THE THEORY AND APPLICATIONS OF THE INNERS

by

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Abstract

In this paper the notion of the Inners of a matrix is fully discussed. The inners applications to Control Theory, Stability theory, Communication Theory, Circuit Theory, Network Theory, Digital Filters, Bioengineering, Sparse Matrix Theory, Quantum Physics and some topics in Mathematics are enumerated and analyzed. It is shown in this survey that the inners concept offers a theoretical as well as computational unification for these applications. In addition the historical background and motivation is presented for the inners approach. The import of the inners notion to education, computation and research in system theory is surveyed and evaluated. Future research problems using this concept are enumerated. Finally, this survey is documented by many past and recent references.

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I. Introduction

The term "inners" is given to certain square submatrices that arise in a square $n \times n$ matrix. This term as well as other definitions connected with it, was first proposed [1] in April, 1970. It appeared first as a journal publication [2] in June, 1971. Since that time many articles on the theory and applications of this concept have been published. Most of this work is discussed in a recently published book [3].

The widespread applications of inners in many fields, as will be mentioned in this survey, have stimulated the writing of this survey to bring to the attention of all the Proceedings readers the present and potential uses of the concept. As an introduction to this work one may mention inners applications in the fields of control theory, network theory, circuit theory, digital filters, communication theory, sparse matrix theory, quantum physics, stability theory, applied mathematics and many others.

In this survey, we will present these various applications without the details and will refer to the published manuscript [3] for complete discussions, proofs, extensions and additional new material. Furthermore, in order to confine the size of this article to reasonable limits, we will dwell mainly on the inners theory and applications without much discussion on general matrix theory applications. Such applications are discussed thoroughly in both mathematical and engineering literature.

The importance of the inners approach lies mainly in the theoretical unification of both continuous time and discrete time theories (especially stability theory), as well as in the computational unification obtained by utilizing only one algorithm for a number of differing applications.

In addition, the inners concept has provided solutions to certain problems for which no solution has previously been known. Such newly solved problems will be discussed in this survey. Furthermore, we shall also indicate open problems for which the inners concept may be an appropriate tool in finding solutions.

Thus it appears that the inners theory is timely, useful and its potentialities still not completely explored. For this and the reason that there are many applications to many fields, it was suggested that the review be written at this time. It is hoped that the reader will benefit from this survey as a source of information to the theory and applications of this concept.

Just as most new concepts are arrived at or motivated by earlier important works and results, so is the inners concept. Many scientists, and especially mathematicians have in fact worked with concepts and generated results which have been important in setting the stage for the inners concept. Because of this we devote the following section to historical background. First, however we need to introduce the new definitions of the inners notion so that comparison and connection with earlier work is facilitated.

<u>Definition 1</u>. Let $\Delta = \Delta_N$ be an $N \times N$ matrix. Form from Δ the matrix Δ_{N-2}, of dimension $N - 2 \times N - 2$, by deleting the first and last rows and first and last columns of Δ; then Δ_{N-2} is called an inner. Now repeat this process on Δ_{N-2} to form Δ_{N-4}. Continue the process until it ends thus forming $\Delta_1, \Delta_3, \Delta_5, \ldots, \Delta_{N-2}$ for N odd and $\Delta_2, \Delta_4, \ldots, \Delta_{N-2}$ for

N even. The appropriate set is called the inners of the matrix Δ.

Remark:

If N is even (larger than or equal to four), the number of inners is (N-2)/2. The inners Δ_2, Δ_4, ... are designated as the first, second, ... inners, respectively. If N is odd, the number of inners is (N-1)/2. The first, second, ..., inners are Δ_1, Δ_3, ... respectively. Note that in this case the first inner, Δ_1, is a one element matrix, that is, a scalar.

Example 1. Let N = 6. The inners of a 6 x 6 matrix are formed as follows

$$\Delta_6 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \tag{1}$$

The two inners are Δ_2 and Δ_4.

Definition 2. If the determinants of all the inners, as well as that of the matrix itself, are positive, we designate the matrix as positive innerwise or (pi). If all the determinants are negative, we designate the matrix as negative innerwise or (ni).

Definition 3. If the determinants of the inners of Δ_N as well as that of Δ_N are zero, we designate Δ_N as null innerwise. If none of the determinants is zero, we designate it as nonnull innerwise. If some of the inners determinants are zero while the remaining determinants are nonzero, we designate it as a semi-innerwise submatrix. We note that the term innerwise matrix referrs to any matrix whose inners determinants are to be determined or have certain significance.

The importance and use of the above definitions will become apparent when we discuss the various applications mentioned above. The relationship between the above definition and past results will be discussed in the following section.

II. Historical Background

When the innerwise matrices discussed in definition 1 take a special form, i.e., a form which is characterized by a left triangle of zeros, then they can be traced to Sylvester [4] and Trudi [5] in the last century. In this section we will briefly review the work of these early investigators and indicate the relationship of their work with the contemporary works of many researchers.

Sylvester's Matrix [4]:

Let $A(z)$ and $B(z)$ be the following polynomials:

$$A(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0 \tag{2}$$

$$B(z) = b_m z^m + b_{m-1}z^{m-1} + \dots + b_0 \tag{3}$$

where z is a complex variable and $\{a_i, b_i\}$ are real or complex. We assume that $m \le n$. A basic result is that the determinant of the $(m + n)$-order Sylvester matrix

$$
\Delta_{n+m} = \left[\begin{array}{ccccccc}
1 & a_{n-1} & a_{n-2} \cdots & a_0 & & 0 \cdots & 0 \\
0 & 1 & a_{n-1} \cdots & a_1 & & a_0 \cdots & 0 \\
\vdots & & & & & & \vdots \\
0 & 0 & & 1 & a_{n-1} & \cdots\cdots\cdots & a_0 \\
\vdots & & & b_m & b_{m-1} & \cdots\cdots\cdots & b_0 \\
\vdots & \cdot & \cdots\cdots & b_m & & & 0 \\
0 & b_m & b_{m-1} & & & & 0 \\
b_m & b_{m-1} & b_{m-2} & & & & 0
\end{array}\right]
\begin{array}{l}
\text{m rows} \\ \\ \\ \\ \\ \text{n rows} \\ \\ \\
\end{array}
\qquad (4)
$$

is nonzero if and only if $A(z)$ and $B(z)$ are relatively prime [that is, no common zeros exist between $A(z)$ and $B(z)$ or equivalently, the greatest common divisor of $A(z)$ and $B(z)$ is a nonzero constant]. The determinant $|\Delta_{n+m}|$ is called the resultant $R[A,B]$ of A and B. The determinants associated with the inners defined earlier are called <u>subresultants</u> or <u>bigradients</u> because of the pattern of two sets of sloping elements.

The Sylvester matrix (also referred to as the Chevron matrix [6]) has a certain pattern of a left triangle of zeros, whereas in making the general definition of the inners of a matrix, no such pattern is postulated. Thus, the above definition is more general and includes the subresultants of Sylvester's pattern as a special case.

If $m < n$ and the coefficients a_i and b_i are real for all i and j, the remainders of the Sturmian division [5] process in the two polynomials may be computed as determinants that are called by Trudi [5] <u>disencumbered remainders</u>. The matrices whose determinants constitute the remainers have the same pattern as the inners of the Sylvester matrix, except that the last column of each inner is varied, with each entry a certain polynomial.

Also Trudi showed that it is possible to form bigradients for any a_i,

b_i that are themselves polynomials. Such polynomial bigradients play an important role in obtaining the greatest common divisor of specified degree of two polynomials. Further earlier work in this direction is related to Netto [7].

Since the early works of Sylvester, Trudi and Netto, the theory of resultant has played an important role in the theory of polynomials. For example resultants can be used to solve two simultaneous polynomial equations in two unknowns and have been incorporated [8] in an algorithm for finding the zeros of a polynomial. They have been used [9] to obtain relative extreme values of a one variable function f(z) as the zeros of another one variable function h(z). Many other mathematical applications appeared in the literature, and the classical treatises of Muir [10]† is an important source of results on the theory of resultants. Recent applications also exist; for example, the theory of resultants was also extended to multivariable polynomials [9,11] to provide a computational tool for solving systems of polynomial equations in many unknowns, by repeated elimination of one variable at a time. This extension is of importance in multidimensional stability problems as will be briefly mentioned in Section III.2.(d).

Application of resultants or bigradients to the study of the stability of linear continuous systems was first made by Fuller [12,13]. Based on Trudi's results, Fuller obtained criteria for a prescribed polynomial to have all zeros within the open often left half of s-plane which turned up to be equivalent to the earlier derived Liénard-Chipart criterion [14]. In this criterion one needs to check the sign condition for either the odd or even Hurwitz minors [12] plus the coefficient (or half of them in a special order) of the characteristic equation. In this work Fuller was able to

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†See also the book of Böcher [171].

transform innerwise matrix to the Hurwitz minor array form. The latter has a special pattern for its generation from the coefficients of the polynomial. It is of interest to note that the matrix whose inners Fuller studied has, like the Sylvester or Trudi matrices, a left triangle of zeros. The inners of the Fuller matrix form are called by him as dotted frames, innermost frames of an array and central minors. In recent years Householder in a series of interesting articles [15-17] has also applied the bigradients to the stability problem. In this case he discussed for a complex polynomial the continuous time stability (or equivalently, the question of whether or not the roots were clustered in the open left half plane). Moreover he was not aware of the earlier work of Fuller. It appears to this writer that Householder was the researcher who was most appreciative of the overall power of subresultants or bigradients, and for a historical review of Trudi's work, Householder's exposition is very enlightening. Other works we might single out for mention include an interesting engineering application to synthesis of a network containing commensurate transmission lines and lumped elements, in which Uruski and Peikarski [18] have applied Trudi's results, with an extension to two variable polynomials to solve their problem. Discussions which are not mainly concerned with the relation between polynomial root distributions and inners include that of Rosenbrock [19] and others [20-22] where a matrix similar to that of Sylvester, is obtained but for matrix polynomials. Conditions on controllability, observability, invertibility can be readily obtained from this innerwise matrix as will be explained later in the discussion of application part. We note too a discussion of resultants is covered in an enlightening way by Barnett [23].

In following the trend of the above historical background, two points emerge. First, it is apparent that subresultants or bigradients have been extensively used by many researchers sometimes independently and sometimes without awareness of the work of others. Second, it is apparent that many names have been given to these central minors arrays and their determinants, which has of itself contributed a barrier to the unifying of many of the concepts. Evidently, a need to unify this earlier work now exists, (and has earlier been especially recognized by Householder [16]).

In my work I arrived at the inners concept from a completely different point of view, stemming from my tackling of problems of discrete-time stability, a topic not considered by all the above mentioned authors. In a series of publications [24-27], I proposed the determinant, table form and division method for tackling the discrete-time stability problem, i.e., checking the root distribution relative to the unit circle of a prescribed polynomial. The determinant method I proposed possessed an innerwise form, but I was not able to transform it in a simple way to the minor array form. This is in contrast to Fuller's work for left half plane stability problem.† Thus for a period of some eight years, I have considered on and off in my mind this contrast between the Hurwitz matrix (the minor form array) and my matrix (the innerwise form). It was in April, 1970 that I decided that probably the Hurwitz form is not necessarily the best formulation for the continuous stability test and that the innerwise form provides a correct formulation for both continuous-time and discrete-time stability problems. This idea was the starting point of my concentrated investigations on this problem which resulted in many publications summarized in the recent book [3].

†This point will be illustrated in detail in the next section.

Because of my work in discrete systems which spans over a quarter century of investigations on stability problems, I was able to pursue vigorously (and, it is hoped effectively) my new approach within recent times. Certainly, I would not, however have been able to pursue with any success such investigations a decade ago. It is equally certain that, with the earlier discrete-time results, it would not have been possible to bring out and to realize effectively the significance and importance of the inners. As a final note in this section, I might mention that when I introduced in 1970 the inners concept, I was not aware of all the preceding works related to the inners, and I have only become acquainted with most of it since my earlier publications on the topic. I also believe that none of the above mentioned authors was aware of all the known works on resultants or bigradiants, with the results that their importance and significance escaped many authors in the past. Householder [15] introduced the inners determinants (bigradients) for the continuous-time case only as an alternate method to the classical minor array form. As with Fuller's inners work, he did not offer (and in fact disclaimed) any significant new results. However, his thorough discussion and the very fact of bringing it to the attention of readers is of much interest.

III. Applications

In this section of this survey we will present the many applications of the inners concept to many diverse areas of control and system theories as well as to other areas of interest to the Proceedings readers. These including communication theory, bioengineering, network theory, digital filtering, quantum physics, and sparse matrix theory. Undoubtedly as time progresses we will find many more applications of the inners concept to other disciplines of science. In the area of control theory, we will emphasize stability theory, positivity and non-negativity of polynomials, controllability, observability and invertibility. The problem of stability is mainly related to the root-clustering properties of a polynomial. Conditions for stability may be represented in terms of positive innerwise matrices. The positivity and non-negativity conditions are related to special root-distribution of polynomials. The conditions are given in terms of the sign patterns of the determinants of inners. The controllability, observability and invertibility conditions are presented in terms of the rank of innerwise matrices. In all the above tests the innerwise matrices have the same unifying pattern, namely the left triangle of zeros. This pattern is of much significance in obtaining a simplified computational algorithm for calculating both the sign and magnitude of the inners determinants. This computational algorithm which is recursive and can be easily implemented on a digital computer and will be discussed in part V.

1. Root-Clustering Problems

In this section we will present the necessary and sufficient condition for the roots of a real or complex polynomial to lie in a certain region in the complex plane. Checking the stability of linear time-invariant dynamic systems (both continuous-time and discrete-time) becomes a special case of the root clustering problem, because it amounts to checking root clustering in the open left half plane and the inside of the unit circle in the complex plane.

The problem of stability has been the subject of extensive investigations by many mathematicians, physicists and engineers during the last century.† The early work of Hermite [28] in 1854 was the pioneering

†A recent enlightening review written by A.T. Fuller, "The Historical and Mathematical Background to Routh's Stability of Motion 1877", University of Cambridge, 1974, is notable reading background reading.

investigation into this problem. This was followed by Routh [29], in 1877 and Hurwitz [30] in 1895. The celebrated result which solves the stability problem for continuous-time systems is now known as the Routh-Hurwitz problem. Further significant results on this problem are provided by the Liénard-Chipart criterion [31] of 1914. The counterpart of these investigations for the discrete case is the work of Schur [32], 1917, and Cohn [33], 1922. Later work on this problem was published by Jury [34],† 1962 and Anderson-Jury [35], 1973. Since the early work of Hermite several hundreds of publications have appeared on the stability problem. It is not the intention of this survey to mention all these publications but merely to indicate one approach in detail that is based on the inners concept. There exists another unifying approach to this problem of root-clustering which is based on the use of quadratic or Hermitian forms. In a related publication [36], the connection between these two approaches has been investigated in detail. These two approaches can be regarded as two sides of a single coin, for they are related to each other by matrix multiplication. The works of Hermite [28], Hankel [37], Markov [38], Lyapunov [39], Schur [32] and others are more closely related to the use of Hermitian forms.

Another related unifying approach based on the companion matrix method is also presented by Barnett [23].†† Furthermore, Barnett††† has also obtained the interrelationship between his approach and the symmetric matrix approach. A detailed discussion of these approaches and their interrelationships are discussed in [3]. In this survey we will concentrate on the inners approach. However the other approaches are also important.

For the root-clustering region we will present the following criteria and indicate, whenever possible, the engineering applications:

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†See also the works of Marden [40].

††See also the survey by Barnett [174].

†††S. Barnett, SIAM J. Appl. Math 22, 84-86, 1972.

a. <u>Root-clustering of a complex polynomial in open left half plane $\mathrm{Re}[s] < 0$</u> [3,40]

Let F(s) be represented as follows:

$$F(s) = s^n + (a'_{n-1} + ja''_{n-1})s^{n-1} + (a'_{n-2} + ja''_{n-2})s^{n-2} + \dots$$
$$+ a'_0 + ja''_0 \qquad (5)$$

The necessary and sufficient condition for the roots of F(s) = 0 to lie in the open left half plane is that the $2n-1 \times 2n-1$ matrix Δ_{2n-1} given by equation (6) be positive innerwise (pi).

$$\Delta_{2n-1} = \left[\begin{array}{ccccccccccccc}
1 & -a''_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & -a''_{n-1} & -a'_{n-2} & a''_{n-3} & a'_{n-4} & -a''_{n-5} & -a'_{n-6} & \cdots & 0 & 0 \\
0 & 0 & & 0 & 1 & -a''_{n-1} & -a'_{n-2} & a''_{n-3} & a'_{n-4} & -a''_{n-5} & \cdots & & 0 \\
0 & 0 & & 0 & 0 & 1 & -a''_{n-1} & -a'_{n-2} & a''_{n-3} & a'_{n-4} & \cdots & & \dagger \\
0 & 0 & & 0 & 0 & 0 & \boxed{a'_{n-1}} & -a''_{n-2} & -a'_{n-3} & a''_{n-4} & \cdots & & \dagger\dagger \\
0 & 0 & \cdots & 0 & 0 & a'_{n-1} & -a''_{n-2} & -a'_{n-3} & a''_{n-4} & a'_{n-5} & \cdots & & 0 \\
0 & 0 & \cdots & 0 & a'_{n-1} & -a''_{n-2} & -a'_{n-3} & a''_{n-4} & a'_{n-5} & -a''_{n-6} & & 0 & 0 \\
0 & 0 & \cdots & a'_{n-1} & -a''_{n-2} & -a'_{n-3} & a''_{n-4} & a'_{n-5} & a''_{n-6} & a'_{n-7} & & 0 & 0 \\
0 & a'_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
a'_{n-1} & -a''_{n-2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{array}\right] \qquad (6)$$

Δ_1 Δ_3 Δ_5 Δ_7

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† $(-1)^{(n+1)/2}\, a''_0$ when n is odd, $(-1)^{n/2}\, a'_0$ when n is even;

†† $(-1)^{(n-1)/2}\, a'_0$ when n is odd, $(-1)^{n/2}\, a''_0$ when n is even.

An alternative form of the necessary and sufficient condition for the roots of F(s) to be in the open left half plane (especially when the coefficient of s^n is not unity) is that a certain matrix Δ_{2n} be positive innerwise (pi). This criterion is called the generalized Routh-Hurwitz criterion [37] and is obtained as follows:

Let F(js) be given as

$$F(js) = (b_n + jc_n)s^n + (b_{n-1} + jc_{n-1})s^{n-1} + \ldots + b_0 + jc_0, \quad c_n \neq 0 \qquad (7)$$

then Δ_{2n} is given as:

$$\Delta_{2n} = \begin{bmatrix} c_n & c_{n-1} & c_{n-2} & \cdots\cdots & 0 & \cdots\cdots & 0 \\ 0 & c_n & c_{n-1} & c_{n-2} & c_{n-3} & & \\ 0 & 0 & c_n & c_{n-1} & c_{n-2} & \cdots\cdots & c_0 \\ 0 & 0 & b_n & b_{n-1} & b_{n-2} & \cdots\cdots & b_0 \\ 0 & b_n & b_{n-1} & b_{n-2} & b_{n-3} & & \\ b_n & b_{n-1} & \cdots\cdots & & 0 & \cdots\cdots & 0 \end{bmatrix} \qquad (8)$$

<u>Remark</u>:

The application of the above criteria lies in the study of relative stability of linear continuous systems. Also it arises in the study of stability of two and multidimensional continuous filters. These will be discussed later . It is also of interest to note as indicated by Householder [16] that $|\Delta_{2n}|$ is the resultant of the real and complex polynomial parts of equation (7).

b. Root-clustering of a real polynomial in the open left half plane [Routh-Hurwitz criterion]:

If we let F(s) in (5) be a real polynomial with $a_i' = a_i$ (and $a_i'' = 0$), the condition that the matrix in (6) should be positive innerwise, becomes equivalent to about two half-sized matrices being positive innerwise.† These two innerwise matrices whihc represent the odd and even inners can be combined into one minor form matrix. This matrix is known to be the Hurwitz matrix [37], and the stability condition reduces to the Hurwitz matrix being positive (i.e., all leading principal minors are positive).†† This discussion can be easily explained by the following example.

Let F(s) be:

$$F(s) = s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \tag{8a}$$

From equation (6), we have:

$$\Delta_9 = \begin{bmatrix} 1 & 0 & -a_3 & 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a_3 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -a_3 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -a_3 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{a_4} & 0 & -a_2 & 0 & a_0 \\ 0 & 0 & 0 & a_4 & 0 & -a_2 & 0 & a_0 & 0 \\ 0 & 0 & a_4 & 0 & -a_2 & 0 & a_0 & 0 & 0 \\ 0 & a_4 & 0 & -a_2 & 0 & a_0 & 0 & 0 & 0 \\ a_4 & 0 & -a_2 & 0 & a_0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

†Such formulation can also be obtained from equation (8).

††This definition is sometimes used in the literature.

Some study of the above matrix will show that it is positive innerwise if and only if the following two matrices are positive innerwise.

$$\Delta_4 = \begin{bmatrix} a_4 & a_2 & a_0 & 0 \\ 0 & a_4 & a_2 & a_0 \\ 0 & 1 & a_3 & a_1 \\ 1 & a_3 & a_1 & 0 \end{bmatrix}, \quad \Delta_5 = \begin{bmatrix} 1 & a_3 & a_1 & 0 & 0 \\ 0 & 1 & a_3 & a_1 & 0 \\ 0 & 0 & a_4 & a_2 & a_0 \\ 0 & a_4 & a_2 & a_0 & 0 \\ a_4 & a_2 & a_0 & 0 & 0 \end{bmatrix} \qquad (10)$$

The innerwise matrices can be combined into one matrix as follows:

$$\Delta_{5H} = \begin{bmatrix} a_4 & a_2 & a_0 & 0 & 0 \\ 1 & a_3 & a_1 & 0 & 0 \\ 0 & a_4 & a_2 & a_0 & 0 \\ 0 & 1 & a_3 & a_1 & 0 \\ 0 & 0 & a_4 & a_2 & a_0 \end{bmatrix}, \qquad (11)$$

The above matrix may be recognizable as the Hurwitz matrix for $n = 5$. In case the coefficient of s^n is a_n an arbitrary positive constant then instead of unity, the entry a_n appears in the above matrix.

It is of interest to note as indicated by Müller [20] that the resultant of the characteristic polynomials $\sum_{i=0}^{n} a_i s^i$ and $\sum_{i=0}^{n} a_i(-s)^i$ is related to the square of the Hurwitz matrix determinant. This can be easily verified from Orlando's relationship [37].† Thus a connection between resultants and Hurwitz matrix determinant can be established.

†See also reference [68].

Liènard-Chipart Criterion [14,31]:

Let F(s) be the real polynomial

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0, \; a_n > 0 \tag{12}$$

The necessary and sufficient conditions for the roots of $F(s) = 0$ to lie in the open left half-plane can be given as follows.

1. The a_i's (or any half of them in a special ordering) be positive.

2. The following matrices Δ^o_{n-1} and Δ^e_{n-1} for n-odd and n-even respectively, are positive innerwise.

$$\Delta^o_{n-1} = \left[\begin{array}{cccccc} a_{n-1} \cdot\cdot\cdot\cdot & a_1 & a_0 & 0 \cdot\cdot\cdot\cdot & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots\cdots & a_0 \\ 0 & a_n & a_{n-1} & \cdots\cdots & a_1 \\ a_n & a_{n-1} & & a_{n-0} \cdot\cdot & 0 \end{array}\right] \tag{13}$$

$$\Delta^e_{n-1} = \left[\begin{array}{cccccc} a_n & a_{n-2} & a_2 & a_0 & 0 \cdot\cdot\cdot\cdot 0 \\ 0 & & & & \\ 0 & & & & \\ & a_n & a_{n-2} & a_{n-4} \cdots\cdots & a_0 \\ & 0 & \boxed{a_{n-1}} & \cdots\cdots & a_1 \\ & a_{n-1} & a_{n-3} & a_{n-5} & \\ 0 & & & & \\ 0 & & & & \\ a_{n-1} & a_{n-3} & a_1 & \cdots\cdots & 0 \end{array}\right] \tag{14}$$

It is of interest to note that the above innerwise matrix is the resultant of the even and odd parts of F(s) [23].

c. <u>Root-clustering of a complex polynomial inside the unit circle,</u> <u>Schur Criterion</u> [32,2]:

Let $F(z)$ be represented as

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \tag{15}$$

The necessary and sufficient condition for the roots of $F(z) = 0$ in (15) with complex coefficients a_k to lie inside the unit circle is that the following matrix Δ_{2n} be positive innerwise (pi).

$$\Delta_{2n} = \begin{bmatrix}
a_n & a_{n-1} & \cdots & \cdots & a_1 & 0 & 0 & \cdots & 0 & a_0 \\
0 & \ddots & & & & & & & \iddots & \vdots \\
\vdots & \vdots & a_n & a_{n-1} & a_{n-2} & 0 & 0 & a_0 & & \vdots \\
\vdots & \vdots & 0 & a_n & a_{n-1} & 0 & a_0 & a_1 & & \vdots \\
\vdots & \vdots & 0 & 0 & a_n & a_0 & a_1 & a_2 & & \vdots \\
\vdots & \vdots & 0 & 0 & \bar{a}_0 & \bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & & \vdots \\
\vdots & \vdots & 0 & \bar{a}_0 & \bar{a}_1 & 0 & \bar{a}_n & \bar{a}_{n-1} & & \vdots \\
0 & & \bar{a}_0 & \bar{a}_1 & \bar{a}_2 & 0 & 0 & \bar{a}_n & & \vdots \\
\bar{a}_0 & \bar{a}_1 & \cdots & \cdots & \bar{a}_{n-1} & 0 & 0 & \cdots & \cdots & \bar{a}_n
\end{bmatrix} \tag{16}$$

The bar indicates complex conjugate.

<u>Remarks</u>:

1. In his earlier publication, Schur [32] presented his criterion as one requiring the determinants of n matrices to be positive. Hence for evaluating the positive sign one needs to find the determinants of n matrices. The above form is a modification of Schur criterion as obtained by Jury [2]. It involves only one matrix whose entries follow a simple pattern. This matrix has a left triangle of zeros as with equations (6) and (8). Hence a unified form for the stability criteria for the left half plane and for

the unit circle is obtained.

2. One application of the above criterion lies in the stability study of two and multidimensional recursive digital filters. This application will be discussed in Section III.2.(d)

3. It may be noted following Schur-Cohn [32,33] the innerwise matrix Δ_{2n} is the resultant[†] of F(z) and $F^*(z)$ where $F^*(z) = z^n\bar{F}(1/z)$ and $\bar{F}(z)$ is the same as F(z) except that all coefficients are replaced by their complex conjugates. Thus it is related to the critical stability constraints [3].[††]

d. Root clustering of a real polynomial inside the unit circle [34,2]:

When the coefficients in (15) are real, then the root-clustering condition within the unit circle can be further simplified, as happens with the Hurwitz criterion, in the following way.

A necessary and sufficient condition for the root of F(z) (with $a_n > 0$) to lie inside the unit circle is:

$$F(1) > 0, \quad (-1)^n F(-1) > 0 \tag{17}$$

and the matrices $\Delta^{\pm}_{n-1} = X_{n+1} \pm Y_{n-1}$ are positive innerwise (pi), where

$$X_{n-1} = \begin{bmatrix} a_n & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 \\ & \ddots & & & & & & \\ & 0 & & & & & \ddots & \\ & & & & & & & a_n \end{bmatrix} \tag{18}$$

---

† This is also recently discussed by Kalman [172] and Barnett [69].

†† See also reference [68].

$$Y_{n-1} = \begin{bmatrix} & 0 & & a_0 \\ & & \diagup & \vdots \\ a_0 & \cdots & & a_{n-2} \end{bmatrix} \qquad (19)$$

We may note that the matrices $\Delta^{\pm}_{n-1}$ are matrices with a left triangle of zeros.

Simplified Determinantal Criterion [35]:

Analogously to the Lienard-Chipart Criterion for left half plane, we also have the following simplified determinental criterion.

The necessary and sufficient condition for the roots of the real polynomial $F(z) = 0$ (with $a_n > 0$) to lie inside the unit circle is given by:

1. The matrix Δ^{-}_{n-1} is positive innerwise (pi) (20)

2. For n-odd, $n \triangleq 2m-1$, either

$$B_{2i} > 0,\ B_{2m-1} > 0, \text{ or } B_{2i+1} > 0,\ B_0 > 0 \qquad (21)$$
$$i = 0,1,\ldots, m-1$$

where

$$B_i \triangleq \sum_{r=0}^{2m-1} [\sum_j (-1)^{r+i-j+1} a_r \binom{r}{j}\binom{2m-1-r}{i-j} \qquad (22)$$

Note that

$$B_{2m-1} = F(1) \quad \text{and} \quad B_0 = -F(-1) \qquad (23)$$

For n-even, $n \triangleq 2m$, either

$$B_{2i} > 0,\ i = 0,1,\ldots,m \qquad (24)$$

or

$$B_{2i+1} > 0,\ B_0 > 0,\ B_{2m} > 0,\ i = 0,1,\ldots,m-1 \qquad (25)$$

where

$$B_i = \sum_{r=0}^{2m} [\sum_j (-1)^{r+i-j} a_r \binom{r}{j}\binom{2m-r}{i-j} \tag{26}$$

[The summation over j is governed by $\max(0,2m-r-i) \leq j \leq \min(i,r)$]

Note

$$B_0 = F(-1), \; B_{2m} = F(1) \tag{27}$$

The form of (24) and (26) arises in the following way:

The coefficients B_i are obtained from the bilinear transformation of the polynomial $F(z)$ to $F_1(w)$, whose roots are in the open left half plane.

Remark:

Having expressed the stability criteria for continuous-time and discrete-time linear time-invariant systems, it should be clear that the innerwise conditions provide a unifying form for stating the stability conditions. This is illustrated in the following example:

Example: Let

$$F(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, \; a_4 > 0 \tag{28}$$

The stability condition in the open left half plane is [12]

(a) $a_k > 0$, $k = 0, 1, 2, 3$ (or half of them as noted by Gantmacher [37]) (29)

(b) In the Hurwitz matrix,

$$\Delta_3 = \begin{bmatrix} \boxed{a_3}_{\,\Delta_1} & a_1 & 0 \\ a_4 & a_2 & a \\ 0 & a_3 & a_1 \end{bmatrix} \tag{30}$$

$$|\Delta_1| > 0, \; |\Delta_3| > 0 \tag{31}$$

The inner form of the preceding Hurwitz matrix can be easily obtained using the inner minor-array transformation [41] as follows:†

$$\Delta_3^1 = \begin{bmatrix} a_4 & a_2 & a_0 \\ 0 & \boxed{a_3}\ {}_{\Delta_1^1} & a_1 \\ a_3 & a_1 & 0 \end{bmatrix} \tag{32}$$

The stability condition as presented above is that Δ_3^1 be positive innerwise (pi) plus condition (a) above.

It is of interest to note that Fuller [12] had reversed the above procedure; he transformed the innerwise form of (32) which he derived into the minor-array form (Hurwitz) as can be easily done. At that time this was a natural thing to do because the Hurwitz form has a simple pattern for its generation (as in the innerwise form) and the Hurwitz form is after all, one of the most popular and well known today in stability study.

For the same polynomial, the stability condition within the unit circle is

$$F(1) > 0, \ F(-1) > 0 \tag{33}$$

and $\Delta_3^{\pm}$ are positive innerwise (or, equivalently, the simplified determinantal criterion).

For instance to obtain Δ_3^{+}, we have

$$\Delta_3^{+} = X_3 + Y_3 = \begin{bmatrix} a_4 & a_3 & a_2 \\ 0 & a_4 & a_3 \\ 0 & 0 & a_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_0 \\ 0 & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{bmatrix} \tag{34}$$

or

†It may be noted that inners can also be transformed to minors (other than leading principal) as done by Barnett [69]. He termed these minors as "Outers."

$$\Delta_3^+ = \begin{bmatrix} a_4 & a_3 & a_2+a_0 \\ 0 & \underset{\Delta_1^+}{\boxed{a_4+a_0}} & a_3+a_1 \\ a & a_1 & a_4+a_2 \end{bmatrix} \tag{35}$$

Δ_3^- can be obtained similarly.

The innerwise matrix Δ_3^+ can be transformed directly into the minor array form [43] whereby the inner Δ_1^+ becomes the leading principal minor array and the determinant values of the matrices are the same. The transformed array is given by

$$\Delta_3^+ = \begin{bmatrix} \underset{\Delta_1^+}{\boxed{a_4+a_0}} & 0 & a_3+a_1 \\ a_3 & a_4 & a_2+a_0 \\ a_1 & a_0 & a_4+a_2 \end{bmatrix} \tag{36}$$

Remarks:

1. In comparing the inner form for the left half-plane and for the unit circle, the first element of the second row in both cases is zero. This is the unifying feature.[†] The minor array form has no such unifying pattern. Furthermore, there exists no inner-minor array transformation that makes the first element of the third row in Δ_3^+ zero, as in the left half-plane. This is also valid for any n and for more general regions in the complex plane.

2. Fuller [12] in his earlier work did not recognize this fact because he was not considering the discrete case. Furthermore, Householder [15], though he recognized that the inner form could result through bilinear transformation from the left half-plane to the inside of the unit circle or certain other regions, did not exhibit it directly from the available

[†]This unifying feature is important in computing the determinants of the inners as will be shown in part IV.

criteria of Schur-Cohn or for the other criteria available for other regions in the complex plane. Furthermore, he worked with complex polynomials and was not aware of the earlier work of Fuller for the real polynomial case (where the simplification of Liénard-Chipart criterion arises). It appears now that in the early works of Fuller and Householder the stability criteria for the discrete case were not available or studied in detail and thus the unifying pattern was clearly not apparent, except through bilinear transformation as indicated by Householder [15].

3. The relationships between the Hermite-Schur-Cohn, Lienard-Chipart and the simplified determinantal matrices were recently obtained through the bilinear transformation [42].

4. The root clustering problem for a complex polynomial f(s) is equivalent to that for a real polynomial of twice the degree, viz $f^*(s)f(s)$.

e. Root-clustering in a certain sector in the left half-plane (Relative stability) [3,43,44]:

The relative stability of dynamic systems is important in obtaining an acceptable transient response. Mathematically, this problem is represented by finding the necessary and sufficient condition for the roots of the system's characteristic equation to lie in a certain sector in the left half of the s-plane. This sector is defined by a certain damping ratio ζ as shown in Figure 1.

Consider the characteristic eqution of the linear system in the form

$$F(s) = \sum_{k=0}^{n} a_k s^k, \; a_n > 0 \tag{37}$$

If the complex variable s is substituted by a function of a variable W

$$s = We^{j(\theta-\pi/2)} \tag{38}$$

the left half of the s-plane that is to the left of the straight line in Figure 1 is mapped into the left half of the W-plane as shown in Figure 2. After the application of the substitution of (38) in $F(s)$ of (37), one obtains the characteristic equation:

$$F_1(W) = \sum_{k=0}^{n} a_k \; e^{jk(\theta-\pi/2)} W^k = 0 \tag{39}$$

Now consider,

$$F_1(jW) = \sum_{k=0}^{n} a_k \; e^{jk\theta} \; W^k = 0 \tag{40}$$

The coefficients $a_k \; e^{jk\theta}$ can be developed in the form:

$$a_k \; e^{jk\theta} = b_k + jc_k \tag{41}$$

To check the conditions for the roots of (39) to lie in the left half of the W-plane, we can apply the Hurwitz form (or the Generalized Routh-Hurwitz Criterion discussed earlier) to $F_1(jW)$. This is represented in equation (7) as being positive innerwise (p i) where,

$b_k = (-1)^k \; a_k T_k(\zeta)$, $T_k(\zeta)$ = Chebyshev function of the first kind;

$c_k = (-1)^{k+1} \; a_k \sqrt{1 - \zeta^2} \;\; U_k(\zeta)$, $U_k(\zeta)$ = Chebyshev function of second kind, with

$T_0(\zeta) = 1$, $T_1(\zeta) = \;\;$, $U_0(\zeta) = 0$, $U_1(\zeta) = 1$.

The damping ratio ζ determines the relative stability or the sector angle in the s-plane. Using the form (5) for $F_1(W)$, the relative stability condition can be more conveniently represented as a requirement that the Δ_{2n-1} matrix of equation (6) be positive innerwise (p.i.).

f. Root-clustering on the negative real axis (Aperiodicity) [13,45,46,47]:

The problem of aperiodicity arises in obtaining a response which has no oscillations or has oscillations of finite number only. Mathematically, this is represented by obtaining the necessary and sufficient condition for all the roots of the characteristic equation to be distinct and on the negative real axis. This would also include stability. These conditions are obtained from equation (12) as follows:

1. All the a_i's are positive.
2. The following $2n-1 \times 2n-1$ matrix Δ_{2n-1} is positive innerwise.

$$\Delta_{2n-1} = \begin{bmatrix} a_n & \cdots\ a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ \cdot & \cdots\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots\ \cdot \\ 0 & \cdots\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots\ 0 \\ 0 & \cdots\ 0 & a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots\ a_0 \\ 0 & \cdots\ 0 & 0 & \boxed{na_n} & (n-1)a_{n-1} & (n-2)a_{n-2} & \cdots\ a_1 \\ 0 & \cdots\ 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & (n-3)a_{n-3} & \cdots\ 0 \\ 0 & \cdots\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & (n-3)a_{n-3} & (n-4)a_{n-4} & \cdots\ 0 \\ \cdot & \cdots\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots\ \cdot \\ na_n & \cdots\ 3a_3 & 2a_2 & a_1 & 0 & 0 & \cdots\ 0 \end{bmatrix} \tag{42}$$

(42)†

†The above matrix is the resultant of $F(z)$ and $\frac{dF(z)}{dz}$.

g. <u>Root-clustering within the unity shifted unit circle (stability of discrete systems)</u> [3,48]:

In the study of discrete systems it is often desirable to present the formulation of stability in the $\psi = z - 1$ plane where the roots of the characteristic equation $F(z) = 0$ in (5) should be within the shifted circle as shown in Figure 3. The necessary and sufficient condition for such root-clustering is formulated in [48]. It is expressed in inners form as follows:

1. For n-even

$$\left.\begin{array}{l} F(0) > 0 \\ F(-2) > 0 \end{array}\right\} \qquad (43)$$

and the following $(n-1) \times (n-1)$ matrix be is positive innerwise in addition to the coefficients $A_{m,\nu}$ (or half of them in a special ordering) given below being positive.

$$\Delta_{n-1} = \begin{bmatrix} A_{0,} & A_{2,\rho-1} & A_{4,\rho-2} & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & A_{0,\rho} & A_{2,\rho-1} & A_{4,\rho-2} & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & A_{0,\rho} & A_{2,\rho-1} & & A_{4,\rho-2} & \cdots & 1 \\ 0 & & 0 & \boxed{A_{1,r}} & & A_{3,r-1} & & A_{n-1,0} \\ \vdots & & A_{1,r} & A_{3,r-1} & & A_{5,r-2} & & \vdots \\ \vdots & A_{1,r} & A_{3,r-1} & A_{5,r-2} & \cdots & \cdots & \cdots & \vdots \\ 0 & & & & & & & \vdots \\ A_{1,r} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \qquad (44)$$

where,

$$\rho = r = \frac{n-2}{2}$$

and

$$A_{m,\nu} = \sum_{\mu=0}^{m} (-1)^{\mu} \binom{\mu+\nu}{\mu} a_{m-\mu} \tag{45}$$

$$m = 0,1,2,\ldots,n$$

$$\nu = 0,1,2,\ldots,\rho \text{ or } r$$

2. For n-odd

$$\left.\begin{aligned} F(0) &> 0 \\ F(-2) &< 0 \end{aligned}\right\} \tag{46}$$

and the following $(n-1) \times (n-1)$ matrix is positive innerwise as well as the coefficients $A_{m,\nu}$ (or half of them in a special ordering) given below being positive.

$$\Delta_{n-1} = \begin{bmatrix} A_{1,r} & & A_{3,r-1} & \cdots & \cdots & 0 \\ 0 & & \vdots & & & \\ \vdots & \ddots & \boxed{\begin{matrix} A_{1,r} & A_{3,r-1} \\ & \\ A_{0,\rho} & A_{2,\rho-1} \end{matrix}} & & \cdots & 1 \\ 0 & 0 & & & & \\ \vdots & 0 & & & & A_{n-1,0} \\ \vdots & \ddots & \vdots & & & \\ 0 & & & & & \\ A_{0,\rho} & & A_{2,\rho-1} & \cdots & \cdots & 0 \end{bmatrix} \tag{47}$$

where

$$r = \frac{n-3}{2}, \quad \rho = r + 1 \tag{48}$$

and

$$A_{m,\nu} = \sum_{\mu=0}^{m} (-1)^{\mu} \binom{\mu+\nu}{\mu} a_{m-\mu}, \quad \text{Note:} \quad A_{0\nu} = a_0 \tag{49}$$

$m = 0,1,2,\ldots,n$

$\nu = 0,1,2,\ldots, \rho$ or r

h. Necessary and sufficient condition for all the roots of a real polynomial to be distinct and to lie on the real axis [3,49]:

The necessary and sufficient condition that (2) has all its roots real and distinct is that Δ_{2n-1} in equation (42) be positive innerwise (p.i.).

System theory applications of this condition can be found in distributed parameters system. In particular the existence and uniqueness of the solution of the partial differential equation which arise as in Lossless Transmission Lines [50] can be tested.

i. Necessary and sufficient condition for root-clustering on the imaginary axis [2,3]:

For a polynomial

$$\hat{F}(z) = a_n z^{2n} + a_{n-1} z^{2n-2} + \ldots + a_1 z^2 + a_0 \tag{50}$$

to have all its roots distinct and pure imaginary, it is necessary and sufficient that

1. All the coefficients of $\hat{F}(z)$ be positive and
2. The matrix Δ_{2n-1} given in equation (42) be positive innerwise (pi).

Application of the above condition lies in obtaining a sufficient condition for a function to be positive real, as will be discussed later.

j. <u>Physical Realization of an RC Passive Network</u> [51]:

The necessary and sufficient conditions for a rational function

$$Z(s) = \frac{g(s)}{f(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}^{\dagger} \tag{51}$$

to be realizable as the driving point impedance of an RC network can be formulated as follows:

1) All the a_i's and b_i's are positive.

2) The following matrix Δ of dimension $(2n-1 \times 2n-1)$ is positive innerwise (pi).

$$\Delta = \begin{bmatrix}
a_n & a_{n-1} & \cdot & \cdot & \cdot & a_2 & a_1 & a_0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & a_n & & & & a_3 & a_2 & a_1 & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & \cdot & & & & \cdot & \cdot & \cdot & & & & \cdot & \cdot \\
\cdot & \cdot & & & & \cdot & \cdot & \cdot & & & & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & a_n & a_{n-1} & a_{n-2} & \cdot & \cdot & \cdot & a_1 & a_0 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & \boxed{b_{n-1}} & b_{n-2} & \cdot & \cdot & \cdot & b_1 & b_0 \\
0 & 0 & \cdot & \cdot & \cdot & b_{n-1} & b_{n-2} & b_{n-3} & \cdot & \cdot & \cdot & b_0 & 0 \\
\cdot & & & & & \cdot & \cdot & \cdot & & & & & \\
 & & & & & \cdot & \cdot & \cdot & & & & & \\
0 & b_{n-1} & \cdot & \cdot & \cdot & b_2 & b_1 & b_0 & \cdot & \cdot & \cdot & 0 & 0 \\
b_{n-1} & b_{n-2} & \cdot & \cdot & \cdot & b_1 & b_0 & 0 & \cdot & \cdot & \cdot & 0 & 0
\end{bmatrix} \tag{52}$$

---

†In case the degrees of the numerator and denominator are the same, write $Z(s) = r + Z_1(s)$ where $Z_1(s)$ is of the form (51), and $r > 0$.

k. Physical Realization of an LC Passive Network [51]:

The necessary and sufficient conditions for

$$Z_1(s) = \frac{b_{n-1}s^{2n-1} + b_{n-2}s^{2n-3} + \cdots + b_0 s}{a_n s^{2n} + a_{n-1}s^{2n-2} + \cdots + a_0} \tag{53}$$

to be realizable as the driving point impedance of an LC passive network are the same as conditions (1) and (2) of (j).

Remark:

The conditions in (j) and (k) are also root-clustering conditions. The condition in j ensures the interlacing of poles and zeros of eqn. (49) on the negative real axis, while the condition in k ensures such interlacing on the imaginary axis.

ℓ. Physical Realizability of the Impedance Function of Short-Circuited Cascade of Uniform Lossless Transmission Lines (SCULL), [52,53]:

The physical realization condition can be expressed by the following theorem:

Theorem 1:

The reflection function $\Psi_0(s) = \dfrac{F(s) - F(-s)e^{su_n}}{F(s) + F(-s)e^{su_n}}$, with $F(s) = \sum_{n=0}^{N} c_n e^{su_n}$ (53a)

is a positive real function of s if it reduces to $\dfrac{\text{zero}}{\text{constant}}$ under repeated application of the algorithm in [54]. When F(s) has integer exponents, the reflection function $\psi_0(s)$ is the impedance function of short circuited cascade of uniform, lossless transmission lines, and hence the test reduces to

$$F(s)\Big|_{s=0} > 0,\ (-1)^N F(s)\Big|_{s=j\pi} > 0 \tag{54}$$

with
$$F(s) = \sum_{k=0}^{N} c_k e^{ks},\ c_N > 0 \tag{55}$$

and,

$\Delta^{\pm}_{n-1} = X_{n-1} \pm Y_{n-1}$, is positive innerwise (p.i.),+ where

$$X_{N-1} = \begin{bmatrix} c_N & c_{N-1} & c_{N-2} & \cdot & \cdot & \cdot & c_2 \\ 0 & c_N & c_{N-1} & \cdot & \cdot & \cdot & c_3 \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & c_N \end{bmatrix}, \tag{56}$$

$$Y_{N-1} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & c_0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & c_0 & \cdot & \cdot & c_{N-4} \\ 0 & c_0 & \cdot & \cdot & \cdot & c_{N-3} \\ c_0 & c_1 & c_2 & \cdot & \cdot & c_{N-2} \end{bmatrix} \tag{57}$$

Remark:

We can also use the above theorem for F(s) in (53a), provided the exponents u_n are rationally related [54]. This would allow us to apply the positive innerwise condition for non-uniformly discrete and distributed parameter systems.

+This matrix has a left triangle of zeros.

m. Root-Clustering Within the Biological Region Γ_B (Study of Bioengineering Modelling) [55,56]:

The application of inners to the study of models which belong to the neuromuscular control systems has been extensively studied [55,56], the field of neuromuscular control systems has made in [55,56]; the salient features necessary to define the root clustering region will be introduced.

Research has shown [56] that linear models which approximate muscles are connections of first order systems; thus all the roots of the model characteristic equation lie on the negative real axis. The vestibular system may also be characterized as having negative real roots. Analysis of the visual eye tracking system has yielded models of varying complexity that have complex and negative real roots. The accomodation system, which controls the power of the lens of the eye, and the pupillary system, which controls the diameter of the pupil, may both be modelled by linear systems with complex and negative roots. The neuromuscular system that controls the hand may be modelled by a highly underdamped second order system. These models define a region Γ_B depicted in Figure 4 within which the roots of most neuromuscular control systems are proposed to lie.

The region Γ_B will be chosen as a biconvex lenticular surface which may be described as the intersection of two circles similar to those depicted in Fig. (5), of equal radius shifted about the real axis to enclose the roots of neuromuscular systems. Performing a transformation to the unit circle in the z-plane for both circles with the subsequent change in the coefficients of the original characteristic equation one may be able to apply the results of part (d) for each circle and ascertain whether the roots of the model characteristic equation lie within Γ_B. The transformation to the unit circle is given by

$$w = (z-z_0)/\rho \qquad (58)$$

Hence the inners test provides an initial check on the appropriateness of the model and increase the probability that it could accurately represent the physical system. This would circumvent the tedious process of simulation of an inaccurate model.

n. Root-Clustering in More Generalized Regions [36,57-60]:

So far in the applications we have discussed specific regions of root-clustering in the complex plane which are of significant engineering application. However, from the mathematical point of view one is interested in more generalized regions for root-clustering problems. Such problems have been recently re-examined [57,58,59,60]. In the following we will present a review of those results [57,36].

Let

$$F(z) \triangleq \sum_{k=0}^{n} a_k z^k \triangleq a_n \prod_{k=1}^{n} (z-\lambda_k) \tag{59}$$

$$a_n \neq 0,\ a_k,\ \lambda_i \in C,\ k = 0, 1, \cdots, n,\ i = 1, 2, \cdots n$$

be a polynomial whose root-clustering is to be investigated. Let the region Γ be given as follows:

$$\Gamma \triangleq \{z | \gamma(z)|^2 - |\delta(z)|^2 > 0\} \tag{60}$$

where $\gamma(z)$ and $\delta(z)$ are given polynomial and

$$|\gamma(z)|^2 - |\delta(z)|^2 \neq \text{constant for all } z \in \Gamma \tag{61}$$

Cases:

1. If $\delta(z) = z - z_0$, $\gamma(z) = r^2$, $r \in R$, then Γ in (60) is a circle

centered at z_0, with radius r. If $z_0 = 0$, $r = 1$, then Γ is the unit circle.

2. If $\gamma(z) = \rho z - \beta$, $\delta(z) = \rho z - \varepsilon$, $\rho, \beta, \varepsilon \in \mathbb{C}$, then Γ is one of the two half-planes divided by a straight line. In particular, if $\gamma(z) = z - 1$, $\delta(z) = z + 1$, then Γ is the open left half plane.

3. If $\gamma(z) = z^2-1$, $\delta(z) = z^2$, then Γ is a hyperbola.

4. If $\gamma(z) = 2z$, $\delta(z) = z^2-1$, then Γ is a certain region (distorted circle) excluding the origin. Similarly, other regions can also be investigated.

Remarks:

1. For the region Γ expressed in (58) one can formulate the root-clustering problems in terms of a positive definite hermitian matrix as shown by Kalman [57] or by a positive innerwise matrix as shown by Jury-Ahn [36].

2. An alternate region Ω that includes half-planes, circles, hyperbolas, ellipses, and parobolas for the root clustering has been recently obtained [58]. This region differs from Γ because it also contains ellipses and parabolas. Again the condition for root-clustering can be formulated in terms of a positive definite hermitian matrix or alternatively in terms of a positive innerwise matrix.

3. The following question still remains open: "What is the largest class of regions in the complex plane where the criteria can be expressed by rational functions of only the coefficients (and their complex conjugates) of the given polynomial?". This author believes that additional constraints are needed for enlarging the regions discussed in this survey. Recent results which stem from decision algebra† formulations, shows that in principle any region bounded by algebraic curves is a possible candidate.

†B.D.O. Anderson, N.K. Bose and E.I. Jury, "Output Feedback Stabilization and Related Problems, Solutions Via Decision Methods. To appear in IEEE Trans. in A-C, Feb., 1975.

o. Positive Innerwise and Positive Definite Symmetric Matrix Relationships [3,36,61]:

In the preceding applications we showed the application of the notion of a positive innerwise matrix to the root-clustering problem. However, this notion can be extended to show its equivalence to any positive definite symmetric matrix.

The test for positive definiteness of a symmetric matrix arises from checking the positivity condition of the following quadratic form in $\underline{x}$.

$$Q = x^T P x, \quad \text{where } x^T = \text{transpose } x. \tag{62}$$

The necessary and sufficient condition for the quadratic form to be positive is that P be positive definite, i.e.

$$P > 0 \tag{63}$$

Without loss of generality we can take P to be symmetric matrix. Using Sylvester's Theorem, the positivity condition is that all the leading principal minors of P are positive.

If we denote these principal minors as D_1, D_2, .. D_n then we obtain the values of n determinants. We can let these determinants be the leading principal minors of Hurwitz matrix associated with a stability test for a real polynomial in relation to the left half plane. Based on results obtained [62,63] for the inverse Hurwitz Problem we can generate the coefficients of the real polynomial. Having obtained them we can easily generate the innerwise form for stability in the open left half plane. Hence we can obtain an innerwise matrix for any symmetric matrix for which positive definiteness is required. Also from a given positive innerwise matrix we can generate a positive definite symmetric matrix and thus the appropriate quadratic form.

Remark:

We can also obtain an equivalent innerwise matrix for a real polynomial whose roots should be inside the unit circle. This can also be done in view of the known results [64] on the inverse Schur-Cohn stability criterion.

2. This correspondence between positive definite symmetric and positive innerwise matrix is of importance not only for the root-clustering problems but probably for other applications. It is hoped that in the future this correspondence can be further utilized and explored.

3. An obvious necessary condition for a matrix to be positive definite is that it be positive innerwise. This condition is obvious and was discussed in [3].

p. Positive Innerwise Matrix in Least Squares Prediction Problems [65,66]:

In a recent publication, Berkhout [65] has shown that certain algorithms for the stability test of linear discrete systems and the algorithms for least-square prediction of stationary discrete-time sequence are closely related. The point is that recursions in Levinson's [67] algorithm for the latter problem are the same as those given by Cohn [33]. In his formulation, Berkhout also obtained a condition involving a positive innerwise matrix. In Levinson, the basic least-squares prediction equation is

$$\sum_{m=0}^{N} f_N(m)R(k-m) = c\delta_{k,0} \quad (k=0,1,\ldots,N) \tag{64}$$

where R represents the autocorrelation function of the stationary sequence. The function $f_N(m)$ represents the impulse response of the optimum "inverse filter."

Equation (64) can be written in a matrix form (choosing c such that $f_N(0) = 1$) as follows: (The star asterisk denotes complex conjugate)

$$\begin{bmatrix}
R(0) & R(1) & \cdots & R(N-2) & R(N-1) & R(N) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
R^*(N-2) & R^*(N-3) & \cdots & R(0) & R(1) & R(2) & 0 & 0 & \cdots & 0 \\
R^*(N-1) & R^*(N-2) & \cdots & R^*(1) & R(0) & R(1) & 0 & 0 & \cdots & 0 \\
R^*(N) & R^*(N-1) & \cdots & R^*(2) & R^*(1) & R(0)\;M_0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & R^*(1) & R(0)\;M_1 & R(1) & \cdots & R(N-1) \\
0 & 0 & \cdots & 0 & 0 & R^*(2) & R^*(1) & R(0)\;M_2 & \cdots & R(N-2) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & & 0 & 0 & R^*(N) & R^*(N-1) & R^*(N-2) & \cdots & R(0)
\end{bmatrix}
\begin{bmatrix} f_N(N) \\ \vdots \\ f_N(2) \\ f_N(1) \\ 1 \\ f_N^*(1) \\ f_N^*(2) \\ \vdots \\ f_N^*(N) \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ c \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{65}$$

or

$$R'F' = C'.$$

From equation (65), Berkhout verified that the determinants of the Inners, $|M_n|$, of R' can be written as:

$$\frac{|M_n|}{|M_{n-1}|} = \frac{|\Delta_{N-n-1}|}{|\Delta_n|}, \quad \text{for } n = 0, 1, \ldots, N-1 \tag{66}$$

with $|M_{-1}| = 1$, and $|\Delta_k|$ is the Schur determinant of order 2k.

Berkhout concluded that the linear discrete system, represented by the characteristic polynomial $P_N(z) = z^n F_N(1/z)$,† is stable if and only if R' in equation (65) is <u>positive innerwise</u>. The above can be also ascertained from the application of part III section (c) and equation (66).

<u>Remarks</u>:

1. The above results indicate an application of the inners concept

---

† The polynomial $F_N(z)$ is defined as $F_N(z) = \sum_{m=0}^{N} F_N(m) z^m$

to problems of communication theory. In an outstanding survey, Kailath [66] had thoroughly discussed Levinson's work as well as its connection with orthogonal polynomials and other properties of least-square prediction problems. Hence, it appears that much can be exploited from this work in future research.

2. The innerwise matrix of equation (65) has no left triangle of zeros. Thus, it is not in the form of Sylvester's matrix. This supports my general definition of innerwise matrices. It is an open question whether the matrix R' can be transformed to an innerwise matrix with left triangle of zeros.

2. Special Root-Distribution Problems:

In this section we will present necessary and sufficient conditions for positivity and non-negativity of polynomials. These conditions are related to the concept of positive real functions, which was first introduced by Brune [70], and which has found many applications in the synthesis of electric networks [71,72]. In recent years this concept has become of importance in many diverse areas, such as the absolute stability [73-75], hyperstability, optimality [76-78] and sensitivity [77] of dynamic systems.

In presenting the inners concept for this problem, we will present the conditions first for regular real polynomials and then for even polynomials that arise in the positive real functions. A complete discussion of these problems appear in [3].

a. Positivity of real polynomials:

Determination of sign definiteness of forms is necessary in stability studies of nonlinear autonomous systems, via the direct method of Lyapunov. The positivity of quadratic forms is simple to determine as discussed in [79]. Results for determining the positive definiteness of binary quartic forms are available [79-81]. In this discussion we will express the positive definiteness condition in terms of signs of the inner determinants [81].

A binary form of degree n is expressed as follows:

$$V(x_1,x_2) = a_n x_1^n + a_{n-1} x_1^{n-1} x_2 + \cdots + a_1 x_1 x_2^{n-1} + a_0 x_2^n \qquad (67)$$

It is a simple matter to show that $V(x_1,x_2) > 0$, for all x_1, x_2 not simultaneously zero if and only if, the single variable non-homogeneous polynomial

$$V_1(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad (68)$$

is (i) devoid of any real root

and (ii) has $V_1(0) = a_0 > 0$, (69)

The test of condition (69) is based on the following theorem [3].

<u>Theorem 2</u>:

The number of distinct real roots, N, of equation [68] with $a_n > 0$ is

$$N = \text{Var.}\ [1, -|\Delta_1^1|, |\Delta_3|, \ldots (-1)^n |\Delta_{2n-1}^1|] - \text{Var.}\ [1, |\Delta_1|, |\Delta_3|, \ldots, |\Delta_{2n-1}|]$$

where "Var." denotes the number of variation of signs, and $|\Delta_i^1|$, $i = 1,3,..,2n-1$ are the inner determinants in the innerwise matrix Δ_1^1 shown below in equation (71), and $|\Delta_{2n-1}^1| \neq 0$. Critical cases when other Δ_1^1s may be zero are handled routinely [37]. When the discriminant [23], $|\Delta_{2n-1}^1| = 0$, the greatest common factor, $V_2(x)$ of $V_1(x)$ and $\frac{dV_1}{dx}$ can be routinely extracted

$$
\Delta^1 = \left[\begin{array}{ccccccccc}
a_n & a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & \cdots & & 0 \\
0 & a_n & a_{n-1} & \cdots & a_1 & a_0 & \cdots & & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & a_n & a_{n-1} & & a_{n-2} & & \cdots & & a_0 \\
 & 0 & \boxed{na_n} \; \Delta_1^1 & & (n-1)a_{n-2} & & & & a_1 \\
 & na_n & (n-1)a_{n-1} & & (n-2)a_{n-2} & \Delta_3^1 & & & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
0 & na_n & (n-1)a_{n-1} & \cdots & a_1 & \cdots & & & 0 \\
na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \cdot & a_1 & 0 & \cdots & & 0
\end{array}\right]_{\Delta_{2n-1}^1} \qquad (71)
$$

using the inners [3] and hence (70) is applied to $V_2(x)$.

Another way the inners can be used to test (68) is to transform $V_1(x)$ of (69) into $V_1(jx)$ and hence the positivity condition reduces to one for the absence of pure imaginary roots of $V_1(jx)$. Now $V_1(jx)$ is a complex polynomial of degree "n" and can be easily used to generate a real polynomial of degree "2n" by coefficient conjugation. The absence of imaginary roots of the "2n" degree real polynomial can be readily ascertained by the inners formulation [3] and its associated algorithm for computing the inners determinants, as will be discussed later on.

Remarks:

1. The condition for positivity requires that $N = 0$ in equation (69). This indicates that there exists many alternative sign patterns for the inners determinants for satisfying this condition. Hence, the positivity test is much more complicated than the stability test, in which case one needs to satisfy one positive innerwise matrix condition.

2. For literal coefficients (i.e. coefficients which are not numbers) in equation (68), one can formulate the positivity conditions for only lower order "n". For higher order n, i.e. $n > 4$, the conditions become extremely complicated [3].

b. Positivity of even real polynomials:

One of the conditions in the test for strictly positive real functions [74] is the requirement that an even real polynomial be positive for all ω, that is,

$$\Pi(\omega^2) > 0 \quad \text{for all } \omega \tag{72}$$

In reference [3], it is shown that the testing of condition (72) reduces to that of checking if $\hat{F}(z)$ as defined by equation (50) is devoid of any positive real roots [74]. This condition can be ascertained using the inners determinants by letting $N = 0$ in the following theorem:

Theorem 3: [3,82]

The number of distinct positive real roots of $\hat{F}(z) = 0$ given in eqn. (50) is

$$\begin{aligned} N = \operatorname{Var}\,[1,\ -|\Delta_2^2|,\ |\Delta_4^2|,\ \ldots,\ (-1)^n|\Delta_{2n}^2|] \\ - \operatorname{Var}\,[1,\ |\Delta_1^1|,\ |\Delta_3^1|\ \ldots.,\ |\Delta_{2n-1}^1|] \end{aligned} \tag{73}$$

where Δ^1 is given in eqn. (71) and Δ^2 of dimension $2n \times 2n$ is given as follows:

$$\Delta^2 = \begin{bmatrix} a_n & \cdots & a_0 & 0 & \cdots & 0 & 0 & \cdot \\ 0 & & & & & & & \cdot \\ 0 & a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & \cdots & a_0 \\ \cdot & 0 & a_n & a_{n-1} & a_{n-2} & \cdots & \cdots & 0 \\ \cdot & 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \cdots & \cdots & \cdot \\ & & & & \Delta^2_2 & & & \\ 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & (n-3)a_{n-3} & & & \\ & & & & & & \Delta^2_4 & \cdot \\ na_n & \cdots & a_1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{\Delta^2_{2n}} \qquad (74)^{\dagger}$$

Remarks:

1. In the counting of N in (73), some of the $|\Delta_k|$'s could be zero, but $|\Delta^2_{2n}| = (-1)^n a_0 |\Delta^1_{2n-1}| \neq 0$. For such a situation, a modification indicated by Gantmacher [37] for the Hurwitz determinants can be readily adopted for this case too. A discussion of these critical cases will be briefly mentioned later and is thoroughly discussed in [3]. It should be noted that $|\Delta^1_{2n-1}|$ can be zero and positivity still holds.

2. A sufficient condition for an even polynomial to be positive is that all its roots are distinct and on the imaginary axis [82]. This condition is presented in part III Section i [1,2]. This fact can be used to establish the connection between positive innerwise (pi), positive definite (pd) and positive real (pr) conditions [82].

3. The general condition required in checking positive realness of a function is that a certain even polynomial be nonnegative.

† It may be noted that this equation represents the Sylvester resultant and Subresultants of $\hat{F}(y)$ and $d/dy\ \hat{F}(y)$, where $y = z^2$ in (50).

$$\Pi(\omega^2) \geq 0 \qquad \text{for all } \omega. \tag{75}$$

Condition (75) is equivalent [74] to that of requiring the roots of $\hat{F}(z) = 0$ in eqn. (50) to be devoid of positive real roots of <u>odd multiplicity</u>. This condition can be also ascertained from eqn. (73) by considering all the critical cases.

4. An alternate and effective approach was suggested and applied by Šiljak [74,3]. In this approach we rotate the real axis by 90° degrees counterclockwise to obtain $\Pi(j\omega)$. The positivity and nonnegativity requirement on $\Pi(\omega^2)$ is now reduced to that of having no pure imaginary roots in $\Pi(j\omega)$ or having pure imaginary roots with even multiplicity. This positivity condition can be achieved by requiring $\Pi(j\omega)$ which is of degree 2n to have n roots with positive real part and thus, (because of symmetry, to also have n roots with negative real part). Hence the Routh Table [29] can be readily applied† and also the inner matrix of eqn. (8) can be also used. This approach seems easier because the critical cases can be readily handled as known from Routh Table and the inners determinants [83].

c. <u>Positivity and nonnegativity of reciprocal polynomials that arise in discrete systems</u> [84]:

Similarly to the continuous case, we can also present the algebraic criterion for positive realness of real rational functions with respect to the unit circle in the complex plane. This criterion can be used in the quantitative analysis of stability and exponential stability of nonlinear discrete systems [85-87]. The algebraic criterion can be formulated as the requirement that the following

†Another form of use of Routh Table for testing positivity was also applied by Fryer [100]. See also Van Vleck [173].

polynomial, (reciprocal with respect to the unit circle)

$$g(z) = \sum_{k=0}^{n} b_k (z^{n+k} + z^{n-k}) \tag{76}$$

have no roots on the unit circle for positivity and roots on the unit circle of even multiplicity for the nonnegativity condition.

Remarks:

1. The positivity and nonnegativity test can be computationally implemented using the table form [26,88] or by using the inners determinants. In this case the innerwise matrix Δ_{2n} of eqn. (16) can be obtained from $[g'(z)]^*$ (the conjugate of the derivative of g(z) with respect to z) [3]. In this way we can determine the root distribution on the unit circle and thus test the needed conditions.

2. The positive realness condition can be extended to more general regions than the imaginary axis or the unit circle. Again the inners concept can be used for ascertaining the required tests.

d. Matrix Generalization [74,84,89,90]:

The conditions for positive realness and strict positive realness for rational matrices in both the continuous and discrete cases are of much importance. The positivity condition arises in the stability study of multinonlinear continuous and discrete control systems as well as other applications [91,92].

The strict positivity conditions arise in the stability tests of two and multidimensional continuous filters and multidimensional recursive digital filters [93-97].

For the continuous case the condition can be formulated as that

$m \times m$ polynomial matrix

$$\Gamma(j\omega) > 0 \qquad \text{for all } \omega \geq 0 \tag{77}$$ †

The matrix in eqn. (77) is Hermitian and hence the strict positivity condition can be ascertained from the discussion of items (a and b). For the discrete case, we require that the following Hermitian matrix

$$\Gamma(z) > 0 \tag{78}$$ ††

for all z on $|z| = 1$

The discussions of (c) explain how to check this condition. The conditions for positivity, i.e. $\Gamma(j\omega) \geq 0$ and $\Gamma(z) \geq 0$ can be also ascertained from the previous discussions. However in this case the test becomes much more complicated.

Remark: [91,98]

In the strict positive realness conditions for both continuous and discrete systems, we can simplify the conditions by assuming that the positivity conditions are satisfied at one point. That is, for certain parameters in $\Gamma(j\omega)$, or $\Gamma(z)$ it is known that $\Gamma(j\omega_1) > 0$ for $\omega_1 \geq 0$ and $\Gamma(z_1) > 0$ for a certain z on the unit circle.

Under the above assumptions the test for strict positivity reduces to

$$|\Gamma(j\omega)| > 0, \qquad \text{for all } \omega \text{ and } \Gamma(0) > 0 \tag{79}$$

for the continuous case, or

$$|\Gamma(z)| > 0, \qquad \text{for all } z\text{:} \quad |z| = 1 \text{ and } \Gamma(1) > 0 \tag{80}$$

†This matrix is defined as $\Gamma(j\omega) \triangleq p^*(j\omega)Q(j\omega) + p(j\omega)Q^*(j\omega)$, where the system transmission matrix $G(s) = \frac{Q(s)}{p(s)}$ and $Q(s)$ is a real polynomial mxm matrix and $p(s)$ is a real scalar polynomial [3].

††This matrix is defined as $\Gamma(z) = p(z)p(z^*)[G^*(z)+G(z)]$, where $G(z) = \frac{Q(z)}{p(z)}$.

for the discrete case.

In concluding this section, one may notice that with the advent of the inners notion, the difficult problem of testing for positivity and nonnegativity can be readily handled in a unified fashion in both continuous and discrete cases. Furthermore, with the development of the computational algorithm for inners the computational burden is much reduced. I believe that inners have aided our understanding of the conditions and thus the Brune problem posed more than four decades ago is now reduced to the use of computational algorithms for inners. Of course other computational algorithms exist for testing the positivity conditions.

3. General Root-Distribution Problems:

In this section we will present certain theorems related to the root distribution of polynomials (real or complex) in certain regions in the complex plane. In this case the sign variation of the inners determinants plays an important role in obtaining information on the number of roots in a certain region in the complex plane. Many excellent mathematical tests and a tremendous number of article publications exist in the literature. We will only present a few of the pertinent theorems which are related to the applications mentioned in this survey. We will also defer the discussion of the critical cases until a later section of this survey.

Theorem 4 [40]:

Consider the polynomial having no pure imaginary zeros

$$F(s) = s^n + (a_{n-1} + jb_{n-1})\, s^{n-1} + (a_{n-1} + jb_{n-2})\, s^{n-2} + \cdots + (a_1 + jb_1)\, s + (a_0 + jb_0) \qquad (81)$$

where a_i, b_i are real

If the $|\Delta_k' s|$ of the innerwise matrix Δ_{2n-1} of eqn. (82) are nonzero for $k = 1,2,\ldots n$, then

$$
\Delta_{2n-1} = \left[\begin{array}{ccccccccccccc}
1 & -b_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & -b_{n-1} & -a_{n-2} & b_{n-3} & a_{n-4} & -b_{n-3} & -a_{n-6} & \cdots & 0 & 0 \\
0 & 0 & & 0 & 1 & -b_{n-1} & -a_{n-2} & b_{n-3} & a_{n-4} & -b_{n-5} & \cdots & & 0 \\
0 & 0 & & 0 & 0 & 1 & -b_{n-1} & -a_{n-2} & -b_{n-3} & a_{n-4} & \cdots & & \dagger \\
0 & 0 & & 0 & 0 & 0 & \boxed{a_{n-1}} & -b_{n-2} & -a_{n-3} & b_{n-4} & \cdots & & \dagger\dagger \\
0 & 0 & \cdots & 0 & 0 & a_{n-1} & -b_{n-2} & -a_{n-3} & b_{n-4} & a_{n-5} & \cdots & & 0 \\
0 & 0 & \cdots & 0 & a_{n-1} & -b_{n-2} & a_{n-3} & b_{n-4} & a_{n-5} & -b_{n-6} & & 0 & 0 \\
0 & 0 & \cdots & a_{n-1} & -b_{n-2} & -a_{n-3} & b_{n-4} & a_{n-5} & -b_{n-6} & -a_{n-7} & & 0 & 0 \\
0 & a_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
a_{n-1} & -b_{n-2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{array}\right] \tag{82}
$$

$$p = \mathrm{Var.}[1, |\Delta_1|, |\Delta_3|, \ldots, |\Delta_{2n-1}|] \tag{83}$$

$$q = \mathrm{Var.}[1, |\Delta_1|, |\Delta_3|, \ldots (-1)^{2n-1}|\Delta_{2n-1}|] \tag{84}$$

$\dagger \quad \left\{\begin{array}{l} (-1)^{\frac{n+1}{2}}\; b_0 \text{ when n is odd} \\ (-1)^{\frac{n}{2}}\; a_0 \text{ when n is even} \end{array}\right\}$ $\quad \dagger\dagger \quad \left\{\begin{array}{l} (-1)^{\frac{n-1}{2}}\; a_0 \text{ when n is odd} \\ (-1)^{\frac{n}{2}}\; b_0 \text{ when n is even} \end{array}\right\}$

where Var. means variation of sign and p and q are the number of zeros of $F(s) = 0$ in the half planes.

$$\text{Re}(s) > 0, \text{ and } \text{Re}(s) < 0 \text{ respectively} \tag{85}$$

The critical cases of the above theorem are discussed in the literature [40] The necessary and sufficient condition for $F(s)$ to have all its zeros (roots) in the open left half plane is that the matrix Δ_{2n-1} be positive innerwise (p.i.).

Remark:

When the polynomial is real, i.e. b_i's are zero, then the same conditions holds provided the various entries of the b_i's = 0 in the innerwise matrix are inserted respectively. Note that when Δ_{2n-1} is (ni) then theorem 4 gives information on the root distribution i.e. in this case $p = 1$ and $q = n-1$.

Theorem 5:

Consider the polynomial

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \text{ with } a_n \neq 0 \tag{86}$$

where $a_n, a_{n-1}, \cdots$ are complex.

If the $|\Delta_k's|$ of the innerwise matrix Δ_{2n} given in eqn. (16) are nonzero for $k = 1,2,\ldots n$, then

$$p = \text{Var}[1, |\Delta_2|, |\Delta_4|, |\Delta_6|, \ldots |\Delta_{2n}|] \tag{87}$$

$$q = \text{Var}[1, -|\Delta_2|, |\Delta_4|, \ldots (-1)^n |\Delta_{2n}| 2 \tag{88}$$

where p and q are the numbers of roots outside and inside the unit circle respectively. Note that Δ_{2n} is (pi) when all the roots are inside the unit circle.

Theorem 6:

When the polynomial F(z) given in eqn. (86) is real (with $a_n > 0$) then the root distribution problem can be simplified by considering the sign variation of two innerwise matrices [3] of size half that of the Schur-Cohn matrix Δ_{2n}.

Theorem 7: [99]

The number of distinct negative real roots of F(z) in Eqn. (86), is:

$$N = \mathrm{Var}[1, -|\Delta_1^1|, |\Delta_3^1|, \ldots (-1)^n |\Delta_{2n-1}^1|] \tag{89}$$

$$- \mathrm{Var}[1, -|\Delta_2^2|, |\Delta_4^2|, \ldots (-1)^n |\Delta_{2n}^2|].$$

where Δ^1 and Δ^2, are given in Eqns. (71) and (74).

Theorem 8: [37,3]

If the complex polynomial F(s) is given in the form of F(js) in equation (7), then the number of its roots p in right half of s-plane is given by:

$$p = \mathrm{Var}[1, |\Delta_2|, |\Delta_4|, |\Delta_6|, \ldots, |\Delta_{2n}|] \tag{90}$$

where the non null innerwise matrix Δ_{2n} is given by equation (8). The number of left half plane roots q is given by

$$q = n-p \tag{91}$$

If F(s) is a real polynomial we can utilize the above theorem (by inserting zeros for the imaginary part) to obtain information on the root distribution. By utilizing the double triangularization procedure (to be discussed) for computation of the inners determinants we can easily accommodate the critical cases, that is some of the $|\Delta_{2k}|$'s are zero.

Remarks:

1. In this section we presented only a few theorems on the root distribution of polynomials. These theorems are of importance in the positivity and nonnegativity tests presented in the preceding section. Also, in the preceding section we presented other theorems on the root distribution.

2. The general problem of root distribution is of much interest in the the mathematical literature and as mentioned before we can only make a passing reference to the vast number of publications in this field. Certainly, one may argue that the root-clustering problem is only a special case of the general root distribution problem. However, following this approach one may not obtain the simplest form computationally. This has been borne out by the development of the Liénard-Chipart and the simplified Schur-Cohn criteria. It is this simplification obtained that tempted this author to discuss the three parts of this section in this chronological order.

4. Magnitudes of Inners Determinants:

In the preceding three applications we obtained the needed conditions by obtaining only the sign of the inner determinants. In this section we will indicate applications of the inners concept where the magnitudes of the inners determinants are required. We will discuss only four major applications, however, there exist others which need further exploration.

a. Integral (sum) of square of signals (continuous and discrete):

The problem of evaluating the total integral square (or sum) of a signal arises in the analysis, and optimization of feedback control systems, both for continuous and discrete systems and for deterministic and stochastic inputs. It also occurs in communication and digital filtering problems. This problem has been investigated during the last quarter of a century. Many methods, similar to those used in the stability problem, have been developed and discussed in the literature [88,101-135]. A complete discussion of this problem is presented in [3]. In the following we will only present a summary which is related to the magnitudes of the inners determinants.

If we let I_n be the integral to be evaluated, then we have

$$I_n = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} G(s)G(-s)ds = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{f(s)}{h(s)h(-s)}ds = \int_0^{\infty} g^2(t)dt \tag{92}$$

where
$$f(s) = b_{n-1}s^{2(n-1)} + b_{n-2}s^{2(n-2)} + \cdots + b_0 \tag{93}$$

$$h(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_n s + a_0 \tag{94}$$

and
$$G(s) = d(s)/h(s)\,,\ d(s)d(-s) = f(s) \tag{95}$$

In equation (92) we assume that G(s) is a stable system.

The formula for I_n in inners form is given as follows: [3]:

$$I_n = \frac{(-1)^{n+1}}{2a_n}\frac{|\Delta_n^b|}{|\Delta_n|} \tag{96}$$

For n-odd

$$\Delta_n = \begin{bmatrix} a_n & \cdots & a_3 & & a_1 & & \cdots & 0 \\ 0 & & & & & & & \\ & & a_n & a_{n-2} & a_{n-4} & \cdots & a_1 & 0 \\ & 0 & & \boxed{a_{n-1}} & a_{n-3} & \cdots & a_2 & a_0 \\ & & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & a_0 & 0 \\ 0 & & & & & & & \\ a_{n-1} & \cdots & & a_2 & a_0 & 0 & \cdots & 0 \end{bmatrix} \tag{97}$$

For n-even

$$\Delta_n \begin{bmatrix} a_{n-1} & \cdots & a_1 & 0 \cdots & 0 \\ 0 & & & & \\ & a_{n-1} & a_{n-3} & \cdots & 0 \\ & a_n & a_{n-2} & \cdots & a_0 \\ 0 & & & & \\ a_n & a_{n-2} \cdots & a_2 & a_0 \cdots & 0 \end{bmatrix} \tag{98}$$

and Δ_n^b is formed from Δ_n by replacing the last row for n-odd and the first row for n-even by

$$[b_{n-1}, b_{n-2}, \ldots, b_1, b_0] \tag{99}$$

It may be noted that for stability we require in addition to the a_i's being positive, that Δ_n be positive innerwise. Thus simultaneous testing for stability and the evaluation of I_n can be carried out in the one algorithm.

Let I_n be the infinite sum of squares of a discrete signal; then

$$I_n = \sum_{k=0}^{\infty} f^2(kT) = \frac{1}{2\pi j} \oint_{\substack{\text{unit} \\ \text{circle}}} F(z)F(z^{-1})z^{-1}dz \tag{100}$$

where F(z) is the z-Transform of f(KT). Assume F(z) is given,

$$F(z) = \frac{\sum_{i=0}^{n} b_i z^i}{\sum_{i=0}^{n} a_i z^i} = \frac{B(z)}{A(z)} \tag{101}$$

and F(z) represents a stable discrete system (i.e. its poles are inside the unit circle), then the integral of (100) is obtained as

$$I_n = \frac{|X_{n+1} + Y_{n+1}|_b}{a_n |X_{n+1} + Y_{n+1}|} \tag{102}$$

where the matrix $[X_{n+1} + Y_{n+1}]_b$ is formed from $X_{n+1} + Y_{n+1}$ by replacing the last row by

$$[2b_n b_0, 2\,\Sigma\, b_i b_{i+n-1}, \ldots, 2\,\Sigma\, b_i b_{i+1}, 2 \sum_{i=0}^{n} b_i^2] \tag{103}$$

and

$$X_{n+1} = \begin{bmatrix} a_n & a_{n-1} & \cdots & & a_0 \\ 0 & a_n & a_{n-1} & \cdots & a_1 \\ \vdots & & & & \\ 0 & \cdot & \cdot & \cdot\ \cdot & a_n \end{bmatrix} \tag{104}$$

$$Y_{n+1} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & a_0 \\ 0 & & 0 & & a_1 \\ \vdots & & \vdots & & \vdots \\ 0 & a_0 & a_1 & \cdots & a_{n-1} \\ a_0 & a_1 & a_2 & \cdots & a_n \end{bmatrix} \tag{105}$$

Let Δ^{+}_{n+1} be defined as

$$\Delta^{+}_{n+1} = X_{n+1} + Y_{n+1} \tag{106}$$

then I_n can be written as:

$$I_n = \frac{|\Delta^{+}_{n+1}|_b}{a_n|\Delta^{+}_{n+1}|} \tag{107}$$

<u>Remarks</u>:

1. The matrix Δ^{+}_{n+1} is an innerwise matrix with a left triangle of zeros as seen from equations (104 and 105).

2. To relate this to a stability test, we can rewrite I_n in (107) as follows [3].

$$I_n = \frac{(-1)^n \, |X_{n+1} + Y_{n+1}|_b}{2a_n A(1)A(-1)\,|\Delta^{-}_{n-1}|} \tag{108}$$

For stability we require (in addition to the bilinearly transformed coefficients as discussed before being positive) that also Δ^{-}_{n-1} be positive innerwise. This is discussed in Section 1.d of this survey. Hence, similar to the continuous case we can simultaneously check for stability and the evaluation of I_n.

3. The inners concept of I_n can also be extended to higher order moments. Again this area has not been completely explored.

b. <u>Synthesis of an RC network and a digital filter</u>: [3,12,51,72,136]

We can synthesize an RC network by calculating the inners determinants of an innerwise matrix [3]. The basic idea is to use the connection between the continued fraction expression coefficients of the input impedance of the network and the inners determinants. This can be accomplished because of the well known relationship between the continued fraction fraction expression coefficients and Hurwitz determinants (or, equivalently, Routh first-column entries). Since theorem (8) in section (3) can be used to established [3] the relationships between the inners determinants and Hurwitz minors or Routh table, the synthesis can be accomplished.

Similarly digital filter synthesis using a ladder network [136] expansion can be obtained. Again, we relate the first column entries of the Routh table to the inners determinants.

<u>Remark</u>:

Synthesis of other types of continuous networks and digital filters, can be also obtained using the inners approach. This writer has not developed this area of research in detail but he believes that much work can be pursued along the discussion of this section. A recent work by Weinberg [137] indicates the use of the inners approach in the synthesis of mixed Lumped-Distributed Networks.

c. <u>Calculation of Chebyshev Functions</u>: [3,73]

It is of interest in this application that we can construct an innerwise matrix for $T_k(\zeta)$ (Chebyshev functions of the first kind) $(0 < \zeta < 1)$. Based on the basic definition of T_0 and T_1, we have

$$\left.\begin{aligned}
|\Delta_1| &= T_0 = 1 \\
|\Delta_3| &= T_1 = \zeta \\
|\Delta_5| &= 2\zeta|\Delta_3| - |\Delta_1| = 2\zeta^2 - 1 = T_2 \\
|\Delta_7| &= 2\zeta|\Delta_5| - |\Delta_3| = 4\zeta^3 - 3\zeta = T_3 \\
&\;\;\vdots \qquad\qquad\quad \vdots \\
|\Delta_{2k+1}| &= 2\zeta|\Delta_{2k-1}| - |\Delta_{2k-3}| = T_k
\end{aligned}\right\} \tag{109}$$

The innerwise matrix is

$$\Delta_{2k+1} = \begin{bmatrix}
2\zeta & -1 & & 0 & & & 0 \\
0 & & & & & & \\
 & 2\zeta & -1 & 0 & 0 & 0 & \\
0 & 0 & \zeta & 0 & 0 & 1 & \\
0 & 0 & 0 & \boxed{1} & 0 & 0 & \\
0 & & & \Delta_1 & & & \\
0 & 0 & 1 & 0 & 1 & 0 & \\
 & & & & \Delta_3 & & \\
0 & 1 & 0 & 0 & 0 & 1 & \\
 & & & & & \Delta_5 & \\
1 & & 0 & & & 1 & 0
\end{bmatrix} \tag{110}$$ †

Similarly we can generate an innerwise matrix for $U_k(\zeta) \triangleq$ Chebyshev function of the second kind, with $0 < \zeta < 1$ [3].

†An alternate form is given by Szego [175]

Remarks:

1. Though the form of (110) is an innerwise matrix with a left triangle of zeros, it is not directly related to the root-distribution problem. This is another example of the extended application of the inners determinants as compared to bigradients or resultants.

2. Recent work has shown that, generally, polynomials orthogonal on the unit circle and on ellipse can be represented in terms of innerwise matrices[+]

d. Response and stability of periodically varying systems: [138]

In the analysis of an RC filter with sinusoidal variation in bandwidth as shown in Fig. (6), one needs to obtain the solution of the following equation:

$$
\begin{bmatrix}
 & \vdots & & & & & \\
\cdots & 1 & \frac{b_1}{b_0 + 2j\omega R} & \frac{b_2}{b_0 + 2j\omega R} & \frac{b_3}{b_0 + 2j\omega R} & \frac{b_4}{b_0 + 2j\omega R} & \cdots \\
 & \frac{b_{-1}}{b_0 + j\omega R} & 1 & \frac{b_1}{b_0 + j\omega R} & \frac{b_2}{b_0 + j\omega R} & \frac{b_3}{b_0 + j\omega R} & \\
 & \frac{b_{-2}}{b_0} & \frac{b_{-1}}{b_0} & \boxed{1} & \frac{b_1}{b_0} & \frac{b_2}{b_0} & \\
 & \frac{b_{-3}}{b_0 - j\omega R} & \frac{b_{-2}}{b_0 - j\omega R} & \frac{b_{-1}}{b_0 - j\omega R} & 1 & \frac{b_1}{b_0 - j\omega R} & \\
\cdots & \frac{b_{-4}}{b_0 - 2j\omega R} & \frac{b_{-3}}{b_0 - 2j\omega R} & \frac{b_{-2}}{b_0 - 2j\omega R} & \frac{b_{-1}}{b_0 - 2j\omega R} & 1 & \cdots \\
 & \vdots & & & & \vdots &
\end{bmatrix}
\begin{bmatrix}
a_2 \\ a_1 \\ a_0 \\ a_{-1} \\ a_{-2}
\end{bmatrix}
=
\begin{bmatrix}
\vdots \\ d_2/b_0 + 2j\omega R \\ d_1/b_0 + j\omega R \\ a_0/b_0 \\ d_1/b_0 - j\omega R \\ a_{-2}/b_0 - 2j\omega R \\ \vdots
\end{bmatrix}
\qquad (111)^{++}
$$

The solution of equation (111) is unique if the innerwise matrix determinant of the doubly infinite matrix is non-singular.

Furthermore the charge, and hence the voltage, can be obtained by direct inversion of the matrix by truncating it to a suitable order. Hence,

[+]See reference [177].

[++]The coefficients b_k, a_k and d_k are the Fourier coefficients of the Fourier Series expansion of the network signals and parameters [138].

the magnitude of the matrix determinant is also required.

Remarks:

1. In the above application the innerwise matrix is doubly infinite and hence by truncating it one must calculate the determinants of the inners indicated in the dotted lines. Therefore the computational algorithm to be introduced in the next part of this survey is of importance.

2. The innerwise matrix of (111) differs from Sylvester's form by the fact that it has no left triangle of zeros. Again this justifies the introduction of the inners notion for general matrices.

5. The Rank of an Innerwise Matrix: [3]

In this section we will indicate four applications of an innerwise matrix whose rank is to be determined. The determination of the rank is readily ascertained by utilizing the double triangularization algorithm developed for determining the inners determinants. In some of the following applications the entries of the innerwise matrices are themselves matrices which are associated with linear systems.

a. Test for controllability:

Since Kalman [139] introduced the concepts of controllability and observability, many publications examining the concept have appeared and are still appearing in the literature. In these publications several tests for controllability are proposed. In the following we will introduce a test which is based on the rank of a certain matrix. This matrix test follows the work of Rosenbrock [140] who laid the groundwork for such a controllability test. In this writer's work [141], the matrix of Rosenbrock is reformulated to become an innerwise matrix and thus the theory of inners can be applied. Details are as follows:

A linear system is described by:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{112}$$

where A and B, C and D are matrices defining the linear time-invariant system.+ The controllability test is given by the following theorem:

<u>Theorem 9</u> [140, 141]:

The linear system is controllable if and only if the following $n(n+\ell-1) \times n^2$ matrix R has rank n^2 where

$$R = \begin{bmatrix} I & -A^T & 0 & \cdots \\ 0 & I & -A^T & \cdots \\ 0 & 0 & I & -A^T \cdot \\ \vdots & \vdots & & \\ 0 & 0 & B^T & 0 \cdot \\ 0 & B^T & 0 & \cdot\cdot \\ B^T & 0 & 0 & \cdot\cdot \end{bmatrix} \tag{113}$$

<u>Remark</u>:

It has been also shown [142] that in the process of double triangularization to check the rank of R, we can also extract the controllable part of an uncontrollable linear system.

b. <u>Test for observability</u> [141]:

Similarly to the controllability test, we can reformulate the Rosenbrook

---

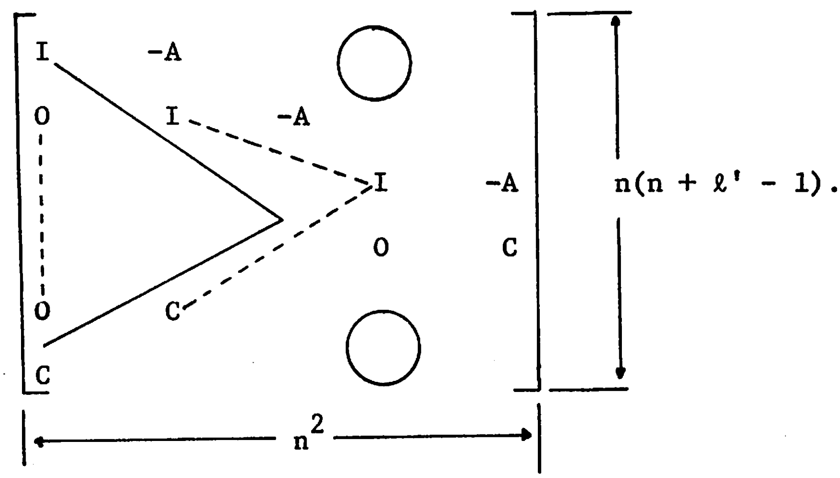
+The dimensions of $A = n \times n$, $B = n \times \ell$, $C = \ell' \times n$, and $D = \ell' \times \ell$.

[140] test by obtaining an innerwise matrix from his matrix. The test is given by the following Theorem:

<u>Theorem 10</u> [141]:

The linear system is observable if and only if the following $n^2 \times n(n+\ell'-1)$ matrix R_0 has rank n^2, where

$R_0 =$



(114)

A similar remark to the controllability test also applies to the observability test [142].

c. <u>Test for Invertibility</u> [143-153]:

Conditions for the invertibility of linear time-invariant dynamical systems are very important. The notion of invertibility arises in many different problems of system theory such as sensitivity theory, filtering and prediction theory [66], pursuit evasion games, decoupling and multivariable control. In the following we will represent the invertibility test in terms of an innerwise matrix. This innerwise matrix is obtained by the author [153] from the matrix of Wang-Davison [143] in a similar fashion to the way in which the controllability matrix is obtained from the Rosenbrock matrix. The test is given by the following theorem:

Theorem 11: [153]

The necessary and sufficient condition for the linear system described in equation (112) to be right invertible is that the following innerwise matrix be of full rank:

$$
M_n \quad \begin{bmatrix}
D & -C & & & & & \\
0 & & 0 & D & -C & & \\
 & & & & \ddots & D & -C \\
 & & & & \iddots & B & -A \\
0 & 0 & B & -A & 0 & & \\
B & -A & 0 & I & & & \\
0 & I & 0 & & & &
\end{bmatrix} \qquad (115)
$$

We can also extract the invertible part similar to (a) and (b) for controllability and observability. A similar condition exists for a left invertible system.†

d. Pole placement by output feedback: [154]

It has been shown by Brasch and Pearson [154] that to obtain an arbitrary pole placement via output feedback a certain matrix Φ must have full rank.

The matrix Φ given in equation (116) is a rotated innerwise matrix and thus the computational algorithm for the inners can also be used to determine its rank. This matrix is given in the following way.

If Φ is of full rank and that given the vector δ, the elements of the feedback matrix K may be determined. The matrix Φ can be related 90° counterclockwise to conform with the usual innerwise matrix with left triangle of zeros.

†For discussion of right invertible and left invertible see reference [143].

$$
\begin{array}{c}
\overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 0}}^{\ell \text{ columns}} \quad \overbrace{\hphantom{0 \quad 0}}^{r(\ell+1)\text{ columns}}
\end{array}
$$

$$
\left[\begin{array}{cccc|cc|c}
1 & 0 & \cdots & 0 & 0 & 0 & \\
\alpha_{n-1} & 1 & \cdots & \vdots & \vdots & \vdots & L \\
\cdot & \alpha_{n-1} & & 0 & \vdots & \vdots & \\
\vdots & & & 0 & \vdots & 0 & \\
\alpha_0 & & & 1 & 0 & & 0 \\
0 & \alpha_0 & & \alpha_{n-1} & & & \cdot \\
\cdot & & & \cdot & & & \cdot \\
\cdot & 0 & & \cdot & & L & \cdot \\
\cdot & \vdots & & \cdot & L & & \\
0 & 0 & & \alpha_0 & & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\delta_1^0 \\
\vdots \\
\delta_\ell^0 \\
\\
\delta_{\ell+1}^1 - \delta_{\ell+1}^0 \\
\vdots \\
\delta_{\ell+1}^r - \delta_{\ell+1}^0 \\
\\
\delta_\ell^1 - \delta_\ell^0 \\
\vdots \\
\delta_\ell^r - \delta_\ell^0 \\
\vdots \\
\delta_1^r - \delta_1^0
\end{array}\right]
$$

$$
= \left[\begin{array}{c}
\beta_{n+\ell-1} - \alpha_{n-1} \\
\beta_{n+\ell-2} - \alpha_{n-2} \\
\vdots \\
\beta_\ell - \alpha_0 \\
\beta_{\ell-1} \\
\vdots \\
\beta_0
\end{array}\right], \quad (116)
$$

or

$$\Phi\ \delta = \beta^*$$

The entries α_i and β_i are defined in ref. [154].

e. The order of the transfer function matrix G(s): [16]

It has been shown by Rowe [6] that the order of the transfer function matrix G(s) arising as

$$G(s) = V(s)\, T^{-1}(s) \tag{117}$$

is given by

$$\text{rank } R_1 - (p\ell-1)\ell \tag{118}$$

where R_1 is an innerwise matrix with left triangle of zeros and given by:

$$R_1 = \underbrace{\begin{bmatrix}
I & T_{p-1} & T_{p-2} & \cdots & T_0 & 0 & \cdots & 0 \\
0 & I & T_{p-1} & \cdots & T_1 & T_0 & \cdots & 0 \\
 & & \ddots & I & T_{p-1} & \cdots & \cdots & T_0 \\
 & 0 & & 0 & V_{p-1} & \cdots & \cdots & V_0 \\
0 & V_{p-1} & \cdots & \cdots & V_1 & V_0 & \cdots & \\
V_{p-1} & V_{p-2} & \cdots & \cdots & V_0 & 0 & \cdots & 0
\end{bmatrix}}_{(p+p\ell-1)\text{ block columns}}
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} p\ell-1 \text{ Blocks rows} \\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} p\ell \text{ blocks rows}
\end{array} \tag{119}$$ †

Where

$$T(s) = I_\ell s^p + T_{p-1}s^{p-1} + \ldots + T_1 s + T_0 \tag{120}$$

$$V(s) = V_{p-1}s^{p-1} + \ldots + V_1 s + V_0 \tag{121}$$

The rank of R_1 can be determined using the double triangularization algorithm to be discussed later. The above application is of importance in the analysis and synthesis of multivariable feedback systems.

† This matrix is referred to as the generalized resultant [16].

Remark:

1. It should be noted that in general the matrices R, R_0, M_2, Φ and R_1 in the above applications are not square matrices. Since we are interested in rank testing this does not cause any difficulties and indeed it represents a major new application of the innerwise matrices. Again these matrices (not square) constitute a departure from the Sylvester matrix, though the left triangle of zeros also exists which indicates some relationship exists.

2. It should be pointed out that Rosenbrock [140] has also recognized the similarity of his matrices to that of Sylvester. Indeed in his book [140] the innerwise matrices appear in various applications.

6. Other applications of innerwise matrices:

In this section, we will present several applications which are not directly related to the five earlier topics. These applications make use of some of the definitions which were presented in part I but were not discussed in the earlier applications . These are listed as follows:

a. Non-null innerwise matrices: [155]

The application of the above definition arises in obtaining the necessary and sufficient condition for the rational function

$$Z(s) \triangleq \frac{g(s)}{f(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \text{ with } a_n \neq 0 \tag{122}$$

to have a certain continued fraction expansion [155]. The condition is that by the following Δ_{2n-1} matrix is non-null.

$$\Delta_{2n-1} \triangleq \begin{bmatrix} a_n & a_{n-1} & \cdots & a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_n & & a_3 & a_2 & a_1 & \cdots 0 & & 0 \\ 0 & & a_n & a_{n-1} & a_{n-2} & \cdots & \cdots & & a_0 \\ \vdots & & 0 & \boxed{b_{n-1}} & b_{n-2} & \cdots & \cdots & & b_0 \\ \vdots & & b_{n-1} & b_{n-2} & b_{n-3} & & & & \\ \vdots & 0 & & & & & & & \\ 0 & & & & & & & & \\ 0 & b_{n-1} & b_{n-2} & & & & & & \\ b_{n-1} & b_{n-2} & \cdots & \cdots & & b_0 & 0 & \cdots & 0 \end{bmatrix} \qquad (123)$$

Other applications of the continued fraction expansions in terms of innerwise matrices are of interest and need to be further explored [3].

A further application of a non-null innerwise matrix is discussed by Berkhout [156]. It is given by the following theorem:

Theorem 12, [156]:

The function $f_N(t)$ is a minimum-phase time function,† if and only if

---

†The sampled signal, with sampling interval Δ, represented by the discrete timefunction

$$s_\Delta(t) = \sum_{n=-\infty}^{\infty} s[n]\delta(t-n\Delta), \quad \text{with } s[n] = \Delta s[n\Delta] \text{ is called}$$

a minimum-phase signal if both $s_\Delta(t)$ and its inverse $f_\Delta(t)$ are energy-bound one sided signals. The function $f_N(t)$ is a general representation of a one-sided discrete time function with duration $(N+1)\Delta$:

$$f_N(t) = f(t)[\Delta \sum_{n=0}^{N} \delta(t-n\Delta)].$$

Note the inverse of $s_\Delta(t)$ is defined by

$$f_\Delta(t) * s_\Delta(t) = \delta(t) \quad \text{for all "t" and * is a convolution.}$$

the following equation has a solution and this solution represents a truncated autocorrelation function.

$$\begin{bmatrix} f_N[N] & f_N[N-1] \ \cdots \ f_n(0) & 0 & \cdots & 0 \\ & \ddots & & & \vdots \\ & & & f_N(0) & \vdots \\ \bigcirc & f_N(N) & \cdots & f_N(0) & 0 \\ & \diagup & & & \vdots \\ f_N^*[0] & \cdots \ f_N^*(N) & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} R(-N) \\ R[N-1] \\ R[-1] \\ R[0] \\ R[1] \\ R[N] \end{bmatrix} = \begin{bmatrix} c' \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \qquad (124)$$

$$F\ R = C \qquad (125)$$

The matrix F is an innerwise matrix with left triangle of zeros.

$R \triangleq$ autocorrelation function of inverse of $f_N(t)$.†

c = a scalar factor,

$f_N^* = f_N$ conjugate.

The application of the above theorem in digital filter design and communication systems is of much importance.

b. Semi-innerwise matrices applications: [3,83]

The application of the above definition arises in studying root distribution with respect to the imaginary axis or the unit circle. The semi innerwise condition arises when there are roots on the imaginary axis, or on the unit circles, or roots which are reciprocals of each other (critical root-distribution pattern). Some applications of such critical cases have been discussed in section 2.

†That is, $f_N(t) * R(t) = c\delta(t)$.

c. Application of innerwise matrices in quantum physics [157]:

In the field of quantum physics, an innerwise matrix with a left triangle of zeros arises in calculating the energy-level pattern for rigid asymmetric rotors used in obtaining the rotational energies of molecules [158]. The innerwise matrix in this case is used to obtain a transformation to a certain symmetrized basis function and is referred to in the literature as the Wang [159] symmetrizing transformation. It is given as follows:

$$X_J = X_J^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{126}$$

d. Application of the innerwise matrix in sparse matrix theory: [160]

In the field of sparse matrix theory, the innerwise matrix is often arrived at in order that no zero (block) matrices can become nonzero because of roundoff error in the process of the Gauss elimination method of computation. In these cases one would like to determine the permutation matrices P and Q such that

$$\tilde{A} = P\ A\ Q$$

where $\tilde{A}$ is an innerwise matrix with a left triangle of zeros and A is a given sparse matrix whose inverse is to be evaluated.

The innerwise matrix $\tilde{A}$ is given as follows:

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,p-1} & A_{1p} \\ 0 & A_{22} & & A_{2,p-1} & A_{2p} \\ & & & & \vdots \\ 0 & A_{p+1,1} & & & \vdots \\ A_{p,1} & A_{p,2} & \cdots & \cdots & A_{pp} \end{bmatrix} \tag{127}$$

We can apply the computational algorithm to be discussed in the next section to evaluate the determinant of $\tilde{A}$.

e. Application of innerwise matrix in power system stability: [161]

In study of the dynamic behaviour of a power system following a disturbance, one needs to study the transient stability. In the course of such a study one needs to determine the non-singularity of a Jacobian matrix which has an innerwise form as follows:

$$J = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & A_{1,m-1} \\ 0 & A_{22} & & & \vdots \\ & & & & \vdots \\ 0 & A_{m,2} & & & \vdots \\ A_{m+1,1} & - & - & - & A_{m+1},A_{m+1} \end{bmatrix} \tag{128}$$

The inners algorithm can be readily used to test the singularity of J. Again this matrix is not directly related to Sylvester's matrix. It may have relationships to controllability or observability matrices.

f. Generation of an innerwise matrix from the last inner: [3]

In this application one usually generates the elements of the matrix edges and by a combinatorial rule we generate all the inners. This has application in obtaining the bilinear transformation matrix [162] or the generation of the Schur symmetric matrix [163] from the entries of the table form for the root distribution with respect to the unit circle [26].

IV. Computational Algorithm for Inners Determinants: [3,164]

In the preceding part we discussed applications of the inners concept to many diverse areas of system theory. In all of these applications we need to evaluate either the sign or the magnitude of the inners determinants. In this part, we develop a computational algorithm to compute the inners determinants in a recursive fashion. This algorithm, which is a variant of the Gaussian elimination algorithm, is readily programmed on a digital computer and can be used for the solution of any of the problems discussed in the preceding parts. It is thoroughly discussed and applied in [3].

In this part we will only mention that the algorithm can be presented in two forms. In the first form we assume that the innerwise matrix A has no zero elements, and we will compute the inner determinants. In the second form we assume that the matrix has a left triangle of zeros. In both forms the inners determinants can also be zeros (critical cases) [3].

To explain the steps involved in the algorithm for the second form, we double triangularize the following (5×5) matrix.

$$\Delta_5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & \boxed{a_{33}} \quad \Delta_1 & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} \quad \Delta_3 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \tag{129}$$

Step 1. Make a_{34} and a_{35} zero by pivoting on a_{33} to obtain

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} - \frac{a_{13}}{a_{33}} a_{34} & a_{15} - \frac{a_{13}}{a_{33}} a_{35} \\ 0 & a_{22} & a_{23} & a_{24} - \frac{a_{23}}{a_{33}} a_{34} & a_{25} - \frac{a_{23}}{a_{32}} a_{35} \\ 0 & 0 & a_{33} & a_{34} - \frac{a_{33}}{a_{33}} a_{34} & a_{35} - \frac{a_{33}}{a_{33}} a_{35} \\ 0 & a_{42} & a_{43} & a_{44} - \frac{a_{43}}{a_{33}} a_{34} & a_{45} - \frac{a_{43}}{a_{33}} a_{35} \\ a_{51} & a_{52} & a_{53} & a_{54} - \frac{a_{53}}{a_{33}} a_{34} & a_{55} - \frac{a_{53}}{a_{33}} a_{35} \end{bmatrix} \tag{130}$$

The matrix in eqn. (130) can be rewritten as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \tilde{a}_{14} & \tilde{a}_{15} \\ 0 & a_{22} & a_{23} & \tilde{a}_{24} & \tilde{a}_{25} \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & a_{42} & a_{43} & \tilde{a}_{44} & \tilde{a}_{45} \\ a_{51} & a_{52} & a_{53} & \tilde{a}_{54} & \tilde{a}_{55} \end{bmatrix} \tag{131}$$

where $\tilde{a}_{ij}$ are the proper entries in (130)

Step 2: Make $\tilde{a}_{24}$ and $\tilde{a}_{25}$ zeros by pivoting on a_{22}, to obtain from Eqn. (131).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \tilde{a}_{14} - \frac{a_{12}}{a_{22}}\tilde{a}_{24} & \tilde{a}_{15} - \frac{a_{12}}{a_{22}}\tilde{a}_{25} \\ 0 & a_{22} & a_{23} & \tilde{a}_{24} - \frac{a_{22}}{a_{22}}\tilde{a}_{24} & \tilde{a}_{25} - \frac{a_{22}}{a_{22}}\tilde{a}_{25} \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & a_{42} & a_{43} & \tilde{a}_{44} - \frac{a_{42}}{a_{22}}\tilde{a}_{24} & \tilde{a}_{45} - \frac{a_{42}}{a_{22}}\tilde{a}_{25} \\ a_{51} & a_{52} & a_{53} & \tilde{a}_{54} - \frac{a_{52}}{a_{22}}\tilde{a}_{24} & \tilde{a}_{55} - \frac{a_{52}}{a_{22}}\tilde{a}_{25} \end{bmatrix} \tag{132}$$

The above matrix can be rewritten as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \tilde{\tilde{a}}_{14} & \tilde{\tilde{a}}_{15} \\ 0 & a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & a_{42} & a_{43} & \tilde{\tilde{a}}_{44} & \tilde{\tilde{a}}_{45} \\ a_{51} & a_{52} & a_{53} & \tilde{\tilde{a}}_{54} & \tilde{\tilde{a}}_{55} \end{bmatrix} \tag{133}$$

where $\tilde{\tilde{a}}_{ij}$ are the proper entries in eqn. (132).

Step 3 and 4: Make $\tilde{\tilde{a}}_{45}$ zero by pivoting on $\tilde{\tilde{a}}_{44}$ and $\tilde{\tilde{a}}_{15}$ zero by pivoting on a_{11}. We finally obtain the double triangularized matrix

$$\tilde{\Delta} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & & 0 \\ 0 & a_{22} & a_{23} & 0 & & 0 \\ 0 & 0 & \boxed{a_{33}} & 0 & & 0 \\ & & & \tilde{\Delta}_1 & & \\ 0 & a_{42} & a_{43} & \tilde{\tilde{a}}_{44} & & 0 \\ & & & & \tilde{\Delta}_3 & \\ a_{51} & a_{52} & a_{53} & \tilde{\tilde{a}}_{54} & & \overset{*}{a}_{55} \end{bmatrix} \tag{134}$$

The determinants of the innerwise matrix Δ_5 of eqn. (129) are

$$\left.\begin{aligned} |\Delta_1| &= a_{33} \\ |\Delta_3| &= a_{22}a_{33}\tilde{\tilde{a}}_{44} = a_{22}|\Delta_1|\,\tilde{\tilde{a}}_{44} \\ |\Delta_5| &= a_{11}a_{22}a_{33}\tilde{\tilde{a}}_{44}\overset{*}{a}_5 = \overset{*}{a}_{55}|\Delta_3|a_{11} \end{aligned}\right\} \tag{135}$$

The algorithm presented in the above example can be illustrated as follows. By choosing a permutation matrix P, form $\bar{\Delta}_5 = P\Delta_5\Delta^T$ such that the determinants of the corresponding minors array of $\bar{\Delta}_5$ are the same as those of the inners of Δ_5. Perform the ordinary Gaussian elimination on $\bar{\Delta}_5$ and obtain $\tilde{\Delta}_5$ (where $\tilde{\Delta}_5$ is directly obtainable from Δ_5 by the double triangularization algorithm [3]).

<u>Remarks</u>:

1. A modification of the above algorithm is suggested [165] in order to lessen the round-off errors arising in the calculations. This modification works by interchanging the columns of the matrix while doing the iteration in order to make it <u>diagonally</u> <u>dominant</u> without destroying the inners of

the original innerwise matrix. This minimizes the round-off error, making the algorithm stable and effective.

2. It has been indicated in application 6 part d that for computational effectiveness of a sparse matrix, it is desirable to transform it to an innerwise matrix with a left triangle of zeros [160]. Thus the above algorithm can be effective for the inversion of the sparse matrix as the round-off error is lessened.

3. When some of the inners determinants are zero (critical case) the above algorithm is modified to account for such a case. In reference [3] a complete discussion of this is presented as well as many computer examples. To confine the length of this paper we presented the discussion of this part of the survey in a very concise and brief manner.

V. Conclusions and Observations:

In this paper a comprehensive survey of the theory and applications of the inners has been presented. In particular, applications have been shown to many diverse fields of interest to both Proceedings readers and others. The historical background of this work including the contribution of many other researchers has also been presented. To help assess the value of this survey as well as the inners concept, the following observations are pertinent.

1. Since the early work of Sturm [166], Cauchy [167], Hermite [28] and others on root distribution in the last century, a tremendous number of publications have appeared in this area (probably over one thousand articles). Because of this large number and because of lack of communication, particularly in earlier times, many of the available stability and other criteria have been rediscovered by different researchers. There is a long history of this

type of duplication. In this survey, I have confined attention because of space limitations and simple ignorance, to only that earlier theory related to inners. Even with such a narrowing it is quite likely that I have not discussed all the available literature in this area. If I have left out reference to many authors (as is very possible), it was because of ignorance, and not ill will, and I offer my apologies for any such an omission.

2. The theory of inners having been presented, it is reasonable to assess its value as well as shortcomings and limitations.

The impact of inners in education is, in the opinion of this author, one of its strong points. The theoretical unification makes it easy for the student to understand, or the teacher to present, the various problems and solutions discussed in this paper in a unified fashion. This saves much time. I have presented the inners in my courses at Berkeley during the last few years and the student response was quite enthusiastic. Also, some of my colleagues have presented it at other universities and again the response was encouraging.

3. The advent of the computational algorithm connected with the inners is another encouraging aspect of this theory. This algorithm is general and can be applied to all the problems presented in this paper. The question has yet not been investigated as to whether the inners technique using this general algorithm is more or less efficient than other computational methods for particular applications. More computational comparison needs to be done and probably it will be several years before a definitive statement can be made. However, the availability of a unifying algorithm is of much interest.

4. The impact of the inners theory on research is of much significance. During the past few years, this author in collaboration with Prof. B.

colleagues in my university and various universities and by my students in developing this work. I wish to offer my gratitude and appreciation to all of these and in particular to Professors B. D. O. Anderson, Thomas Kailath, S. Barnett and N. Bose for offering constructive suggestions in this survey. I also wish to sincerely thank the editorial board of the Proceedings for inviting me to present the survey of my recent work.

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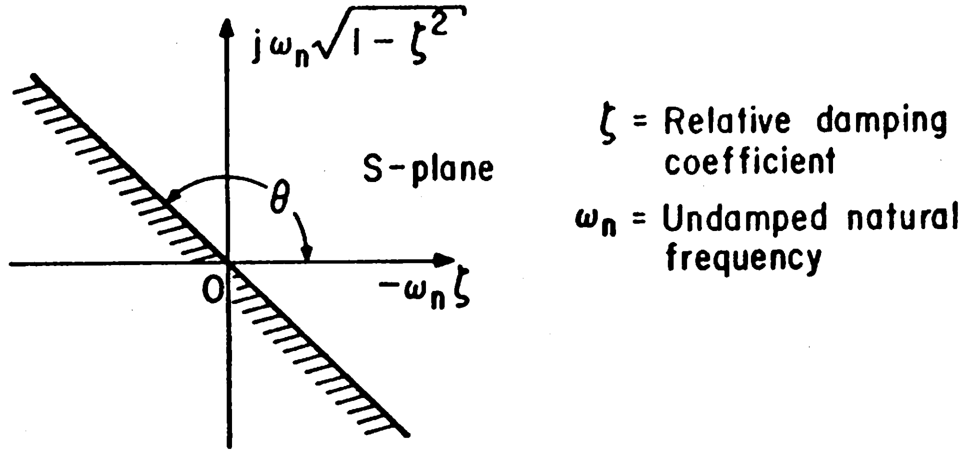

Fig. 1. Relative stability boundary in s-plane.

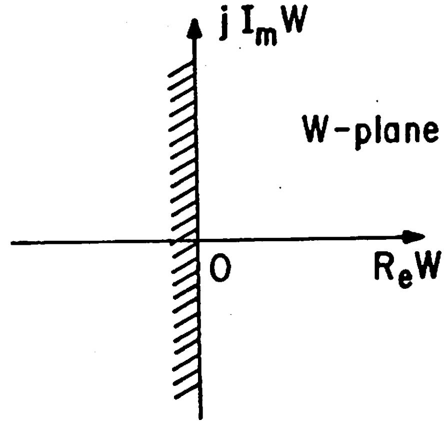

Fig. 2. Relative stability boundary in W-plane.

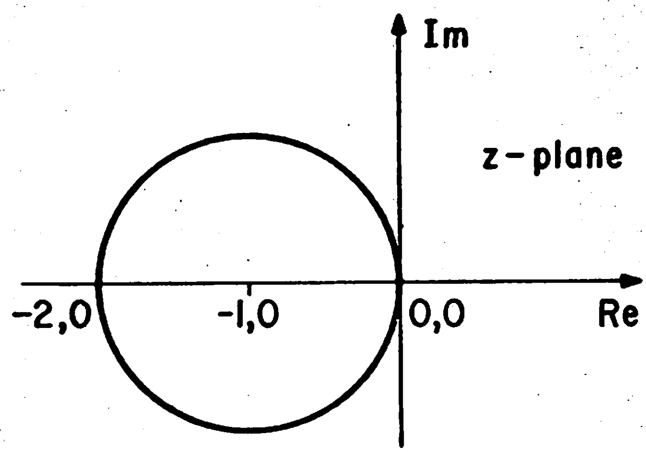

Fig. 3. Unity-shifted unit circle.

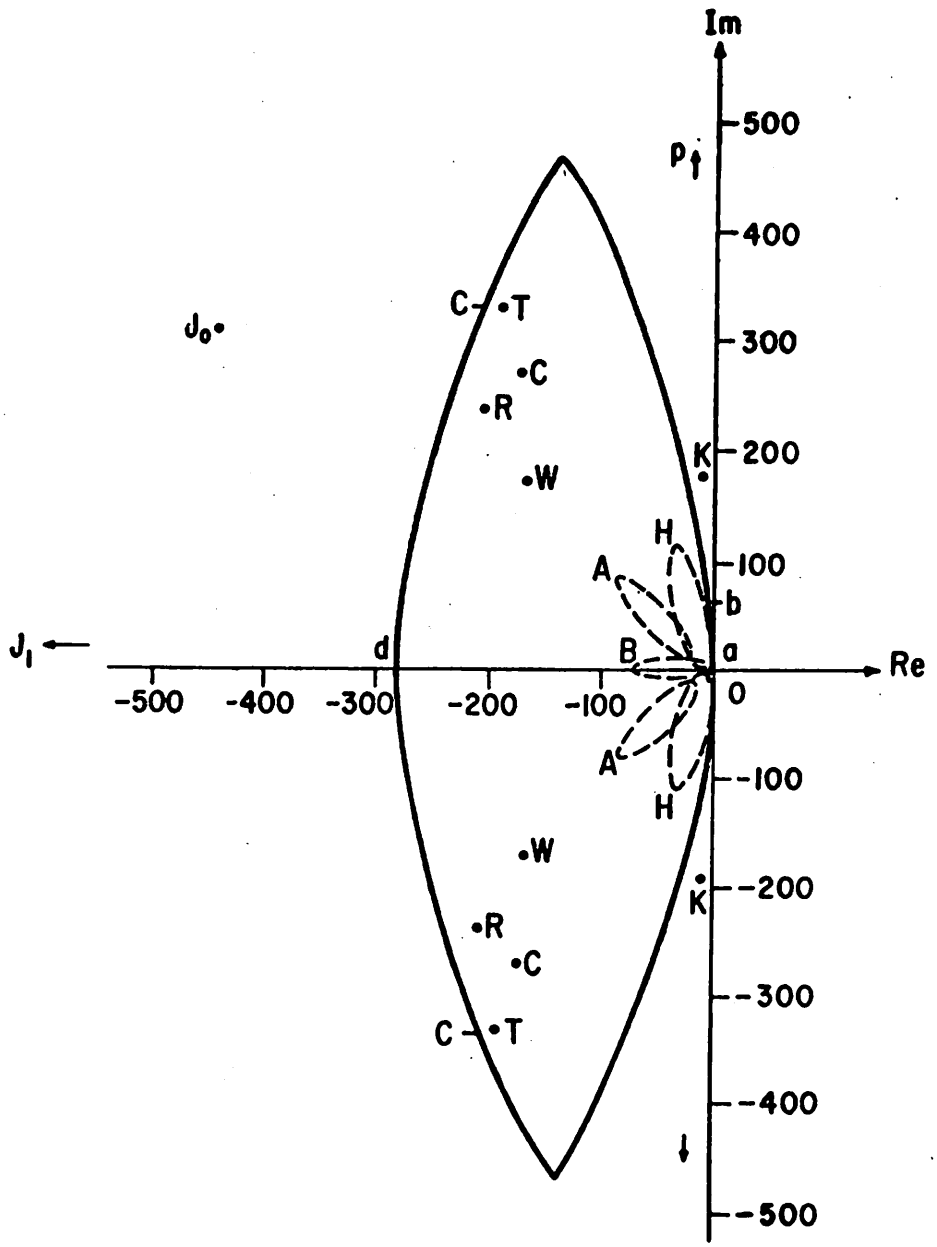

Fig. 4. An approximation to Γ_B, the region within which the roots of models for most neuromuscular systems lie. It is represented as the intersection of two circles. Roots lying within Γ_B: a-origin; b-most underdamped root for the model of the hand; c-most underdamped root for the model of the eye; d-activation-deactivation neural time constants; A-roots of second order models for the eye; B- negative real roots of models of muscle, accomodation, disparity vergence, pupil, semi circular canals, and vestibular system; H-roots of models for the hand; W-second order model for the eye; R,T- complex roots of fourth order models for the eye; C-complex roots of sixth order model for the eye. Roots lying outside Γ_B: K-roots of linearization of model C; P-imaginary roots of models of insect song and flight muscle; J_0-original roots of a model for the eye; J_1-revised roots for model of the eye with original roots J_0.

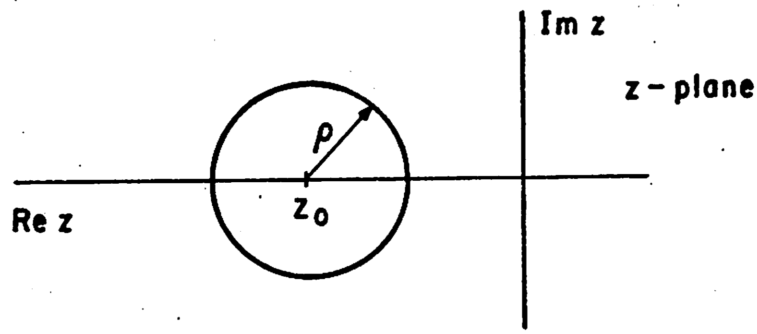

Fig. 5. A shifted circle in the z-plane.

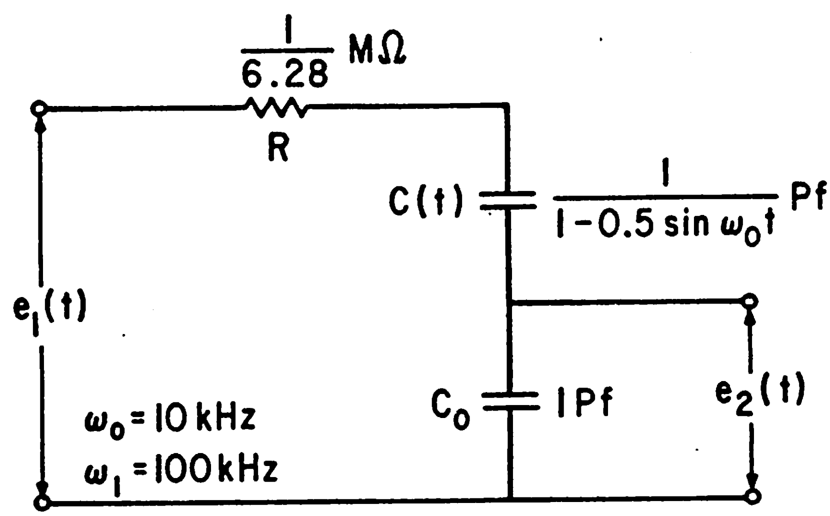

Fig. 6. Periodically Time-Varying R-C network.