THE REVIVAL OF CITIES IN MEDIEVAL EUROPE

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Alistair I. Mees

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Alistair I. Mees

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

ABSTRACT

In early medieval times, a great change came over Europe. One of its most noticeable aspects was the growth of cities which had been static or declining for centuries. This paper discusses the possibility that the growth was due to the fact that trade was gradually becoming easier. Using the notions of catastrophe theory, it is shown that even a slow improvement in communications could result in a sudden change in the nature of the system, with regions which had previously been unspecialized and self-contained changing over to active concentration on manufacturing or farming. The resulting improvement in living standard could explain the increase in total population which happened concurrently with the growth of the cities.

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The author is on leave from Kings College, Cambridge, England.

§1 INTRODUCTION

Why did cities throughout Europe suddenly start to grow in the Middle Ages after a long period of decline? One of the best-known attempts to answer this question is Pirenne's [1]: he draws attention to the "striking resemblance" between the rise of cities and an expansion of trade which took place at the same time. Some aspects of his analysis are controversial [2] but there seems to be little disagreement with his two main claims: that there was a large increase in population - and especially urban population - from the eleventh to the thirteenth centuries [3], and that the existence of cities was intimately connected with trading opportunities [4, p.181].

In outline, Pirenne's argument is that, beginning in the late seventh century, the expansion of the Islamic Empire restricted and eventually almost closed the Mediterranean to European trade. At around the same time, Norsemen controlled the North Sea and the Baltic and made it dangerous even to live near major rivers. The difficulty of overland travel forced on Europe an isolationist "economy of no trade" in which there were many small, self-sufficient regions (demesnes) which did not exchange goods over long distances, except perhaps in times of famine. With certain exceptions such as Venice and Constantinople, cities shrank until they consisted of little more than a castle or a monastery.

By the year 1000 or so, things were beginning to change: the Islamic Empire had stopped expanding and the Norsemen had settled in the lands they had conquered. Trade routes began to open up again. Soon, cities began to grow and within a century or so, a thriving medieval society had almost entirely replaced the earlier static one.

This paper presents a simple mathematical model for the effect of increasing trade opportunities on the urban and rural populations. The simplicity is deliberate: it makes the concepts clear and it does not try to conceal the fact that in the present state of knowledge, a model can be no more than a caricature of reality if it involves human behavior. For example, how does one include the fact that the world did not end in 1000 A.D. as many people expected it would? Yet the resulting change in mental horizons must have been a factor in the vigor that characterized the Middle Ages.

First, we have to suppose that there is some way of measuring how attractive are the town and the countryside as places to live in. (For definiteness, a method based on utility functions is described later, but the results do not depend directly on this.) We then look at migrations within a single region, blurring the rest of the world into a featureless environment which communicates with the region only through trade. As trade opportunities improve (measured in our utility framework by decreasing prices), a point is reached beyond which the very nature of the system's behavior changes. The change is called a catastrophe [5] and the point to notice is that it occurs no matter how slowly trade improves. In the present context, the post-catastrophe behavior is that regions tend to specialize in what they do best, be it farming or manufacturing. The resulting population shifts take place rather quickly, so that if one were to look at the region just before the catastrophe and again just after it, one might be tempted to conclude wrongly that trade costs were not a cause of the specialization, since they had only changed by a small amount.

Finally, we look at how these changes could affect the total population size: it turns out that a region that was declining before the catastrophe might begin to grow after it, if we assume that a higher living standard can lead to a decreased death rate, an increased birth rate, or immigration from outside the region.

§2 THE MODEL

Any attempt to use mathematics to describe a real situation involves abstracting features which are thought to be important. In the present paper, we deal with two levels of modelling: one concerning population dynamics and another underlying "fast" level which drives the first. It is the first level that mainly concerns us and we will deal with it here, taking the underlying level as given; §3 describes a simple model for the underlying level, though no claim is made that it is the best or the only way to do the job.

§2.1 Trade and Catastrophe

Suppose there is a region with population P. We initially take P to be fixed in time (though this assumption is relaxed in §2.2). At time t, $p_f(t)$ of the people belong to a group f ("farmers") and $p_g(t)$ belong to a group g ("city dwellers"). The population sizes p_f and p_g - as, indeed, all other quantities in the model - will be taken as real numbers rather than integers; this enormously simplifies the analysis at the expense of a slight loss in realism. We ignore the small number of people such as landowners or travelling merchants who may not fit into either category, so that

$$p_f + p_g = P \tag{1}$$

at all times.

Some underlying process, operating very quickly by comparison with changes at the present level, determines the attractiveness U_f of belonging to group f and U_g of belonging to group g at any time. "Attractiveness" can be identified with utility, as will be done in §3, or with real income or some more abstract concept related the the idea of living standard. Whatever the interpretation of U_f and U_g, we assume they can be expressed as real numbers in an essentially unique way* and their values at a given time depend on the current values of p_f, p_g and an exogenous parameter τ measuring the current difficulty of trade between the region and the rest of the world. Other factors affecting U_f and U_g either have a negligible effect or remain effectively fixed during the time the region is studied.

Over a period of several years, people tend to move from a less attractive situation to a more attractive one, in such a way that the overall effect is described by

$$\dot{p}_f = D(U_f, U_g, p_f, p_g) \tag{2}$$

where $\dot{p}_f$ means $\frac{dp_f}{dt}$, time being measured in appropriate units, and D( ) is a continuous function. Because of (1), we can immediately say that $\dot{p}_g = -\dot{p}_f$. Clearly, D ought to obey the conditions

$$
\begin{array}{lll}
D \geq 0 & \text{if} & U_f > U_g \\
D \leq 0 & \text{if} & U_f < U_g \\
D = 0 & \text{if} & U_f = U_g \\
D \geq 0 & \text{if} & p_f = 0 \\
D \leq 0 & \text{if} & p_g = 0 \quad ;
\end{array}
$$

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*i.e., up to one-one transformations.

these are just restatements of the requirements that there is no net movement from a more attractive situation to a less attractive one, that if both groups are equally attractive, the system remains in balance, and that populations cannot become negative. A simple form of the D function is given by

$$\dot{p}_f = p_f\, p_g(U_f - U_g) = - \dot{p}_g \qquad (3)$$

and we shall deal entirely with (3), returning in §4 to notice that our conclusions are qualitatively unchanged if a different D is used.

We now state certain properties of the U_f and U_g functions found in §3 and show how equation (3) behaves. It will be argued in §4 that more general attractiveness functions give similar results, so the final claim is that for a broad range of attractiveness functions and for any "reasonable" dynamics, the results we are about to obtain remain true.

For the utility functions of §3, then, we find that U_f and U_g each depend only on p_f/p_g. Further, when τ is large, $U_f > U_g$ for small p_f, $U_f = U_g$ at a unique value of p_f/p_g and $U_f < U_g$ when p_g is small. This leads to the picture in Fig. 1, where rate of change of population is plotted against the current population using (3). (Note that the same diagram does for $\dot{p}_g$ and p_g, by taking $p_f = P$ as the origin and measuring p_g backwards and $\dot{p}_g$ downwards). There is only one stable equilibrium point, namely E_m, and it corresponds to a mixed region which has both farmers and town dwellers. E_g, for example, is unstable since a slight increase in p_f makes $\dot{p}_f$ positive so p_f starts to grow.

When τ is somewhat smaller, it turns out that if p_f/p_g is less than a certain value (which increases as τ decreases), U_f and U_g are constant; there is a similar critical value for p_g/p_f. This leads to the representation of (3) shown in Fig. 2. Here the section b between lines ① and ② is the same as the corresponding section of Fig. 1. Sections a and c join continuously at the boundaries. There is no effective difference between this and Fig. 1, E_m being the only stable equilibrium as before. However, as τ decreases still further, lines ① and ② move towards E_m as shown in Fig. 3a. Which line reaches E_m first depends on the values of various parameters; suppose it is line ① , as in Fig. 3b. Once τ has decreased a little further, Fig. 3c holds. What has happened is that the stable equilibrium E_m has disappeared and the previously unstable E_g has become stable. Since $\dot{p}_f$ is now always negative, everyone eventually moves from the countryside to the city: the region specializes in city activities.

If the parameter values had been different - for example, if the farmlands were particularly productive - line ② would have annihilated E_m and a pure farming economy E_f would result as in Fig. 4.

If one accepts the arguments in §4 that even if they are not simple utilities, U_f and U_g will have properties similar to the above, we have substantiated the claim that increasing trade opportunities lead to specialization. In particular, since the mixed-economy equilibrium E_m would be likely to occur at small values of p_g, cities much larger than the previous norm would appear rather suddenly (and in widely separated regions) as trade costs went through a critical range.

§2.2 Population Growth

If the region's total population is assumed to vary with attractiveness as explained at the end of §1, we can look at overall growth. Each group suffers an autonomous growth or decline in addition to changes due to migration. The simplest assumption is that (3) is replaced by

$$\dot{p}_f = p_f\, p_g(U_f - U_g) + Rp_f(U_f - A)$$
$$\dot{p}_g = p_f\, p_g(U_g - U_f) + Rp_g(U_g - A) \qquad (4)$$

where A is a constant.

The rate R will be small: the autonomous growth will take place over a generation or so if it is related to an imbalance between births and deaths, while if it is due to migration between the region and the rest of the world, it should still be much slower than internal movement. This observation saves us from the non-trivial task of analyzing (4) in the p_f-p_g plane as τ varies; instead, we repeat the earlier trick of widely differing time scales. Adding the two equations in (4) gives

$$\dot{P} = R(p_f U_f + p_g U_g - PA) \ . \qquad (5)$$

If R is small, P can be taken to be constant while p_f and p_g vary due to internal migration as in (3); this is exactly the problem we sоlved in §2.1. Conversely, in (5) we can assume that p_f and p_g vary so quickly that they reach their equilibrium values in a negligible time. Thus everyone has the same utility U, and the r.h.s. of (5) becomes RP(U-A). In particular, it may be that $\dot{P}$ is negative when the region is isolated but that trade improves the standard of living, so that U is larger than A once trade starts; in this case, the region declines until the catastrophe and grows after it (Fig. 5). Of course, depending on the values of parameters there can be any combination of growth or decline before the catastrophe with growth or decline after it, though growth followed by decline is unlikely because this requires that increased trade results in decreased attractiveness, in which case the region would choose to remain isolated.

By a stroke of good luck, this analysis works even if R is not small, since scaling (4) by

$$p_f \mapsto Rp_f \ , \quad p_g \mapsto Rp_g \ , \quad t \mapsto t/R$$

results in the same equation with R=1. Thus all positive values of R give (4) the same qualitative behavior.* Our argument has really taken R to be zero, so we ought to prove that the global flow (the set of all solutions) depends continuously on R at R = 0. Instead of giving a proof, we merely appeal to intuition.

However, this lack of dependence on R is special to the form of (4), with U_f and U_g depending on p_f/p_g. An appropriate generalization of (2) to the case where P varies does require the assumption that R be small, so in this most general model we have 3 levels of dynamics: a fast underlying level where attractiveness is determined (e.g., by day to day buying decisions as in §3), an intermediate level of internal migration (with a typical time of a few years) and a slow level of overall growth (with a time scale of a generation or so).

§3 UTILITY FUNCTIONS

We now give an interpretation of attractiveness in terms of utility functions; the reader may prefer to look on this section as an existence proof rather than a genuine model of the underlying dynamics, though one can argue that the results obtained here are not so far from reality as might be supposed at first. The calculations are essentially the same as in a standard case [6] so some of the algebraic details are omitted.

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*One says that solutions of (4) for different values of R>0 are orbit conjugate since there is a homeomorphism between the solutions which only changes the time parameterization.

Let us ignore spatial effects such as availability and ownership of land. (In fact, at the time we are talking about, the population density was not very great and the inflexible legal system made rents less important than they might have been in the early stages of city growth [1, p. 185, 195]). Suppose all the farmers produce a single perishable commodity F and all the city dwellers produce a single perishable commodity G: even if G is really durable, we may assume it has been available for so long that it is only replacement which matters rather than initial setup. The existence of services (banking, mercantile law, entertainment) in the city can be allowed for by assuming that a constant fraction of the city population is engaged in providing such services.

Suppose F and G are the only commodities consumed in the region and their prices are π_f and π_g. Everyone distributes his income so as to maximize his present utility U(F,G) - the same function for everybody - subject to the requirement that he consume a minimum amount of food F_o. We will use a modification of a Cobb-Douglas utility function [6]:

$$U = (F - F_o)^{\alpha} G^{\beta} \qquad (6)$$

where $0 < \alpha < 1$, $0 < \beta < 1$.

The rest of the world has an infinite supply of, and demand for, F and G. The region is too small to affect the world prices π_f^w and π_g^w and can trade with the world provided the region pays fees τ_f and τ_g per unit of F or G transferred into or out of the region. This is intended to reflect the markup by a travelling merchant whose prices will depend on the difficulty of travel at the time.

If the region is isolated - either because it chooses not to trade or because the cost of exporting a commodity is more than it can be sold for - then prices are fixed by supply and demand. We assume people are price-takers [7], i.e., everyone is unwilling or unable to influence prices or is unaware that he could do so. The price-setting mechanism operates on the basis of day-to-day buying decisions and does not require adjustment of supply; consequently, prices may be assumed always to be in equilibrium as far as the dynamics in §2 are concerned.

Finally, everyone in group f produces the same amount μ_f of F while everyone in group g produces μ_g of G. Migration as in §2 does not affect μ_f and μ_g because the timescale of the migration is longer than the time for a new trade to be learned or for crops to grow. Initially, μ_f and μ_g will be taken as constants.

Under these assumptions, the problem faced by a farmer is to buy amounts F_f of food and G_f of city goods so as to maximize

$$\left.\begin{aligned} & U_f = (F_f - F_o)^\alpha \, G_f^\beta \\ \text{subject to} \qquad & F_f \geq F_o \quad , \quad G_f \geq 0 \, , \\ & \pi_f \mu_f = \pi_f F_f + \pi_g G_f \, . \end{aligned}\right\} \qquad (7)$$

The equality constraint above states that income equals expenditure, since we are assuming people don't save. Problem (7) is a standard constrained nonlinear optimization problem and is equivalent [8] to maximizing

$$L_f \triangleq U_f + \lambda(\pi_f(\mu_f - F_f) - \pi_g G_f) + k_1(F_f - F_o) + k_2 G_f$$

over F_f, F_g and λ without constraints, where the k_i (i=1,2) are zero if the factors they multiply are positive and positive otherwise. Differentiating L_f with respect to F_f, G_f and λ gives 3 equations from which λ can be eliminated, leaving

$$F_f = \frac{1}{\alpha+\beta}\,[\alpha\mu_f + \beta F_o]$$

$$G_f = \frac{\beta/\sigma_f}{\alpha+\beta}\,[\mu_f - F_o]$$

where $\sigma_f = \pi_f/\pi_g$. These expressions hold provided $\mu_f \geq F_o$. If we adopt the convention that $U = 0$ if $F \leq F_o$ or $G \leq 0$ then

$$U_f = \frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}\quad \sigma_f^{\beta}\,(\mu_f - F_o)^{\alpha+\beta} \tag{8}$$

provided $\mu_f \leq F_o$, $\sigma_f \geq 0$; and $U_f = 0$ otherwise.

The city dweller's problem of maximizing U_g is solved in the same way to give

$$U_g = \frac{\alpha^{\alpha}\beta^{\beta}}{(\alpha+\beta)^{\alpha+\beta}}\quad \sigma_g^{\alpha}\,(\mu_g - F_o/\sigma_g)^{\alpha+\beta} \tag{9}$$

provided $\mu_g \geq F_o/\sigma_g$, $\sigma_g \geq 0$; and $U_g = 0$ otherwise. Here, of course, $\sigma_g = \pi_g/\pi_f = 1/\sigma_f$.

Suppose that $\tau_f < \pi_f^w$ so that the farmers can consider trading. Then the farmers sell their food to the merchants at a price $\pi_f^w - \tau_f$ and buy goods at a price $\pi_g^w + \tau_g$. Thus

$$\sigma_f^w = \frac{\pi_f^w - \tau_f}{\pi_g^w + \tau_g}$$

which increases as τ decreases. City dwellers will be happy to accept this price structure, since the farmers are paying all the trading costs, so

$$\sigma_g = 1/\sigma_f^w\ .$$

Of course, a city dweller can be sure of selling as much as he wants of his output by charging an arbitrarily small amount less than the trade price; if the farmers could not then afford to buy all the locally produced G, they would not be trying to import G, i.e., they would not be trading.

If the city dwellers trade then

$$\sigma_g^w = \frac{\pi_g^w - \tau_g}{\pi_f^w + \tau_f}$$

and $\sigma_f = 1/\sigma_g^w$. Clearly, if one group decides to trade the other will find it more profitable not to do so.

However, we must also look at what happens when there is no trade. In this case, we have the extra condition that supply equals demand for each commodity. The conditions that income equals expenditure make one supply/demand equation redundant, so we need only take, say, the one for food:

$$p_f \mu_f = p_f F_f + p_g F_g .$$

Substituting the expressions for F_f and F_g used in finding U_f and U_g gives

$$\sigma_f = 1/\sigma_g = \frac{\alpha}{\beta} \frac{p_g \mu_g}{p_f \mu_f - PF_o} . \quad (10)$$

This internally determined price ratio can now be put into (8) and (9) and the conditions following them. Before discussing the properties of the U_f and U_g which result, we must decide when trade will take place.

Since people are assumed to be utility maximizers, they will choose to trade or not according as the relevant σ value determined by (10) is less or greater than the value achieved by trading. Thus farmers will trade when

$$\frac{p_f}{p_g} > \frac{\frac{\alpha\mu_g}{\beta} \frac{1}{\sigma_f^w} + F_o}{\mu_f - F_o}$$

i.e., for large p_f; as τ decreases (so that σ_f^w increases) the value of p_f at which trade will begin decreases. Obviously, U_f and U_g are continuous at the change over point since σ_f is continuous. The same argument applies to trade decisions by city dwellers: they will trade when p_g is large since the small farming population does not provide an adequate market for their goods.

The properties of the utility functions can now be analyzed. Begin with the case where $\tau_f > \pi_f^w$ and $\tau_g > \pi_g^w$ so trade is impossible: this represents the demesnial society of pre-medieval times. If there were no minimum food requirement, i.e., $F_o = 0$, (8) and (9) would become

$$U_f = \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha+\beta} \left(\frac{p_g}{p_f}\right)^{\beta} \mu_f^{\alpha} \mu_g^{\beta} \tag{8'}$$

$$U_g = \left(\frac{\beta}{\alpha+\beta}\right)^{\alpha+\beta} \left(\frac{p_f}{p_g}\right)^{\alpha} \mu_f^{\alpha} \mu_g^{\beta} \tag{9'}$$

which gives the graph of Fig. 1 for the dynamics in §2. The effect of F_o is only important when the total food output is small, either because of a bad harvest decreasing μ_f or because p_f is small. As the total food output $p_f \mu_f$ decreases towards the total minimum requirement PF_o, the price ratio σ_f rises causing U_g to fall and - when σ_f is large enough - U_f to rise. In other words, the need for a minimum amount of food means that food shortages affect food producers less badly than goods producers. The effect this has on the dynamics in Fig. 1 is that once $p_f \mu_f$ drops below F_o, everyone's utility becomes zero: the city dwellers cannot buy food and the farmers cannot buy goods; the picture changes to that of Fig. 6. The economy still has one stable state E_m to which all other states tend, but if the city is too large then no one can survive (i.e., everyone's utility is zero).

Once trade becomes profitable enough for city dwellers to be able to import F_o, the city can defend itself against food shortages and it is the farmers who suffer when the harvest is bad. This is because trading reduces interdependence: a city dweller's utility only depends on his own productivity μ_g when he trades, whereas when he does not trade he is affected by μ_f, μ_g and p_f/p_g. Once τ_f and τ_g are small enough, therefore, both Fig. 1 and Fig. 6 are supplanted by Figs. 2-4: in Fig. 2, for example, region a represents the case where it is profitable for the city to trade, so U_f and U_g are constant. The graph is in fact a parabola in this region, since U_f-U_g is a positive constant. Similarly in region c U_f-U_g is a negative constant and the farmers are trading.

§4 FROM THE PARTICULAR TO THE GENERAL

The three levels of dynamics mentioned at the end of §2 are central to the discussion and we retain them in describing possible increases in generality.

Most of our attention is focused on the fast level of which one possible model was given in §3. There are two approaches one can take: either the concept of utility can be accepted and models technically superior but conceptually similar to the one in §3 can be obtained, or one can look for a completely different explanation of what makes one situation more attractive than another.

The former approach - that of technical improvement - is exemplified by allowing for more products and looking for more general utility functions. Even a complete Walrasian framework, however [7], seems inadequate since growth provides opportunities for entirely new occupations: banking, for example, would hardly be necessary in an isolated demesne but would become important as trade grew, while the production of fine clothing would not be possible without a wide range of materials to choose from. To discuss this problem of innovations here would take us too far afield; it is the subject of current research. A different technical improvement might be to introduce more sophisticated production functions. For example, manufacturing presumably tends to concentrate into cities because of some sort of increasing returns to scale. If we assume that μ_g increases with p_g, so that U_g also increases with p_g, Fig. 2 can be replaced by Fig. 7. When the city is large enough (to the left of X) the streets are paved with gold: even though at present trade prices, a region with a smaller city is stuck at the mixed-economy equilibrium point E_m, a region which has somehow (perhaps for political reasons) got a large city will specialize in manufacturing since E_g is now a stable equilibrium point. In Fig. 7, as trade gets easier X moves towards E_m and it becomes easier for the barrier XE_m to be passed by temporary economic distortions. Ultimately the catastrophe of Fig. 3 will occur, or perhaps the catastrophe of Fig. 4, in which case the end result is not Fig. 4c but Fig. 8.

Amusing though such technical improvements may be, they are only relevant if the general economic framework of utility functions is accepted. The deliberately vague concept of attractiveness used in §2 allows for other causes of migration. It is still necessary to assume that the cause is a tangible material one, but this is surely uncontroversial when it is realized that our slow dynamics dealing with population shifts and growth is intended to represent an average effect. An analogy in physics is thermodynamics, which was understood before the details of molecular forces were known and was later shown to be a correct description of the averaged behavior of the molecules: in the same way, we might hope that given a sufficiently well-defined situation, the reaction of people as a group can be described in a deterministic way even though individual behavior is much too complex to be understood at present. For a wide range of measures of attractiveness U_f and U_g, it seems likely that, because trade opportunities allow economic interdependence to be weakened, U_f will not depend strongly on p_g when τ is small, nor will U_g depend strongly on p_f. On the other hand, when τ is large U_f and U_g will depend on both p_f and p_g. Consequently, when τ is large, there will be at least one stable equilibrium as in Fig. 1 because of mutual interdependence, but there is no reason to suppose that this equilibrium will persist for small τ. At some point as τ gets smaller there will be a catastrophe; its outcome depends on how U_f and U_g vary with τ, but the argument that a slow change in τ can result in a sudden change in p_f and p_g is not altered.

Similarly, the second and third levels of the dynamics can be generalized: any function D( ) satisfying the conditions in §2.1 will allow stable equilibria only when $U_f = U_g$ or one of p_f and p_g is zero. The dynamics may be as in Fig. 9, for example, but there is clearly no qualitative difference between this and Fig. 2. None of this changes the paper's conclusions which are based on the extremely general notions of catastrophe theory: because the attractiveness of living in the town or the country changes with trade availability, a sudden movement from one to the other is likely as trade gets easier. Furthermore, if attractiveness affects overall population growth, the improvement in everyone's situation due to the migration may cause the population to increase even if it was static or falling before the catastrophic change.

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FIGURE CAPTIONS

Fig. 1. Dynamics when there is no trade.

Fig. 2. Dynamics with trade at some population values.

Fig. 3. Catastrophic change as trade costs decrease: farming dies out.

Fig. 4. Alternative catastrophe where city disappears.

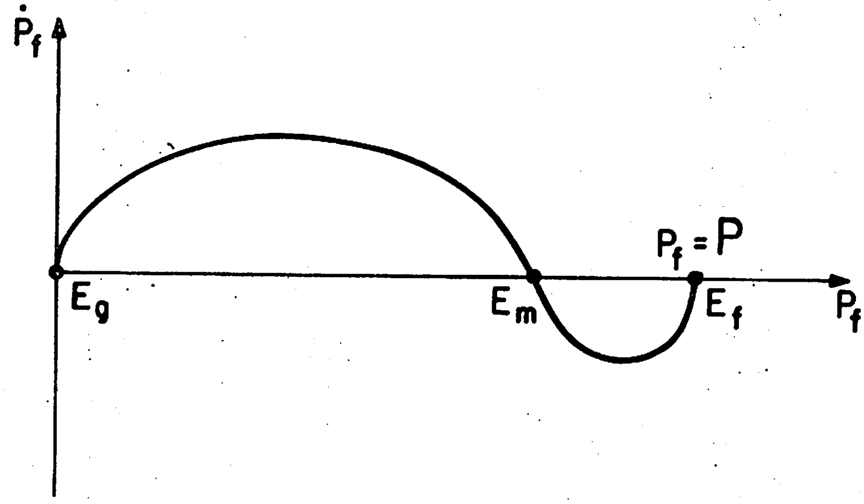
Fig. 5. Specializing may allow a declining region to grow.

Fig. 6. Dynamics in an isolated region with a minimum food requirement.

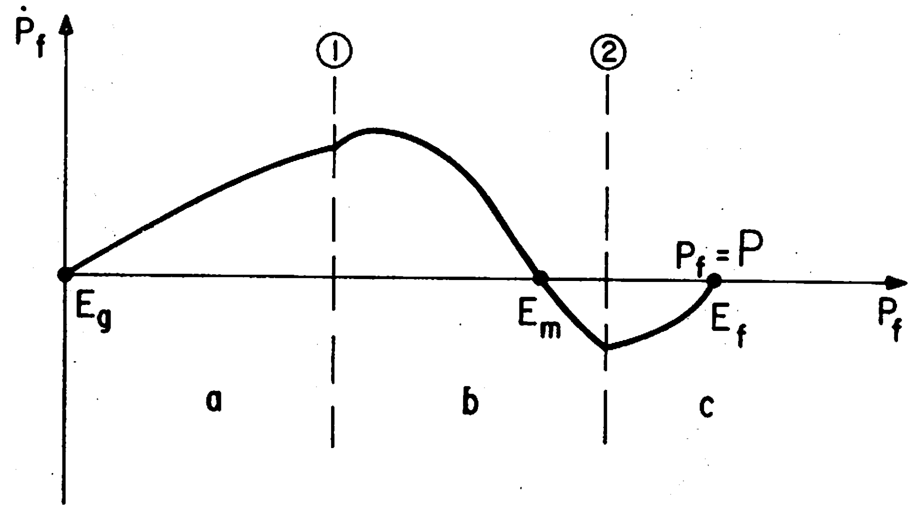
Fig. 7. Effect of increasing returns to scale in G production.

Fig. 8. End result of a catastrophe after which either farming or manufacturing is viable.

Fig. 9. More general dynamics.



$\dot{P}_f$
$P_f = P$
E_g
E_m
E_f
P_f



$\dot{P}_f$
①
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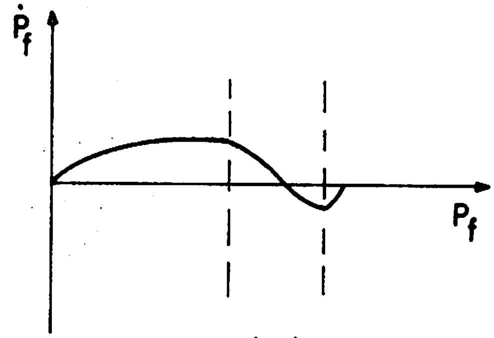

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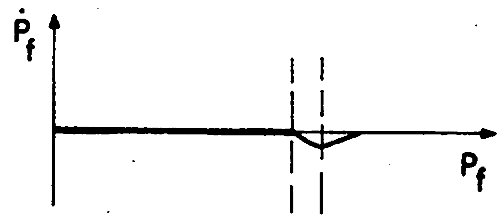
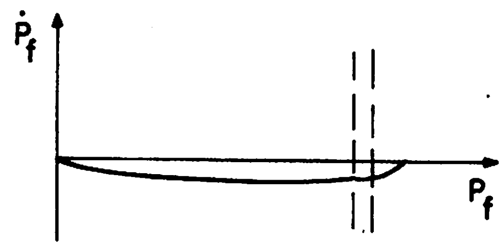


(b)



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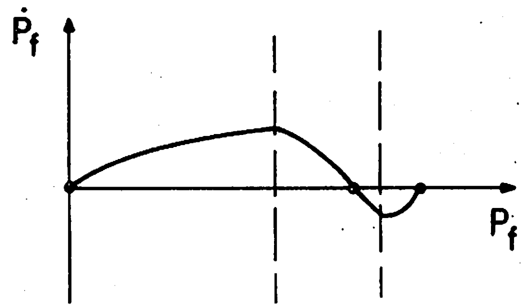

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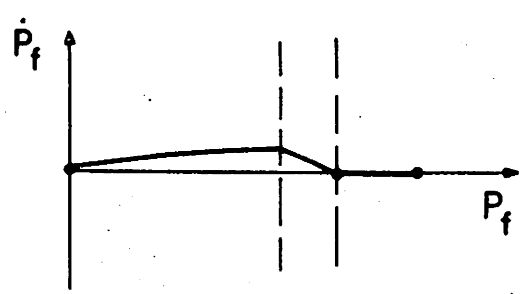

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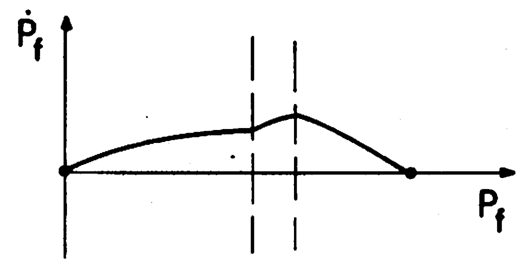

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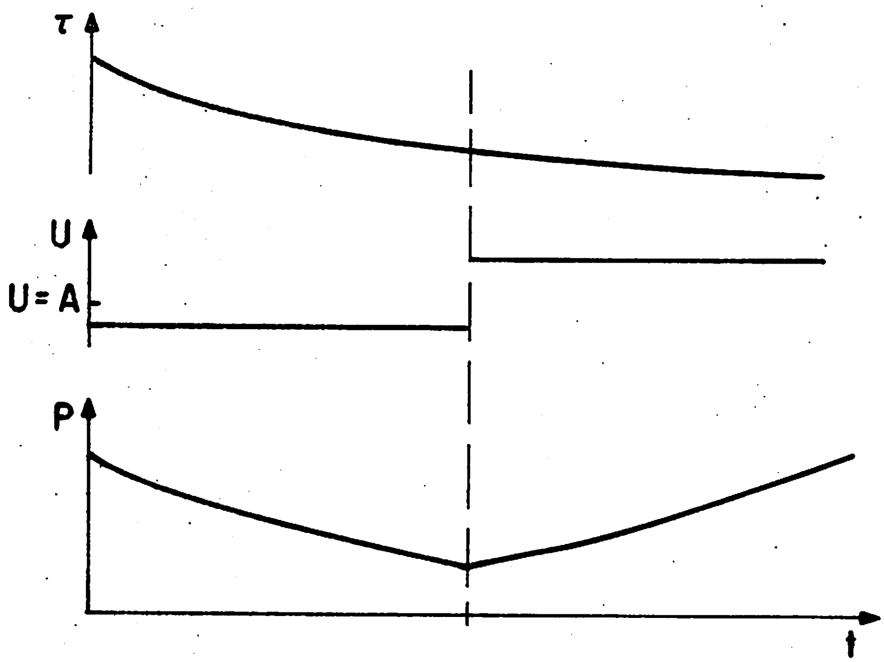
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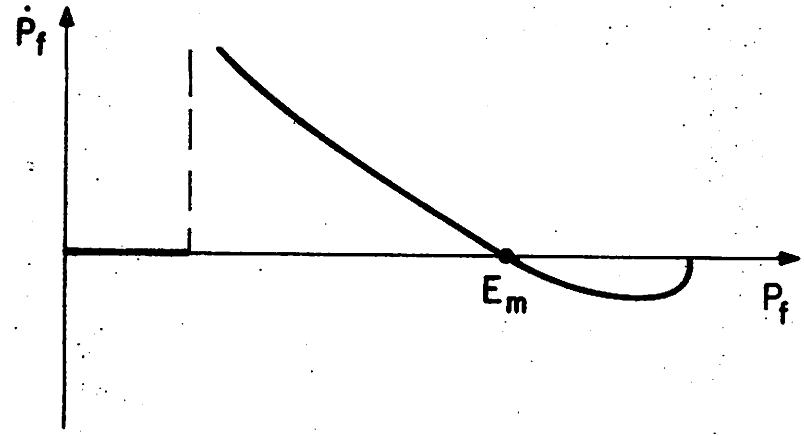

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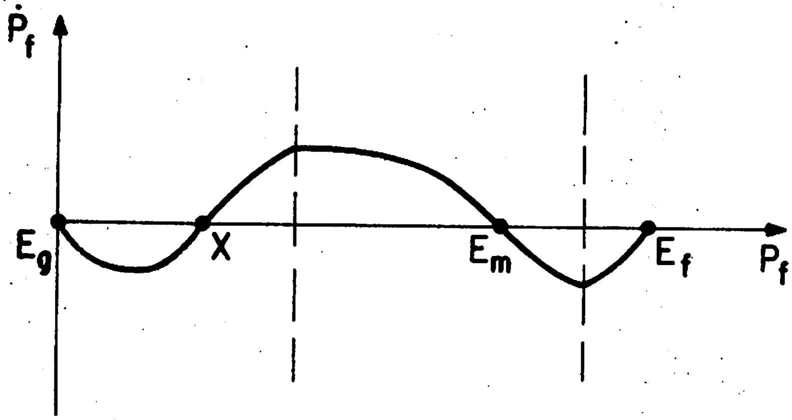
E_m
P_f



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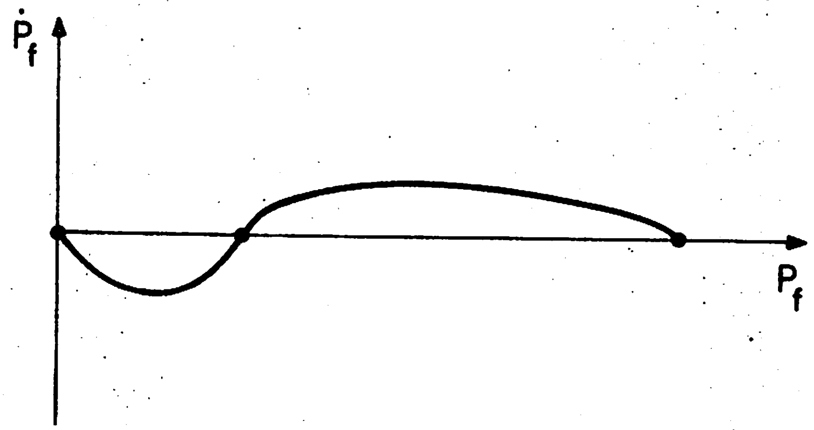
E_g
X
E_m
E_f
P_f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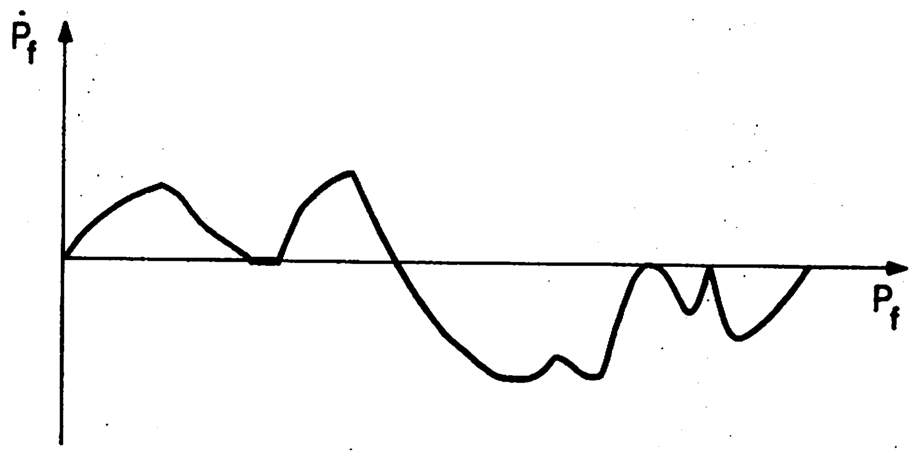
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