_t = \int_0^t a dX_s$$

$$\hat{a}_t X_t = \int_0^t \hat{a}_s dX_s + \int_0^t X_{s-} d\hat{a}_s + \int_0^t \psi_1 dX_s$$

Equating $E(d(aX_t)/\mathcal{Z}_t) = E(d(\hat{a}_t X_t)/\mathcal{Z}_t)$, we find:

$$\psi_1 = \frac{\widehat{a\lambda_t(a)} - \hat{a}_t \hat{\lambda}_t}{\hat{\lambda}_t} \tag{89}$$

In the same way, we obtain:

$$\psi_2 = \frac{\widehat{a\mu_t(a)} - \hat{a}_t \hat{\mu}_t}{\hat{\mu}_t} \tag{90}$$

and the expression (88) becomes:

$$d\hat{a}_t = [\widehat{a\lambda_t(a)} - \hat{a}_t \hat{\lambda}_t]\hat{\lambda}_t^{-1}(dX_t - \hat{\lambda}_t dt)$$

$$+ [\widehat{a\mu_t(a)} - \hat{a}_t \hat{\mu}_t]\hat{\mu}_t^{-1}(dY_t - \hat{\mu}_t dt) \tag{91}$$

$$\hat{a}_o = E(a)$$

Using the same approximations done in [Snyder,1973a], that is:

(a) Take:

$$\lambda_t(a) = \lambda_t(\hat{a}_t) + \left(\frac{\partial \lambda_t(\hat{a}_t)}{\partial \hat{a}_t}\right)^T (a - \hat{a}_t) + \text{higher order error } (a - \hat{a}_t)$$

- terms.

(b) Replace $\hat{\lambda}_t$ by $\lambda_t(\hat{a}_t)$

With the approximations, we find:

$$\widehat{[a\lambda_t} - \hat{a}\hat{\lambda}_t \;] = \widehat{[a-\hat{a}_t]\lambda_t(a)}$$

$$= E[(a-\hat{a}_t)(\lambda_t(\hat{a}_t) + \left(\frac{\partial\lambda_t(\hat{a}_t)}{\partial\hat{a}_t}\right)(a-\hat{a}_t)^T + \ldots)/\mathcal{Z}_t)$$

$$\cong \left(\frac{\partial\lambda_t(\hat{a}_t)}{\partial\hat{a}_t}\right)E[(a-\hat{a}_t)(a-\hat{a}_t)^T/\mathcal{Z}_t] \tag{92}$$

In the same way we will get:

$$\widehat{a\mu_t(a)} - \hat{a}_t\hat{\mu}_t \cong \left(\frac{\partial\mu_t(\hat{a}_t)}{\partial\hat{a}_t}\right) E[(a-\hat{a}_t)(a-\hat{a}_t)^T/\mathcal{Z}_t] \tag{93}$$

The equation for the approximated estimator a_t^* is:

$$da_t^* = -\sum_t^* \frac{\partial\lambda_t(a_t^*)}{\partial a_t^*}dt - \sum_t^* \frac{\partial\mu_t(a_t^*)}{\partial a_t^*} 1(Z_t)dt$$

$$+ \sum_t^* \frac{\partial(\ln\lambda_t(a_t^*)}{\partial a_t^*}dX_t + \sum_t^* \frac{\partial(\ln\mu_t(a_t^*)}{\partial a_t^*} 1(Z_{t-})dY_t \tag{94}$$

where $\sum_t^* = E[(a-\hat{a}_t)(a-\hat{a}_t)^T/\mathcal{Z}_t]$ (95)

Repeating the same procedures, we find:

$$d\sum_t^* = -\sum_t^* \left(\frac{\partial\lambda(a_t^*)}{\partial a_t^*}\right)\left(\frac{\partial\lambda(a_t^*)}{\partial a_t^*}\right)^T \sum_t^* \lambda_t^{-2}\;(a_t^*)dX_t$$

$$-\sum_t^* \left(\frac{\partial\mu(a_t^*)}{\partial a_t^*}\right)\left(\frac{\partial\mu(a_t^*)}{\partial a_t^*}\right)^T \sum_t^* \mu_t^{-2}\;(a_t^*)1(Z_{t-})dY_t$$

$$+ \sum_t^* \left(\frac{\partial^2 \lambda(a_t^*)}{\partial (a_t^*)^2} \right) \sum_t^* \lambda_t^{-1} (a_t^*) dX_t + \sum_t^* \left(\frac{\partial^2 \mu(a_t^*)}{\partial (a_t^*)^2} \right) \sum_t^* \mu_t^{-1}(a_t^*) 1(Z_t) dY_t$$

$$- \sum_t^* \left(\frac{\partial^2 \lambda(a_t^*)}{\partial (a_t^*)^2} \right) \sum_t^* dt - \sum_t^* \left(\frac{\partial^2 \mu(a_t^*)}{\partial (a_t^*)^2} \right) \sum_t^* 1(Z_{s-}) dt \tag{96}$$

The euqation (94) and (96) are recursive equations for the approximated estimator a_t^* and the covariance matrix $\sum_t^*$.

CHAPTER 7

DISSCUSSIONS AND CONCLUSIONS

The goal of this dissertation was to apply the theory of martingales and stochastic integrals and, at the same time, compare this approach with the classical one. We think that we made our point in showing that the martingale approach is really both more general and intuitive. The advantages are apparent in optimal control, estimation and identification. Besides that, in relation with the calculation of the characteristics of the queueing process, we gave a procedure to obtain them for the case were both the interarrival and service times are i.i.d.. This same procedure can be modified in order to solve other particular problems.

The main results in chapter 3 are the representation of the queue length in a semi-martingale form and the subsequent extension for a more general class of processes. The new points presented are that we don't need to assume Markov properties for the queueing processes and we can deal with the processes more directly.

In chapter 4 we calculated some characteristics of the queueing systems using the transformation of measures and compared it with the classical results. In doing that we were able to find more general results, besides checking some already known expressions. Contrasting with usual methods, we had not to use Laplace transforms and complex variable results and we were able to use the same kind of approach for different kinds of service and arrivals.

Chapter 5 and 6 are the ones in which the theory of martingales and stochastic integration proves indeed its usefulness. We could formulate the optimal control problem, obtaining a Hamilton-Jacobi equation for the Markov case and applying it to cases where the cost function were linear and quadratic. In estimation, some results in recursive filtering were obtained.

We think that the major areas of future research could be:

a. Application of the martingale approach to other types of queues.

b. Calculation of the conditions for the existence of a steady state behaviour for the single-server single queue and the solution of optimal control problems, estimation and identification in this case.

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