NON-NEGATIVITY TEST OF MULTIDIMENSIONAL HERMITIAN MATRICES

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Memorandum No. ERL-M486

28 May 1974

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Abstract

Arguments are advanced to point out that simplification analogous to that in the computational test for positive definiteness of multidimensional Hermitian matrices is not possible when the Hermitian matrix is to be tested for non-negative definiteness.

Research sponsored by the National Science Foundation, Grant GK-38205.

1. Introduction

With the increasing research activities in the areas of multidimensional filtering and multivariable realizability theory, it is often necessary to implement tests for positivity [1], [2] and non-negativity [3] of multidimensional Hermitian matrices. Such tests are usually of great computational complexity and the simplifications, whenever possible are of great value. A useful result, quoted below as a lemma, was, to the author's best knowledge, first proposed in [4], and recently used in [5].

Lemma 1 Let $H(j\omega)$ be a Hermitian matrix, with elements which are polynomials (or rational functions) in ω having complex coefficients. Then $H(j\omega) > 0$, $-\infty \leq \omega \leq \infty$, if and only if

$$\text{(i)} \quad H(0) > 0, \text{ and (ii) } \det H(j\omega) > 0, \ \forall\omega \tag{1}$$

In some engineering problems, nonnegativity (not positivity) of multivairable Hermitian matrices, with elements which are polynomials of not one but several variables and having complex coefficients, has to be checked. For example [3], a matrix, $Z(p_1,p_2,\ldots,p_n)$, with elements which are rational functions of the complex variables, $p_1,p_2,\ldots,p_n$ is positive real if

(i) $Z(p_1,p_2,\ldots,p_n)$ is real for real $p_1,p_2,\ldots,p_n$,

(ii) $Z(p_1,p_2,\ldots,p_n)$ is holomorphic in Re $p_1 > 0$, Re $p_2 > 0$, ..., Re $p_n > 0$ ('Re' denotes 'real part of')

and

(iii) $Z(p_1,p_2,\ldots,p_n) + \tilde{Z}^*(p_1,p_2,\ldots,p_n)$ is non-negative definite in

$$\text{Re } p_1 > 0, \text{ Re } p_2 > 0, \ldots, \text{ Re } p_n > 0,$$

where the superscript 'tilda' and 'star' represent 'transpose' and 'complex conjugate,' respectively. (i) is simple to test for, (ii) can be tested using procedure similar to that discussed for the $n = 2$ case in [6], (where test for non-negative definiteness of Hermitian matrices with polynomial elements are required) and in the test for (iii) also one questions whether simplification analogous to that in lemma 1 and lemma 2 (given below) can be implemented in testing a Hermitian multi-dimensional matrix for non-negativity. Scopes for application of such results also exist in optimality problems [7].

2. Main Results

First, the positivity result summarized in lemma 1 is written below for a multivariable Hermitian matrix.

Lemma 2 Let $H(j\omega_1 j\omega_2,\ldots,j\omega_n)$ be a $m \times m$ Hermitian matrix, with elements which are multivariable polynomials in ω_i, $1 \leq i \leq n$, and having complex coefficients. Then, $H(j\omega_1,j\omega_2,\ldots,j\omega_n) > 0$, $-\infty \leq \omega_i \leq \infty$, and $1 \leq i \leq n$, if and only if,

$$\text{(i) } H(0,0,\ldots,0) > 0, \text{ and (ii) } \det H(j\omega_1,j\omega_2,\ldots,j\omega_n) > 0, \ \forall \omega_i, \ 1 \leq i \leq n \tag{2}$$

The proof of lemma 2 is simple and is similar to that advanced in [4], [5], after observing that the eigenvalues, λ_j, $(j = 1,2,\ldots,m)$ of $H(j\omega_1 j\omega_2,\ldots,j\omega_n)$ which are algebraic functions of ω_i, $1 \leq i \leq n$, are real and continuous. Realness follows from the Hermitian nature of H and continuity properties can be proved using arguments in [8].

For $H(j\omega_1, j\omega_2, \ldots, j\omega_n)$ to be ≥ 0, $-\infty \leq \omega_i \leq \infty$, $1 \leq i \leq n$ additional conditions including those given in (3a) and (3b) are required

$$H(0,0,\ldots,0) \geq 0 \tag{3a}$$

$$\det H(j\omega_1, j\omega_2, \ldots, j\omega_n) \geq 0, \ \forall\omega_i, \ 1 \leq 1 \leq n \tag{3b}$$

This follows from the fact that (3a) and (3b) could be satisfied, and yet there might be an <u>even</u> number of eigenvalues of $H(j\omega_1, j\omega_2, \ldots, j\omega_n)$ that might be negative for at least some values of the variables. For example,[†]

the matrix, $h(j\omega_1) = \begin{bmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_1^4 \end{bmatrix}$ satisfies (3a) and (3b)

but is not non-negative definite. Therefore, $H(j\omega_1, j\omega_2, \ldots, j\omega_n)$ is a non-negative definite Hermitian matrix if and only if it satisfies not only conditions (3a), (3b) but also (3c), given below.

> $H(j\omega_1, j\omega_2, \ldots, j\omega_n)$ does not have an even number of eigenvalues which are negative for at least some real values of $\omega_1, \omega_2, \ldots, \omega_n$. (3c)

However, tests for conditions (3a),(3b) and (3c) are seen to be no simpler than that necessitated by the definition for non-negative definiteness given below.

<u>Definition</u> A Hermitian matrix $H(j\omega_1, j\omega_2, \ldots, j\omega_n)$ is non-negative definite $\forall\omega_i$, $1 \leq i \leq n$, if and only if all its eigenvalues are non-negative, $\forall\omega_i$.

[†]This example was supplied to the author by Professor B. D. O. Anderson.

In this context, it is pointed out that for non-negativity tests for multi-dimensional Hermitian matrices, it is necessary and sufficient to determine non-negativity of all principal minors and not only the leading principal minors as appears to be in [6].

Conclusions

It is pointed out that computational simplification analogous to that in the test for positive definiteness of multidimensional Hermitian matrices is not possible when a Hermitian matrix is to be tested for non-negative definiteness as is required in various applied problems arising in multivariable network theory [3], optimality [7], etc.

Acknowledgement

The author is pleased to acknowledge Professor B. D. O. Anderson's valuable comments as well as the fruitful discussions with Professor E. I. Jury regarding portions of this article.

<u>References</u>

[1] B. D. O. Anderson and E. I. Jury, "Stability test for two-dimensional recursive filters," IEEE Trans. Audio Electroacoustics, Vol. AU-21, Aug. 1973, pp. 366-372.

[2] N. K. Bose and E. I. Jury, "Positivity and stability tests for multidimensional filters (discrete-continuous)," IEEE Trans. Acoustics, Speech, and Signal <u>Processing</u>, Vol. ASSP-22, June 1974, pp. 174-180.

[3] N. K. Bose and R. W. Newcomb, "Tellegen's theorem and multivariable realizability theory," Int. of Electronics, Vol. 36, Mar. 1974, pp. 417-425.

[4] E. I. Jury and B. W. Lee, "A stability theory for multinonlinear control systems," paper no. 289, Third IFAC Congress, London, England, June 1966, pp. 28A.1-28A.11.

[5] D. D. Siljak, "Algebraic criteria for positive realness relative to the unit circle," Journal of Franklin Institute, April 1973, pp. 469-476.

[6] H. G. Ansell, "On certain two-variable generalizations of circuit theory with applications to networks of transmission lines and lumped reactances," IEEE Trans. Circuit Theory, Vol. CT-11, June 1964, pp. 214-223.

[7] B. D. O. Anderson, "The inverse problem of optimal control," Rept. SEL - 66-038 (TR-6560-3), Stanford Electronics Laboratories Stanford, Calif. 1966.

[8] G. A. Bliss, "<u>Algebraic Functions</u>," Dover Publications Inc., N. Y., 1966, pp. 24-29.