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A MARTINGALE APPROACH TO QUEUES

by

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Abstract

This paper explores an approach to queueing problems using the recently developed calculus of martingales. Specific results developed in this paper include a general formula or virtual waiting time, and the formulation of an optimal control problem on queues. A problem of optimal control with quadratic cost is solved.

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1. Introduction

Standard approaches to problems in queues and congestion have been primarily analytical rather than stochastic. By this, we mean that the analysis is focused at a very early stage on distributions and averages rather than on the processes themselves. Recent advances in martingales theory[1-4] have made available a stochastic calculus for jump processes, which in turn makes possible a new approach to queueing problems emphasizing the underlying processes.[5]

In this paper we shall introduce this approach by using it on a number of relatively simple problems in queues. We should emphasize that the main advantage of this approach is not so much in getting closed-form solutions as in allowing more general problems to be formulated and analyzed. In particular, problems involving feedback can be formulated with ease, thus allowing optimal control for waiting line problems to be considered in a natural way.

2. Martingales and Poisson Process

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing family of sub-σ-fields. A stochastic process $\{X_t, t \geq 0\}$ is said to be adapted to $\{\mathcal{F}_t, t \geq 0\}$ if for each t X_t is $\mathcal{F}_t$-measurable. We say $\{X_t, \mathcal{F}_t, t \geq 0\}$ is a martingale if

$$(2.1) \qquad E(X_{t+s} | \mathcal{F}_t) = X_t \qquad \text{with probability 1}$$

for every t, s ≥ 0. It will be convenient to assume all processes to be right continuous.

We say $\{N_t, t \geq 0\}$ is a counting process if $N_0 = 0$, and N is constant except for a finite number of jumps of size 1 in each finite interval.

We say $\{N_t, \mathcal{F}_t, t \geq 0\}$ is a <u>standard Poisson process</u> if N is a counting process and $\{N_t - t, \mathcal{F}_t, t \geq 0\}$ is a martingale.

A process $\{f_t, t \geq 0\}$ is said to be adapted to $\{\mathcal{F}_t\}$ if f_t is $\mathcal{F}_t$-measurable for every t. It is said to be $\{\mathcal{F}_t\}$ predictable if as an (ω,t) function it is measurable with respect to the σ-field of (ω,t) sets generated by all left-continuous and adapted processes. Let $\{f_t, t \geq 0\}$ be an integrable $\{\mathcal{F}_t\}$ predictable process. Then

$$X_t = \sum_{s \leq t} f_s - \int_0^t f_s \, ds \tag{2.2}$$

where the summation is taken over the jumps of N, is an $\{\mathcal{F}_t\}$ martingale. It is convenient to introduce the martingale

$$q_t = N_t - t \tag{2.3}$$

and write (2.2) as a stochastic integral

$$X_t = \int_0^t f_s \, dq_s \tag{2.4}$$

The martingale property of X then follows immediately from the definition of stochastic integrals.

One can easily verify that the martingale definition of a Poisson process is consistent with the usual definition of a Poisson process. For example, write

$$e^{uN_t} - 1 = \sum_{s \leq t} \left(e^{uN_s} - e^{uN_{s-}}\right)$$

$$= \sum_{s \leq t} e^{uN_{s-}} \left(e^{u} - 1\right)$$

$$= \int_0^t e^{uN_{s-}} \left(e^{u} - 1\right) dq_s + \int_0^t e^{uN_{s-}} \left(e^{u} - 1\right) ds$$

If we define $F(u,t) = Ee^{uN_t}$ then we get

$$F(u,t) - 1 = \left(e^{u} - 1\right) \int_0^t F(u,s-)ds$$

which yields the generating function

$$F(u,t) = e^{(e^{u} - 1)t} = \bar{e}^{t} \sum_{n=o}^{\infty} e^{un} \frac{t^n}{n!}$$

3. A Model for Single-Server First-Come-First-Served Queues

Let X_t, Y_t, $t \geq 0$ be a pair of independent standard Poisson processes. We assume that both X and Y are right continuous and denote by $\mathcal{F}_{xt}$, $\mathcal{F}_{yt}$ and $\mathcal{F}_t$ the respective σ-fields generated by $\{X_s, s \leq t\}$, $\{Y_s, s \leq t\}$ and the two together.

Consider a single-server queue with Poisson arrivals at rate 1, and with exponentially distributed service times, again at unit rate. Let the system begin at t = 0 with no one in the system, and let Z_t denote the total number of customers (waiting and being served) in the system at time t. The process Z can now be represented in terms of a pair of independent standard Poisson processes (X,Y) as follows: Let X represent the arrival process so that the positive jumps of Z are given by the jumps of X. Similarly, the negative jumps of Z can be represented by a Poisson process Y except when Z is zero. Hence,

$$(3.1) \qquad Z_t = X_t - \int_0^t 1(Z_{s-})dY_s$$

where $1(z) = 1$ or 0 according as z is greater than 0 or equal to 0. More generally, let $f(t,z)$ be continuously differentiable in t, and let $\dot{f}(t,z) = \frac{\partial}{\partial t} f(t,z)$. Then

$$f(t,Z_t) - f(0,0) = \int_0^t \dot{f}(s,Z_s)ds + \sum_{s \leq t} [f(s,Z_s) - f(s,Z_{s-})]$$

when the sum is taken over all jumps of Z. Separating the positive and negative jumps, we get

$$(3.2) \qquad f(t,Z_t) - f(0,0) = \int_0^t \dot{f}(s,Z_s)ds + \int_0^t [f(s,Z_{s-} + 1) - f(s,Z_{s-})]dX_s$$

$$+ \int_0^t [f(s,Z_{s-}-1) - f(s,Z_{s-})]1(Z_{s-})dY_s$$

Although the integrals in (3.2) can be interpreted as Stieltjes integrals, they are more usefully interpreted as stochastic integrals. If we introduce the martingales

$$q_{xt} = X_t - t$$

and

$$q_{yt} = Y_t - t$$

then (3.2) can be rewritten as

$$(3.3) \qquad f(t,Z_t) - f(0,0) = \int_0^t \dot{f}(s,Z_s)ds + \int_0^t [f(s,Z_{s-}+1) - f(s,Z_{s-})]ds$$

$$+ \int_0^t [f(s,Z_{s-}-1) - f(s,Z_{s-})]1(Z_{s-})ds$$

$$+ \int_0^t [f(s,Z_{s-}+1) - f(s,Z_{s-})]dq_{xs}$$

$$+ \int_0^t [f(s,Z_{s-}-1) - f(s,Z_{s-})]1(Z_{s-})dq_{ys}$$

where the last two integrals yield martingales.

If we take $f(Z_t) = e^{\alpha Z_t}$ and $G(\alpha,t) = Ef(Z_t)$, then (3.3) immediately yields

$$G(\alpha,t) - 1 = (e^{\alpha} - 1)\int_0^t G(\alpha,s)ds + (e^{-\alpha} - 1)\int_0^t G(\alpha,s)ds$$

$$- (e^{-\alpha} - 1)\int_0^t P_0(s)ds$$

where $P_0(s) = \text{Prob}(Z_s = 0)$. The unknown $P_0(t)$ can be determined by requiring that $G(\ell nz,t)$ be analytic on the open disk $|z| < 1$. This procedure yields the generating function for Z_t, which is well known and of no particular interest in this paper.

The process Z_t defined by (3.1) can be viewed as the standard queueing process from which other queueing process considered in this paper will be derived by a transformation of the probability measure.

4. Transformation of Probability

The objective here is to define a queueing process corresponding to general arrival and service rates. We shall do this via the standard process introduced in the last section.

Let $(\Omega, \mathcal{F}, P_0)$ be a probability space on which a pair of independent standard Poisson processes (X,Y) is defined. Let $\mathcal{F}_t = \sigma(X_s, Y_s, s \leq t)$. A process ϕ_t is said to be adapted to $\{\mathcal{F}_t\}$ if for every t ϕ_t is $\mathcal{F}_t$-measurable. A process $\{\phi_t, t \geq 0\}$ is said to be predictable (w.r.t. $\{\mathcal{F}_t\}$) if as an (ω, t) function it is measurable with respect to the σ-field generated by all left-continuous adapted processes.

A positive random variable τ is said to be a stopping time (of $\{\mathcal{F}_t\}$) if $\{\omega: \tau(\omega) \leq t\}$ is $\mathcal{F}_t$-measurable for every t. Let $L_{\ell oc}$ denote the set of all predictable processes ϕ for which an increasing sequence of stopping times τ_n exists such that $\lim_{n\uparrow\infty} \tau_n = \infty$ a.s. and for each n

$$E \int_0^{\tau_n} |\phi_t| dt < \infty$$

Let λ and μ be two positive processes in $L_{\ell oc}$ and define

$$(4.1) \qquad \Lambda_t = \prod_{\tau \leq t} \lambda_\tau \prod_{s \leq t} \mu_s \exp\{-\int_0^t (\lambda_s + \mu_s - 2) ds\}$$

where τ and s denote the times of jump for the two processes X and Y. From the work of Doleans-Dade [5], we know that Λ_t is the unique solution to the integral equation

$$(4.2) \qquad \Lambda_t = 1 + \int_0^t \Lambda_{s-}[(\lambda_s - 1)(dX_s - ds) + (\mu_s - 1)(dY_s - ds)]$$

$$= 1 + \int_0^t \Lambda_{s-}[(\lambda_s - 1)dq_{xs} + (\mu_s - 1)dq_{ys}]$$

which implies that $\{\Lambda_t, \mathcal{F}_t\}$ is a local martingale, i.e., there exists an increasing sequence of stopping time τ_n such that $\tau_n \uparrow \infty$ and for each n $\{\Lambda_{t\wedge\tau_n}, \mathcal{F}_t\}$ is a martingale.

Let $\Lambda = \lim_{t\to\infty} \Lambda_t$ and assume $E_o\Lambda = 1$. Then we can define a new probability measure $\mathcal{P}$ via the transformation

$$\frac{d\mathcal{P}}{d\mathcal{P}_o} = \Lambda \tag{4.3}$$

in which use $\Lambda_t = E_o(\Lambda | \mathcal{F}_t)$ are just the likelihood ratios corresponding to $\mathcal{F}_t$. Brémaud [1] first suggested such a transformation as a means for constructing generalized Poisson process, and he showed that under the $\mathcal{P}$-measure

$$X_t - \int_0^t \lambda_s ds$$

and

$$Y_t - \int_0^t \mu_s ds$$

are local martingales. This justifies the interpretation of λ and μ as rates for the processes X and Y under $\mathcal{P}$.

Once X and Y are constructed, (3.1) again generates a queueing process Z with X and Y representing the arrival and service processes respectively. Since the arrival rate λ and the service rate μ are allowed to be any $L_{\ell oc}$ process for which $E_o\Lambda_\infty = 1$, we have defined a queueing process of considerable generality. In particular, the fact that λ_t and μ_t can depend on $\{Z_s, s \le t\}$ allows control problems to be considered in a natural way.

Let $\mathcal{F}_{zt} = \sigma(Z_s, s \le t)$ and denote

(4.4) $$L_t = E_o(\Lambda_t | \mathcal{F}_{zt})$$

It was shown in [2] that L satisfies the integral equation

(4.5) $$L_t = 1 + \int_0^t L_{s-} \, [(\hat{\lambda}_{s-} - 1)dq_{xs} + (\hat{\mu}_{s-} - 1)1(Z_{s-})dq_{ys}]$$

where

(4.6) $$\hat{\lambda}_t = E(\lambda_t | \mathcal{F}_{zt})$$

and

(4.7) $$\hat{\mu}_t = E(\mu_t | \mathcal{F}_{zt})$$

are predictable processes by construction.

If $\hat{\lambda}_t$ and $\hat{\mu}_t$ are known functions of $\{Z_s, s \leq t\}$, then L_t is a known function of $\{Z_s, s \leq t\}$. Indeed, by the formula of Doléans-Dade, we have

(4.8) $$L_t = \prod_{\tau,\sigma} \hat{\lambda}_\tau \hat{\mu}_\sigma \exp \{-\int_0^t [(\hat{\lambda}_s - 1) + (\hat{\mu}_s - 1)1(Z_{s-})]ds\}$$

where τ and σ denote the times of the positive and negative jumps of Z respectively. Given any $\mathcal{F}_{zt}$-measurable random variable ϕ, the formula

$$E\phi = E_o L_t \phi$$

provides, at least in principle, a means of computing $E\phi$ by using the known distribution of Z relative to the $\mathcal{P}_o$ measure.

Alternatively, (4.5) can be combined with (3.2) to yield

(4.9) $$f(Z_t)L_t - f(0) = \int_0^t L_{s-} \; \hat{\lambda}_s[f(Z_{s-} + 1) - f(Z_{s-})]dq_{xs}$$

$$+ \hat{\mu}_s[f(Z_{s-} - 1) - f(Z_{s-})]dq_{ys}$$

$$+ \int_0^t L_{s-} \{\hat{\lambda}_s[f(Z_{s-} + 1) - f(Z_{s-})]$$

$$+ \hat{\mu}_s[f(Z_{s-} -1) - f(Z_{s-})]1(Z_{s-})\}ds$$

which in turn yields the formula

$$\text{(4.10)} \quad Ef(Z_t) = f(0) + \int_0^t E\{\hat{\lambda}_s[f(Z_{s-} + 1) - f(Z_{s-})] + \hat{\mu}_s[f(Z_{s-} - 1) - f(Z_{s-})]1(Z_{s-})\}ds$$

If $\hat{\lambda}_s$ and $\hat{\mu}_s$ are functions of only Z_{s-} then (4.10) gives rise to the Kolmogorov differential-difference equation usually encountered in the queueing literature.

5. A Formula on Virtual Waiting Time

Let η_t be the time that a customer would wait before he is served if he joins the queue at time t. η_t is called the virtual waiting time [6]. Suppose that at time t there are n persons in the system. Then η_t is of the form

$$\text{(5.1)} \quad \eta_t = \sum_{j=1}^{n-1} \tau_j + \tau_n - (t - t_n)$$

where $\tau_1, \tau_2, \cdots, \tau_n$ are the service times of the n persons in the system at t, and t_n is the time at which service began for the person being served at t. A sample function of η is illustrated in Figure 1. The jumps of η occur at the arrivals, i.e., the jumps of X. Between jumps η_t decays with slope - 1. Therefore, we can write

(5.2) $$\eta_t = \sum_{s \leq t} \alpha_{s-} - \int_0^t 1(Z_{s-})\,ds$$

where α_s is the service time for one arriving at s and the summation is taken over the jumps of X. It is tempting to write (5.2) as

(5.3) $$\eta_t = \int_0^t \alpha_{s-}\,dX_s - \int_0^t 1(Z_{s-})\,ds$$

However, α_t is not $\mathcal{F}_t$-measurable and it is not clear that the first integral in (5.3) can be interpreted as a stochastic integral. Of course, it can always be interpreted as a Stieltjes integral (which is just (5.21)) this does not make available the martingale calculus for computing $E\eta_t$.

<u>Theorem 5.1</u> Let $G_t = \mathcal{F}_{y\infty} \vee \mathcal{F}_{xt}$. Let $\mathcal{P}_o$ and $\mathcal{P}$ be defined by (4.3) and (4.2). Suppose that: (a) μ_t is ${}_{yt}$-measurable, and (b) $E\alpha_{t-}\lambda_t < \infty$. Then η is a $(\mathcal{P},\{G_t\})$ martingale and

(5.4) $$E\eta_t = \int_0^t P_0(s)\,ds - t + \int_0^t E(\alpha_{s-}\lambda_s)\,ds$$

where $P_0(t) = \mathcal{P}(\{\omega:\ Z_t(\omega) = 0\})$

Remark (a) Equation (5.4) generalizes similar formulas derived under more restrictive conditions. (See e.g., [6].)

(b) The condition that μ_t is $\mathcal{F}_{yt}$-measurable is a non-trivial condition. It means that the service rate μ_t cannot depend on the arrivals. In particular, this condition excludes the case where μ involves feedback so that μ_t depends on $\{Z_s, s \leq t\}$.

Proof: First, we observe that α_t is the time interval between the Z_t - th and $(Z_t + 1)$ - th jumps of Y after t. Hence, α_t is G_t-predictable. Under $\mathcal{P}_o$, X and Y are independent standard Poisson processes, so that $(X_t - t)$ is a $(\mathcal{P}_o, G_t)$ martingale.

Now, $\mathcal{P}$ is defined by (4.3), and we can write

$$\frac{d\mathcal{P}}{d\mathcal{P}_o} = \Lambda = \Lambda_{x\infty}\Lambda_{y\infty}$$

where

$$\Lambda_{xt} = \exp\{\int_0^t [\ln\lambda_s dX_s - (\lambda_s - 1)ds]\}$$

and

$$\Lambda_{yt} = \exp\{\int_0^t [\ln\mu_s dY_s - (\mu_s - 1)ds]\}$$

Under condition (a), $\Lambda_{y\infty}$ is G_t-measurable for all t. Hence,

$$g_t = E_o(\Lambda|G_t) = \Lambda_{y\infty}E_o(\Lambda_{x\infty}|G_t)$$

Because $\Lambda_{x\infty}$ satisfies

$$\Lambda_{x\infty} = 1 + \int_0^\infty \Lambda_{xs}(\lambda_s - 1)d(X_s - s)$$

and $(X_t - t)$ is a $(\mathcal{P}_o, G_t)$ martingale,

$$E_o(\Lambda_{x\infty}|G_t) = E_o(\Lambda_{x\infty}|\mathcal{F}_t) = \Lambda_{xt}$$

Therefore, g_t satisfies

$$g_t = \Lambda_{y\infty} \Lambda_{xt} = \Lambda_{y\infty} [1 + \int_0^t \Lambda_{xs} (\lambda_s - 1) d(X_s - s)]$$

$$= g_0 + \int_0^t g_s(\lambda_s - 1) d(X_s - s)$$

It follows from the results of VanSchuppen and Wong [7] that $X_t - \int_0^t \lambda_s ds$ is a $(\mathcal{P}, G_t)$ local martingale. Let (5.3) be rewritten as

$$Z_t = \int_0^t \alpha_{s-}(dX_s - \lambda_s ds) + \int_0^t \alpha_{s-}\lambda_s ds - \int_0^t 1(Z_{s-}) ds$$

The first integral is a $(\mathcal{P}, G_t)$ martingale. Therefore,

$$EZ_t = \int_0^t E(\alpha_{s-}\lambda_s) ds - \int_0^t E[1(Z_{s-})] ds$$

and (5.4) follows.

6. Optimal Control for Queueing Processes

In this section we shall consider the problem of controlling a queue by observing the past of the queue-length and by varying the service rate. For simplicity we will consider the case of a constant arrival rate, but generalization to any Markovian arrival process poses no difficulty. Related results have also been obtained by Boel and Varaiya [8].

Let $(\Omega, \mathcal{F}, \mathcal{P}_o)$ be a probability space. Let X, Y be a pair of processes representing the arrivals and the service processes respectively. Let Z be the queue-length process defined as in (3.1). We assume that under $\mathcal{P}_o$, X and Y are independent standard Poisson processes. Define

(6.1) $$\rho_s^t = \prod_{s \le t_i \le t} \lambda_{t_i-} \prod_{s \le s_j \le t} \mu_{s_j-} \exp[-\int_s^t (\lambda_\tau + \mu_\tau - 2)d\tau]$$

where $\lambda_t = \lambda$ has been assumed to be constant for $t \in [0,1]$.

Let $\mathcal{F}_{zt} = \sigma(Z_s, s \le t)$. A nonnegative process μ with value in $\Sigma \subset R$ is said to be an admissible control if it is $\mathcal{F}_{zt}$-predictable and satisfies

(6.2) $$E_o \rho_0^1(\mu) = 1$$

We denote the set of all admissible controls by $\mathcal{N}$. We say an admissible control is Markov if there exists a measurable function f such that

(6.3) $$\mu_t = f(t, Z_{t-})$$

We denote the set of all Markov controls by $\mathcal{M}$.

If μ is an admissible control then we can define a probability measure $\mathcal{P}_\mu$ by

(6.4) $$\frac{d\mathcal{P}_\mu}{d\mathcal{P}_o} = \rho_0^1(\mu)$$

The cost is then given by

(6.5) $$J(\mu) = E_\mu \int_0^1 c(s, Z_{s-}, \mu_s) ds$$
$$= E_o[\rho_0^1 \int_0^1 c(s, Z_{s-}, \mu_s) ds]$$

The control problem is to find an admissible μ^* such that

$$J(\mu^*) = \inf_{\mu \in \mathcal{N}} J(\mu)$$

First, we shall try to determine an optimal Markov control. Define

$$(6.6)\quad W_t = \inf_{\mu \in \mathcal{M}} E_\mu [\int_t^1 c(s,Z_{s-},\mu_s)ds \mid \mathcal{F}_{zt-}]$$

Since $\frac{d\mathcal{P}_\mu}{d\mathcal{P}_o} = \rho_0^1(\mu)$, we have

$$(6.7)\quad W_t = \inf_{\mu \in \mathcal{M}} \frac{E_o\{\rho_0^1(\mu)\int_t^1 c(s,Z_{s-},\mu_s)ds \mid \mathcal{F}_{zt-}\}}{E_o\{\rho_0^1(\mu) \mid \mathcal{F}_{zt-}\}}$$

$$= \inf_{\mu \in \mathcal{M}} E_o\{\rho_t^1(\mu)\int_t^1 c(s,Z_{s-},\mu_s)ds \mid \mathcal{F}_{zt-}\}$$

Since μ is Markov in (6.7) and Z is Markov under $\mathcal{P}_o$, W_t is a function of Z_{t-} and not of its past, i.e.,

$$(6.8)\quad W_t = V(t,Z_{t-})$$

From (6.6) we can write

$$V(t,Z_{t-}) = \inf_{\mu \in \mathcal{M}} E_\mu\{\int_t^{t+h} c(s,Z_{s-},\mu_s)ds + \int_{t+h}^1 c(s,Z_{s-},\mu_s)ds \mid \mathcal{F}_{zt-}\}$$

$$= \inf_{\mu \in \mathcal{M}} E_\mu\{\int_t^{t+h} c(s,Z_{s-},\mu_s)ds + V(t+h,Z_{t+h}) \mid \mathcal{F}_{zt-}\}$$

or

$$(6.9)\quad \inf_{\mu \in \mathcal{M}} E_\mu\{[V(t+h,Z_{t+h}) - V(t,Z_{t-})] + \int_t^{t+h} c(s,Z_{s-},\mu_s)ds \mid \mathcal{F}_{zt-}\} = 0$$

The differentiation rule (3.2) now yields

$$V(t+h,Z_{t+h}) - V(t,Z_{t-}) = \int_t^{t+h} (\mathcal{L}_\mu V)(s,Z_{s-})ds + (M^\mu_{t+h} - M^\mu_t) \tag{6.10}$$

where we have adopted the notation

$$(\mathcal{L}_\mu V)(t,z) = \frac{\partial}{\partial t} V(t,z) + \lambda[V(t,z+1) - V(t,z)] + \mu_t[V(t,z-1) - V(t,z)]1(z) \tag{6.11}$$

and

$$M^\mu_t = \int_0^t [V(s,Z_{s-} + 1) - V(s,Z_{s-})](dX_s - \lambda ds) + \int_0^t [V(s,Z_{s-} - 1) - V(s,Z_{s-})]1(Z_{s-})(dY_s - \mu_s ds)$$

We note that M^μ_t is a local martingale respect to $(\mathcal{P}_\mu, \mathcal{F}_{zt})$. Therefore, using (6.10) in (6.9), we get formally

$$\inf_{\mu \in \mathcal{M}} E_\mu\{\int_t^{t+h} [(\mathcal{L}_\mu V)(s,Z_{s-}) + c(s,Z_{s-},\mu_s)]ds | \mathcal{F}_{zt-}\} = 0$$

or

$$\inf_{\mu \in \mathcal{M}} \{(\mathcal{L}_\mu V)(t,Z_{t-}) + c(t,Z_{t-},\mu_t) + E_\mu \frac{1}{h}\int_t^{t+h} [(\mathcal{L}_\mu V + c_\mu)(s,Z_{s-}) - (\mathcal{L}_\mu V + c_\mu)(t,Z_{t-})]ds\}$$

Hence, continuity of $\mathcal{L}_\mu V + c_\mu$ yields the Hamilton-Jacobi equation

$$\inf_{\mu \in \mathcal{M}} \{(\mathcal{L}_\mu V + c_\mu)(t,z)\} = 0 \tag{6.13}$$

It turns out that if (6.13) has a solution then it is not only an optimal Markov control but optimal in general.

Theorem 6.1 Suppose that $V(z,t)$ satisfies

$$\mathcal{L}_v V(t,z) + c(t,z,v) \geq 0 \text{ for all } v \in \Sigma$$

$$\mathcal{L}_v V(t,z) + c(t,z,v) = 0 \text{ for } v = \mu^*(t,z) \tag{6.14}$$

$$V(1,z) = 0$$

where c is a non-negative function. Then, $\mu^*(t,Z_{t-})$ yields an optimal control, i.e.,

$$J(\mu^*) = \min_{\mu \in \mathcal{N}} J(\mu) = V(0,z)$$

Proof: Let J_M denote $V(o,z)$. Then

$$- J_M = E_\mu \int_0^1 dV(s,Z_{s-})$$

$$= E_\mu\{\int_0^1 (\mathcal{L}_\mu V)(s,Z_{s-})ds + (M_1^\mu)\}$$

or

$$J_M = - E_\mu\{\int_0^1 \mathcal{L}_\mu V(s,Z_{s-})ds + M_1^\mu\}$$

Using (6.14) and the fact that M^μ is a local martingale, we get

$$- E_\mu\{\int_0^{1\wedge\tau_n} \mathcal{L}_\mu V(s,Z_{s-})ds + M_{1\wedge\tau_n}^\mu\}$$

$$= - E_\mu\left[\int_0^{1\wedge\tau_n} \mathcal{L}_\mu V(s,Z_{s-})\,ds\right] \leq E_\mu\left[\int_0^{1\wedge\tau_n} c(s,Z_{s-},\mu_s)\,ds\right]$$

for an increasing sequence of stopping time $\tau_n \uparrow \infty$. Letting $n \to \infty$ yields

$$J_M \leq E_\mu \int_0^1 c(s,Z_{s-1}\mu_s)\,ds$$

with equality for $\mu = \mu^*$. q.e.d.

7. An Example

Consider a cost function which is quadratic in the control, viz.,

$$c(t,z,\mu) = \frac{\mu^2}{\lambda^2} + f(t,z) \tag{7.1}$$

Equation (6.14) becomes for this case

$$\frac{\partial}{\partial t} V(t,z) + \lambda[V(t,z+1) - V(t,z)] + f(t,z)$$

$$+ \min\{v[V(t,z-1) - V(t,z)]\, 1(z) + \frac{v^2}{\lambda^2}\} = 0$$

which yields

$$\mu^*(t,z) = -\frac{1}{2}\lambda^2[V(t,z-1) - V(t,z)]1(z)$$

and V must satisfy the differential equation

$$\frac{\partial}{\partial t} V(t,z) + \lambda[V(t,z+1) - V(t,z)] + f(t,z) - \frac{1}{4}\lambda^2 1(z)\, [V(t,z-1) - V(t,z)]^2 = 0 \tag{7.2}$$

with $V(1,z) = 0$.

We observe that if

$$f(t,z) = a(t)z^2 + b(t)z + c(t) + 1(z)d(t)$$

then (7.2) yields a solution of the form

$$V(t,z) = \alpha(t)z^2 + \beta(t)z + \gamma(t)$$

provided that d(t) bears a certain relationship to a and b. Specifically, by equating like terms, we get

$$\dot{\alpha} - \lambda^2\alpha^2 + a = 0$$

$$\dot{\beta} - \lambda^2\alpha\beta + (b-1) + (\alpha\lambda+1)^2 = 0$$

$$\dot{\gamma} + \lambda\beta + c = 0$$

$$d = \frac{1}{4}\lambda^2(\beta-\alpha)^2$$

The first two equations can be solved for α and β in terms of a and b. The next two equations determine γ and d. The optimal control is then given by

$$\mu\ (t,z) = \frac{1}{2}\lambda^2 1(z)[2\alpha z + (\beta-\alpha)]$$

For example, suppose that a = 1, b = c = 0. Then we find

$$\alpha(t) = -\frac{1}{\lambda}\tanh\lambda(t-1)$$

$$\beta(t) = \int_t^1 \frac{\cosh\lambda(t'-1)}{\cosh\lambda(t-1)}[\lambda^2\alpha^2(t') + 2\lambda\alpha(t')]dt'$$

$$= -\frac{1}{\lambda \cosh \lambda(t-1)} \{\sinh \lambda(t-1) - \tan^{-1}[\sinh \lambda(t-1)] + 2 - 2\cosh \lambda(t-1)\}$$

$$\gamma(t) = \int_t^1 \lambda\beta(t')dt'$$

$$= \frac{1}{2\lambda} \ln[\cosh \lambda(t-1)] - \frac{1}{2\lambda} [\tan^{-1} \sinh \lambda(t-1)]^2 + \frac{2}{\lambda} \tan^{-1} \sinh \lambda(t-1) + 2(1-t)$$

and

$$d(t) = \frac{\lambda^2}{4} (\beta-\alpha)^2$$

$$= \frac{1}{4\cosh^2 \lambda(t-1)} [\tan^{-1}(\sinh \lambda(t-1)) + 2\cosh \lambda(t-1)-2]^2$$

If the average number of arrivals in the interval [0,1] is large, say $\lambda \geq 10$, then $d(t)$ is nearly zero throughout the interval [0,1], which means that the control

$$\mu^v(t,z) = \frac{\lambda^2}{2} \lambda^2 1(z)[2\alpha z + (\beta-\alpha)]$$

is very nearly optimal for the quadratic cost function

$$c(t,z,\mu) = \mu^2 + z^2$$

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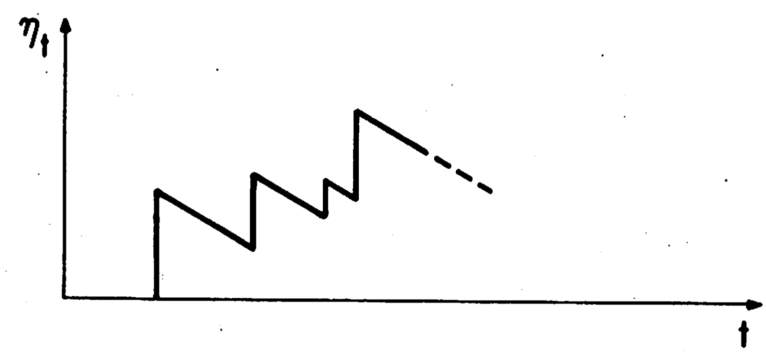
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