THE FEEDBACK INTERCONNECTION OF MULTIVARIABLE SYSTEMS: SIMPLIFYING THEOREMS FOR STABILITY

by

C. A. Desoer and W. S. Chan

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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C. A. Desoer and W. S. Chan

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

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Introduction

This paper may be viewed as a first step toward a general input-output theory for arbitrary interconnections of multi-input multi-output subsystems. In contrast to [1] it does allow, in several results, unstable subsystems. It is closely related to [2] which gives necessary and sufficient conditions for stability allowing for unstable subsystems. The thrust of the paper is towards finding conditions under which stability tests are greatly simplified. The results below constitute an extension of results presented at the 1974 Allerton Conference [18]. The discrete-time extension is described in section IV.

The point of view adopted in the paper is that pioneered by Sandberg and Zames [3,4]. This approach to stability problems has been developed in many papers [5-9] and books [10-12]. A slightly different but closely related approach is to be found in [13-16].

In the first section of the paper we describe the system under consideration and review the pertinent definitions and facts needed to state our results. The second section presents two basic examples which are needed to understand some basic points related to the new results. The third section states precisely the three basic theorems and tries to describe the nature and interrelationships of the results. All the proofs are relegated to the Appendix.

<u>Notations</u>. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$, $\mathcal{A}$ denote, respectively, the fields of real numbers, complex numbers, rational functions with real coefficients, and the convolution algebra defined in [5] and [6]. Superscripts n and n×n are used to denote the corresponding classes of ordered n-tuples (e.g. $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathcal{A}^n$) and n×n arrays (e.g. $\mathbb{R}(s)^{n\times n}$), respectively. Laplace transforms are denoted by a ^; Z-transforms by a ∿. Operators and matrix-transfer-functions are denoted by capitals (e.g. G_1, $\hat{G}_1$). Scalar transfer functions are denoted by lower case letters, (e.g. $\hat{g}(s)$). The abbreviations MIMO and SISO denote "multiple-input multiple-output" and "single-input single-output", respectively. $\mathbb{C}_+$ and $\mathring{\mathbb{C}}_+$ denote the <u>closed</u> and the <u>open</u> right half-plane.

I. System Description and Preliminary Definitions

We consider a feedback system S whose inputs, outputs, etc. are defined on $\mathcal{T} \subset \mathbb{R}$: typically $\mathcal{T} = \mathbb{R}_+$, for continuous-time systems, and $\mathcal{T} = \mathbb{Z}_+$, (the nonnegative integers) for discrete-time systems. Let $\mathcal{F} = \{f: \mathcal{T} \to \mathcal{V}\}$ where $\mathcal{V}$ is a normed space with norm $\|\cdot\|$. For any $T \in \mathcal{T}$, $f_T(t) = f(t)$ if $t \leq T$, and zero for $t > T$. Using the usual definitions of addition and scalar product, we define the vector space

$$\mathcal{L}_e = \{f \in \mathcal{F} \mid \forall T \in \mathcal{T}, \|f_T\| < \infty\}$$

To avoid long concatenations of subscripts, we shall write

$\|f\|_T$ for $\|f_T\|$.

The feedback system S is made up of two subsystems as shown in Fig. I. If $\mathcal{V} = \mathbb{R}^n$, then the two subsystems are n-input n-output subsystems. The inputs u_i, errors e_i, outputs y_i belong to $\mathcal{L}_e$.

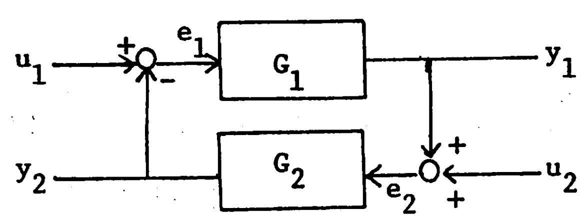

Fig. I

We define for $i = 1,2$

$$G_i : \mathcal{L}_e \to \mathcal{L}_e$$

$$y_i = G_i(e_i) = G_i e_i \qquad (1)$$

The equations are then

$$e_1 = u_1 - G_2 e_2 \tag{2}$$

$$e_2 = G_1 e_1 + u_2 \tag{3}$$

We make a general existence assumption which will hold throughout the paper: $\forall (u_1, u_2) \in \mathcal{L}_e \times \mathcal{L}_e$, $\exists (e_1, e_2) \in \mathcal{L}_e \times \mathcal{L}_e$ which satisfy the equations (2), (3) of the system. For general existence criteria see [4, 11, 12]. Note that uniqueness is not required. If uniqueness holds, there is a map, denoted by H_e such that

$$H_e: \quad (u_1, u_2) \longmapsto (e_1, e_2)$$

If uniqueness does not hold, H_e becomes a relation [17].

G_i is said to be $\mathcal{L}$-stable iff

$$\exists k < \infty \ni \forall x \in \mathcal{L}_e, \quad \forall T \in \mathcal{T} \tag{4}$$

$$\|G_i x\|_T \leq k \|x\|_T$$

The gain of G_i is defined to be the infimum of all such k; it is denoted by $\gamma(G_i)$. Calculations of the gain for SISO and MIMO systems can be found in [3, 4, 11, 12]. The incremental gain of G_i, $\tilde{\gamma}(G_i)$, is defined as

$$\tilde{\gamma}(G_i) \triangleq \inf\{\gamma \in \mathbb{R}_+ | \; \forall x_1, x_2 \in \mathcal{L}_e, \; \forall T \in \mathcal{T},$$

$$\|G_i x_1 - G_i x_2\|_T \leq \gamma \|x_1 - x_2\|_T\}. \tag{5}$$

For linear system $\gamma(G_i) = \tilde{\gamma}(G_i)$.

Let u, e, and y denote the ordered pairs (u_1, u_2), (e_1, e_2), and (y_1, y_2), respectively. We also have the map H_y: $u \longmapsto y$. It is important to note

Using ^ to denote Laplace transformed quantities, we have

$$\hat{y}_i = \hat{G}_i \cdot \hat{e}_i .$$

In the linear, time-invariant, distributed case, we introduce the Banach algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ as follows (see [5], [6], [12])

$$\mathcal{A} \triangleq \{f: \ \mathbb{R}_+ \to \mathbb{R} \,|\, f(t) = \sum_{i=o}^{\infty} f_i \delta(t-t_i) + f_a(t) \text{ where } \sum_{i=o}^{\infty} |f_i| < \infty, \ t_i \geq 0 \ \forall i, \ f_a \in L_1\} \tag{10}$$

$A \in \mathcal{A}^{n\times n}$ means that each element of the matrix $A \in \mathcal{A}$. $\hat{\mathcal{A}}^{n\times n} = \{\hat{A} | A \in \mathcal{A}^{n\times n}\}$. It is well known that if $\hat{G}_1, \hat{G}_2 \in \hat{\mathcal{A}}^{n\times n}$, then $\hat{G}_1 + \hat{G}_2$, $\hat{G}_1 \hat{G}_2 \in \hat{\mathcal{A}}^{n\times n}$ and $\hat{G}_1^{-1} \in \hat{\mathcal{A}}^{n\times n} \Leftrightarrow \inf_{s\in\mathbb{C}_+} |\det \hat{G}_1(s)| > 0$.

If $h \in \mathcal{A}$

$$\|h\|_a \triangleq \sum_{i=o}^{\infty} |h_i| + \int_0^{\infty} |h_a(t)|dt \tag{11}$$

and if $H \in \mathcal{A}^{n\times n}$

$$\|H\|_a \triangleq \max \sum_{j=1}^{n} \|h_{ij}\|_a . \tag{12}$$

Then if $1 \leq p \leq \infty$, $u \in L_p^n$ and $H \in \mathcal{A}^{n\times n}$ then

$$\|H*u\|_p \leq \|H\|_a \cdot \|u\|_p , \tag{13}$$

where $\|\cdot\|_p$ denotes the p^{th} norm [12].

Two elements $\hat{\mathcal{N}}, \hat{\mathcal{D}}$ of $\hat{\mathcal{A}}^{n\times n}$ are said to be pseudo right coprime, abbr. p.r.c., (resp. pseudo left coprime, abbr. p.l.c.) [12, 19] iff that if we define $J: \mathcal{L}_e \times \mathcal{L}_e \to \mathcal{L}_e \times \mathcal{L}_e$ by $Ju = (u_2, -u_1)$, then

$$H_y = J(H_e - I) \text{ and } H_e = I - JH_y, \tag{6}$$

where I denotes the identity.

If both G_1 and G_2 are linear maps, the map H_e: $u \longmapsto e$ takes the form

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (I+G_2G_1)^{-1} & -G_2(I+G_1G_2)^{-1} \\ G_1(I+G_2G_1)^{-1} & (I+G_1G_2)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7}$$

where G_1G_2 denotes the composition of G_1 with G_2.

The map, (or the relation), H_e is said to be $\mathcal{L} \times \mathcal{L}$-stable iff $\exists k < \infty$ such that $\forall u_1, u_2 \in \mathcal{L}_e$, $\forall T \in \mathcal{T}$, for $i = 1,2$

$$\|e_i\|_T \leq k(\|u_1\|_T + \|u_2\|_T). \tag{8}$$

In other words, if in the product space we choose the norm $\|u\| = \|u_1\| + \|u_2\|$, then we see that (8) is equivalent to $\gamma(H_e) < \infty$. From (6), $\gamma(H_e) < \infty$ if and only if $\gamma(H_y) < \infty$.

For the continuous-time, linear, time-invariant case, for $i = 1,2$, we define

$$G_i: \mathbb{R}_+ \to \mathbb{R}^{n\times n}$$

by a convolution. To alleviate notation, we also use G_i to denote the kernel of the convolution operator, thus

$$y_i \triangleq G_i * e_i. \tag{9}$$

$\exists \hat{\mathcal{U}}, \hat{\mathcal{V}}, \hat{\mathcal{W}} \in \hat{\mathcal{A}}^{n\times n} \ni$

(i) $\det \hat{\mathcal{W}}(s) \neq 0 \quad \forall s \in \mathbb{C}_+$

and (ii) $\hat{\mathcal{U}}\hat{\mathcal{N}} + \hat{\mathcal{V}}\hat{\mathcal{D}} = \hat{\mathcal{W}}$ (resp. $\hat{\mathcal{N}}\hat{\mathcal{U}} + \hat{\mathcal{D}}\hat{\mathcal{V}} = \hat{\mathcal{W}}$)

Given a function $\hat{G}: \mathbb{C}_+ \to \mathbb{C}^{n\times n}$

$(\hat{\mathcal{N}}, \hat{\mathcal{D}})$ is said to be a p.r.c. factorization, abbr. p.r.c.f. (resp. p.l.c. factorization, abbr. p.l.c.f.) of $\hat{G}$ iff

(i) $\hat{G} = \hat{\mathcal{N}}\hat{\mathcal{D}}^{-1}$ (resp. $\hat{G} = \hat{\mathcal{D}}^{-1}\hat{\mathcal{N}}$)

(ii) $\hat{\mathcal{N}}, \hat{\mathcal{D}}$ are p.r.c. (resp. $\hat{\mathcal{N}}, \hat{\mathcal{D}}$ are p.l.c.)

and (iii) $\forall$ sequences $(s_i)_{i=1}^{\infty} \subset \mathbb{C}_+$ and $|s_i| \to \infty$

$$\liminf_{i\to\infty} |\det \hat{\mathcal{D}}(s_i)| > 0.$$

The following fact has been established in [12]. If $\hat{G} \in \hat{\mathcal{A}}^{n\times n}$ and $(\hat{\mathcal{N}}, \hat{\mathcal{D}})$ is a p.r.c.f. or a p.l.c.f. of $\hat{G}$

then $p \in \mathbb{C}_+$ is a pole of $\hat{G}$

$\Leftrightarrow$ $p \in \mathbb{C}_+$ is a zero of $\det \hat{\mathcal{D}}$.

If $\hat{G} \in \mathbb{R}(s)^{n\times n}$ and $\hat{G}$ is proper, then $\hat{G}$ has both a left- and a right-coprime factorization.

In the linear, time-invariant, lumped case, $\hat{G}_1, \hat{G}_2 \in \mathbb{R}(s)^{n\times n}$ and $\hat{G}_i$ is said to be proper iff all its elements are bounded at infinity, and

$\hat{G}_i$ is said to be exponentially stable (abbr. exp. st.) iff it is proper and has all its poles in $\mathring{\mathbb{C}}_-$, (the open left half plane).

II. Instructive Example

In the linear case, H_e is given by (7): H_e splits into four partial maps: $u_i \mapsto e_j$, $i,j = 1,2$. Each one of these four partial maps may be $\mathcal{L}$-stable or not: this gives $16 = 2^4$ possible patterns of instability; this number is further reduced to 10 by interchanging subscripts 1 and 2. In view of the fact that each of the four partial maps depends on the same two functions G_1 and G_2, one might expect that not all possible patterns of instability might occur and hence that one might prove the $\mathcal{L} \times \mathcal{L}$-stability of H_e by studying only a proper subset of the four partial maps. This is, in fact, not so. Consider the following two linear time-invariant examples.

Example 1. If $\hat{g}_1(s) = 1/s$, $\hat{g}_2(s) = s/(s+1)$, then all submatrices of $\hat{H}_e$ are exp. stable except $\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1}$ which has a pole at $s = 0$.

Example 2. If $\hat{G}_1(s) = \begin{bmatrix} s/(s+1) & 1/s \\ 0 & 1/s \end{bmatrix}$, $\hat{G}_2(s) = \begin{bmatrix} 1/(s+1) & 1/s \\ 0 & 1/(s+1) \end{bmatrix}$

then all submatrices of $\hat{H}_e$ are exp. stable except $(I+\hat{G}_2\hat{G}_1)^{-1}$ which has a pole at $s = 0$. A detailed study of all 10 possibilities is reported in [18].

In conclusion, even in the lumped, linear, time-invariant case, in order to prove that H_e is $\mathcal{L} \times \mathcal{L}$ stable, one must investigate the stability of each of the four partial maps $u_i \longmapsto e_j$, $i,j = 1,2$.

III. The Simplifying Theorems

In most design procedures and stability considerations one assumes $u_2 \equiv 0$ and studies the stability of the map $u_1 \mapsto y_1$, namely, $G_1(I+G_2G_1)^{-1}$. An interesting question is then: under what general conditions does the $\mathcal{L}$-stability of $G_1(I+G_2G_1)^{-1}$ imply the $\mathcal{L}\times\mathcal{L}$-stability of H_e? The following theorem answers the question for a broad class of nonlinear systems:

Theorem 1.(†) (Nonlinear time-varying MIMO)

Let G_1 be defined as in (1). If G_2 and $G_1(I+G_2G_1)^{-1}$ are $\mathcal{L}$-stable, and if the incremental gain of G_2, $\tilde{\gamma}(G_2)$, is finite, then H_e is $\mathcal{L}\times\mathcal{L}$ stable.

In particular, if as in most practical cases the feedback subsystem, G_2, is linear, then the condition $\tilde{\gamma}(G_2) < \infty$ is equivalent to that G_2 be $\mathcal{L}$-stable.

If G_2 is unbiased (i.e. $G_2 0 = 0$), choosing $x_2 = 0$ in (5) and comparing with (4), we see that $\gamma(G_2) \leq \tilde{\gamma}(G_2)$. Hence, we have the following

Corollary 1.1. (Nonlinear time-varying MIMO)

If $G_1(I+G_2G_1)^{-1}$ is $\mathcal{L}$-stable, if G_2 is unbiased and if G_2 has a finite incremental gain, $\tilde{\gamma}(G_2)$, then H_e is $\mathcal{L}\times\mathcal{L}$ stable.

In order to bring to bear analytical tools, we restrict ourselves to linear time-invariant distributed systems. An important feature of Theorem 2 and its corollaries, is that they do not impose any stability conditions on either G_1 or G_2. This is in contrast to Theorem 1 which requires that $\tilde{\gamma}(G_2) < \infty$.

(†) All proofs are in the appendix.

Theorem 2 (Linear time-invariant distributed MIMO)

Let G_1 and G_2 be represented by convolution operators as in (9). Suppose that $\hat{G}_1$ has p.l.c.f. and $\hat{G}_2$ has p.r.c.f. or $\hat{G}_1$ has p.r.c.f. and $\hat{G}_2$ has p.l.c.f.. Suppose that $\forall$ sequences $(s_i)_{i=1}^{\infty} \subset \mathbb{C}_+$ and $|s_i| \to \infty$

$$\liminf_{i\to\infty} |\det[I + \hat{G}_1(s_i)\hat{G}_2(s_i)]| > 0 \tag{14}$$

U.t.c. if (a) $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$, $\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$ are in $\hat{\mathcal{A}}^{n\times n}$, and (b) $\hat{G}_1$, $\hat{G}_2$ have no common $\mathbb{C}_+$ pole, then $\hat{H}_e \in \hat{\mathcal{A}}^{2n\times 2n}$.

Comment: this conclusion implies that H_e is L_p-stable for all $p \in [1,\infty]$, see (13).

Corollary 2.1 (Linear time-invariant lumped MIMO)

Let, for $i = 1,2$, G_i be a convolution operator, $\hat{G}_i(s) \in \mathbb{R}(s)^{n\times n}$ and be proper. Let $\det(I+\hat{G}_1\hat{G}_2)(\infty) \neq 0$. U.t.c., if $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$, $\hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}$ are exp. st. and if $\hat{G}_1$ and $\hat{G}_2$ have no common $\mathbb{C}_+$ pole, then $\hat{H}_e$ is exp. st.

The condition $\det(I+\hat{G}_1\hat{G}_2)(\infty) \neq 0$ is related to well-posedness [11, 15]: with the $\hat{G}_i(s) \in \mathbb{R}(s)^{n\times n}$ and proper, this determinantal condition is violated if and only if $(I+\hat{G}_1\hat{G}_2)^{-1}$ and $(I+\hat{G}_2\hat{G}_1)^{-1}$ have a pole at infinity, i.e. the closed loop system transfer function $\hat{H}_e$ includes differentiators!

Corollary 2.2 (Linear time-invariant lumped SISO)

Let, for $i = 1,2$, g_i be a convolution operator, $\hat{g}_i(s) \in \mathbb{R}(s)$ and be proper. U.t.c. if $\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1}$ and $\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ are exp. st., then $\hat{H}_e$ is exp. st.

Note that for the SISO case the requirement that the transfer functions have no common right-half plane poles is dropped. That this condition is indispensable for the MIMO case is shown by Example 2 above.

The basic algebraic reason is that in the algebra $\mathbb{R}(s)^{n\times n}$ the cancellation law does not hold, whereas it does in the algebra $\mathbb{R}(s)$. More precisely $\mathbb{R}(s)^{n\times n}$ is a noncommutative ring which includes divisors of zero; $\mathbb{R}(s)$ is a field [17].

For a similar reason, Theorem 2 simplifies to the following corollary in the SISO case.

Corollary 2.3 (Linear time-invariant distributed SISO)

Let, for $i = 1,2$, G_i be SISO, hence denoted by g_i and let it be a convolution operator. Let $\hat{g}_1, \hat{g}_2$ have p.c.f.. U.t.c. if $\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1}$ and $\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ are in $\hat{\mathcal{A}}$ then $\hat{H}_e \in \hat{\mathcal{A}}^{2\times 2}$.

Theorem 3 and its corollary are more restrictive: they exploit the properties of the algebras $\hat{\mathcal{A}}^{n\times n}$ and $\mathbb{R}(s)^{n\times n}$, resp. and impose some stability requirement on G_2,

Theorem 3 (Linear time-invariant distributed MIMO)

If $\hat{G}_2$ and $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ are in $\hat{\mathcal{A}}^{n\times n}$, then $\hat{H}_e$ is in $\hat{\mathcal{A}}^{2n\times 2n}$.

Since the proof of Theorem 3 is purely algebraic, it obviously extends almost verbatim to the lumped case.

Corollary 3.1. (Linear time-invariant lumped MIMO)

If $\hat{G}_2$ and $\hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1}$ are exponentially stable, then so is $\hat{H}_e$.

Note that it is this corollary which justifies the common design procedures and the elementary discussions of MIMO feedback systems.

IV. The Discrete-time Case

The results above except for Theorem 1 and its corollary are stated for the continuous-time case. A study of the proofs would easily show that they extend easily to the discrete-time case. The required changes are listed in the Table I: $B(0,1)$ and $B(0,1)^c$ denote the open unit ball centered on 0 in $\mathbb{C}$ and its complement, resp.; ℓ_1 denotes the convolution algebra of absolutely convergent sequences:

$\ell_1 = \{(z_i)_0^\infty \subset \mathbb{C} \mid \sum_0^\infty |z_i| < \infty\}$, (for details see [12]).

Table I

Laplace transform	→	Z-transform
$\mathcal{A}$	→	ℓ_1
$\mathcal{A}^{n\times n}$	→	$\ell_1^{n\times n}$
$\mathbb{C}_-$	→	$B(0,1)$
$\mathbb{C}_+$	→	$B(0,1)^c$
$s \to \infty$	→	$z \to \infty$
$\mathbb{R}(s)^{n\times n}$	→	$\mathbb{R}(z)^{n\times n}$

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APPENDIX: PROOFS

<u>Proof of Theorem 1</u> $\forall u_2 \in \mathcal{L}_e$ and $\forall e_1 \in \mathcal{L}_e$, let

$\tilde{u} = G_2 G_1 e_1 - G_2(u_2 + G_1 e_1)$; then

$\forall T \in \mathcal{T}, \ \forall u_2 \in \mathcal{L}_e$

$$\|\tilde{u}\|_T \leq \tilde{\gamma}(G_2)\|G_1 e_1 - (u_2 + G_1 e_1)\|_T$$

$$= \tilde{\gamma}(G_2)\|u_2\|_T$$

From the systems equations (2) and (3), we have

$$u_1 = e_1 + G_2(u_2 + G_1 e_1)$$

$$\therefore \quad u_1 + \tilde{u} = e_1 + G_2 G_1 e_1$$

Hence $\quad e_1 = (I + G_2 G_1)^{-1}(u_1 + \tilde{u})$

and $\quad G_1 e_1 = G_1(I + G_2 G_1)^{-1}(u_1 + \tilde{u})$

The assumed $\mathcal{L}$-stability of $G_1(I + G_2 G_1)^{-1}$ implies

$$\exists k_1 < \infty \ni \forall T \in \mathcal{T}, \ \forall u_1, u_2 \in \mathcal{L}_e$$

$$\|G_1 e_1\|_T \leq k_1 \|u_1 + \tilde{u}\|_T$$

$$\leq k_1(\|u_1\|_T + \|\tilde{u}\|_T)$$

$$\leq k_1(\|u_1\|_T + \tilde{\gamma}(G_2)\|u_2\|_T)$$

Letting $k_2 = \max\{k_1, k_1 \tilde{\gamma}(G_2)\}$, we have

$$\|G_1 e_1\|_T \leq k_2(\|u_1\|_T + \|u_2\|_T) \tag{A1}$$

Using (3) and (A1), we conclude that

$$\exists k_2 < \infty \ni \forall T \in \mathcal{T}, \quad \forall u_1, u_2 \in \mathcal{L}_e$$

$$\|e_2\|_T \leq (1+k_2)(\|u_1\|_T + \|u_2\|_T) \tag{A2}$$

The assumed $\mathcal{L}$-stability of G_2 and (A2) imply that

$$\exists k_3 < \infty \ni \forall T \in \mathcal{T}, \quad \forall u_1, u_2 \in \mathcal{L}_e$$

$$\|G_2 e_2\|_T \leq k_3(\|u_1\|_T + \|u_2\|_T) \tag{A3}$$

Using (2) and (A3), we conclude that

$$\exists k_3 < \infty \ni \forall T \in \mathcal{T}, \quad \forall u_1, u_2 \in \mathcal{L}_e$$

$$\|e_1\|_T \leq (1+k_3)(\|u_1\|_T + \|u_2\|_T) \tag{A4}$$

(A2) and (A4) imply that H_e is $\mathcal{L} \times \mathcal{L}$ stable.

Q.E.D.

<u>Proof of Theorem 2</u>

Case 1: Suppose $\hat{G}_1$ has p.l.c.f. $(\hat{\mathcal{N}}_1, \hat{\mathcal{D}}_1)$ and $\hat{G}_2$ has p.r.c.f. $(\hat{\mathcal{N}}_2, \hat{\mathcal{D}}_2)$

$$I - \hat{G}_1 \cdot \hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1} = I - \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1} \cdot \hat{G}_2 = (I+\hat{G}_1\hat{G}_2)^{-1} \tag{A5}$$

Assumptions (a) and (b) imply that

the three expressions (A5) have no $\mathbb{C}_+$ pole (A6)

Note the equalities

$$(I+\hat{G}_1\hat{G}_2)^{-1} = (I+\hat{\mathcal{D}}_1^{-1}\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2\hat{\mathcal{D}}_2^{-1})^{-1} = \hat{\mathcal{D}}_2(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)^{-1}\hat{\mathcal{D}}_1 \tag{A7}$$

Now $\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2, (\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2) \in \hat{\mathcal{A}}^{n\times n}$ and (A7) imply successively,

$$p \text{ is a } \mathbb{C}_+ \text{ pole of } (I+\hat{G}_1\hat{G}_2)^{-1}$$

$$\Leftrightarrow \quad p \text{ is a } \mathbb{C}_+ \text{ pole of } (\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)^{-1}$$

$$\Leftrightarrow \quad p \text{ is a } \mathbb{C}_+ \text{ zero of } \det(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)$$

hence by (A6), we have

$$\det(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)(s) \neq 0 \quad \forall s \in \mathbb{C}_+ \tag{A8}$$

From (A7),

$$\det(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2) = \det\hat{\mathcal{D}}_1 \times \det\hat{\mathcal{D}}_2 \times \det(I+\hat{G}_1\hat{G}_2)$$

by definition of p.c.f. and the assumption (14), we have

$$\forall \text{ sequences } (s_i)_{i=1}^{\infty} \subset \mathbb{C}_+ \text{ and } |s_i| \to \infty$$

$$\liminf_{i\to\infty} |\det(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)(s_i)| > 0$$

this, together with (A8), imply $\inf_{s\in\mathbb{C}_+}|\det(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)(s)| > 0$

Hence $(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)^{-1} \in \hat{\mathcal{A}}^{n\times n}$ and in view of (A7), so is $(I+\hat{G}_1\hat{G}_2)^{-1}$.

The fact that $(I+\hat{G}_2\hat{G}_1)^{-1} \in \hat{\mathcal{A}}^{n\times n}$ follows immediately by observing

$$\begin{aligned}(I+\hat{G}_2\hat{G}_1)^{-1} &= I - \hat{G}_2(I+\hat{G}_1\hat{G}_2)^{-1}\hat{G}_1 \\ &= I - \hat{\mathcal{N}}_2(\hat{\mathcal{D}}_1\hat{\mathcal{D}}_2+\hat{\mathcal{N}}_1\hat{\mathcal{N}}_2)^{-1}\hat{\mathcal{N}}_1\end{aligned}$$

The last two conclusions together with assumption (a), imply that $\hat{H}_e \in \hat{\mathcal{A}}^{2n\times 2n}$.

Case 2: Suppose $\hat{G}_1$ has p.r.c.f. $(\hat{\mathcal{N}}_1, \hat{\mathcal{D}}_1)$ and $\hat{G}_2$ has p.l.c.f. $(\hat{\mathcal{N}}_2, \hat{\mathcal{D}}_2)$

The proof follows in the same manner as in Case 1 by interchanging subscripts 1 and 2 throughout.

Q.E.D.

Proof of Corollary 2.1

By Cramer's Rule, $(I+\hat{G}_1\hat{G}_2)^{-1} = \frac{\mathrm{adj}(I+\hat{G}_1\hat{G}_2)}{\det(I+\hat{G}_1\hat{G}_2)}$

Hence, for the conditions under consideration,

$$(I+\hat{G}_1\hat{G}_2)^{-1} \text{ is proper} \qquad \text{(A9)}$$

For $i = 1,2$, $\hat{G}_i \in \mathbb{R}(s)^{n\times n}$ and $\hat{G}_i$ is proper, hence $\hat{G}_i$ has a coprime factorization.

Now (A5), (A6) are valid; this together with (A9), imply that $(I+\hat{G}_1\hat{G}_2)^{-1}$ is exp. stable.

The fact that $(I+\hat{G}_2\hat{G}_1)^{-1}$ is also exp. stable follows in the same manner by interchanging subscripts 1 and 2 throughout.

These conclusions, together with the assumption, imply that $\hat{H}_e$ is exp. st.

Q.E.D.

Proof of Corollary 2.2

$$(1+\hat{g}_1\hat{g}_2)^{-1} = 1 - \hat{g}_1 \cdot \hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$$

By assumption, $\hat{g}_1, \hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ are proper.

Hence $(1+\hat{g}_1\hat{g}_2)^{-1}$ is proper (A18)

Suppose, for sake of contradiction, $(1+\hat{g}_1\hat{g}_2)^{-1}$ has a $\mathbb{C}_+$ pole, say, p. The assumptions of the corollary imply that $\hat{g}_1(p) = \hat{g}_2(p) = 0$, hence $(1+\hat{g}_1\hat{g}_2)^{-1}(p) = 1$, which is a contradiction. Hence $(1+\hat{g}_1\hat{g}_2)^{-1}$ has no

$\mathbb{C}_+$ pole.

This, together with (A18) imply that

$$(1+\hat{g}_1\hat{g}_2)^{-1} = (1+\hat{g}_2\hat{g}_1)^{-1} \text{ is exp. st.}$$

This, combined with the assumptions of the corollary guarantee that $\hat{H}_e$ is exp. st.

Q.E.D.

Lemma.

Let $\hat{g}_1, \hat{g}_2$ be meromorphic functions mapping $\mathbb{C}_+$ into $\mathbb{C}$.
If $\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1}$ and $\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ are bounded on $\mathbb{C}_+$ then

$$\inf_{s\in\mathbb{C}_+} |(1+\hat{g}_1\hat{g}_2)(s)| > 0.$$

Proof:

Suppose, for sake of contradiction, $\inf_{s\in\mathbb{C}_+}|(1+\hat{g}_1\hat{g}_2)(s)| = 0$

$\therefore \exists$ a sequence $(s_i)_{i=1}^{\infty}$ in $\mathbb{C}_+ \ni$

$$|(1+\hat{g}_1\hat{g}_2)(s_i)| \to 0 \text{ as } i \to \infty \tag{A10}$$

But $\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ is bounded on $\mathbb{C}_+$, hence

$$\therefore \quad \hat{g}_2(s_i) \to 0 \text{ as } i \to \infty$$

Similarly from $\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1} = \hat{g}_1(1+\hat{g}_1\hat{g}_2)^{-1}$, we have

$$\hat{g}_1(s_i) \to 0 \quad \text{as } i \to \infty$$

Hence $|(1+\hat{g}_1\hat{g}_2)(s_i)| \to 1$ as $i \to \infty$

which contradicts (A10). Hence the proof is complete.

Q.E.D.

Proof of Corollary 2.3

Suppose $\hat{g}_1, \hat{g}_2$ have p.c.f. (n_1,d_1), (n_2,d_2), respectively

$$(1+\hat{g}_1\hat{g}_2)^{-1} = (1+\hat{g}_2\hat{g}_1)^{-1} = \frac{d_1d_2}{n_1n_2+d_1d_2} \qquad \text{(A11)}$$

$$\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1} = \frac{n_1d_2}{n_1n_2+d_1d_2} \qquad \text{(A12)}$$

$$\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1} = \frac{n_2d_1}{n_1n_2+d_1d_2} \qquad \text{(A13)}$$

The assumption implies $\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1}$, $\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ are bounded on $\mathbb{C}_+$.

Hence, by lemma,

$$\inf_{s\in\mathbb{C}_+}|(1+\hat{g}_1\hat{g}_2)(s)| = \inf_{s\in\mathbb{C}_+}\left|\left(\frac{n_1n_2+d_1d_2}{d_1d_2}\right)(s)\right| > 0 \qquad \text{(A14)}$$

By definition of p.c.f.,

for $i = 1,2$, n_i, d_i have no common $\mathbb{C}_+$ zero. (A15)

$\hat{g}_1(1+\hat{g}_2\hat{g}_1)^{-1}$ is bounded on $\mathbb{C}_+$, hence by (A12) and (A15),

n_2, d_1 have no common $\mathbb{C}_+$ zero. (A16)

Similarly,

$\hat{g}_2(1+\hat{g}_1\hat{g}_2)^{-1}$ is bounded on $\mathbb{C}_+$, hence by (A13) and (A15)

n_1, d_2 have no common $\mathbb{C}_+$ zero. (A17)

(A15), (A16) and (A17) imply

n_1n_2, d_1d_2 have no common $\mathbb{C}_+$ zero.

$\therefore$ $n_1n_2 + d_1d_2$, d_1d_2 have no common $\mathbb{C}_+$ zero.

Hence, it follows from (A14) that $\inf_{s\in\mathbb{C}_+}|(n_1n_2+d_1d_2)(s)| > 0$.

$\therefore \quad (n_1 n_2 + d_1 d_2)^{-1} \in \hat{\mathcal{A}}$ and in view of (A11)

so is $(1+\hat{g}_1\hat{g}_2)^{-1} = (1+\hat{g}_2\hat{g}_1)^{-1}$

This together with the assumption, imply that $\hat{H}_e \in \hat{\mathcal{A}}^{2\times 2}$

Q.E.D.

Proof of Theorem 3. Since

$$(I+\hat{G}_1\hat{G}_2)^{-1} = I - \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1} \cdot \hat{G}_2 \tag{A20}$$

we have $(I+\hat{G}_1\hat{G}_2)^{-1} \in \hat{\mathcal{A}}^{n\times n}$. Consequently, we also have $\hat{G}_2 \cdot (I+\hat{G}_1\hat{G}_2)^{-1} \in \hat{\mathcal{A}}^{n\times n}$. Since

$$(I+\hat{G}_2\hat{G}_1)^{-1} = I - \hat{G}_2 \cdot \hat{G}_1(I+\hat{G}_2\hat{G}_1)^{-1} \tag{A21}$$

we have $(I+\hat{G}_2\hat{G}_1)^{-1} \in \hat{\mathcal{A}}^{n\times n}$. Hence

$$\hat{H}_e \in \hat{\mathcal{A}}^{2n\times 2n}$$

Q.E.D.

Proof of Corollary 3.1

Copy the proof of Theorem 3 and replace " $\in \hat{\mathcal{A}}^{n\times n}$" and "$\in \hat{\mathcal{A}}^{2n\times 2n}$" by "is exp. st.".

Q.E.D.