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FUZZY LOGIC AND APPROXIMATE REASONING

by

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In memory of Grigore Moisil

Fuzzy Logic and Approximate Reasoning

L. A. Zadeh*

Abstract

The term *fuzzy logic* is used in this paper to describe an imprecise logical system, FL, in which the truth-values are fuzzy subsets of the unit interval with linguistic labels such as *true*, *false*, *not true*, *very true*, *quite true*, *not very true and not very false*, etc. The truth-value set, $\mathcal{T}$, of FL is assumed to be generated by a context-free grammar, with a semantic rule providing a means of computing the meaning of each linguistic true-value in $\mathcal{T}$ as a fuzzy subset of [0,1].

Since $\mathcal{T}$ is not closed under the operations of negation, conjunction, disjunction and implication, the result of an operation on truth-values in $\mathcal{T}$ requires, in general, a linguistic approximation by a truth-value in $\mathcal{T}$. As a consequence, the truth tables and the rules of inference in fuzzy logic are (i) inexact and (ii) dependent on the meaning associated with the primary truth-value *true* as well as the modifiers *very*, *quite*, *more or less*, etc.

Approximate reasoning is viewed as a process of approximate solution of a system of relational assignment equations. This process is formulated as a compositional rule of inference which subsumes *modus ponens* as a special case. A characteristic feature of approximate reasoning is the fuzziness and nonuniqueness of consequents of fuzzy premisses. Simple examples of approximate reasoning are: (a) *Most* men are *vain*; Socrates is a man; therefore, it is *very likely* that Socrates is *vain*. (b) x is *small*; x and y are *approximately equal*; *therefore* y is *more or less small*, where italicized words are labels of fuzzy sets.

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1. Introduction

It is a truism that much of human reasoning is approximate rather than precise in nature. As a case in point, we reason in approximate terms when we decide on how to cross a traffic intersection, which route to take to a desired destination, how much to bet in poker and what approach to use in proving a theorem. Indeed, it could be argued, rather convincingly, that only a small fraction of our thinking could be categorized as precise in either logical or quantitative terms.

Perhaps the simplest way of characterizing fuzzy logic is to say that it is a logic of approximate reasoning. As such, it is a logic whose distinguishing features are (i) fuzzy truth-values expressed in linguistic terms, e.g., _true_, _very true_, _more or less true_, _rather true_, _not true_, _false_, _not very true and not very false_, etc.; (ii) imprecise truth tables; and (iii) rules of inference whose validity is approximate rather than exact. In these respects, fuzzy logic differs significantly from standard logical systems ranging from the classical Aristotelian logic [1] to inductive logics [2] and many-valued logics with set-valued truth-values [3].

An elementary example of approximate reasoning in fuzzy logic is the following variation on a familiar Aristotelian syllogism.

A_1 : _Most_ men are _vain_ (1.1)

A_2 : Socrates is a man

A_3 : It is _likely_ that Socrates is _vain_

or

A_3' : It is _very likely_ that Socrates is _vain_

In this example, both A_3 and A'_3 are admissible approximate consequents of A_1 and A_2, with the degree of approximation depending on the definitions of the terms most, likely and very as fuzzy subsets of their respective universes of discourse. For example, assume that most and likely are defined as fuzzy subsets of the unit interval by compatibility functions[1] of the form shown in Fig. 1, and let very be defined as a modifier which squares the compatibility function of its operand. Then A'_3 is a better approximation than A_3 to the exact consequent of A_1 and A_2 provided very likely, as a fuzzy subset of [0,1], is a better approximation than likely to the fuzzy subset most. This is assumed to be the case in Fig. 1.

Additional examples of approximate reasoning in fuzzy logic are the following. (u_1 and u_2 are numbers.)

A_1 : u_1 is small (1.2)

A_2 : u_1 and u_2 are approximately equal

A_3 : u_2 is more or less small

A_1 : (u_1 is small) is very true (1.3)

A_2 : (u_1 and u_2 are approximately equal) is very true

A_3 : (u_2 is more or less small) is true

The italicized words in these examples represent labels of fuzzy sets. Thus, a fuzzy proposition of the form "u_1 is small," represents the assign-

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[1] If U is a universe of discourse and F is a fuzzy subset of U, then the compatibility function (or, equivalently, membership function) of F is a mapping $\mu_F : U \rightarrow [0,1]$ which associates with each $u \in U$ its compatibility (or grade of membership) $\mu_F(u)$, $0 \leq \mu_F(u) \leq 1$, [4].

ment of a fuzzy set (or, equivalently, a unary fuzzy relation) labeled small as a value of u_1. Similarly, the fuzzy proposition "u_1 and u_2 are approximately equal," represents the assignment of a binary fuzzy relation approximately equal to the ordered pair (u_1,u_2). And, the nested fuzzy proposition "(u_1 is small) is very true," represents the assignment of a fuzzy truth-value very true to the fuzzy proposition (u_1 is small).

As will be seen in Sec. 3, the above examples may be viewed as special instances of a model of reasoning in which the process of inference involves the solution of a system of relational assignment equations. Thus, in terms of this model, approximate reasoning may be viewed as the determination of an approximate solution of a system of relational assignment equations in which the assigned relations are generally, but not necessarily, fuzzy rather than nonfuzzy subsets of a universe of discourse.

In what follows, we shall outline in greater detail some of the main ideas which form the basis for fuzzy logic and approximate reasoning. Our presentation will be informal in nature.

2. Fuzzy Logic

A fuzzy logic, FL, may be viewed, in part, as a fuzzy extension of a nonfuzzy multi-valued logic which constitutes a base logic for FL. For our purposes, it will be convenient to use as a base logic for FL the standard Lukasiewicz logic L_1 (abbreviated from $L_{\mathrm{A}\ell\mathrm{eph}_1}$) in which the truth-values are real numbers in the interval [0,1] and[2]

$$v(\neg p) \triangleq 1 - v(p) \tag{2.1}$$

$$v(p \vee q) \triangleq \max(v(p), v(q)) \tag{2.2}$$

[2] The symbol $\triangleq$ stands for "is defined to be," or "denotes."

$$v(p \,\Delta\, q) = \min\,(v(p),\, v(q)) \tag{2.3}$$

$$v(p \Rightarrow q) = \min\,(1,\, 1 - u(p) + v(q)) \tag{2.4}$$

where $v(p)$ denotes the truth-value of a proposition p, $\neg$ is the negation, $\wedge$ is the conjunction, $\vee$ is the disjunction and $\Rightarrow$ is the implication. In what follows, however, it will be more convenient to denote the negation, conjunction and disjunction by <u>not</u>, <u>and</u> and <u>or</u>, respectively, reserving the symbols $\neg$, $\wedge$ and $\vee$ to denote operations on truth-values, with $\wedge \triangleq \min$ and $\vee \triangleq \max$.

<u>The truth-value set of FL</u>

The truth-value set of FL is assumed to be a countable set $\mathcal{T}$ of the form

$$\mathcal{T} = \{\underline{\text{true}},\ \underline{\text{false}},\ \underline{\text{not}}\ \underline{\text{true}},\ \underline{\text{very}}\ \underline{\text{true}},\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}},\ \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{true}},\ \underline{\text{rather}}\ \underline{\text{true}},\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}},\ldots\} \tag{2.5}$$

Each element of this set represents a fuzzy subset of the truth-value set of L_1, i.e., $[0,1]$. Thus, the <u>meaning</u> of a linguistic truth-value, τ, in $\mathcal{T}$ is assumed to be a fuzzy subset of $[0,1]$.

More specifically, let $\mu_\tau : [0,1] \to [0,1]$ denote the compatibility (or membership) function of τ. Then the meaning of τ, as a fuzzy subset of $[0,1]$, is expressed by

$$\tau = \int_0^1 \mu_\tau(v)/v \tag{2.6}$$

where the integral sign denotes the union of fuzzy singletons $\mu_\tau(v)/v$, with $\mu_\tau(v)/v$ signifying that the compatibility of the numerical truth-value v with the linguistic truth-value τ is $\mu_\tau(v)$, or, equivalently, that the grade of membership of v in the fuzzy set labeled τ is $\mu_\tau(v)$.

If the support[3] of τ is a finite subset $\{v_1,\ldots,v_n)$ of $[0,1]$, τ may be expressed as

$$\tau = \mu_1/v_1 + \ldots + \mu_n/v_n. \tag{2.7}$$

or more simply as

$$\tau = \mu_1 v_1 + \ldots + \mu_n v_n \tag{2.8}$$

when no confusion between μ_i and v_i in a term of the form $\mu_i v_i$ can arise. Note that + in (2.7) plays the role of the union rather than the arithmetic sum.

As a simple illustration, suppose that the meaning of <u>true</u> is defined by

$$\begin{aligned} \mu_{\underline{true}}(v) &= 0 \qquad\qquad\qquad \text{for} \quad o \leq v \leq \alpha \\ &= 2\left(\frac{v-\alpha}{1-\alpha}\right)^2 \quad \text{for} \quad \alpha \leq v \leq \frac{\alpha+1}{2} \\ &= 1 - 2\left(\frac{v-\alpha}{1-\alpha}\right)^2 \quad \text{for} \quad \frac{\alpha+1}{2} \leq v \leq 1 \end{aligned} \tag{2.9}$$

where α is a point in $[0,1]$.

Then, we may write

$$\underline{true} \triangleq \int_{\alpha}^{\frac{\alpha+1}{2}} 2\left(\frac{v-\alpha}{1-\alpha}\right)^2 / v + \int_{\frac{\alpha+1}{2}}^{1} \left(1 - 2\left(\frac{v-\alpha}{1-\alpha}\right)^2\right) / v \tag{2.10}$$

If $v_1 = 0$, $v_2 = 0.1$, ..., $v_{11} = 1$, then <u>true</u> might be defined by, say,

$$\underline{true} = 0.3/0.6 + 0.5/0.7 + 0.7/0.8 + 0.9/0.9 + 1/1 \tag{2.11}$$

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[3]The <u>support</u> of a fuzzy subset, A, of U is the set of points in U at which $\mu_A(u) > 0$. The <u>crossover</u> points of A are the points of U at which $\mu_A(u) = 0.5$.

In terms of the meaning of true, the truth-value false may be defined as

$$\underline{\text{false}} \triangleq \int_0^1 \mu_{\underline{\text{true}}}(1-v)/v \tag{2.12}$$

while not true is given by

$$\underline{\text{not}}\ \underline{\text{true}} = \int_0^1 (1 - \mu_{\underline{\text{true}}}(v))/v \tag{2.13}$$

Thus, as a fuzzy set, not true is the complement of true whereas false is the truth-value of the proposition not p, if true is the truth-value of p. In the case of (2.11), this implies that

$$\underline{\text{false}} \triangleq \neg\ \underline{\text{true}} = 1/0 + 0.9/0.1 + 0.7/0.2 + 0.5/0.3 + 0.3/0.4 \tag{2.14}$$

and

$$\underline{\text{not}}\ \underline{\text{true}} \triangleq \underline{\text{true}}' = 1/(0 + 0.1 + 0.2 + 0.3 + 0.4 + 0.5) + 0.7/0.6 + 0.5/0.7 + 0.3/0.8 + 0.1/0.9 \tag{2.15}$$

where $\neg$ stands for negation and ' denotes the complement (see footnote 5).

More generally, the truth-value set of FL is characterized by two rules: (i) a syntactic rule, which we shall assume to have the form of a context-free grammar G such that

$$\mathcal{T} = L(G) \tag{2.16}$$

that is, $\mathcal{T}$ is the language generated by G; and (ii) a semantic rule, which is an algorithmic procedure for computing the meaning of the elements of $\mathcal{T}$. Generally, we shall assume that $\mathcal{T}$ contains one or more primary terms (e.g., true) whose meaning is specified a priori and which form the basis for the computation of the meaning of the other terms in $\mathcal{T}$. The truth-values in $\mathcal{T}$ are referred to as linguistic truth-values in order to differentiate them from the numerical truth-values of L_1.

Example 2.17. As a simple illustration, suppose that $\mathcal{T}$ is of the form

$$\mathcal{T} = \{\underline{\text{true}},\ \underline{\text{false}},\ \underline{\text{not}}\ \underline{\text{true}},\ \underline{\text{very}}\ \underline{\text{true}},\ \underline{\text{very}}\ \underline{\text{very}}\ \underline{\text{true}},\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}},\ \underline{\text{not}}\ \underline{\text{true}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{false}},\ \underline{\text{true}}\ \underline{\text{and}}\ (\underline{\text{not}}\ \underline{\text{false}}\ \underline{\text{or}}\ \underline{\text{not}}\ \underline{\text{true}}),\ \ldots\ \} \tag{2.18}$$

It can readily be verified that $\mathcal{T}$ can be generated by a context-free grammar G whose production system is given by

$$\begin{array}{ll} T \rightarrow A & C \rightarrow D \\ T \rightarrow T\ \underline{\text{or}}\ A & C \rightarrow E \\ A \rightarrow B & D \rightarrow \underline{\text{very}}\ D \\ A \rightarrow A\ \underline{\text{and}}\ B & E \rightarrow \underline{\text{very}}\ E \\ B \rightarrow C & D \rightarrow \underline{\text{true}} \\ B \rightarrow \underline{\text{not}}\ C & E \rightarrow \underline{\text{false}} \\ C \rightarrow (T) & \end{array} \tag{2.19}$$

In this grammar, T, A, B, C, D and E are nonterminals; and true, false, very, not, and, or, (,) are terminals. Thus, a typical derivation yields

$$\begin{aligned} T &\Rightarrow A \Rightarrow A\ \underline{\text{and}}\ B \Rightarrow B\ \underline{\text{and}}\ B \Rightarrow \underline{\text{not}}\ C\ \underline{\text{and}}\ B \Rightarrow \underline{\text{not}}\ E\ \underline{\text{and}}\ B \\ &\Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ E\ \underline{\text{and}}\ B \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}}\ \underline{\text{and}}\ B \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}} \\ &\quad \underline{\text{and}}\ \underline{\text{not}}\ C \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}}\ \underline{\text{and}}\ \underline{\text{not}}\ D \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}}\ \underline{\text{and}} \\ &\quad \underline{\text{not}}\ \underline{\text{very}}\ D \Rightarrow \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}}. \end{aligned} \tag{2.20}$$

If the syntactic rule for generating the elements of $\mathcal{T}$ is expressed as a context-free grammar, then the corresponding semantic rule may be conveniently expressed by a system of productions and relations in which each production in G is associated with a relation between the fuzzy subsets representing the meaning of the terminals and nonterminals [5].[4]

[4] This technique is related to Knuth's method of synthesized attributes [6].

For example, the production A → A and B induces the relation

$$A_L = A_R \cap B_R \tag{2.21}$$

where A_L, A_R, and B_R represent the meaning of A and B as fuzzy subsets of [0,1] (the subscripts L and R serve to differentiate between the symbols on the left- and right-hand sides of a production), and $\cap$ denotes the intersection.[5] Thus, in effect, (2.21) defines the meaning of the connective and.

With this understanding, the dual system corresponding to (2.19) may be written as

$$\begin{array}{lcl}
T \to A & : & T_L = A_R \\
T \to T \ \underline{\text{or}} \ A & : & T_L = T_R \cup A_R \\
A \to B & : & A_L = B_R \\
A \to A \ \underline{\text{and}} \ B & : & A_L = A_R \cap B_R \\
B \to C & : & B_L = C_R \\
B \to \underline{\text{not}} \ C & : & B_L = C'_R \\
C \to (T) & : & C_L = T_R \\
C \to D & : & C_L = D_R \\
C \to E & : & C_L = E_R \\
D \to \underline{\text{very}} \ D & : & D_L = (D_R)^2 \\
E \to \underline{\text{very}} \ E & : & E_L = (E_R)^2 \\
D \to \underline{\text{true}} & : & D_L = \underline{\text{true}} \\
E \to \underline{\text{false}} & : & E_L = \underline{\text{false}}
\end{array} \tag{2.22}$$

---

[5] If A and B are fuzzy subsets of $U = \{u\}$ with respective compatibility functions μ_A and μ_B, then the complement, A', of A is defined by $\mu_{A'}(u) = 1 - \mu_A(u)$; the intersection of A and B, $A \cap B$, is defined by $\mu_{A\cap B}(u) = \mu_A(u) \wedge \mu_B(u)$; the union of A and B (denoted by $A \cup B$ or A+B) is defined by $\mu_{A+B}(u) = \mu_A(u) \vee \mu_B(u)$; the product of A and B is defined by $\mu_{AB}(u) = \mu_A(u)\mu_B(u)$; and A^α is defined by $\mu_{A^\alpha}(u) = (\mu_A(u))^\alpha$. If $A \subset U$ and $B \subset V$, then the cartesian product of A and B, $A \times B$, is defined by $\mu_{A\times B}(u,v) = \mu_A(u) \wedge \mu_B(v)$ [7],[8].

This dual system is employed in the following manner to compute the meaning of a composite truth-value in $\mathcal{T}$.

1. The truth-value in question, e.g., <u>not very true and not very false</u>, is parsed by the use of an appropriate parsing algorithm for G [9], yielding a syntax tree such as shown in Fig. 2. The leaves of this syntax tree are (a) primary terms whose meaning is specified a priori; (b) names of modifiers, connectives and negation; and (c) markers such as parentheses which serve as aids to parsing.

2. Starting from the bottom, the primary terms are assigned their meaning and, using the equations of (2.22), the meaning of nonterminals connected to the leaves is computed. Then, the subtrees which have these nonterminals as their roots are deleted, leaving the nonterminals in question as the leaves of the pruned tree. This process is repeated until the meaning of the term associated with the root of the syntax tree is derived.[6]

In applying this procedure to the syntax tree shown in Fig. 2, we first assign to <u>true</u> and <u>false</u> the meaning expressed by (2.6) and (2.12). Then, we obtain in succession

$$\begin{aligned} D_7 &= \underline{\text{true}} \\ E_{11} &= \underline{\text{false}} \\ D_6 = D_7^2 &= \underline{\text{true}}^2 \\ E_{10} = E_{11}^2 &= \underline{\text{false}}^2 \end{aligned} \tag{2.23}$$

[6]It should be noted that in the case of truth-values of the form (2.18), this process is similar to the familiar procedure for evaluating Boolean and arithmetic expressions.

$$C_5 = D_6 = \underline{\text{true}}^2$$

$$C_9 = E_{10} = \underline{\text{false}}^2$$

$$B_4 = C_5' = (\underline{\text{true}}^2)'$$

$$B_8 = C_9' = (\underline{\text{false}}^2)'$$

$$A_3 = B_4 = (\underline{\text{true}}^2)'$$

$$A_2 = A_3 \cap B_8 = (\underline{\text{true}}^2)' \cap (\underline{\text{false}}^2)'$$

and finally,

$$\underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{true}}\ \underline{\text{and}}\ \underline{\text{not}}\ \underline{\text{very}}\ \underline{\text{false}} = (\underline{\text{true}}^2)' \cap (\underline{\text{false}}^2)' \tag{2.24}$$

where $\mu_{\underline{\text{true}}^2} = (\mu_{\underline{\text{true}}})^2$ and likewise for $\mu_{\underline{\text{false}}^2}$. (See footnote 5.)

It should be noted that the truth-values in (2.18) involve just one modifier, very, whose meaning is characterized by (2.22). As defined by (2.22), very has the effect of squaring the compatibility function of its operand. This simple approximation should not be viewed, of course, as an accurate representation of the complex and rather varied ways in which very modifies the meaning of its operands in a natural language discourse.

In addition to very, the more important of the modifiers which may be of use in generating the linguistic truth-values in $\mathcal{T}$ are: more or less, rather, quite, essentially, completely, somewhat, and slightly. As in the case of very, the meaning of such modifiers may be defined - as a first approximation - in terms of a set of standardized operations on the fuzzy sets representing their operands.[7] Better approximations, however, would require the use of algorithmic techniques in which a definition is expressed as a fuzzy recognition algorithm which has the form of a branching question-

[7] A more detailed discussion of linguistic modifiers and hedges may be found in [10],[11] and [4].

naire [12].

What is the rationale for using the linguistic truth-values of FL in preference to the numerical truth-values of L_1? At first glance, it may appear that we are moving in a wrong direction, since it is certainly easier to manipulate the real numbers in [0,1] than the fuzzy subsets of [0,1]. The answer is two-fold. First, the truth-value set of L_1 is a continuum whereas that of FL is a countable set. More importantly, in most applications to approximate reasoning, a small finite subset of the truth-values of FL would, in general, be sufficient because each truth-value of FL represents a fuzzy subset rather than a single element of [0,1]. Thus, we gain by trading the large number of simple truth-values of L_1 for the small number of less simple truth-values of FL.

The second and related point is that approximate reasoning deals, for the most part, with propositions which are fuzzy rather than precise, e.g., "Vera is <u>highly intelligent</u>," Douglas is <u>very inventive</u>," Berkeley is <u>close</u> to San Francisco," "It is <u>very likely</u> that Jean-Paul will <u>succeed</u>," etc. Clearly, the fuzzy truth-values of FL are more commensurate with the fuzziness of such propositions than the numerical truth-values of L_1.

<u>Operations on linguistic truth-values</u>

So far, we have focused our attention on the structure of the truth-value set of FL. We turn next to some of the basic questions relating to the manipulation of linguistic truth-values which are labels of fuzzy subsets of [0,1].

To extend the definitions of negation, conjunction, disjunction and implications in L_1 to those of FL, it is convenient to employ an extension principle for fuzzy sets which may be stated as follows [4].[8]

Let f be a mapping from V to W and let A be a fuzzy subset of V expressed as

$$A = \int_V \mu_A(v)/v \tag{2.25}$$

or, in the finite case, as

$$A = \mu_1 v_1 + \ldots + \mu_n v_n \tag{2.26}$$

where μ_A is the compatibility function of A, with $\mu_A(v)$ and μ_i denoting, respectively, the compatibilities of v and v_i, $i = 1, \ldots, n$, with A.

Then, the image of A under f is a fuzzy subset, f(A), of W defined by

$$f(A) = \int_W \mu_A(v)/f(v) \tag{2.27}$$

or, in the case of (2.26),

$$f(A) = \mu_1\ f(v_1) + \ldots + \mu_n\ f(v_n) \tag{2.28}$$

where $w = f(v)$ is the image of v under f. In effect, (2.27) and (2.28) extend the domain of definition of f from points in V to fuzzy subsets of V.

More generally, let * denote a mapping (or a relation) from the cartesian product $U \times V$ to W. Thus, expressed in infix form, we have

$$w = u * v, \qquad u \in U, \quad v \in V, \quad w \in W \tag{2.29}$$

where w is the image of u and v under *.

[8]In the context of operations on linguistic truth-values, this principle may be viewed as an extension to fuzzy-set-valued logics of the expansion techniques used in quasi-truth-functional systems [3].

Let A and B be fuzzy subsets of U and V, respectively, expressed as

$$A = \int_U \mu_A(u)/u \tag{2.30}$$

$$B = \int_V \mu_B(v)/v \tag{2.31}$$

or

$$A = \mu_1 u_1 + \ldots + \mu_m u_m \tag{2.32}$$

$$B = \nu_1 v_1 + \ldots + \nu_n v_n \tag{2.33}$$

Then, the image of A×B under * is a fuzzy subset, A*B, of W defined by

$$A*B = \int_{U\times V} (\mu_A(u) \wedge \mu_B(v))/(u,v) \tag{2.34}$$

or, in the case of (2.32) and (2.33),

$$A*B = \sum_{i,j} (\mu_i \wedge \nu_j)/u_i * v_j \ , \quad i=1,\ldots,m, \ j=1,\ldots,n \tag{2.35}$$

provided u and v are noninteractive [4] in the sense that the assignment of a value to u does not affect the values that may be assigned to v, and vice-versa. A convenient feature of (2.35) is that the expression for A*B may be obtained quite readily through term by term multiplication of (2.32) and (2.33), and employing the identities

$$(\mu_i u_i) * (\nu_j v_j) = (\mu_i \wedge \nu_j)/u_i * v_j \tag{2.36}$$

and

$$\alpha_k w_k + \beta_k w_k = (\alpha_k \vee \beta_k) w_k \tag{2.37}$$

for combination and simplification.

To apply the extension principle to the definition of negation, conjunction, disjunction and implication in FL, it is expedient to use (2.8), since it is easy to extend the resulting definitions to the case where the truth-values are of the form (2.6).

Specifically, let p and q be fuzzy propositions whose truth-values are fuzzy sets of the form

$$v(p) = \mu_1 v_1 + \ldots + \mu_m v_m \tag{2.38}$$

$$v(q) = \nu_1 v_1 + \ldots + \nu_n v_n \tag{2.39}$$

For example, p might be "Eugenia is <u>very</u> <u>kind</u>," with the truth-value of p being <u>very</u> <u>true</u>, while q might be "Fania was <u>very</u> <u>healthy</u>," with the truth-value of q being <u>more</u> <u>or</u> <u>less</u> <u>true</u>.

Applying (2.28) to (2.38), the expression for the truth-value of the proposition <u>not</u> p is found to be

$$v(\underline{\text{not}}\ p) = \mu_1/(1-v_1) + \ldots + \mu_m/(1-v_m) \tag{2.40}$$

For example, if <u>true</u> is defined by (2.11) and $v(p) = \underline{\text{very}}\ \underline{\text{true}}$, then

$$\underline{\text{very}}\ \underline{\text{true}} = 0.09/0.6 + 0.25/0.7 + 0.49/0.8 + 0.81/0.9 + 1/1 \tag{2.41}$$

and

$$v(\underline{\text{not}}\ p) = 0.09/0.4 + 0.25/0.3 + 0.49/0.2 + 0.81/0.1 + 1/0 \tag{2.42}$$

which in view of (2.14) may be expressed as

$$v(\underline{\text{not}}\ p) = \underline{\text{very}}\ \underline{\text{false}} \tag{2.43}$$

In this example, the truth-value of <u>not</u> p is an element of $\mathcal{T}$. In general, however, this will not be the case, so that a fuzzy truth-value, ϕ, obtained as a result of application of (2.28) or (2.35) would normally have to be approximated by a linguistic truth-value, ϕ^*, which is in $\mathcal{T}$.

The relation between ϕ^* and ϕ will be expressed as

$$\phi^* = LA[\phi] \tag{2.44}$$

where LA is an abbreviation for <u>linguistic approximation</u>. Note that a linguistic approximation to a given ϕ will not, in general, be unique.[9]

At present, there is no simple or general technique for finding a "good" linguistic approximation to a given fuzzy subset of V. In most cases, such an approximation has to be found by ad hoc procedures, without a precisely defined criterion of the "goodness" of approximation. In view of this, the standards of precision in computations involving linguistic truth-values are, in general, rather low. This, however, is entirely consistent with the imprecise nature of fuzzy logic and its role in approximate reasoning.

Turning to the definitions of conjunction, disjunction and implication[10] in FL, we obtain on application of (2.35) to (2.2), (2.3) and (2.4)

$$\begin{aligned} v(p \text{ \underline{and} } q) &= LA[v(p) \wedge v(q)] \\ &= LA[(\mu_1 u_1 + \ldots + \mu_m u_m) \wedge (\nu_1 v_1 + \ldots + \nu_n v_n)] \\ &= LA[\sum_{i,j} (\mu_i \wedge \nu_j)/u_i \wedge v_j] \end{aligned} \tag{2.45}$$

$$\begin{aligned} v(p \text{ \underline{or} } q) &= LA[v(p) \vee v(q)] \\ &= LA[(\mu_1 u_1 + \ldots + \mu_m u_m) \vee (\nu_1 v_1 + \ldots + \nu_n v_n)] \\ &= LA[\sum_{i,j} (\mu_i \wedge \nu_j)/u_i \vee v_j] \end{aligned} \tag{2.46}$$

---

[9] It should be noted that the inexactness of the truth tables of FL is a consequence of the application of linguistic approximation to expressions of the form $v(p) * v(q)$, where $*$ is the tabulated operation on the linguistic truth-values of fuzzy propositions p and q.

[10] As defined here, these operations are tacitly assumed to be noninteractive. A more detailed discussion of interactivity of fuzzy variables may be found in [4] and [12].

and similarly

$$v(p \Rightarrow q) = LA[\sum_{i,j} (\mu_i \wedge \nu_j)/(1 \wedge (1 - (u_i - v_j)))] \tag{2.47}$$

As an illustration, suppose that

$$v(p) = \underline{true} = 0.6/0.8 + 0.9/0.9 + 1/1 \tag{2.48}$$

and

$$v(q) = \underline{not}\ \underline{true} = 1/(0 + 0.1 +\ldots+ 0.7) + 0.4/0.8 + 0.1/0.9 \tag{2.49}$$

Then

$$v(\underline{not}\ p) = 0.6/0.2 + 0.9/0.1 + 1/0 \tag{2.50}$$

and

$$\begin{aligned} v(p\ \underline{and}\ q) &= LA[v(p) \wedge v(q)] \\ &= LA[1/(0 +\ldots+ 0.7) + 0.4/0.8 + 0.1/0.9] \\ &= \underline{not}\ \underline{true} \end{aligned} \tag{2.51}$$

Applying the same technique to the computation of the truth-value of the proposition very p (e.g., if p $\triangleq$ Evan is very smart, then very p $\triangleq$ Evan is very very smart), we have

$$\begin{aligned} v(\underline{very}\ p) &= LA[v^2(p)] \\ &= LA[(\mu_1 u_1 +\ldots+ \mu_m u_m)^2] \\ &= LA[\mu_1 u_1^2 +\ldots+ \mu_m u_m^2] \end{aligned} \tag{2.52}$$

and for the particular case where

$$\begin{aligned} v(p) &= \underline{true} \\ &= 0.6/0.8 + 0.9/0.9 + 1/1 \end{aligned} \tag{2.53}$$

(2.52) yields

$$v(\underline{\text{very}}\ p) = LA[0.6/0.64 + 0.9/0.81 + 1/1] \qquad (2.54)$$

$$\cong \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{true}} \qquad (2.55)$$

if the modifier <u>more</u> <u>or</u> <u>less</u> is defined by

$$\mu_{\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ A} = (\mu_A)^{1/2} \qquad (2.56)$$

where A is a fuzzy subset of U, and μ_A and $\mu_{\underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ A}$ are the compatibility functions of A and <u>more</u> <u>or</u> <u>less</u> A, respectively. It should be noted that the approximation of the bracketed expression in (2.54) by (2.55) is low in precision.

3. Approximate Reasoning

It is rather illuminating as well as convenient to view the process of reasoning as the solution of a system of relational assignment equations. Specifically, consider a fuzzy proposition, p, of the form

$$p \triangleq u \text{ is } A \qquad (3.1)$$

or, more concretely

$$p \triangleq \text{Mark is } \underline{\text{tall}} \qquad (3.2)$$

in which A is a fuzzy subset of a universe of discourse U and u is an element of a possibly different universe V. Conventionally, p would be interpreted as "u is a member of A," (e.g., "Mark is a member of the class of <u>tall</u> men"). However, if A is a fuzzy rather than a nonfuzzy subset of U, then it is not meaningful to assert that u is a member of A - if "is a member of" is interpreted in its usual mathematical sense.

We can get around this difficulty by interpreting "u is A" as the assignment of a unary fuzzy relation A as the value of a variable which corresponds to an implied attribute of u. For example, "Mark is <u>tall</u>," would be interpreted as the assignment equation

$$\text{Height(Mark)} = \underline{\text{tall}} \tag{3.3}$$

in which Height(Mark) is the name of a variable and tall is its assigned linguistic value. Similarly, the proposition

$$p \triangleq \text{Mark is } \underline{\text{tall}} \text{ and Jacob is } \underline{\text{not}}\ \underline{\text{heavy}} \tag{3.4}$$

is equivalent to two assignment equations

$$\text{Height(Mark)} = \underline{\text{tall}} \tag{3.5}$$

and

$$\text{Weight(Jacob)} = \underline{\text{not}}\ \underline{\text{heavy}} \tag{3.6}$$

in which both tall and not heavy are fuzzy subsets of the real line which may be characterized by their respective compatibility functions $\mu_{\underline{\text{tall}}}$ and $\mu_{\underline{\text{not}}\ \underline{\text{heavy}}}$.

As a further example, consider the proposition

$$p \triangleq \text{Mark is } \underline{\text{much}}\ \underline{\text{taller}} \text{ than Mac} \tag{3.7}$$

In this case, the relational assignment equation may be expressed as

$$\text{(Height(Mark), Height(Mac))} = \underline{\text{much}}\ \underline{\text{taller}}\ \underline{\text{than}} \tag{3.8}$$

in which the linguistic value on the right-hand-side represents a binary fuzzy relation in R×R (R $\triangleq$ real line) which is assigned to the variable on the left-hand-side of (3.8).

More generally, let $U_1,\dots,U_n$ be a collection of universes of discourse, and let $(u_1,\dots,u_n)$ be an n-tuple in the cartesian product $U_1 \times\dots\times U_n$. By a restriction on $(u_1,\dots,u_n)$, denoted by $R(u_1,\dots,u_n)$, is meant a fuzzy relation in $U_1 \times\dots\times U_n$ which defines the compatibility with $R(u_1,\dots,u_n)$ of values that are assigned to $u_1,\dots,u_n$.[11] As a

[11]The concept of a restriction on a fuzzy variable may be viewed as a generalization of the concept of the range of a nonfuzzy variable. A more detailed discussion of this concept may be found in [4].

simple example, if u is a real number and R(u) is the fuzzy set

$$R(u) = 1/0 + 1/1 + 0.8/2 + 0.5/3 + 0.2/4 \tag{3.9}$$

then 2 may be assigned as a value to u with compatibility 0.8.

Now if p is a proposition of the form

$$p \triangleq (u_1,\dots,u_n) \text{ is } A \tag{3.10}$$

where A is an n-ary fuzzy relation in $U_1 \times\dots\times U_n$, then (3.10) may be interpreted as the assignment equation

$$R(u_1,\dots,u_n) = A \tag{3.11}$$

which for simplicity may be written as

$$(u_1,\dots,u_n) = A \tag{3.12}$$

In this sense, a collection of propositions of the form

$$p_i \triangleq (u_{i_1},\dots,u_{i_k}) \text{ is } A_i \tag{3.13}$$

where $(i_1,\dots,i_k)$ is a subsequence of the index sequence $(1,\dots,n)$, translates into a collection of assignment equations of the form

$$R(u_{i_1},\dots,u_{i_k}) = A_i \ , \quad i = 1,2,\dots \tag{3.14}$$

or more simply

$$(u_{i_1},\dots,u_{i_k}) = A_i. \tag{3.15}$$

For example, the propositions

$$p \triangleq u_1 \text{ is small} \tag{3.16}$$

and

$$q \triangleq u_1 \text{ and } u_2 \text{ are } \underline{\text{approximately}}\ \underline{\text{equal}} \tag{3.17}$$

translate into the relational assignment equations

$$u_1 = \underline{\text{small}} \tag{3.18}$$

and

$$(u_1, u_2) = \underline{\text{approximately}}\ \underline{\text{equal}} \tag{3.19}$$

As was stated in the Introduction, the process of inference may be viewed as the solution of a system of relational assignment equations. In the case of (3.18) and (3.19), for example, solving these equations for u_2 yields

$$u_2 = \text{LA}[\underline{\text{small}}\ \text{o}\ \underline{\text{approximately}}\ \underline{\text{equal}}] \tag{3.20}$$

where o denotes the composition of fuzzy relations[12] and LA stands for linguistic approximation. Thus, if

$$\underline{\text{small}} = 1/1 + 0.6/2 + 0.2/3 \tag{3.21}$$

and

$$\begin{aligned}\underline{\text{approximately}}\ \underline{\text{equal}} = 1/((1,1) + (2,2) + (3,3) + (4,4)) + \\ + 0.5/((1,2) + (2,1) + (2,3) + (3,2) \\ + (3,4) + (4,3))\end{aligned} \tag{3.22}$$

then by expressing (3.20) as the max-min product of the relation matrices for small and approximately equal, we obtain

$$\begin{aligned} u_2 &= \text{LA}[1/1 + 0.6/2 + 0.5/3 + 0.2/4] \\ &\cong \underline{\text{more}}\ \underline{\text{or}}\ \underline{\text{less}}\ \underline{\text{small}}\end{aligned} \tag{3.23}$$

as a rough linguistic approximation to the bracketed fuzzy set. This explains the way in which the consequent "u_2 is more or less small," was

[12]The composition of a unary relation A with a binary relation B is defined by $\mu_{A \circ B}(u_2) \triangleq \vee_{u_1} \mu_A(u_1) \wedge \mu_B(u_1, u_2)$, where A and B are fuzzy subsets of U_1 and $U_1 \times U_2$, respectively, and $\vee_{u_1}$ is the supremum over $u_1 \in V_1$. If A and B are fuzzy subsets of $U_1 \times U_2$ and $U_2 \times U_3$, respectively, then $\mu_{A \circ B}(u_1, u_3) \triangleq \vee_{u_2} \mu_A(u_1, u_2) \wedge \mu_B(u_2, u_3)$, where μ_A and μ_B are the compatibility functions of A and B.

inferred in the second example in the Introduction.

Stated in somewhat more general terms, the compositional rule of inference expressed by (3.18), (3.19) and (3.20) may be summarized as follows.[13]

$$\begin{array}{lcl} A_1 & : & u_1 \text{ is } A \\ A_2 & : & u_1 \text{ and } u_2 \text{ are } B \\ \hline A_3 & : & u_2 \text{ is } LA[A \circ B] \end{array} \tag{3.24}$$

and

$$\begin{array}{lcl} A_1 & : & u_1 \text{ and } u_2 \text{ are } A \\ A_2 & : & u_2 \text{ and } u_3 \text{ are } B \\ \hline A_3 & : & u_1 \text{ and } u_3 \text{ are } LA[A \circ B] \end{array} \tag{3.25}$$

where A and B are fuzzy relations expressed in linguistic terms and $LA[A \circ B]$ is a linguistic approximation to their composition.

The rationale for the compositional rule of inference can readily be understood by viewing the composition of A and B as the projection on U_2 of the intersection of B with the cylindrical extension of A. More specifically, if $R(u_{i_1},\dots,u_{i_k})$ is a fuzzy relation in $U_{i_1} \times\dots\times U_{i_k}$, then its cylindrical extension, $\bar{R}(u_{i_1},\dots,u_{i_k})$, is a fuzzy relation in $U_1 \times\dots\times U_n$ defined by

$$\bar{R}(u_{i_1},\dots,u_{i_k}) = R(u_{i_1},\dots u_{i_k}) \times U_{j_1} \times\dots\times U_{j_\ell} \tag{3.26}$$

where $(j_1,\dots,j_\ell)$ is the index sequence complementary to $(i_1,\dots i_k)$ (E.g., if $n = 6$ and $(i_1,i_2,i_3,i_4) = (2,4,5,6)$, then $(j_1,j_2) = (1,3)$.)

[13] As pointed out in [4], modus ponens may be viewed as a special case of (3.24).

Now suppose that we have translated a given set of propositions into a system of relational assignment equations each of which is of the form

$$(u_{r_1}, \ldots, u_{r_s}) = R_r \tag{3.27}$$

where R_r is a fuzzy relation in $U_{r_1} \times \ldots \times U_{r_s}$. To solve this system for, say, $(u_{j_1}, \ldots, u_{j_\ell})$, we form the intersection of the cylindrical extensions of the R_r and project[14] the resulting relation on $U_{j_1} \times \ldots \times U_{j_\ell}$. Thus, in symbols,

$$(u_{j_1}, \ldots, u_{j_\ell}) = \mathrm{Proj}_{U_{j_1} \times \ldots \times U_{j_\ell}} \bigcap_r \bar{R}_r \tag{3.28}$$

which subsumes (3.24) and (3.25) as special cases. In this sense, as stated in the Introduction, the process of inference may be viewed as the solution of a system of relational assignment equations.

In the foregoing discussion, we have limited our attention to propositions of the form "u is A." How, then, could we treat nested propositions of the form

$$p_1 \triangleq (u \text{ is } A_1) \text{ is } \tau \tag{3.29}$$

e.g., (Lisa is <u>young</u>) is <u>very</u> <u>true</u>, where A_1 is a fuzzy subset of U and τ is a linguistic truth-value?

In can readily be shown [12] that a proposition of the form (3.29) implies

$$p_2 \triangleq u \text{ is } A_2 \tag{3.30}$$

---

[14] If R is a fuzzy relation in $U_1 \times \ldots \times U_n$, then its <u>projection</u> <u>on</u> $U_{j_1} \times \ldots \times U_{j_\ell}$ is a fuzzy relation in $U_{j_1} \times \ldots \times_{j_\ell}$ defined by

$$\mu_{\mathrm{Proj}\ R\ \mathrm{on}\ U_{j_1} \times \ldots \times U_{j_\ell}}(u_{j_1}, \ldots, u_{j_\ell}) = \vee_{u_{i_1}, \ldots, u_{i_k}} \mu_R(u_1, \ldots u_n)$$

where $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_\ell)$ are complementary index sequences, and $\vee_{u_{i_1}, \ldots, u_{i_k}}$ is the supremum over $(u_{i_1}, \ldots, u_{i_k}) \in U_{i_1} \times \ldots \times U_{i_k}$.

where A_2 is given by the composition

$$A_2 = \mu_{A_1}^{-1} \circ \tau$$

in which μ_{A_1} is the compatibility function of A and $\mu_{A_1}^{-1}$ is its inverse (Fig. 3). It is this relation between p_2 and p_1 that in conjunction with the compositional rule of inference provides the basis for the approximate inference

$$\begin{array}{l} (u_1 \text{ is A}) \text{ is } \tau_1 \\ (u_1 \text{ and } u_2 \text{ are B}) \text{ is } \tau_2 \\ \hline (u_2 \text{ is C}) \text{ is } \tau_3 \end{array} \tag{3.31}$$

where A,B,C are fuzzy relations; τ_1,τ_2 and τ_3 are linguistic truth-values; and C and τ_3 satisfy the approximate equality

$$\mu_C^{-1} \circ \tau_3 \cong (\mu_A^{-1} \circ \tau_1) \circ (\mu_B^{-1} \circ \tau_2) \tag{3.32}$$

between the fuzzy set $\mu_C^{-1} \circ \tau_3$, on the one hand, and the composition of $\mu_A^{-1} \circ \tau_1$, and $\mu_B^{-1} \circ \tau_2$, on the other.

An illustration of (3.31) is provided by the last example in the Introduction.

4. Concluding Remarks

In spirit as well as in substance, fuzzy logic and approximate reasoning represent a rather sharp departure from the traditional approaches to logic and the mathematization of human reasoning. Thus, in essence, fuzzy logic may be viewed as an attempt at accomodation with the pervasive reality of fuzziness and vagueness in human cognition. In this sense, fuzzy logic represents a retreat from what may well be an unrealizable objective, namely, the construction of a rigorous mathematical foundation for human reasoning and rational behavior.

In our brief discussion of fuzzy logic and approximate reasoning in the present paper, we have not considered many interesting as well as significant issues. Among these are inferences that are fuzzy-probabilistic in nature; the concept of a fuzzy "proof" of a fuzzy assertion; modal logics with linguistic truth-values and fuzzy modal operators of the form _it is quite possible that_, _it is very necessary that_, etc.; and methods of translating a given complex fuzzy proposition into a system of relational assignment equations - a problem which is related to the case-grammar approach to deep structure and conceptual dependency [24]-[26].

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Figure Captions

Fig. 1. Compatibility functions (not to scale) of most, likely, very likely, unlikely, few and very unlikely. Note that unlikely and likely are symmetric with respect to $u = 0.5$; very likely is the square of likely; and very unlikely is the square of unlikely.

Fig. 2. Syntax tree for the linguistic truth-value not very true and not very false.

Fig. 3. The compatibility function associated with the nested proposition $p \triangleq$ (Lisa is young) is very true, where $\mu_{\underline{young}_2} = \mu^{-1}_{\underline{young}_1} \circ$ very true.

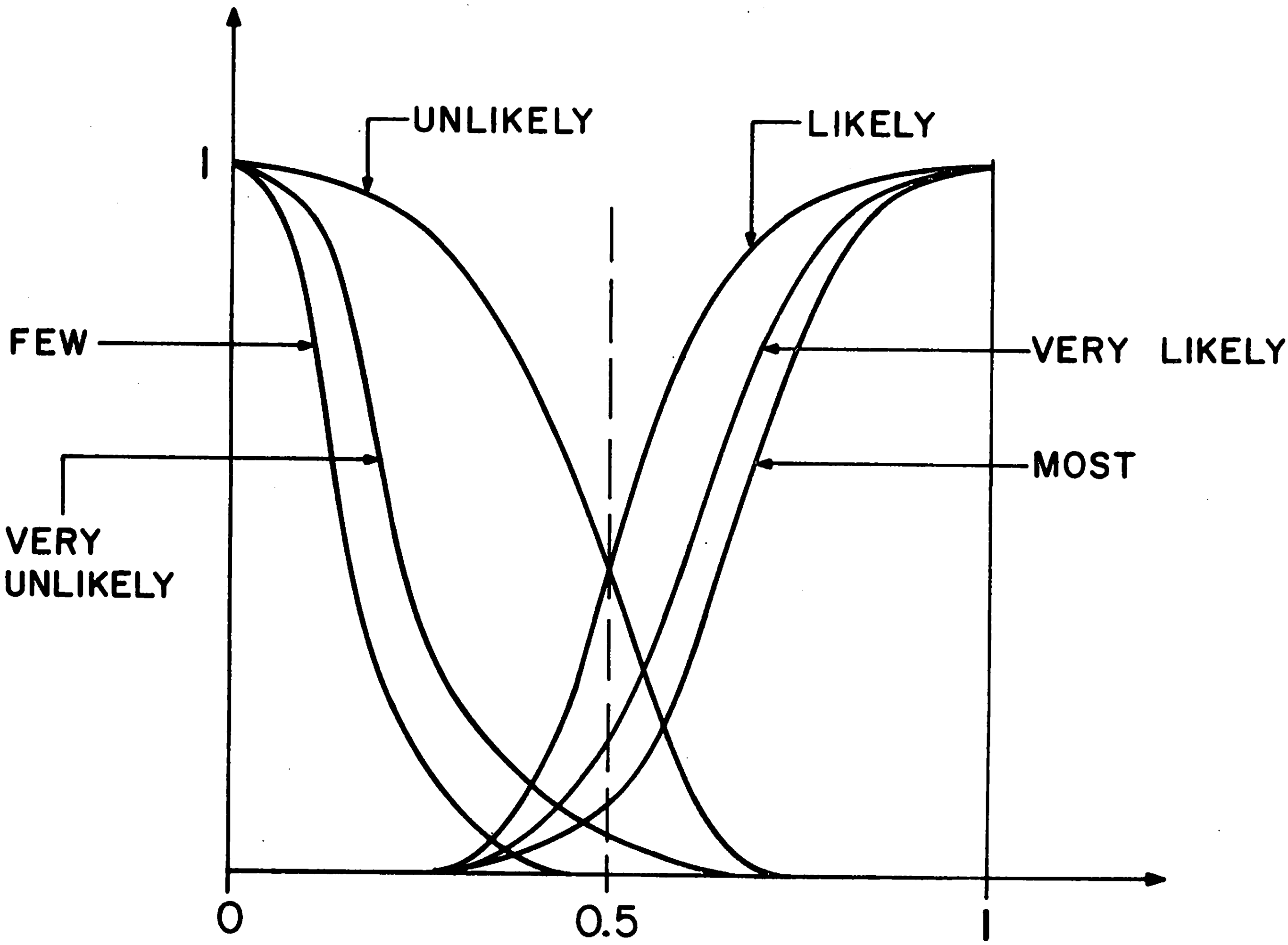

Fig. 1. Compatibility functions (not to scale) of <u>most</u>, <u>likely</u>, <u>very likely</u>, <u>unlikely</u>, <u>few</u> and <u>very unlikely</u>. Note that <u>unlikely</u> and <u>likely</u> are symmetric with respect to $u = 0.5$; <u>very likely</u> is the square of <u>likely</u>; and <u>very unlikely</u> is the square of <u>unlikely</u>.

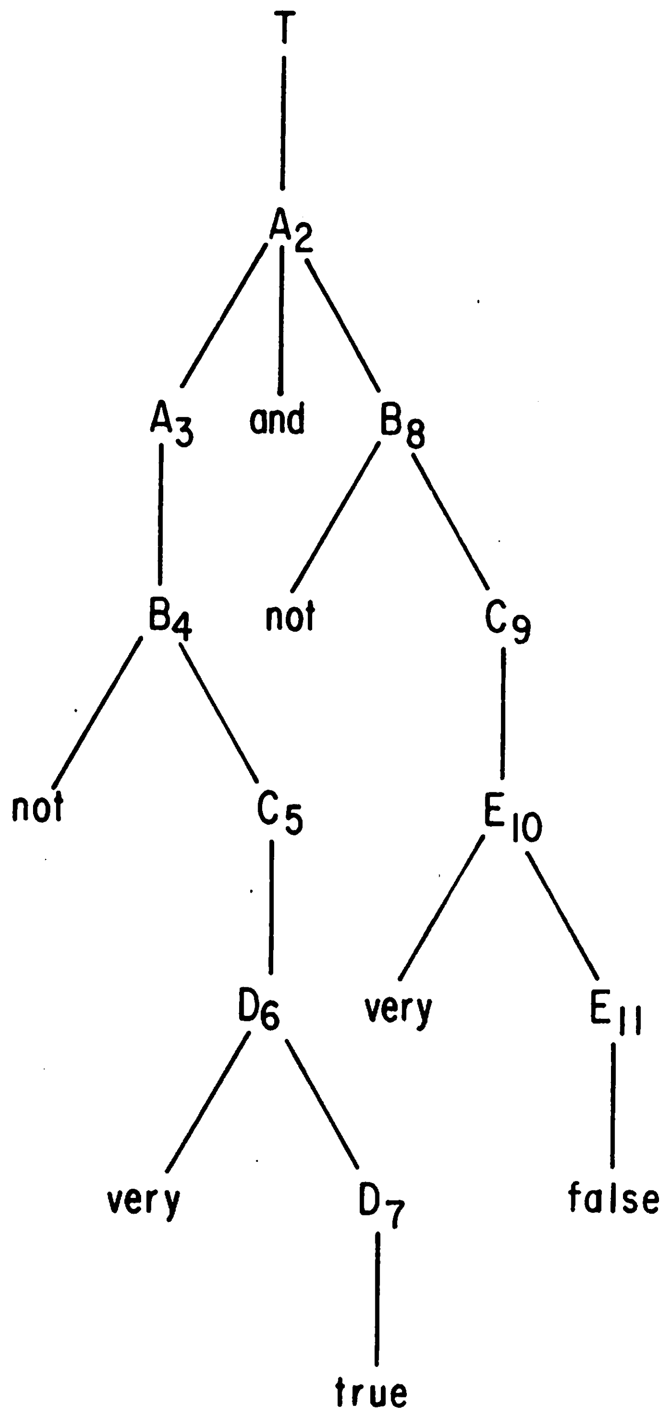

Fig. 2. Syntax tree for the linguistic truth-value <u>not</u> <u>very</u> <u>true</u> <u>and</u> <u>not</u> <u>very</u> <u>false</u>.

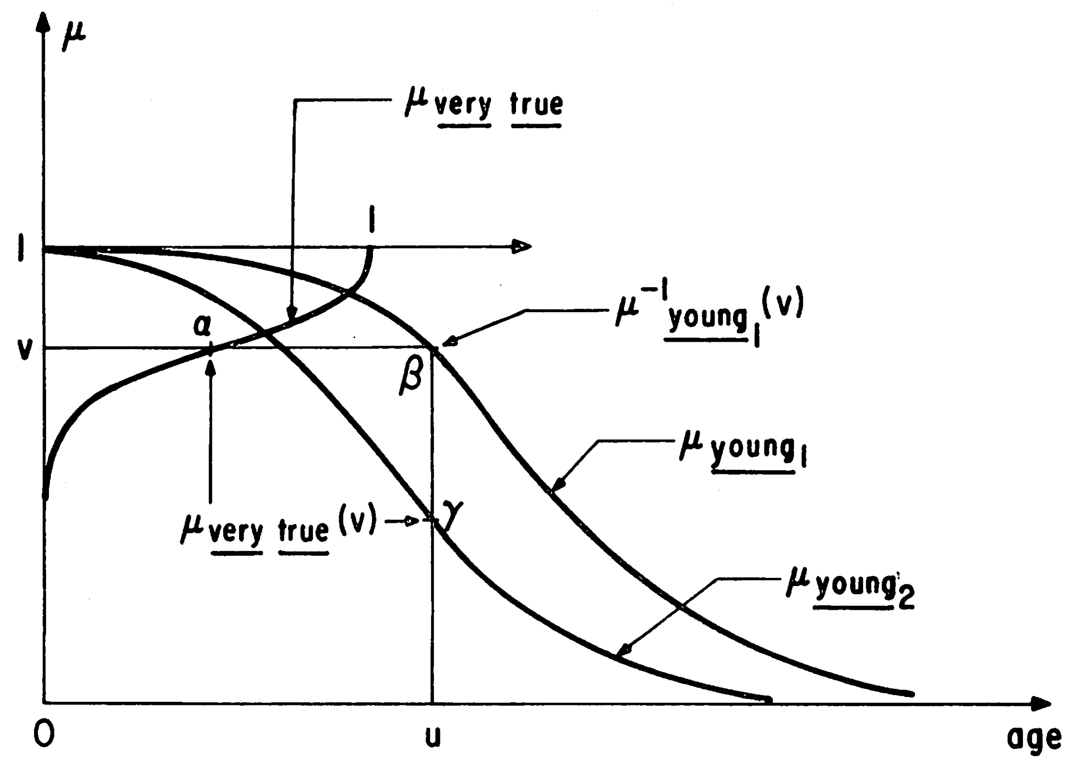

Fig. 3. The compatibility function associated with the nested proposition p $\triangleq$ (Lisa is young) is very true, where $\mu_{\underline{young}_2} = \mu^{-1}_{\underline{young}_1} \circ \underline{very}\ \underline{true}$.